

# A GENERAL FRAMEWORK FOR HOMOTOPIC DESCENT AND CODESCENT

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ABSTRACT. In this paper we elaborate a general homotopy-theoretic framework in which to study problems of descent and completion and of their duals, codescent and cocompletion. Our approach to homotopic (co)descent and to derived (co)completion can be viewed as  $\infty$ -category-theoretic, as our framework is constructed in the universe of simplicially enriched categories, which are a model for  $(\infty, 1)$ -categories.

We provide general criteria, reminiscent of Mandell's theorem on  $E_\infty$ -algebra models of  $p$ -complete spaces, under which homotopic (co)descent is satisfied. Furthermore, we construct general descent and codescent spectral sequences, which we interpret in terms of derived (co)completion and homotopic (co)descent.

We prove that Baum-Connes and Farrell-Jones-type isomorphism conjectures for assembly can be expressed in the language of derived cocompletion and show that a number of very well-known spectral sequences, such as the unstable and stable Adams spectral sequences, the Adams-Novikov spectral sequence and the descent spectral sequence of a map, are examples of general (co)descent spectral sequences. There is also a close relationship between the Lichtenbaum-Quillen conjecture and homotopic descent along the Dwyer-Friedlander map from algebraic K-theory to étale K-theory.

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## 1. INTRODUCTION

The notions of descent and completion have long played a significant role in algebraic geometry, number theory, category theory and homotopy theory. In this paper we elaborate a general homotopy-theoretic framework in which to study problems of descent and completion and of their duals, codescent and cocompletion. Our approach to homotopic (co)descent and to derived (co)completion can be viewed as  $\infty$ -category-theoretic, as our framework is constructed in the universe of simplicially enriched categories, which are a model for  $(\infty, 1)$ -categories (or  $\infty$ -categories, as homotopy theorists often call them).

We provide general criteria, reminiscent of the Main Theorem in [17], under which homotopic (co)descent is satisfied (Theorems 5.3 and 5.8). Furthermore, we construct general descent and codescent spectral sequences, which we interpret in terms of derived (co)completion and homotopic (co)descent.

To illustrate the breadth and flexibility of the framework built here, we prove that Baum-Connes and Farrell-Jones-type isomorphism conjectures for assembly can be expressed in the language of derived cocompletion (Corollary 4.32). We show moreover that a number of very well-known spectral sequences, such as the unstable and stable Adams spectral sequences (sections 5.3.1 and 6.1.1), the Adams-Novikov spectral sequence (section 6.1.2) and the descent spectral sequence of a map (section 6.2), are examples of general (co)descent spectral sequences. Finally, we describe the close relationship between the Lichtenbaum-Quillen conjecture and homotopic descent along the Dwyer-Friedlander map from algebraic K-theory to étale K-theory (section 6.1.3).

We begin in section 2 by recalling the framework of classical descent and codescent theory, expressed in terms of monads and comonads and of their associated Eilenberg-Moore categories of algebras and coalgebras (Definitions 2.3 and 2.7). We also remind the reader of important well-known examples: Grothendieck descent theory along a ring homomorphism and codescent along a continuous map, with its application to bundle theory. We generalize both of these examples to arbitrary monoidal categories, describing descent along a morphism of monoids and codescent along a morphism of comonoids. We emphasize the description of Grothendieck descent theory in terms of the descent co-ring associated to the fixed monoid morphism (Example 2.18) and its dual in terms of the codescent ring associated to the fixed comonoid morphism (Example 2.24).

Section 3 is devoted to the study of simplicial structures related to descent and codescent. We begin by recalling the well-known cobar construction associated to a monad and bar construction associated to a comonad, proving a technical homotopy-theoretic result about each of these constructions (Lemmas 3.3 and 3.7) required at two critical junctures later in the paper. We then discuss how simplicial enrichment of a category induces simplicial enrichment on associated Eilenberg-Moore categories of algebras and coalgebras and explore various properties of these induced enrichments. In particular, we prove that if a (co)monad is appropriately compatible with the simplicial enrichment of the category on which it acts, then the associated canonical (co)descent data functor (Definitions 2.11 and 2.14) is simplicial (Proposition 3.13), and certain of its components are actually isomorphisms (Proposition 3.15), which turns out to be of crucial importance in the next sections.

Derived completion along a monad and cocompletion along a comonad are the subjects of section 4. Given a simplicial model category  $\mathbf{M}$  on which a monad  $\mathbb{T}$  acts compatibly with the simplicial structure, we define notions of  $\mathbb{T}$ -equivalence and of  $\mathbb{T}$ -complete objects (Definitions 4.2 and 4.4), and provide examples. We define the  $\mathbb{T}$ -completion of an object in  $\mathbf{M}$  in terms of the totalization of the  $\mathbb{T}$ -cobar construction and prove its homotopy invariance. We then introduce the related notion of strongly  $\mathbb{T}$ -complete objects, which is also formulated in terms of the  $\mathbb{T}$ -cobar construction. We give examples and show that strongly  $\mathbb{T}$ -complete, fibrant objects are  $\mathbb{T}$ -complete. Finally, we point out that strongly  $\mathbb{T}$ -complete objects may be best understood as  $\infty$ - $\mathbb{T}$ -algebras, i.e., as  $\mathbb{T}$ -algebras, up to an infinite family of higher structure maps. To conclude the section, we dualize all of our theory of derived completion, to formulate a theory of derived cocompletion along a comonad.

In section 5 we are finally ready to define homotopic (co)descent for simplicially enriched categories (Definitions 5.1 and 5.6). In the case of a (co)monad acting on a simplicial model category, we provide criteria for homotopic (co)descent (Theorems 5.3 and 5.8), which are strongly analogous to the Main Theorem in Mandell's paper [17]. The criteria are formulated in terms of strong (co)completion, motivating our introduction of this notion. We then explain how to construct the (co)descent spectral sequence that arises naturally from the action of a (co)monad on a simplicial model category as special cases of the extended homotopy spectral sequence of Bousfield and Kan [4, X.6]. The notions of both derived (co)completion and homotopic (co)descent play a role in interpreting these spectral sequences, which we can view as interpolating backwards along the simplicial canonical (co)descent data functor.

The examples motivating our work are studied in section 6. We begin by developing the general theory of homotopic Grothendieck descent and of derived completion along a morphism of monoids in a monoidal model category, then sketch a number of intriguing concrete applications, which will be further developed in later articles. In particular, we indicate how both the stable Adams spectral sequence and the Adams-Novikov spectral sequence arise as descent spectral sequences associated to unit maps of ring spectra. Fixing some prime  $\ell$  and a noetherian  $\mathbb{Z}[\frac{1}{\ell}]$ -algebra  $A$ , we then describe the close relationship between the Lichtenbaum-Quillen conjecture for  $A$  and homotopic descent along the Dwyer-Friedlander map from the algebraic K-theory spectrum of  $A$  to its étale K-theory spectrum. Finally, we sketch an application of homotopic Grothendieck descent to the construction of a spectral sequence converging from Quillen homology of a ring spectrum to homotopy.

Turning all the arrows around, we then develop a theory of homotopic Grothendieck codescent and derived cocompletion along any morphism in a cartesian model category. We show in particular that the comonad associated to pulling back over a Kan fibration of simplicial sets admitting a section satisfies homotopic codescent.

In the appendices the reader will find the proof of Theorem 4.31, as well as a brief review of the various simplicial and model structures that come in handy in this article.

### 1.1. Perspectives.

- We have chosen in this paper to work with simplicial model categories, both because they are a tractable model for  $\infty$ -categories and because, as Rezk, Schwede and Shipley proved in [22], a large class of model categories are at least Quillen equivalent to simplicial model categories. It would, however, be very interesting to develop analogous theories of homotopic descent and codescent for dg-categories (categories enriched over chain complexes) and for spectral categories (categories enriched over spectra). We suspect that much of the theory would translate directly into either of these contexts and open the door to new and significant applications.
- It seems likely that many well-known (co)descent-type spectral sequences and descent-related notions, beyond those treated in section 6, can be expressed within the framework developed in this paper. For example, the homotopy fixed point spectral sequence, as set up by Davis in [6], is almost certainly a descent spectral sequence for an appropriate monad, while the étale cohomological descent spectral sequence developed by Jardine [15, §6.1] is most probably a special case of a codescent spectral sequence for a well-chosen comonad.
- Motivated by the close links between descent theory and Galois theory, we intend to explore the relationship between homotopic Hopf-Galois theory, as developed in [10], and homotopic descent theory.

### 1.2. Notation and conventions.

- Let  $\mathbf{C}$  be a category, and let  $A, B \in \text{Ob } \mathbf{C}$ . In these notes, the class of morphisms from  $A$  to  $B$  is denoted  $\mathbf{C}(A, B)$ . The identity morphism on an object  $A$  is usually denoted  $A$  as well.
- If  $C$  is an object of a category  $\mathbf{C}$ , then  $C/\mathbf{C}$  denotes the under category of morphisms with domain  $C$ , while  $\mathbf{C}/C$  denotes the overcategory of morphisms with codomain  $C$ .
- Arbitrary ordinary categories are denoted  $\mathbf{C}$  or  $\mathbf{D}$ , while simplicially enriched categories are called  $\mathbf{S}$  or  $\mathbf{S}'$ . Finally, we write  $\mathbf{M}$  or  $\mathbf{M}'$  for model categories, whether or not they are simplicial.
- If  $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$  is a pair of adjoint functors, then  $F$  (for “free”) is the left adjoint, while  $U$  (for “underlying”) is the right adjoint. We refer to this adjoint pair as the  $(F, U)$ -adjunction.
- Let  $\Delta$  denote the category with  $\text{Ob } \Delta = \mathbb{N}$ , where  $\Delta(m, n)$  is the set of order-preserving maps from  $\{0, \dots, m\}$  to  $\{0, \dots, n\}$ . The standard  $n$ -simplex is denoted  $\Delta[n]$ .

If  $\mathbf{C}$  is any other category, then  $\mathbf{C}^{\Delta^{op}}$  (respectively,  $\mathbf{C}^{\Delta}$ ) is the category of simplicial (respectively, cosimplicial) objects in  $\mathbf{C}$ . We write  $\mathbf{sSet}$  for  $\mathbf{Set}^{\Delta^{op}}$ .

- If  $X$  is an object in  $\mathbf{C}$ , then  $cc^{\bullet}X$  denotes the constant cosimplicial object that is  $X$  in each level, where all cofaces and codegeneracies are identity maps. Similarly,  $cs_{\bullet}X$  denotes the constant simplicial object that is  $X$  in each level.
- If  $\mathbf{M}$  is a model category, we always assume that the categories  $\mathbf{M}^{\Delta}$  and  $\mathbf{M}^{\Delta^{op}}$  are endowed with their Reedy model category structure [11, Ch 15]. The details of their Reedy structure that are necessary for this paper can be found in the appendix B.1.

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## 2. CLASSICAL DESCENT AND CODESCENT

Here we recall the classical theory of descent and codescent, as expressed in terms of monads and comonads. We take some care in doing so, to establish clearly notation and terminology that we need throughout the paper and to ensure that the reader understands exactly what we are generalizing when we define homotopic descent and codescent in section 5.

We refer the reader to [19] for further details.

### 2.1. Foundations of the theory of (co)descent.

#### 2.1.1. (Co)monads and their (co)algebras.

**Definition 2.1.** Let  $\mathbf{C}$  be a category. A *monad* on  $\mathbf{C}$  consists of an endofunctor  $T : \mathbf{C} \rightarrow \mathbf{C}$ , together with natural transformations  $\mu : T \circ T \rightarrow T$  and  $\eta : Id_{\mathbf{C}} \rightarrow T$  such that  $\mu$  is appropriately associative and unital. In other words,  $\mathbb{T} = (T, \mu, \eta)$  is a monoid in the category of endofunctors of  $\mathbf{C}$ , which is monoidal under composition.

Dually, a *comonad* on  $\mathbf{C}$  an endofunctor  $K : \mathbf{C} \rightarrow \mathbf{C}$ , together with natural transformations  $\Delta : K \rightarrow K \circ K$  and  $\varepsilon : K \rightarrow Id_{\mathbf{C}}$  such that  $\Delta$  is appropriately coassociative and counital, i.e.,  $\mathbb{K} = (K, \Delta, \varepsilon)$  is a comonoid in the category of endofunctors of  $\mathbf{C}$ .

*Example 2.2.* If  $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$  is a pair of adjoint functors, with unit  $\eta : Id_{\mathbf{C}} \rightarrow UF$  and counit  $\varepsilon : FU \rightarrow Id_{\mathbf{D}}$ , then  $(UF, U\varepsilon F, \eta)$  is a monad on  $\mathbf{C}$  and  $(FU, F\varepsilon U, \varepsilon)$  is a comonad on  $\mathbf{D}$ .

To any monad we can associate a useful category of “algebras.”

**Definition 2.3.** Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on a category  $\mathbf{C}$ . The objects of the *Eilenberg-Moore category of  $\mathbb{T}$ -algebras*, denoted  $\mathbf{C}^{\mathbb{T}}$ , are pairs  $(A, m)$ , where  $A \in \text{Ob } \mathbf{C}$  and  $m \in \mathbf{C}(TA, A)$ , which is appropriately associative and unital, i.e.,

$$m \circ Tm = m \circ \mu_A \quad \text{and} \quad m \circ \eta_A = Id_A.$$

A morphism in  $\mathbf{C}^{\mathbb{T}}$  from  $(A, m)$  to  $(A', m')$  is a morphism  $f : A \rightarrow A'$  in  $\mathbf{C}$  such that  $m' \circ Tf = f \circ m$ .

The category  $\mathbf{C}^{\mathbb{T}}$  of  $\mathbb{T}$ -algebras is related to the underlying category  $\mathbf{C}$  as follows.

*Remark 2.4.* Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on a category  $\mathbf{C}$ . The forgetful functor  $U^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \rightarrow \mathbf{C}$  admits a left adjoint

$$F^{\mathbb{T}} : \mathbf{C} \rightarrow \mathbf{C}^{\mathbb{T}},$$

called the *free  $\mathbb{T}$ -algebra functor*, which is defined on objects by

$$F^{\mathbb{T}}(X) = (TX, \mu_X)$$

and on morphisms by

$$F^{\mathbb{T}}(f) = Tf.$$

Note that  $\mathbb{T}$  itself is the monad associated to the  $(F^{\mathbb{T}}, U^{\mathbb{T}})$ -adjunction. On the other hand, the comonad associated to the  $(F^{\mathbb{T}}, U^{\mathbb{T}})$ -adjunction is

$$\mathbb{K}^{\mathbb{T}} = (F^{\mathbb{T}}U^{\mathbb{T}}, F^{\mathbb{T}}\eta U^{\mathbb{T}}, \varepsilon^{\mathbb{T}}),$$

where  $\varepsilon^{\mathbb{T}}$  is the counit of the  $(F^{\mathbb{T}}, U^{\mathbb{T}})$ -adjunction, which is given by

$$(\varepsilon^{\mathbb{T}})_{(A, m)} = m : (TA, \mu_A) \rightarrow (A, m).$$

In the case of a monad  $\mathbb{T}$  arising from an adjunction  $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$ , it is natural to wonder about the relationship between  $\mathbf{D}$  and the category of  $\mathbb{T}$ -algebras, which is mediated by the following comparison functor.

**Definition 2.5.** Let  $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$  be a pair of adjoint functors, with unit  $\eta : Id_{\mathbf{C}} \rightarrow UF$  and counit  $\varepsilon : FU \rightarrow Id_{\mathbf{D}}$ . Let  $\mathbb{T}$  denote the associated monad. The *canonical  $\mathbb{T}$ -algebra functor*

$$\text{Can}^{\mathbb{T}} : \mathbf{D} \rightarrow \mathbf{C}^{\mathbb{T}}$$

is defined on objects by

$$\text{Can}^{\mathbb{T}}(D) = (UD, U\varepsilon_D)$$

and on morphisms by

$$\text{Can}^{\mathbb{T}}(f) = Uf.$$

If  $\mathbf{D}$  admits coequalizers, then  $\text{Can}^{\mathbb{T}}$  has a left adjoint, the “indecomposables” functor

$$Q^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \rightarrow \mathbf{D},$$

which is defined on objects by

$$Q^{\mathbb{T}}(A, m) = \text{coequal}(FUFA \begin{array}{c} \xrightarrow{Fm} \\ \xrightarrow{\varepsilon_{FA}} \end{array} FA).$$

The functor  $U$  is *monadic* if  $\text{Can}^{\mathbb{T}}$  is an equivalence of categories.

*Remark 2.6.* For any adjunction  $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$  with associated monad  $\mathbb{T}$ , the diagram

$$\begin{array}{ccc} \mathbf{C} & \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{U} \end{array} & \mathbf{D} \\ \parallel & & \downarrow \text{Can}^{\mathbb{T}} \\ \mathbf{C} & \begin{array}{c} \xrightarrow{F^{\mathbb{T}}} \\ \xleftarrow{U^{\mathbb{T}}} \end{array} & \mathbf{C}^{\mathbb{T}} \end{array}$$

commutes, i.e.,  $\text{Can}^{\mathbb{T}} \circ F = F^{\mathbb{T}}$  and  $U^{\mathbb{T}} \circ \text{Can}^{\mathbb{T}} = U$ .

The dual situation for comonads can be described as follows.

**Definition 2.7.** Let  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a comonad on  $\mathbf{D}$ . The objects of the *Eilenberg-Moore category of  $K$ -coalgebras*, denoted  $\mathbf{D}_{\mathbb{K}}$ , are pairs  $(D, \delta)$ , where  $D \in \text{Ob } \mathbf{D}$  and  $\delta \in \mathbf{D}(D, KD)$ , which is appropriately coassociative and counital, i.e.,

$$K\delta \circ \delta = \Delta_D \circ \delta \quad \text{and} \quad \varepsilon_D \circ \delta = \text{Id}_D.$$

A morphism in  $\mathbf{D}_{\mathbb{K}}$  from  $(D, \delta)$  to  $(D', \delta')$  is a morphism  $f : D \rightarrow D'$  in  $\mathbf{D}$  such that  $Kf \circ \delta = \delta' \circ f$ .

The category  $\mathbf{D}_{\mathbb{K}}$  of  $\mathbb{K}$ -coalgebras is related to the underlying category  $\mathbf{D}$  as follows.

*Remark 2.8.* Let  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a comonad on  $\mathbf{D}$ . The forgetful functor  $U_{\mathbb{K}} : \mathbf{D}_{\mathbb{K}} \rightarrow \mathbf{D}$  admits a right adjoint

$$F_{\mathbb{K}} : \mathbf{D} \rightarrow \mathbf{D}_{\mathbb{K}},$$

called the *cofree  $\mathbb{K}$ -coalgebra functor*, which is defined on objects by

$$F_{\mathbb{K}}(X) = (KX, \Delta_X)$$

and on morphisms by

$$F_{\mathbb{K}}(f) = Kf.$$

Note that that  $\mathbb{K}$  itself is the comonad associated to the  $(U_{\mathbb{K}}, F_{\mathbb{K}})$ -adjunction. On the other hand, the monad associated to the  $(U_{\mathbb{K}}, F_{\mathbb{K}})$ -adjunction is

$$\mathbb{T}_{\mathbb{K}} = (F_{\mathbb{K}}U_{\mathbb{K}}, F_{\mathbb{K}}\varepsilon U_{\mathbb{K}}, \eta_{\mathbb{K}}),$$

where  $\eta_{\mathbb{K}}$  is the unit of the  $(U_{\mathbb{K}}, F_{\mathbb{K}})$ -adjunction, which is given by

$$(\eta_{\mathbb{K}})_{(D, \delta)} = \delta : (D, \delta) \rightarrow (KD, \Delta_D).$$

As in the monad case, if the comonad  $\mathbb{K}$  arises from an adjunction  $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$ , then a comparison functor, defined below, mediates between  $\mathbf{C}$  and  $\mathbf{D}_{\mathbb{K}}$ .

**Definition 2.9.** Let  $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$  be a pair of adjoint functors, with unit  $\eta : \text{Id}_{\mathbf{C}} \rightarrow UF$  and counit  $\varepsilon : FU \rightarrow \text{Id}_{\mathbf{D}}$ . Let  $\mathbb{K}$  denote the associated comonad. The *canonical  $\mathbb{K}$ -coalgebra functor*

$$\text{Can}_{\mathbb{K}} : \mathbf{C} \rightarrow \mathbf{D}_{\mathbb{K}}$$

is defined on objects by

$$\text{Can}_{\mathbb{K}}(C) = (FC, F\eta_C)$$

and on morphisms by

$$\text{Can}_{\mathbb{K}}(f) = Ff.$$

If  $\mathbf{C}$  admits equalizers, then  $\text{Can}_{\mathbb{K}}$  has a right adjoint, the “primitives” functor

$$\text{Prim}_{\mathbb{K}} : \mathbf{D}_{\mathbb{K}} \rightarrow \mathbf{C},$$

which is defined on objects by

$$\text{Prim}_{\mathbb{K}}(D, \delta) = \text{equal}(UD \begin{array}{c} \xrightarrow{U\delta} \\ \xrightarrow{\eta_{UD}} \end{array} UFUD).$$

The functor  $F$  is *comonadic* if  $\text{Can}_{\mathbb{K}}$  is an equivalence of categories.

*Remark 2.10.* For any adjunction  $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$  with associated comonad  $\mathbb{K}$ , the diagram

$$\begin{array}{ccc} \mathbf{C} & \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{U} \end{array} & \mathbf{D} \\ \text{Can}_{\mathbb{K}} \downarrow & & \parallel \\ \mathbf{C}_{\mathbb{K}} & \begin{array}{c} \xrightarrow{U_{\mathbb{K}}} \\ \xleftarrow{F_{\mathbb{K}}} \end{array} & \mathbf{D} \end{array}$$

commutes, i.e.,  $U_{\mathbb{K}} \circ \text{Can}_{\mathbb{K}} = F$  and  $\text{Can}_{\mathbb{K}} \circ U = F_{\mathbb{K}}$ .

2.1.2. *Monads and descent.* When the adjunction from which we start is

$$F^{\mathbb{T}} : \mathbf{C} \rightleftarrows \mathbf{C}^{\mathbb{T}} : U^{\mathbb{T}}$$

itself, for some monad  $\mathbb{T}$  on a category  $\mathbf{C}$ , we can formulate the well-known notion of descent in terms of algebras over a monad and coalgebras over a comonad.

**Definition 2.11.** The *descent category* of a monad  $\mathbb{T}$  on a category  $\mathbf{C}$ , denoted  $\mathcal{D}(\mathbb{T})$ , is the category  $(\mathbf{C}^{\mathbb{T}})_{\mathbb{K}^{\mathbb{T}}}$  of  $\mathbb{K}^{\mathbb{T}}$ -coalgebras in the category of  $\mathbb{T}$ -algebras. The objects of  $\mathcal{D}(\mathbb{T})$  are called *descent data*.

The monad  $\mathbb{T}$  satisfies *descent* if

$$\text{Can}_{\mathbb{K}^{\mathbb{T}}} : \mathbf{C} \rightarrow \mathcal{D}(\mathbb{T})$$

is fully faithful. If  $\text{Can}_{\mathbb{K}^{\mathbb{T}}}$  is an equivalence of categories, i.e., if the forgetful functor  $U^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \rightarrow \mathbf{C}$  is monadic, then  $\mathbb{T}$  satisfies *effective descent*.

To formulate a correct homotopical generalization of the notion of descent, we need to unfold this rather intricate definition. We begin by analyzing the objects in  $\mathcal{D}(\mathbb{T})$ .

*Remark 2.12.* Let  $\mathbb{T} = (T, \mu, \eta)$  be any monad on  $\mathbf{C}$ . A descent datum for  $\mathbb{T}$  is a triple  $(H, m, \delta)$ , where  $m : TH \rightarrow H$  (respectively  $\delta : H \rightarrow TH$ ) is a morphism in  $\mathbf{C}$  that is associative and unital (respectively, coassociative and counital) and

$$\begin{array}{ccc} TH & \xrightarrow{m} & H \\ T\delta \downarrow & & \downarrow \delta \\ T^2H & \xrightarrow{\mu_H} & TH \end{array}$$

commutes. We use the symbol  $H$  for the underlying object in  $\mathbf{C}$ , as descent data are of a Hopf-like nature, being endowed with compatible multiplication and comultiplication.

For any object  $C$  in  $\mathbf{C}$ , the canonical descent datum associated to  $C$  is

$$\text{Can}_{\mathbb{K}^{\mathbb{T}}}(C) = (TC, \mu_C, T\eta_C).$$



It follows that for any object  $C$ ,

$$F_{\mathbb{T}} F^{\mathbb{T}}(C) = (T^2 C, \mu_{TC}, T\eta_{TC}) = \text{Can}_{\mathbb{T}}(TC).$$

We next examine in more detail the notion of (effective) descent.

*Remark 2.13.* If  $\mathbb{T} = (T, \mu, \eta)$  is a monad on a category  $\mathbf{C}$  admitting equalizers, then for all descent data  $(H, m, \delta)$ ,

$$\text{Prim}_{\mathbb{T}}(H, m, \delta) = \text{equal}(H \begin{array}{c} \xrightarrow{\delta} \\ \xrightarrow{\eta_H} \end{array} TH),$$

where the equalizer is computed in  $\mathbf{C}$ . In particular, for all objects  $C$  in  $\mathbf{C}$ ,

$$\text{Prim}_{\mathbb{T}} \text{Can}_{\mathbb{T}}(C) = \text{equal}(TC \begin{array}{c} \xrightarrow{T\eta_C} \\ \xrightarrow{\eta_{TC}} \end{array} T^2 C).$$

It follows that, if  $\mathbb{T}$  satisfies descent, then the natural map

$$C \rightarrow \text{equal}(TC \begin{array}{c} \xrightarrow{T\eta_C} \\ \xrightarrow{\eta_{TC}} \end{array} T^2 C),$$

which is the unit of the  $(\text{Can}_{\mathbb{T}}, \text{Prim}_{\mathbb{T}})$ -adjunction, must be an isomorphism for all  $C$ , i.e., the “primitives” of the canonical descent datum associated to  $C$  are naturally isomorphic to  $C$  itself. If  $\mathbb{T}$  satisfies effective descent, then, in addition, any descent datum  $(H, m, \delta)$  must be naturally isomorphic to the canonical descent datum associated to its object of “primitives”, i.e.,

$$(H, m, \delta) \cong \text{Can}_{\mathbb{T}} \text{Prim}_{\mathbb{T}}(H, m, \delta).$$

2.1.3. *Comonads and codescent.* Starting now from the adjunction

$$U_{\mathbb{K}} : \mathbf{D}_{\mathbb{K}} \rightleftarrows \mathbf{D} : F_{\mathbb{K}}$$

for some monad  $\mathbb{K}$  on a category  $\mathbf{D}$ , we present the elements of the theory of codescent.

**Definition 2.14.** The *codescent category* of a comonad  $\mathbb{K}$  on a category  $\mathbf{D}$ , denoted  $\mathcal{D}^{co}(\mathbb{K})$ , is the category  $(\mathbf{D}_{\mathbb{K}})^{\mathbb{T}_{\mathbb{K}}}$  of  $\mathbb{T}_{\mathbb{K}}$ -algebras in the category of  $\mathbb{K}$ -coalgebras. The objects of  $\mathcal{D}^{co}(\mathbb{K})$  are called *codescent data*.

The comonad  $\mathbb{K}$  satisfies *codescent* if

$$\text{Can}^{\mathbb{T}_{\mathbb{K}}} : \mathbf{D} \rightarrow \mathcal{D}^{co}(\mathbb{K})$$

is fully faithful. If  $\text{Can}^{\mathbb{T}_{\mathbb{K}}}$  is an equivalence of categories, i.e., if the forgetful functor  $U_{\mathbb{K}} : \mathbf{D}_{\mathbb{K}} \rightarrow \mathbf{D}$  is comonadic, then  $\mathbb{K}$  satisfies *effective codescent*.

We again spell out more explicitly what this definition means, in order to generalize it correctly later.

*Remark 2.15.* Let  $\mathbb{K} = (K, \Delta, \varepsilon)$  be any comonad on  $\mathbf{D}$ . A codescent datum for  $\mathbb{K}$  is a triple  $(H, \delta, m)$ , where  $\delta : H \rightarrow KH$  (respectively  $m : KH \rightarrow H$ ) is a morphism in  $\mathbf{D}$  that is coassociative and counital (respectively, associative and unital) and

$$\begin{array}{ccc} KH & \xrightarrow{m} & H \\ \Delta_H \downarrow & & \downarrow \delta \\ K^2 H & \xrightarrow{Km} & KH \end{array}$$

commutes. We again use the symbol  $H$  for the underlying object in  $\mathbf{D}$ , as codescent data are also of a Hopf-like nature.

For any object  $D$  in  $\mathbf{D}$ , the canonical codescent datum associated to  $D$  is

$$\text{Can}^{\mathbb{K}}(D) = (KD, \Delta_D, K\varepsilon_D).$$

In particular, for all  $D \in \text{Ob } D$ ,

$$F^{\mathbb{K}}F_{\mathbb{K}}(D) = (K^2D, \Delta_{KD}, K\eta_{KD}) = \text{Can}^{\mathbb{K}}(KD).$$

Finally, we examine in more detail the notion of (effective) codescent.

*Remark 2.16.* If  $\mathbb{K} = (K, \Delta, \varepsilon)$  is a comonad on a category  $\mathbf{D}$  admitting coequalizers, then for all codescent data  $(H, \delta, m)$ ,

$$\mathbb{Q}^{\mathbb{K}}(H, \delta, m) = \text{coequal}(KH \begin{array}{c} \xrightarrow{m} \\ \xrightarrow{\varepsilon_{KH}} \end{array} H),$$

where the coequalizer is computed in  $\mathbf{D}$ . In particular, for all objects  $D$  in  $\mathbf{D}$ ,

$$\mathbb{Q}^{\mathbb{K}}\text{Can}^{\mathbb{K}}(D) = \text{coequal}(K^2D \begin{array}{c} \xrightarrow{K\varepsilon_D} \\ \xrightarrow{\varepsilon_{KD}} \end{array} KD).$$

It follows that, if  $\mathbb{K}$  satisfies codescent, then the natural map

$$\text{coequal}(K^2D \begin{array}{c} \xrightarrow{K\varepsilon_D} \\ \xrightarrow{\varepsilon_{KD}} \end{array} KD) \rightarrow D,$$

which is the counit of the  $(\mathbb{Q}^{\mathbb{K}}, \text{Can}^{\mathbb{K}})$ -adjunction, must be an isomorphism for all  $D$ , i.e., the “indecomposables” of the canonical codescent datum associated to  $D$  are naturally isomorphic to  $D$  itself. If  $\mathbb{K}$  satisfies effective codescent, then, in addition, any codescent datum  $(H, \delta, m)$  must be naturally isomorphic to the canonical codescent datum associated to its object of “indecomposables”, i.e.,

$$(H, \delta, m) \cong \text{Can}^{\mathbb{K}}\mathbb{Q}^{\mathbb{K}}(H, \delta, m).$$

## 2.2. Grothendieck descent and its dual.

2.2.1. *Grothendieck descent for modules.* The example of descent that we present here is a slight generalization of classical Grothendieck descent for modules over rings.

Let  $(\mathbf{M}, \wedge, I)$  be a monoidal category. Let  $\varphi : B \rightarrow A$  be a morphism of monoids in  $\mathbf{M}$ , which induces an adjunction

$$-\wedge_B A : \mathbf{Mod}_B \rightleftarrows \mathbf{Mod}_A : \varphi^*$$

and therefore a monad  $\mathbb{T}_\varphi$  with underlying endofunctor

$$\varphi^*(-\wedge_B A) : \mathbf{Mod}_B \rightarrow \mathbf{Mod}_B$$

and a comonad  $\mathbb{K}_\varphi$  with underlying endofunctor

$$\varphi^*(-)\wedge_B A : \mathbf{Mod}_A \rightarrow \mathbf{Mod}_A.$$

The central problem of Grothendieck descent theory for modules is to determine when  $-\wedge_B A$  is comonadic, i.e., when

$$\text{Can}_{\mathbb{K}_\varphi} : \mathbf{Mod}_B \rightarrow (\mathbf{Mod}_A)_{\mathbb{K}_\varphi}$$

is an equivalence of categories. Since, as is well known (cf. [19, §8]),  $\varphi^*$  is always monadic, i.e.,

$$\mathrm{Can}^{\mathbb{T}\varphi} : \mathbf{Mod}_A \xrightarrow{\cong} (\mathbf{Mod}_B)^{\mathbb{T}\varphi}$$

is an equivalence, it follows that  $\mathbb{T}\varphi$  is of effective descent if and only if  $- \underset{B}{\wedge} A$  is comonadic.

A great deal is known about conditions under which  $- \underset{B}{\wedge} A$  is comonadic. For example, if  $\mathbf{M}$  is the category of abelian groups and  $\varphi$  is a homomorphism of commutative rings, then  $- \underset{B}{\wedge} A$  is comonadic if and only if  $A$  is pure as a  $B$ -module.

There is a useful alternate description of effective descent for modules in monoidal categories, in terms of the following generalization of an important notion from ring theory. We suppose henceforth that the functors

$$X \wedge -, - \wedge X : \mathbf{M} \rightarrow \mathbf{M}$$

preserve colimits for all objects  $X$  in  $\mathbf{M}$ , so that  $- \underset{A}{\wedge} -$  defines a monoidal structure on the category  ${}_A\mathbf{Mod}_A$  of  $A$ -bimodules.

**Definition 2.17.** Let  $A$  be a monoid. An  $A$ -co-ring is a comonoid in the monoidal category  $({}_A\mathbf{Mod}_A, - \underset{A}{\wedge} -, A)$  of  $A$ -bimodules. In other words, an  $A$ -co-ring is an  $A$ -bimodule  $W$  that is endowed with a coassociative, counital comultiplication  $\psi : W \rightarrow W \underset{A}{\wedge} W$  that is a morphism of  $A$ -bimodules.

*Example 2.18.* Let  $\varphi : B \rightarrow A$  be any morphism of monoids in  $\mathbf{M}$ . The *descent co-ring associated to  $\varphi$* , denoted  $W_\varphi$  and also known as the *canonical co-ring on  $\varphi$* , has as underlying  $A$ -bimodule  $A \underset{B}{\wedge} A$ , endowed with a comultiplication  $\psi_{can}$ , which is equal to the composite

$$A \underset{B}{\wedge} A \cong A \underset{B}{\wedge} B \underset{B}{\wedge} A \xrightarrow{A \underset{B}{\wedge} \varphi \underset{B}{\wedge} A} A \underset{B}{\wedge} A \underset{B}{\wedge} A \cong (A \underset{B}{\wedge} A) \underset{A}{\wedge} (A \underset{B}{\wedge} A).$$

The morphism  $\bar{\mu} : A \underset{B}{\wedge} A \rightarrow A$  induced by the multiplication map of  $A$  is the counit of  $\psi_{can}$ .

*Remark 2.19.* The descent co-ring  $W_\varphi$  is a Hopf algebroid over  $A$  (generally, without antipode), where the left and right units are

$$\varphi \underset{B}{\wedge} A : A \rightarrow W_\varphi$$

and

$$A \underset{B}{\wedge} \varphi : A \rightarrow W_\varphi.$$

**Definition 2.20.** Let  $A$  be a monoid, and let  $(W, \psi, \epsilon)$  be an  $A$ -co-ring. The category  $\mathbf{M}_A^W$  is the category of  $W$ -comodules in the category of right  $A$ -modules. In other words, an object of  $\mathbf{M}_A^W$  is a right  $A$ -module  $M$  together with a morphism  $\theta : M \rightarrow M \underset{A}{\wedge} W$  of right  $A$ -modules such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\theta} & M \underset{A}{\wedge} W \\ \downarrow \theta & & \downarrow \theta \underset{A}{\wedge} W \\ M \underset{A}{\wedge} W & \xrightarrow{M \underset{A}{\wedge} \psi} & M \underset{A}{\wedge} W \underset{A}{\wedge} W \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\theta} & M \underset{A}{\wedge} W \\ & \searrow = & \downarrow M \underset{A}{\wedge} \epsilon \\ & & M \end{array}$$

commute. Morphisms in  $\mathbf{M}_A^W$  are morphisms of  $A$ -modules that respect the  $W$ -coactions.

*Example 2.21.* Let  $\varphi : B \rightarrow A$  be any morphism of monoids in  $\mathbf{M}$ , and let

$$W_\varphi = (A \wedge_B A, \psi_{can}, \epsilon_{can}),$$

the descent co-ring associated to  $\varphi$ . If the monoid  $A$  is augmented, the category  $\mathbf{M}_A^{W_\varphi}$  is isomorphic to  $\mathcal{D}(\mathbb{T}_\varphi)$ , the descent category associated to  $\mathbb{T}_\varphi$ . Analyzing the definition of descent data in this specific case, we see that an object of  $\mathcal{D}(\mathbb{T}_\varphi)$  is a right  $A$ -module  $M$  endowed with a morphism  $\theta : M \rightarrow M \wedge_B A$  of right  $A$ -modules such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\theta} & M \wedge_B A \\ \downarrow \theta & & \downarrow \theta \wedge_B A \\ M \wedge_B A & \xrightarrow{M \wedge_B \varphi \wedge_B A} & M \wedge_B A \wedge_B A \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\theta} & M \wedge_B A \\ & \searrow = & \downarrow \bar{r} \\ & & M \end{array}$$

commute, where  $\bar{r}$  is induced by the right  $A$ -action on  $M$ , and we have suppressed the restriction of scalars,  $\varphi^*$ , from the notation. The morphisms in  $\mathcal{D}(\mathbb{T}_\varphi)$  are  $A$ -module morphisms respecting the structure maps.

The key to showing that  $\mathbf{M}_A^{W_\varphi}$  and  $\mathcal{D}(\mathbb{T}_\varphi)$  are isomorphic is the observation that

$$M \wedge_A W_\varphi = M \wedge_A A \wedge_B A \cong M \wedge_B A$$

for all right  $A$ -modules  $M$ .

*Remark 2.22.* It is easy to check that  $\mathcal{D}(\mathbb{T}_\varphi) \cong (\mathbf{Mod}_A)_{\mathbb{K}_\varphi}$ . Moreover, under this isomorphism

$$\text{Can}_\varphi := \text{Can}_{\mathbb{K}_\varphi} : \mathbf{Mod}_B \rightarrow \mathcal{D}(\mathbb{T}_\varphi)$$

is defined on objects by  $\text{Can}_\varphi(M) = (M \wedge_B A, \theta_M)$ , with  $\theta_M = M \wedge_B \varphi \wedge_B A$ . This is a particularly nice descent situation, where the descent category is *Tannakian*, i.e., equivalent to a category of comodules over a coalgebra in a monoidal category. We return to a general consideration of this special descent framework in an upcoming article. We expect Hovey's recent work on a homotopic version of the Eilenberg-Watts theorem [13] to prove useful in this context.

If  $\mathbf{Mod}_B$  admits equalizers, then the right adjoint of the functor  $\text{Can}_\varphi$  is

$$\text{Prim}_\varphi : \mathcal{D}(\mathbb{T}_\varphi) \rightarrow \mathbf{Mod}_B,$$

where

$$\text{Prim}_\varphi(N, \theta) = \text{equal}(N \begin{array}{c} \xrightarrow{\theta} \\ \xrightarrow{N \wedge_B \varphi} \end{array} N \wedge_B A).$$

**2.2.2. Dualizing Grothendieck descent.** We now dualize the results of the previous section, studying codescent associated to a morphism of comonoids. We again suppose that  $(\mathbf{M}, \wedge, I)$  is a monoidal category.

Let  $C$  be a comonoid in  $\mathbf{M}$ . If  $(M, \rho)$  is a right  $C$ -comodule and  $(N, \lambda)$  is a left  $C$ -comodule, then their cotensor product is defined to be

$$M \square_C N = \text{equal}(M \otimes N \begin{array}{c} \xrightarrow{\rho \otimes N} \\ \xrightarrow{M \otimes \lambda} \end{array} M \otimes C \otimes N),$$

which is an object in  $\mathbf{M}$ . We assume that  $-\square_C-$  is naturally associative, giving rise to a monoidal structure on  ${}_C\mathbf{Comod}_C$ , the category of  $C$ -bicomodules.

It is helpful to keep in mind the special class of examples in which the monoidal product  $\wedge$  is the categorical product  $\times$ , so that  $\mathbf{M}$  is *cartesian*. If  $\mathbf{M}$  is cartesian, then any object  $X$  has a natural comonoid structure given by the diagonal morphism  $\Delta_X$ . Furthermore, a right (or left) coaction of  $(X, \Delta_X)$  on an object  $Y$  is determined by a morphism  $Y \rightarrow X$  in  $\mathbf{M}$ . More precisely,

$$\mathbf{Comod}_X \cong \mathbf{M}/X \cong {}_X\mathbf{Comod},$$

where  $\mathbf{M}/X$  denotes the slice category (or overcategory or comma category or category of bundles over  $X$ ...), of which the objects are morphisms in  $\mathbf{M}$  with target  $X$  and the morphisms are commuting triangles. Note that the cotensor product of comodules corresponds to pullback of morphisms under this identification, which gives rise to a monoidal structure on  ${}_X\mathbf{Comod}_X$ .

We consider here the following sort of adjunction. Let  $\varphi : C \rightarrow D$  be a morphism of comonoids in  $\mathbf{M}$ , which induces an adjunction

$$\varphi_! : \mathbf{Comod}_C \rightleftarrows \mathbf{Comod}_D : -\square_D C,$$

where  $\varphi_!$  is the “extension of scalars” functor. The associated monad  $\mathbb{T}_\varphi$  has underlying endofunctor

$$\varphi_!(-)\square_D C : \mathbf{Comod}_C \rightarrow \mathbf{Comod}_C,$$

while the endofunctor underlying the associated comonad  $\mathbb{K}_\varphi$  is

$$\varphi_!(-\square_D C) : \mathbf{Comod}_D \rightarrow \mathbf{Comod}_D.$$

Just as the “restriction of scalars” functor is monadic in the case of a morphism of monoids, the functor

$$\varphi_! : \mathbf{Comod}_C \rightarrow \mathbf{Comod}_D$$

is comonadic, i.e.,

$$\text{Can}_{\mathbb{K}_\varphi} : \mathbf{Comod}_C \rightarrow (\mathbf{Comod}_D)_{\mathbb{K}_\varphi}$$

is an equivalence of categories. Consequently,  $\mathbb{K}_\varphi$  is of effective codescent if and only if  $-\square_D C$  monadic, i.e., if and only if

$$\text{Can}^{\mathbb{T}_\varphi} : \mathbf{Comod}_D \rightarrow (\mathbf{Comod}_C)^{\mathbb{T}_\varphi}$$

is an equivalence of categories.

As in the module case, there is a useful alternate description of effective codescent.

**Definition 2.23.** Let  $C$  be a comonoid. A  $C$ -ring is a monoid in  $({}_C\mathbf{Comod}_C, -\square_C-, C)$ .

In other words, a  $C$ -ring is an  $C$ -bicomodule  $V$  that is endowed with an associative, unital multiplication  $\gamma : V\square_C V \rightarrow V$  that is a morphism of  $C$ -bicomodules.

*Example 2.24.* Let  $\varphi : C \rightarrow D$  be any morphism of comonoids in  $\mathbf{M}$ . The *codescent ring associated to  $\varphi$* , denoted  $V^\varphi$ , has as underlying  $C$ -bicomodule  $C\square_D C$ , endowed with a multiplication  $\gamma_{can}$ , which is equal to the composite

$$(C\square_C C)\square_C(C\square_C C) \cong C\square_D C\square_D C \xrightarrow{C\square_D \varphi\square_D C} C\square_D D\square_D C \cong C\square_D C.$$

The morphism  $\bar{\Delta} : C \rightarrow C \square_D C$  induced by the comultiplication map  $\Delta$  of  $C$  is the unit of  $\gamma_{can}$ .

**Definition 2.25.** Let  $C$  be a comonoid, and let  $(V, \gamma, \eta)$  be a  $C$ -ring. The category  $\mathbf{M}_V^C$  is the category of  $V$ -modules in the category of right  $C$ -comodules. In other words, an object of  $\mathbf{M}_V^C$  is a right  $C$ -comodule  $M$  together with a morphism  $\theta : M \square_C V \rightarrow M$  of right  $C$ -comodules such that the diagrams

$$\begin{array}{ccc} M \square_C V \square_C V & \xrightarrow{\theta \square_C V} & M \square_C V \\ \downarrow M \square_C \gamma & & \downarrow \theta \\ M \square_C V & \xrightarrow{\theta} & M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{M \square_C \eta} & M \square_C V \\ & \searrow = & \downarrow \theta \\ & & M \end{array}$$

commute. Morphisms in  $\mathbf{M}_V^C$  are morphisms of  $C$ -comodules that respect the  $V$ -actions.

*Example 2.26.* Let  $\varphi : C \rightarrow D$  be any morphism of comonoids in  $\mathbf{M}$ , and let

$$V^\varphi = (C \square_D C, \gamma_{can}, \eta_{can}),$$

the codescent ring associated to  $\varphi$ . If  $C$  is coaugmented, then the category  $\mathbf{M}_{V^\varphi}^C$  is isomorphic to the category of codescent data  $\mathcal{D}^{co}(\mathbb{K}_\varphi)$ . An object of  $\mathcal{D}^{co}(\mathbb{K}_\varphi)$  is a right  $C$ -comodule  $M$  endowed with a morphism  $\theta : M \square_D C \rightarrow M$  such that the diagrams

$$\begin{array}{ccc} M \square_D C \square_D C & \xrightarrow{\theta \square_D C} & M \square_D C \\ \downarrow M \square_D \varphi \square_D C & & \downarrow \theta \\ M \square_D C & \xrightarrow{\theta} & M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\bar{\rho}} & M \square_D C \\ & \searrow = & \downarrow \theta \\ & & M \end{array}$$

commute, where  $\bar{\rho}$  is induced by the right  $C$ -coaction on  $M$ , and we have suppressed the extension of scalars,  $\varphi_!$ , from the notation. The morphisms in  $\mathcal{D}^{co}(\mathbb{K}_\varphi)$  are  $C$ -comodule morphisms respecting the structure maps.

The key to showing that  $\mathbf{M}_{V^\varphi}^C$  and  $\mathcal{D}^{co}(\mathbb{K}_\varphi)$  are isomorphic is the observation that

$$M \square_C V^\varphi = M \square_C C \square_D C \cong M \square_D C$$

for all right  $C$ -comodules  $M$ .

*Remark 2.27.* It is easy to check that  $\mathcal{D}^{co}(\mathbb{K}_\varphi) \cong (\mathbf{Comod}_C)^{\mathbb{T}^\varphi}$ . Moreover, under this isomorphism

$$\text{Can}^\varphi := \text{Can}^{\mathbb{T}^\varphi} : \mathbf{Comod}_D \rightarrow \mathcal{D}^{co}(\mathbb{K}_\varphi),$$

is defined on objects by  $\text{Can}^{\mathbb{T}^\varphi}(M) = (M \square_D C, \theta_M)$ , with  $\theta_M = M \square_D \varphi \square_D C$ . This is a particularly nice codescent situation, where the codescent category is *coTannakian*, i.e., equivalent to a category of modules over an algebra in a monoidal category. We return to a general consideration of this special codescent framework in an upcoming article. We expect Hovey's recent work on a homotopic version of the Eilenberg-Watts theorem [13] to prove useful in this context.

If  $\mathbf{Comod}_D$  admits coequalizers, then the left adjoint of the functor  $\text{Can}^\varphi$  is

$$\text{Q}^\varphi(N, \theta) = \text{coequal}(N \square_D C \begin{array}{c} \xrightarrow{\theta} \\ \xrightarrow{N \square_D \varphi} \end{array} N).$$

### 3. SIMPLICIAL STRUCTURES AND (CO)DESCENT

In order to define homotopic (co)descent, we need the simplicial structures associated to any (co)monad that we present here. We begin with the well-known generalized cobar and bar constructions associated to a monad and a comonad, proving in each case an important technical lemma that holds when the underlying category is a model category. We then discuss simplicial enrichment of Eilenberg-Moore categories of (co)algebras and establish crucial simplicial adjunction isomorphisms (Proposition 3.15).

**3.1. Generalized cobar and bar constructions.** Given a monad on a category  $\mathbf{C}$ , we can canonically associate to any object in  $\mathbf{C}$  a very important and well-known object in  $\mathbf{C}^\Delta$ , which plays a crucial role in our discussion of derived completion and homotopic descent.

**Definition 3.1.** Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on a category  $\mathbf{C}$ . The *cosimplicial  $\mathbb{T}$ -cobar construction* is a functor

$$\Omega_{\mathbb{T}}^\bullet : \mathbf{C} \rightarrow \mathbf{C}^\Delta$$

defined for  $C \in \text{Ob } \mathbf{C}$  by

$$\Omega_{\mathbb{T}}^n C = T^{n+1}(C)$$

and for all  $0 \leq i \leq n$ ,

$$d^i = T^i \eta_{T^{n-i} C} : \Omega_{\mathbb{T}}^n C \rightarrow \Omega_{\mathbb{T}}^{n+1} C,$$

while for all  $0 \leq j \leq n$ ,

$$s^j = T^j \mu_{T^{n-j} C} : \Omega_{\mathbb{T}}^{n+1} C \rightarrow \Omega_{\mathbb{T}}^n C,$$

where  $T^0 := \text{Id}_{\mathbf{C}}$ .

*Remark 3.2.* The  $\mathbb{T}$ -cobar construction is naturally coaugmented by  $\eta_C : C \rightarrow TC$  for all  $C \in \text{Ob } \mathbf{C}$ , which gives rise in the obvious way to a morphism of cosimplicial objects

$$\eta_C^\bullet : cc^\bullet C \rightarrow \Omega_{\mathbb{T}}^\bullet C.$$

The following technical result is a crucial ingredient of the proofs of Theorems 5.3 and 5.16.

**Lemma 3.3.** *Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on a model category  $\mathbf{M}$  such that  $U^{\mathbb{T}} : \mathbf{M}^{\mathbb{T}} \rightarrow \mathbf{M}$  right-induces a model category structure on  $\mathbf{M}^{\mathbb{T}}$  and  $U_{\mathbb{K}\mathbb{T}} : \mathcal{D}(\mathbb{T}) \rightarrow \mathbf{M}^{\mathbb{T}}$  then left-induces a model category structure on  $\mathcal{D}(\mathbb{T})$  (cf. appendix C).*

*If  $\Omega_{\mathbb{T}}^\bullet(Y)$  is Reedy fibrant in  $\mathbf{M}^\Delta$ , then  $\text{Can}_{\mathbb{K}\mathbb{T}}^\Delta \Omega_{\mathbb{T}}^\bullet(Y)$  is Reedy fibrant in  $\mathcal{D}(\mathbb{T})$ .*

*Proof.* Note that the hypotheses on model category structure imply that

$$\mathbf{M} \begin{array}{c} \xrightarrow{F^{\mathbb{T}}} \\ \xleftarrow{U^{\mathbb{T}}} \end{array} \mathbf{M}^{\mathbb{T}} \quad \text{and} \quad \mathcal{D}(\mathbb{T}) \begin{array}{c} \xrightarrow{U_{\mathbb{K}\mathbb{T}}} \\ \xleftarrow{F_{\mathbb{K}\mathbb{T}}} \end{array} \mathbf{M}^{\mathbb{T}}$$

are both Quillen pairs.

Throughout this proof, we use the notation of appendix B.1. Since  $T = U^\mathbb{T}F^\mathbb{T}$ , it is immediate that if  $j \geq 1$ , then there exists  $t^j \in \text{Mor } \mathbf{M}^\mathbb{T}$  such that  $s^j = U^\mathbb{T}t^j$ . Moreover, since

$$\begin{array}{ccc} T^3Z & \xrightarrow{T\mu_Z} & T^2Z \\ \mu_{T^2Z} \downarrow & & \downarrow \mu_Z \\ T^2Z & \xrightarrow{\mu_Z} & TZ \end{array}$$

commutes for all objects  $Z$  in  $\mathbf{M}$ , the multiplication map  $\mu_Z : T^2Z \rightarrow TZ$  is a morphism of  $\mathbb{T}$ -algebras for all  $Z$ , whence the existence of  $t^0 \in \text{Mor } \mathbf{M}^\mathbb{T}$  such that  $s^0 = U^\mathbb{T}t^0 : T^mY \rightarrow T^{m-1}Y$  as well, for all  $m$ .

It follows that

$$\begin{aligned} M_n \Omega_{\mathbb{T}}^\bullet Y &= \prod_{0 \leq i \leq n} \Omega_{\mathbb{T}}^n Y \xrightarrow[\psi_2]{\psi_1} \prod_{0 \leq i < j \leq n-1} \Omega_{\mathbb{T}}^{n-1} Y \\ &= U^\mathbb{T} \left( \prod_{0 \leq i \leq n} F^\mathbb{T}(T^{n-1}Y) \xrightarrow[v_2]{v_1} \prod_{0 \leq i < j \leq n-1} F^\mathbb{T}(T^{n-2}Y) \right) =: U^\mathbb{T} \mathcal{J}_n(Y), \end{aligned}$$

where if

$$\prod_{0 \leq i \leq n} F^\mathbb{T}(T^{n-1}Y) \xrightarrow{p_i} F^\mathbb{T}(T^{n-1}Y) \quad \text{and} \quad \prod_{0 \leq i < j \leq n-1} F^\mathbb{T}(T^{n-2}Y) \xrightarrow{p_{i,j}} F^\mathbb{T}(T^{n-2}Y)$$

are the obvious projections, then

$$p_{i,j} v_1 = t^i p_j \quad \text{and} \quad p_{i,j} v_2 = t^{j-1} p_i.$$

Thus, since  $\Omega_{\mathbb{T}}^\bullet Y$  is Reedy fibrant by hypothesis and the model category structure on  $\mathbf{M}^\mathbb{T}$  is right-induced by  $U^\mathbb{T}$ , the natural map

$$F^\mathbb{T}(T^n Y) \rightarrow \lim \mathcal{J}_n Y,$$

which is induced by  $t_0, \dots, t_{n-1} : F^\mathbb{T}(T^n Y) \rightarrow F^\mathbb{T}(T^{n-1}Y)$ , is a fibration for all  $n \geq 0$ . Note that we are also using here that  $U^\mathbb{T}$  commutes with limits, as it is a right adjoint.

Applying the right Quillen functor  $F_{\mathbb{K}^\mathbb{T}} : \mathbf{M}^\mathbb{T} \rightarrow \mathcal{D}(\mathbb{T})$ , we obtain fibrations

$$F_{\mathbb{K}^\mathbb{T}} F^\mathbb{T}(T^n Y) \rightarrow F_{\mathbb{K}^\mathbb{T}}(\lim \mathcal{J}_n Y) \cong \lim F_{\mathbb{K}^\mathbb{T}}(\mathcal{J}_n(Y))$$

for all  $n$ . On the other hand, as already observed above,

$$F_{\mathbb{K}^\mathbb{T}} F^\mathbb{T}(T^n Y) = \text{Can}_{\mathbb{K}^\mathbb{T}}(\Omega_{\mathbb{T}}^n Y),$$

which generalizes easily to

$$F_{\mathbb{K}^\mathbb{T}}(\mathcal{J}_n(Y)) = M_n \text{Can}_{\mathbb{K}^\mathbb{T}}^\Delta(\Omega_{\mathbb{T}}^\bullet Y).$$

We conclude that  $\text{Can}_{\mathbb{K}^\mathbb{T}}^\Delta(\Omega_{\mathbb{T}}^\bullet Y)$  is Reedy fibrant.  $\square$

*Remark 3.4.* When considering derived completion and homotopic descent later in this paper, we will be particularly interested in monads  $\mathbb{T}$  such that  $\Omega_{\mathbb{T}}^\bullet$  sends fibrant objects to fibrant objects. This fibrancy condition, though constraining, is not impossible to fulfill. For example, if  $\mathbf{M} = \mathbf{sSet}$  and, for all simplicial sets  $X$ ,  $TX$  is the simplicial set underlying a simplicial group, then  $\Omega_{\mathbb{T}}^\bullet X$  is always Reedy fibrant [4, X.4.10].

Of course, there is an equally important dual construction for comonads.



**Definition 3.5.** Let  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a comonad on  $\mathbf{D}$ . The *simplicial  $\mathbb{K}$ -bar construction* is a functor

$$\mathcal{B}_{\bullet}^{\mathbb{K}} : \mathbf{D} \rightarrow \mathbf{D}^{\Delta^{op}}$$

defined for  $D \in \text{Ob } \mathbf{D}$  by

$$\mathcal{B}_n^{\mathbb{K}} D = K^{n+1}(D)$$

and for all  $0 \leq i \leq n$ ,

$$d_i = K^i \varepsilon_{K^{n-i} D} : \mathcal{B}_n^{\mathbb{K}} D \rightarrow \mathcal{B}_{n-1}^{\mathbb{K}} D,$$

while for all  $0 \leq j \leq n$ ,

$$s_j = K^j \Delta_{K^{n-j} D} : \mathcal{B}_n^{\mathbb{K}} D \rightarrow \mathcal{B}_{n+1}^{\mathbb{K}} D,$$

where  $K^0 := Id_{\mathbf{D}}$ .

*Remark 3.6.* The  $\mathbb{K}$ -bar construction is naturally augmented by  $\varepsilon_D : KD \rightarrow D$  for all  $D \in \text{Ob } \mathbf{D}$ , which gives rise in the obvious way to a morphism of simplicial objects

$$(\varepsilon_D)_{\bullet} : \mathcal{B}_{\bullet}^{\mathbb{K}} D \rightarrow cs_{\bullet} D.$$

To prove Theorems 5.8 and 5.23, we need the following lemma. The proof is strictly dual to that of Lemma 3.3.

**Lemma 3.7.** *Let  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a monad on a model category  $\mathbf{M}$  such that  $U_{\mathbb{K}} : \mathbf{M}_{\mathbb{K}} \rightarrow \mathbf{M}$  left-induces a model category structure on  $\mathbf{M}_{\mathbb{K}}$  and  $U^{\mathbb{K}} : \mathcal{D}^{co}(\mathbb{K}) \rightarrow \mathbf{M}_{\mathbb{K}}$  then right-induces a model category structure on  $\mathcal{D}^{co}(\mathbb{K})$  (cf. section C).*

*If  $\mathcal{B}_{\bullet}^{\mathbb{K}} X$  is Reedy cofibrant in  $\mathbf{M}^{\Delta^{op}}$ , then  $(\text{Can}^{\mathbb{K}})^{\Delta^{op}}(\mathcal{B}_{\bullet}^{\mathbb{K}} X)$  is Reedy cofibrant in  $\mathcal{D}^{co}(\mathbb{K})$ .*

*Remark 3.8.* An analysis of the cofibrancy constraint on  $\mathcal{B}_{\bullet}^{\mathbb{K}} X$ , dual to that in Remark 3.4, shows that it is not an unreasonable condition to impose. For example, every object  $\mathbf{sSet}^{\Delta^{op}}$  is Reedy cofibrant [9, IV.32], so in particular if  $\mathbb{K}$  is any comonad on  $\mathbf{sSet}$ , then  $\mathcal{B}_{\bullet}^{\mathbb{K}} X$  is Reedy cofibrant for all simplicial sets  $X$ .

**3.2. Simplicial enrichments of Eilenberg-Moore categories.** We begin by observing that simplicial enrichment of a category induces simplicial enrichment of its associated Eilenberg-Moore categories of (co)algebras over a (co)monad with underlying simplicial endofunctor. We refer the reader to section B.3 for the terminology and notation we use here.

**Lemma 3.9.** *If  $\mathbf{S}$  is a simplicially enriched category,  $\mathbb{T} = (T, \mu, \eta)$  is a monad on  $\mathbf{S}$  such that  $T$  is a simplicial functor, and  $\mathbb{K} = (K, \Delta, \varepsilon)$  is a comonad on  $\mathbf{S}$  such that  $K$  is a simplicial functor, then  $\mathbf{S}^{\mathbb{T}}$  and  $\mathbf{S}_{\mathbb{K}}$  are also naturally simplicially enriched. In particular,*

- (1)  $\text{Map}_{\mathbf{S}^{\mathbb{T}}} : (\mathbf{S}^{\mathbb{T}})^{op} \times \mathbf{S}^{\mathbb{T}} \rightarrow \mathbf{sSet}$  is defined for any  $\mathbb{T}$ -algebras  $(A, m)$  and  $(A', m')$  by

$$\text{Map}_{\mathbf{S}^{\mathbb{T}}}\left((A, m), (A', m')\right) = \text{equal}\left(\text{Map}_{\mathbf{S}}(A, A') \begin{array}{c} \xrightarrow{m^*} \\ \xrightarrow{m'_* \circ T_{A, A'}} \end{array} \text{Map}_{\mathbf{S}}(TA, A')\right);$$

and

(2)  $\text{Map}_{\mathbf{S}_{\mathbb{K}}} : (\mathbf{S}_{\mathbb{K}})^{op} \times \mathbf{S}_{\mathbb{K}} \rightarrow \mathbf{sSet}$  is defined for any  $\mathbb{K}$ -coalgebras  $(C, \delta)$  and  $(C', \delta')$  by

$$\text{Map}_{\mathbf{S}_{\mathbb{K}}}((C, \delta), (C', \delta')) = \text{equal}\left(\text{Map}_{\mathbf{S}}(C, C') \begin{array}{c} \xrightarrow{\delta'_*} \\ \xrightarrow{\delta^* \circ K_{C, C'}} \end{array} \text{Map}_{\mathbf{S}}(C, KC')\right).$$

*Remark 3.10.* In the lemma above, equalizers of simplicial maps are understood to be constructed levelwise using the following explicit model for the equalizer of two set maps.

$$\text{equal}\left(X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y\right) = \{x \in X \mid f(x) = g(x)\}.$$

*Proof.* (1) For any objects  $X, Y, Z$  in  $\mathbf{S}$ , let

$$c : \text{Map}_{\mathbf{S}}(Y, Z) \times \text{Map}_{\mathbf{S}}(X, Y) \rightarrow \text{Map}_{\mathbf{S}}(X, Z)$$

denote the composition map of the simplicial enrichment, and let

$$i_X : * \rightarrow \text{Map}_{\mathbf{S}}(X, X)$$

denote the unit map.

As defined above,  $\text{Map}_{\mathbf{S}^{\mathbb{T}}}((A, m), (A'm'))$  is a simplicial subset of  $\text{Map}_{\mathbf{S}}(A, A')$ , for all  $\mathbb{T}$ -algebras  $(A, m)$  and  $(A', m')$ . Since it is obvious that

$$\text{Map}_{\mathbf{S}^{\mathbb{T}}}((A, m), (A'm'))_0 = S^{\mathbb{T}}((A, m), (A', m')),$$

to prove that the definition above of  $\text{Map}_{\mathbf{S}^{\mathbb{T}}}(-, -)$  gives a simplicial enrichment of  $\mathbf{S}^{\mathbb{T}}$ , it suffices to check:

- (a)  $c(f', f) \in \text{Map}_{\mathbf{S}^{\mathbb{T}}}((A, m), (A''m''))_n$  for all  $f \in \text{Map}_{\mathbf{S}^{\mathbb{T}}}((A, m), (A'm'))_n$  and  $f' \in \text{Map}_{\mathbf{S}^{\mathbb{T}}}((A', m'), (A''m''))_n$ ; and
- (b)  $i_A(*) \in \text{Map}_{\mathbf{S}^{\mathbb{T}}}((A, m), (A, m))_0$  for all possible multiplications  $m : TA \rightarrow A$ .

To prove (a), note that  $f \in \text{Map}_{\mathbf{S}^{\mathbb{T}}}((A, m), (A'm'))_n$  and  $f' \in \text{Map}_{\mathbf{S}^{\mathbb{T}}}((A', m'), (A''m''))_n$  if and only if

$$m^*(f) = m'_*(T_{A, A'}(f)) \quad \text{and} \quad (m')^*(f') = m''_*(T_{A', A''}(f')).$$

Lemma B.3 therefore implies that

$$m^*(c(f', f)) = m''_*(c(T_{A', A''}(f'), T_{A, A'}(f))).$$

On the other hand, since  $T$  is a simplicial functor,

$$c(T_{A', A''}(f'), T_{A, A'}(f)) = T_{A, A''}(c(f', f)),$$

and we can conclude that  $c(f', f) \in \text{Map}_{\mathbf{S}^{\mathbb{T}}}((A, m), (A''m''))_n$ .

The following sequence of equalities, which are true for all  $\mathbb{T}$ -algebras  $(A, m)$ , establishes that condition (b) holds as well.

$$\begin{aligned} m^*(i_A(*)) &= c(i_A(*), m) = m \\ &= c(m, i_{TA}(*)) = c(m, T_{A, A}(i_A(*))) = m_*(T_{A, A}(i_A(*))). \end{aligned}$$

The proof of (2) is strictly dual to the proof of (1) and therefore left to the reader. □

Tensoring and cotensoring over  $\mathbf{sSet}$  can also sometimes carry over to Eilenberg-Moore categories of (co)algebras.

**Lemma 3.11.** *Let  $\mathbf{S}$  be a simplicially enriched category that is tensored and cotensored over  $\mathbf{sSet}$ . Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on  $\mathbf{S}$  and  $\mathbb{K} = (K, \Delta, \varepsilon)$  a comonad on  $\mathbf{S}$ .*

- (1) *If  $T$  is a simplicial functor, then the Eilenberg-Moore category  $\mathbf{S}^{\mathbb{T}}$  of  $\mathbb{T}$ -algebras is cotensored over  $\mathbf{sSet}$ , via*

$$(-)_{\mathbb{T}}^{(-)} : \mathbf{S}^{\mathbb{T}} \times \mathbf{sSet}^{op} \rightarrow \mathbf{S}^{\mathbb{T}} : ((A, m), L) \mapsto (A^L, m^L \circ \tau_{A,L}).$$

- (2) *If  $K$  is a simplicial functor, then the Eilenberg-Moore category  $\mathbf{S}_{\mathbb{K}}$  of  $\mathbb{K}$ -coalgebras is tensored over  $\mathbf{sSet}$ , via*

$$- \overset{\mathbb{K}}{\otimes} - : \mathbf{S}_{\mathbb{K}} \times \mathbf{sSet} \rightarrow \mathbf{S}_{\mathbb{K}} : ((C, \delta), L) \mapsto (C \otimes L, \theta_{C,L} \circ (\delta \otimes L)).$$

The proof of this lemma is quite elementary and thus left to the reader. In particular, it is easy to check that there are natural isomorphisms

$$\mathrm{Map}_{\mathbf{S}^{\mathbb{T}}}((A, m), (A', m')_{\mathbb{T}}^L) \cong \mathrm{Map}_{\mathbf{sSet}}\left(L, \mathrm{Map}_{\mathbf{S}^{\mathbb{T}}}\left((A, m), (A', m')\right)\right)$$

for all  $\mathbb{T}$ -algebras  $(A, m)$  and  $(A', m')$  and all simplicial sets  $L$ , as well as

$$\mathrm{Map}_{\mathbf{S}_{\mathbb{K}}}\left((C, \delta) \overset{\mathbb{K}}{\otimes} L, (C', \delta')\right) \cong \mathrm{Map}_{\mathbf{sSet}}\left(L, \mathrm{Map}_{\mathbf{S}_{\mathbb{K}}}\left((C, \delta), (C', \delta')\right)\right)$$

for all  $\mathbb{K}$ -coalgebras  $(C, \delta)$  and  $(C', \delta')$  and all simplicial sets  $L$ .

When  $\mathbf{S}$  is cotensored (respectively, tensored) over  $\mathbf{sSet}$ , as well as simplicially enriched, then the mapping spaces for  $\mathbb{T}$ -algebras (respectively,  $\mathbb{K}$ -coalgebras) satisfy a very useful simplicial adjunction property.

**Lemma 3.12.** *Let  $\mathbf{S}$  be a simplicially enriched category, and let  $\mathbb{T} = (T, \mu, \eta)$  and  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a monad and a comonad on  $\mathbf{S}$  such that  $T$  and  $K$  are simplicial functors.*

- (1) *If  $\mathbf{S}$  is cotensored over  $\mathbf{sSet}$ , then for all objects  $X$  in  $\mathbf{S}$  and all  $\mathbb{T}$ -algebras  $(A, m)$ , there is a natural isomorphism*

$$\mathrm{Map}_{\mathbf{S}}(X, U^{\mathbb{T}}(A, m)) \cong \mathrm{Map}_{\mathbf{S}^{\mathbb{T}}}(F^{\mathbb{T}}X, (A, m)).$$

- (2) *If  $\mathbf{S}$  is tensored over  $\mathbf{sSet}$ , then for all objects  $Y$  in  $\mathbf{S}$  and all  $\mathbb{K}$ -coalgebras  $(C, \delta)$ , there is a natural isomorphism*

$$\mathrm{Map}_{\mathbf{S}}(U_{\mathbb{K}}(C, \delta), Y) \cong \mathrm{Map}_{\mathbf{S}_{\mathbb{K}}}\left((C, \delta), F_{\mathbb{K}}Y\right).$$

*Proof.* (1) Recall the definition of the cotensoring of  $\mathbf{S}^{\mathbb{T}}$  over  $\mathbf{sSet}$  from Lemma 3.11. For all  $n$ ,

$$\begin{aligned} \mathrm{Map}_{\mathbf{S}^{\mathbb{T}}}(F^{\mathbb{T}}X, (A, m))_n &= \mathbf{S}^{\mathbb{T}}(F^{\mathbb{T}}X, (A, m)^{\Delta[n]}) \\ &\cong \mathbf{S}\left(X, U^{\mathbb{T}}\left((A, m)^{\Delta[n]}\right)\right) \\ &= \mathbf{S}(X, A^{\Delta[n]}) \\ &= \mathrm{Map}_{\mathbf{S}}(X, A)_n. \end{aligned}$$

The proof of (2) is dual to the proof above, using the tensoring of  $\mathbf{S}$ , rather than the cotensoring.  $\square$

We now show that the simplicial enrichments we have defined on categories of algebras and coalgebras are compatible with the enrichment of the underlying category, in the following sense.

**Proposition 3.13.** *Let  $\mathbf{S}$  be a simplicially enriched category that is tensored and cotensored over  $\mathbf{sSet}$ . If  $\mathbb{T} = (T, \mu, \eta)$  is a monad on  $\mathbf{S}$  such that  $T$  is a simplicial functor, and  $\mu$  and  $\eta$  are simplicial natural transformations, then the free  $\mathbb{T}$ -algebra functor  $F^{\mathbb{T}} : \mathbf{S} \rightarrow \mathbf{S}^{\mathbb{T}}$  and the canonical descent data functor  $\text{Can}_{\mathbb{K}^{\mathbb{T}}} : \mathbf{S} \rightarrow \mathcal{D}(\mathbb{T})$  are both simplicial functors.*

*Dually, if  $\mathbb{K} = (K, \Delta, \varepsilon)$  is a comonad on  $\mathbf{S}$  such that  $K$  is a simplicial functor, and  $\Delta$  and  $\varepsilon$  are simplicial natural transformations, then the free  $\mathbb{K}$ -coalgebra functor  $F_{\mathbb{K}} : \mathbf{S} \rightarrow \mathbf{S}_{\mathbb{K}}$  and the canonical codescent data functor  $\text{Can}^{\mathbb{K}} : \mathbf{S} \rightarrow \mathcal{D}^{\text{co}}(\mathbb{K})$  are both simplicial functors.*

*Proof.* We treat the monad case and leave the strictly dual, comonad case to the reader. The idea underlying this proof is that if  $f : X \rightarrow Y$  is any morphism in  $\mathbf{S}$ , then  $Tf$  is a morphism of  $\mathbb{T}$ -algebras between  $F^{\mathbb{T}}X$  and  $F^{\mathbb{T}}Y$  and also a morphism of descent data from  $\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X)$  to  $\text{Can}_{\mathbb{K}^{\mathbb{T}}}(Y)$ . We simply generalize this argument to simplices of positive dimension.

Note that  $T^2$  is a simplicial functor because  $T$  is, so that it makes sense to require that  $\mu$  be a simplicial natural transformation.

That  $\mu$  is a simplicial natural transformation means that there is a family of simplicial maps

$$\{\mu_X : * \rightarrow \text{Map}_{\mathbf{S}}(T^2X, TX) \mid X \in \text{Ob } \mathbf{S}\}$$

such that

$$(\mu_Y)_* \circ T_{TX, TY} \circ T_{X, Y} = (\mu_Y)_* \circ T_{X, Y}^2 = (\mu_X)_* \circ T_{X, Y}.$$

It follows

$$T_{X, Y} : \text{Map}_{\mathbf{S}}(X, Y) \rightarrow \text{Map}_{\mathbf{S}}(TX, TY)$$

factors through  $\text{Map}_{\mathbf{S}^{\mathbb{T}}}(F^{\mathbb{T}}X, F^{\mathbb{T}}Y)$ . Let

$$F_{X, Y}^{\mathbb{T}} : \text{Map}_{\mathbf{S}}(X, Y) \rightarrow \text{Map}_{\mathbf{S}^{\mathbb{T}}}(F^{\mathbb{T}}X, F^{\mathbb{T}}Y)$$

denote the corestriction of  $T_{X, Y}$ . The compatibility of  $F_{-, -}^{\mathbb{T}}$  with simplicial composition and identities is an immediate consequence of the same properties of  $T_{-, -}$ .

To treat the canonical descent data functor, recall from Remark 2.12 that for all  $X \in \text{Ob } \mathbf{S}$ ,

$$\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X) = (TX, \mu_X, T\eta_X),$$

so that

$$\begin{aligned} \text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \text{Can}_{\mathbb{K}^{\mathbb{T}}}(Y))_n = \\ \{g \in \text{Map}_{\mathbf{S}^{\mathbb{T}}}(F^{\mathbb{T}}X, F^{\mathbb{T}}Y) \mid (T\eta_X)^*(T_{TX, TY}(g)) = (T\eta_Y)_*(g)\}. \end{aligned}$$

Exactly as above, the fact that  $\eta : Id \rightarrow T$  is a simplicial natural transformation tells us that  $F_{X, Y}^{\mathbb{T}}$  factors through  $\text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \text{Can}_{\mathbb{K}^{\mathbb{T}}}(Y))$ . We can therefore let

$$(\text{Can}_{\mathbb{K}^{\mathbb{T}}})_{X, Y} : \text{Map}_{\mathbf{S}}(X, Y) \rightarrow \text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \text{Can}_{\mathbb{K}^{\mathbb{T}}}(Y))$$

denote the corestriction of  $F_{X, Y}^{\mathbb{T}}$  and conclude that  $\text{Can}_{\mathbb{K}^{\mathbb{T}}}$  is a simplicial functor.  $\square$

*Remark 3.14.* Under the hypotheses of Proposition 3.13,  $U^{\mathbb{T}} : \mathbf{S}^{\mathbb{T}} \rightarrow \mathbf{S}$  and  $U_{\mathbb{K}} : \mathbf{S}_{\mathbb{K}} \rightarrow \mathbf{S}$  are obviously simplicial functors: for any  $(A, m), (A', m') \in \text{Ob } \mathbf{S}^{\mathbb{T}}$ ,

$$U_{(A,m),(A',m')}^{\mathbb{T}} : \text{Map}_{\mathbf{S}^{\mathbb{T}}}((A, m), (A', m')) \rightarrow \text{Map}_{\mathbf{S}}(A, A')$$

is just the inclusion. The components of  $U_{\mathbb{K}}$  are also inclusions.

The next result, in which we further probe the nature of  $\text{Can}_{\mathbb{K}^{\mathbb{T}}}$  and  $\text{Can}^{\mathbb{T}_{\mathbb{K}}}$  as simplicial functors, is crucial to the proof of our criteria for homotopic (co)descent (Theorems 5.3 and 5.8) in section 5.

**Proposition 3.15.** *Let  $\mathbf{S}$  be a simplicially enriched category. If  $\mathbb{T} = (T, \mu, \eta)$  is a monad on  $\mathbf{S}$  such that  $T$  is a simplicial functor, and  $\mu$  and  $\eta$  are simplicial natural transformations, then*

$$(\text{Can}_{\mathbb{K}^{\mathbb{T}}})_{X, TY} : \text{Map}_{\mathbf{S}}(X, TY) \rightarrow \text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \text{Can}_{\mathbb{K}^{\mathbb{T}}}(TY))$$

is an isomorphism for all  $X, Y \in \text{Ob } \mathbf{S}$ .

Dually, if  $\mathbb{K} = (K, \Delta, \varepsilon)$  is a comonad on  $\mathbf{S}$  such that  $K$  is a simplicial functor, and  $\Delta$  and  $\varepsilon$  are simplicial natural transformations, then

$$(\text{Can}^{\mathbb{T}_{\mathbb{K}}})_{KX, Y} : \text{Map}_{\mathbf{S}}(KX, Y) \rightarrow \text{Map}_{\mathcal{D}^{co}(\mathbb{K})}(\text{Can}^{\mathbb{T}_{\mathbb{K}}}(KX), \text{Can}^{\mathbb{T}_{\mathbb{K}}}(Y))$$

is an isomorphism for all  $X, Y \in \text{Ob } \mathbf{S}$ .

*Proof.* As usual, we prove the monad case and leave the comonad case to the reader. The idea of the proof is to use the  $(F^{\mathbb{T}}, U^{\mathbb{T}})$ -adjunction and the  $(U_{\mathbb{K}^{\mathbb{T}}}, F_{\mathbb{K}^{\mathbb{T}}})$ -adjunction to prove the existence of isomorphisms

$$\text{Map}_{\mathbf{S}}(X, TY) \xrightarrow[\cong]{\alpha} \text{Map}_{\mathbf{S}^{\mathbb{T}}}(F^{\mathbb{T}}X, F^{\mathbb{T}}Y) \xleftarrow[\cong]{\beta} \text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \text{Can}_{\mathbb{K}^{\mathbb{T}}}(TY))$$

such that  $\alpha = \beta \circ (\text{Can}_{\mathbb{K}^{\mathbb{T}}})_{X, TY}$ , implying that  $(\text{Can}_{\mathbb{K}^{\mathbb{T}}})_{X, TY}$  is also an isomorphism, as desired.

It is not enough simply to apply Lemma 3.12, as we need an explicit description of the isomorphism  $\alpha$ . Moreover, since  $\mathbf{S}^{\mathbb{T}}$  may not be tensored over  $\mathbf{sSet}$  even if  $\mathbf{S}$  is, we cannot suppose that  $\mathcal{D}(\mathbb{T})$  is tensored and therefore cannot necessarily apply Lemma 3.12 to obtain the isomorphism  $\beta$ .

The simplicial map  $\alpha$  is defined by

$$\alpha(g) = (\mu_Y)_*(F_{X, TY}^{\mathbb{T}}(g)),$$

for any  $g \in \text{Map}_{\mathbf{S}}(X, TY)$  and its inverse  $\alpha' : \text{Map}_{\mathbf{S}^{\mathbb{T}}}(F^{\mathbb{T}}X, F^{\mathbb{T}}Y) \rightarrow \text{Map}_{\mathbf{S}}(X, TY)$  by

$$\alpha'(h) = (\eta_X)^*(h),$$

for any  $h \in \text{Map}_{\mathbf{S}^{\mathbb{T}}}(F^{\mathbb{T}}X, F^{\mathbb{T}}Y)$ . To see that  $\alpha'$  is indeed the inverse of  $\alpha$ , observe that

$$\begin{aligned} \alpha' \circ \alpha(g) &= (\eta_X)^*(\mu_Y)_*(F_{X, TY}^{\mathbb{T}}(g)) \\ &\stackrel{(1)}{=} (\mu_Y)_*(\eta_X)^*(F_{X, TY}^{\mathbb{T}}(g)) \\ &\stackrel{(2)}{=} (\mu_Y)_*(\eta_{TY})_*(g) \\ &= (\mu_Y \eta_{TY})_*(g) \\ &= g. \end{aligned}$$

Equality (1) follows from the associativity of simplicial composition, while equality (2) holds since  $\eta$  is a simplicial natural transformation. Furthermore,

$$\begin{aligned}
\alpha \circ \alpha'(h) &= (\mu_Y)_* F_{X,TY}^{\mathbb{T}}((\eta_X)^*(h)) \\
&\stackrel{(3)}{=} (\mu_Y)_*(\eta_{TX})^* F_{TX,TY}^{\mathbb{T}}(h) \\
&\stackrel{(4)}{=} (\eta_{TX})^*(\mu_Y)_* F_{TX,TY}^{\mathbb{T}}(h) \\
&\stackrel{(5)}{=} (\eta_{TX})^*(\mu_X)^*(h) \\
&= (\mu_X \eta_{TX})^*(h) \\
&= h.
\end{aligned}$$

Equality (3) holds because the simplicial functor  $F^{\mathbb{T}}$  respects composition. Associativity of simplicial composition implies equality (4), while equality (5) is a consequence of the fact that  $h$  is not just an element of  $\text{Map}_{\mathbf{S}}(TX, TY)$ , but actually an element of its simplicial subset  $\text{Map}_{\mathbf{S}^{\mathbb{T}}}(F^{\mathbb{T}}X, F^{\mathbb{T}}Y)$ .

As for the simplicial map  $\beta$ , it is given by

$$\beta(k) = (\mu_Y)_*(k),$$

for all  $k \in \text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \text{Can}_{\mathbb{K}^{\mathbb{T}}}(TY))$ , while its inverse  $\beta'$  satisfies

$$\beta'(h) = (\eta_{TX})^*((F_{\mathbb{K}^{\mathbb{T}}})_{F^{\mathbb{T}}X, F^{\mathbb{T}}Y}(g)).$$

This definition of  $\beta$  makes sense, since

$$\text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \text{Can}_{\mathbb{K}^{\mathbb{T}}}(TY)) \subset \text{Map}_{\mathbf{S}^{\mathbb{T}}}(F^{\mathbb{T}}X, F^{\mathbb{T}}(TY))$$

by construction, and the object in  $\mathbf{S}$  underlying  $F^{\mathbb{T}}(TY)$  is  $T^2Y$ . Moreover,  $\mu_Y$  is itself a map of  $\mathbb{T}$ -algebras. On the other hand, since  $F_{\mathbb{K}^{\mathbb{T}}}F^{\mathbb{T}}Y = \text{Can}_{\mathbb{K}^{\mathbb{T}}}(TY)$  and, as is easily checked,  $\eta_{TX} : TX \rightarrow T^2X$  underlies a morphism of descent data  $\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X) \rightarrow \text{Can}_{\mathbb{K}^{\mathbb{T}}}(TX)$ , the definition of  $\beta'$  is also acceptable. The proof that  $\beta$  and  $\beta'$  are mutually inverse strongly resembles the proof above that  $\alpha' = \alpha^{-1}$ , and we therefore omit it.

Since the components of the simplicial functor  $\text{Can}_{\mathbb{K}^{\mathbb{T}}}$  are corestrictions of the components of  $T$ , it is clear that  $\alpha = \beta \circ (\text{Can}_{\mathbb{K}^{\mathbb{T}}})_{X, TY}$ , and we can therefore conclude.  $\square$

#### 4. DERIVED COMPLETION AND COCOMPLETION

In this section we introduce the notion of the derived (co)completion along a (co)monad, which is clearly strongly influenced by the definition of Bousfield-Kan  $p$ -completion of simplicial sets [4], as well as by Carlsson's derived completion of module spectra along ring spectra [5]. Derived (co)completion plays an important role both in the formulation of conditions under which homotopic (co)descent holds (cf. Theorems 5.3 and 5.8) and in the analysis of the (co)descent spectral sequences (cf. Theorems 5.16 and 5.23). To conclude this section, we describe the very close relationship between assembly maps and derived cocompletion (Corollary 4.32).

##### 4.1. Derived completion along a monad.

*Convention 4.1.* Unless otherwise specified,  $\mathbb{T} = (T, \mu, \eta)$  denotes a monad on a simplicial model category  $\mathbf{M}$  such that  $U^{\mathbb{T}} : \mathbf{M}^{\mathbb{T}} \rightarrow \mathbf{M}$  right-induces a model category structure on  $\mathbf{M}^{\mathbb{T}}$  and  $U_{\mathbb{K}^{\mathbb{T}}} : \mathcal{D}(\mathbb{T}) \rightarrow \mathbf{M}^{\mathbb{T}}$  then left-induces a model category

structure on  $\mathcal{D}(\mathbb{T})$  (cf. section C). Assume that  $T$  is a simplicial functor, and that  $\mathbf{M}^{\mathbb{T}}$  and  $\mathcal{D}(\mathbb{T})$  are endowed with the simplicial enrichment of Lemma 3.9.

Note that it follows from the convention above that

$$\mathbf{M} \begin{array}{c} \xrightarrow{F^{\mathbb{T}}} \\ \xleftarrow{U^{\mathbb{T}}} \end{array} \mathbf{M}^{\mathbb{T}}, \quad \mathcal{D}(\mathbb{T}) \begin{array}{c} \xrightarrow{U_{\mathbb{K}^{\mathbb{T}}}} \\ \xleftarrow{F_{\mathbb{K}^{\mathbb{T}}}} \end{array} \mathbf{M}^{\mathbb{T}}, \quad \text{and} \quad \mathbf{M} \begin{array}{c} \xrightarrow{\text{Can}_{\mathbb{K}^{\mathbb{T}}}} \\ \xleftarrow{\text{Prim}_{\mathbb{K}^{\mathbb{T}}}} \end{array} \mathcal{D}(\mathbb{T})$$

are all Quillen pairs. Moreover,  $U^{\mathbb{T}}$  and  $U_{\mathbb{K}^{\mathbb{T}}}$  both preserve all weak equivalences.

By analogy with the definition of  $p$ -complete topological spaces, where  $p$  is a prime, we formulate the following definitions. We are also inspired by Carlsson's definition of the derived completion of module spectra along commutative ring spectra [5]. We begin by enlarging the class of morphisms considered to be equivalences.

**Definition 4.2.** Let  $X$  and  $Y$  be cofibrant objects in  $\mathbf{M}$ . A morphism  $f : X \rightarrow Y$  of in  $\mathbf{M}$  is a  $\mathbb{T}$ -equivalence if the induced morphism of simplicial sets

$$\text{Map}_{\mathbf{M}}(Y, U^{\mathbb{T}}A) \xrightarrow{f^*} \text{Map}_{\mathbf{M}}(X, U^{\mathbb{T}}A)$$

is a weak equivalence for all fibrant  $\mathbb{T}$ -algebras  $A$ .

*Remark 4.3.* Suppose that the simplicial enrichment on  $\mathbf{M}^{\mathbb{T}}$  satisfies the extra axiom of Definition B.8. Since  $F^{\mathbb{T}}$  is a left Quillen functor, it preserves weak equivalences between cofibrant objects. It follows that a weak equivalence  $f : X \rightarrow Y$  of cofibrant objects in  $\mathbf{M}$  induces a weak equivalence of simplicial sets

$$(F^{\mathbb{T}}f)^* : \text{Map}_{\mathbf{M}^{\mathbb{T}}}(F^{\mathbb{T}}Y, A) \xrightarrow{\sim} \text{Map}_{\mathbf{M}^{\mathbb{T}}}(F^{\mathbb{T}}X, A)$$

for all fibrant  $\mathbb{T}$ -algebras  $A$ . Lemma 3.12 (1) implies then that

$$f^* : \text{Map}_{\mathbf{M}}(Y, U^{\mathbb{T}}A) \xrightarrow{\sim} \text{Map}_{\mathbf{M}}(X, U^{\mathbb{T}}A)$$

must be a weak equivalence of simplicial sets for all fibrant  $\mathbb{T}$ -algebras  $A$ , i.e.,  $f$  is a  $\mathbb{T}$ -equivalence.

More generally, if  $f : X \rightarrow Y$  is any morphism of cofibrant objects in  $\mathbf{M}$  such that  $Tf$  is a weak equivalence, then  $f$  is a  $\mathbb{T}$ -equivalence. To prove this, recall that  $T = U^{\mathbb{T}}F^{\mathbb{T}}$  and that the model category structure on  $\mathbf{M}^{\mathbb{T}}$  is defined so that a morphism  $g$  of  $\mathbb{T}$ -algebras is a weak equivalence if and only if  $U^{\mathbb{T}}g$  is a weak equivalence. Thus, since  $Tf$  is a weak equivalence in  $\mathbf{M}$ ,  $F^{\mathbb{T}}f$  is a weak equivalence of cofibrant  $\mathbb{T}$ -algebras, and we can conclude by the same argument as above.

**Definition 4.4.** A fibrant object  $Z$  in  $\mathbf{M}$  is  $\mathbb{T}$ -complete if for every  $\mathbb{T}$ -equivalence  $f : X \rightarrow Y$ , the induced morphism of simplicial sets

$$\text{Map}_{\mathbf{M}}(Y, Z) \xrightarrow{f^*} \text{Map}_{\mathbf{M}}(X, Z)$$

is a weak equivalence.

*Remark 4.5.* Obviously, if  $A$  is a fibrant  $\mathbb{T}$ -algebra, then  $U^{\mathbb{T}}A$  is a  $\mathbb{T}$ -complete object of  $\mathbf{M}$ . In particular, if  $TX$  is a fibrant object of  $\mathbf{M}$ , then it is  $\mathbb{T}$ -complete, since  $F^{\mathbb{T}}X$  is then a fibrant  $\mathbb{T}$ -algebra.

A more interesting class of  $\mathbb{T}$ -complete objects can be formed as follows, using the  $\mathbb{T}$ -cobar construction.

**Lemma 4.6.** *Let  $Z$  be an object of  $\mathbf{M}$ . If the  $\mathbb{T}$ -cobar construction  $\Omega_{\mathbb{T}}^{\bullet}Z$  is Reedy fibrant, then  $\text{Tot} \Omega_{\mathbb{T}}^{\bullet}Z$  is  $\mathbb{T}$ -complete.*

*Proof.* Since  $\Omega_{\mathbb{T}}^{\bullet}Z$  is Reedy fibrant, its totalization is a fibrant object of  $\mathbf{M}$ . Moreover,  $\Omega_{\mathbb{T}}^n Z$  must be fibrant for all  $n$ . Consequently,  $F^{\mathbb{T}}(T^n Z)$  is a fibrant  $T$ -algebra for all  $n$  because  $\Omega_{\mathbb{T}}^n Z = T^{n+1}Z = U^{\mathbb{T}}F^{\mathbb{T}}(T^n Z)$  and  $U^{\mathbb{T}}$  right-induces the model category structure on  $\mathbf{M}^{\mathbb{T}}$ .

It follows that if  $f : X \rightarrow Y$  is a  $\mathbb{T}$ -equivalence, then

$$\mathrm{Map}_{\mathbf{M}}(Y, \Omega_{\mathbb{T}}^n Z) \xrightarrow{f^*} \mathrm{Map}_{\mathbf{M}}(X, \Omega_{\mathbb{T}}^n Z)$$

is a weak equivalence of simplicial sets for all  $n$ . Furthermore, since  $X$  and  $Y$  are cofibrant, the cosimplicial simplicial sets  $\mathrm{Map}_{\mathbf{M}}(X, \Omega_{\mathbb{T}}^{\bullet}Z)$  and  $\mathrm{Map}_{\mathbf{M}}(Y, \Omega_{\mathbb{T}}^{\bullet}Z)$  are both Reedy fibrant. The morphism of cosimplicial simplicial sets induced by  $f$ ,

$$f^{\bullet} : \mathrm{Map}_{\mathbf{M}}(Y, \Omega_{\mathbb{T}}^{\bullet}Z) \rightarrow \mathrm{Map}_{\mathbf{M}}(X, \Omega_{\mathbb{T}}^{\bullet}Z),$$

is thus a levelwise weak equivalence of Reedy fibrant objects. Applying  $\mathrm{Tot}$ , we obtain a weak equivalence of simplicial sets

$$\mathrm{Tot} f^{\bullet} : \mathrm{Tot} \mathrm{Map}_{\mathbf{M}}(Y, \Omega_{\mathbb{T}}^{\bullet}Z) \rightarrow \mathrm{Tot} \mathrm{Map}_{\mathbf{M}}(X, \Omega_{\mathbb{T}}^{\bullet}Z).$$

Recall that  $\mathrm{Tot}$  can be calculated as an equalizer. Moreover, if  $G : \mathbf{D} \rightarrow \mathbf{M}$  is a functor from any small category to a simplicial model category, then

$$\lim_{\mathbf{D}} \mathrm{Map}_{\mathbf{M}}(X, G) \cong \mathrm{Map}_{\mathbf{M}}(X, \lim_{\mathbf{D}} G).$$

Thus,

$$\mathrm{Tot} \mathrm{Map}_{\mathbf{M}}(W, \Omega_{\mathbb{T}}^{\bullet}Z) \cong \mathrm{Map}_{\mathbf{M}}(W, \mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet}Z)$$

for all objects  $W$ , and we can conclude.  $\square$

Lemma 4.6 justifies the next definition.

**Definition 4.7.** Let  $\widehat{X}$  be a fibrant replacement of an object  $X$  in  $\mathbf{M}$ . If  $\Omega_{\mathbb{T}}^{\bullet}\widehat{X}$  is Reedy fibrant, then  $\mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet}\widehat{X}$  is a *model of the derived  $\mathbb{T}$ -completion of  $X$* .

Homotopy invariance of derived  $\mathbb{T}$ -completion is guaranteed under the following conditions.

**Lemma 4.8.** *Let  $X$  and  $Y$  be objects in  $\mathbf{M}$ , and let  $\mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet}\widehat{X}$  and  $\mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet}\widehat{Y}$  be models of their derived  $\mathbb{T}$ -completions. If  $T$  preserves weak equivalences between fibrant objects, then any weak equivalence  $X \xrightarrow{\sim} Y$  induces a weak equivalence  $\mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet}\widehat{X} \xrightarrow{\sim} \mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet}\widehat{Y}$ .*

*Proof.* An elementary argument using the lifting axiom of a model category shows that a weak equivalence  $w : X \xrightarrow{\sim} Y$  induces a weak equivalence  $\widehat{w} : \widehat{X} \xrightarrow{\sim} \widehat{Y}$  between the fibrant replacements of  $X$  and  $Y$ . The hypotheses on  $T$  imply that  $T^n \widehat{w}$  is a weak equivalence for all  $n$  and therefore that

$$T^{\bullet+1} \widehat{w} : \Omega_{\mathbb{T}}^{\bullet}\widehat{X} \rightarrow \Omega_{\mathbb{T}}^{\bullet}\widehat{Y}$$

is a levelwise weak equivalence of Reedy fibrant cosimplicial objects. Consequently,

$$\mathrm{Tot} T^{\bullet+1} \widehat{w} : \mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet}\widehat{X} \rightarrow \mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet}\widehat{Y}$$

is a weak equivalence in  $\mathbf{M}$ .  $\square$

Lemmas 4.6 and 4.8 imply that all models of the derived  $\mathbb{T}$ -completion of an object of  $\mathbf{M}$  are indeed  $\mathbb{T}$ -complete, as well as weakly equivalent to each other if  $T$  preserves weak equivalences between fibrant objects, whence the notation introduced below.



*Notation 4.9.* Let  $X$  be an object in  $\mathbf{M}$ . If  $\mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet} \widehat{X}$  is a model of its derived  $\mathbb{T}$ -completion, then, abusing notation slightly, we write

$$X_{\mathbb{T}}^{\wedge} := \mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet} \widehat{X}.$$

*Remark 4.10.* If  $\mathbb{R}$  is the monad on  $\mathbf{sSet}$  of [4, I.2], then for all simplicial sets  $X$ , our notion of the derived  $\mathbb{R}$ -completion of  $X$  agrees with that of Bousfield and Kan.

A stronger, purely categorical notion of  $\mathbb{T}$ -completeness, which makes sense for a monad acting on almost any category, is necessary to our discussion of homotopic descent.

**Definition 4.11.** Let  $\mathbb{T}$  be a monad on a category  $\mathbf{C}$  that admits all finite limits and colimits. An object  $Z$  in  $\mathbf{M}$  is *strongly  $\mathbb{T}$ -complete* if the coaugmentation  $\eta^{\bullet} : cc^{\bullet}Z \rightarrow \Omega_{\mathbb{T}}^{\bullet}(Z)$  fits into an external cosimplicial SDR

$$cc^{\bullet}Z \begin{array}{c} \xrightarrow{\eta^{\bullet}} \\ \xleftarrow{\rho^{\bullet}} \end{array} \Omega_{\mathbb{T}}^{\bullet}(Z) \circlearrowleft h$$

(cf. Definition B.18).

We first show that the terminology chosen above is justified: under reasonable hypotheses, strong  $\mathbb{T}$ -completeness implies  $\mathbb{T}$ -completeness for fibrant objects.

**Lemma 4.12.** *Under the hypotheses of Convention 4.1, let  $Z$  be a fibrant object in  $\mathbf{M}$  such that  $\Omega_{\mathbb{T}}^{\bullet}(Z)$  is Reedy fibrant. If  $Z$  is strongly  $\mathbb{T}$ -complete, then it is  $\mathbb{T}$ -complete.*

*Proof.* By Corollary B.15, since  $Z$  is strongly  $\mathbb{T}$ -complete,

$$\mathrm{Tot} \eta^{\bullet} : \mathrm{Tot} cc^{\bullet}Z \rightarrow \mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet}(Z) = Z_{\mathbb{T}}^{\wedge}$$

is a weak equivalence, with fibrant source and target, since  $\mathrm{Tot}$  preserves fibrancy. On the other hand, since  $cc^{\bullet}Z$  is Reedy fibrant, it follows from Theorem 18.7.4 (2) in [11] that the Bousfield-Kan map

$$\mathrm{Tot} cc^{\bullet}Z \rightarrow \mathrm{holim} cc^{\bullet}Z$$

is a weak equivalence. Moreover, Example 18.3.8 and Theorem 18.4.16 (2) in [11] together imply that the natural map

$$Z \cong \lim cc^{\bullet}Z \rightarrow \mathrm{holim} cc^{\bullet}Z$$

is also a weak equivalence.

There is therefore a zig-zag of weak equivalences of fibrant objects

$$Z \xrightarrow{\sim} \mathrm{holim} cc^{\bullet}Z \xleftarrow{\sim} \mathrm{Tot} cc^{\bullet}Z \xrightarrow{\sim} Z_{\mathbb{T}}^{\wedge},$$

which implies that  $Z$  is  $\mathbb{T}$ -complete, since  $Z_{\mathbb{T}}^{\wedge}$  is  $\mathbb{T}$ -complete by Lemma 4.6.  $\square$

Strong  $\mathbb{T}$ -completeness is not an overly restrictive condition.

**Lemma 4.13.** *Suppose that  $\mathbb{T}$  is the monad associated to an adjunction*

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : U,$$

where  $\mathbf{C}$  admits all finite limits and colimits. If  $Y$  is any object of  $\mathbf{D}$ , then  $UY$  is strongly  $\mathbb{T}$ -complete.

*Remark 4.14.* When applied to the adjunction  $F^{\mathbb{T}} : \mathbf{C} \rightleftarrows \mathbf{C}^{\mathbb{T}} : U^{\mathbb{T}}$ , for some monad  $\mathbb{T}$  on  $\mathbf{C}$ , this lemma implies that  $U^{\mathbb{T}}(A, m)$  is strongly  $\mathbb{T}$ -complete for all  $\mathbb{T}$ -algebras  $(A, m)$ . In particular,  $TX$  is strongly  $\mathbb{T}$ -complete for all objects  $X$  in  $\mathbf{C}$ , since  $TX = U^{\mathbb{T}}F^{\mathbb{T}}X$ .

*Proof.* The argument applied here follows a well-trodden path. Let  $\varepsilon$  denote the counit of the  $(F, U)$ -adjunction. The coaugmented cosimplicial object

$$UY \xrightarrow{\eta_{UY}} \Omega_{\mathbb{T}}^{\bullet}(UY)$$

admits an “extra codegeneracy” at each level, i.e., is contractible. More explicitly, from level  $n$  to level  $n - 1$  we have

$$s^n = T^n(U\varepsilon_Y) : T^{n+1}(UY) \rightarrow T^n(UY),$$

commuting appropriately with the cofaces and the other codegeneracies of the  $\mathbb{T}$ -cobar construction. We leave this straightforward calculation to the reader.

By Proposition B.17, there is thus an external cosimplicial SDR

$$cc^{\bullet}UY \begin{array}{c} \xrightarrow{\eta^{\bullet}} \\ \xleftarrow{\rho^{\bullet}} \end{array} \Omega_{\mathbb{T}}^{\bullet}(UY) \circlearrowleft h,$$

i.e.,  $UY$  is strongly  $\mathbb{T}$ -complete.  $\square$

*Remark 4.15.* Let  $\mathbb{T}$  be a monad on a category  $\mathbf{C}$  admitting all finite limits and colimits. A careful analysis of Definition 4.11 leads one naturally to view strongly  $\mathbb{T}$ -complete objects as “ $\infty$ - $\mathbb{T}$ -algebras”. Fitting the coaugmentation  $\eta^{\bullet} : cc^{\bullet}Z \rightarrow \Omega_{\mathbb{T}}^{\bullet}(Z)$  into an external SDR

$$cc^{\bullet}Z \begin{array}{c} \xrightarrow{\eta^{\bullet}} \\ \xleftarrow{\rho^{\bullet}} \end{array} \Omega_{\mathbb{T}}^{\bullet}(Z) \circlearrowleft h$$

is equivalent to constructing a family

$$\{m_n : T^n Z \rightarrow T^{n-1} Z \mid n \geq 1\} \subset \text{Mor } \mathbf{C}$$

such that for all  $n \geq 1$  and all  $0 \leq i \leq n - 1$ ,

$$m_n \circ T^i \eta_{T^{n-i-1}Z} = \begin{cases} T^i \eta_{T^{n-i-2}Z} \circ m_{n-1} & : 0 \leq i \leq n - 2 \\ Id_{T^{n-1}Z} & : i = n - 1, \end{cases}$$

while

$$m_n \circ T^i \mu_{T^{n-i-1}Z} = \begin{cases} T^i \mu_{T^{n-i-2}Z} \circ m_{n+1} & : 0 \leq i \leq n - 2 \\ m_n \circ m_{n+1} & : i = n - 1 \end{cases}$$

The first two identities are the transcription of the relationship between the “extra codegeneracy” and the cofaces of the  $\mathbb{T}$ -cobar construction, while the last two specify the relationship of the “extra codegeneracy” to the other codegeneracies.

It is evident from the definition of a  $\mathbb{T}$ -algebra that if  $(Z, m)$  is a  $\mathbb{T}$ -algebra, then we can set

$$m_n = T^{n-1}m : T^n Z \rightarrow T^{n-1}Z$$

for all  $n \geq 1$ , and the identities above will be satisfied. Consequently, if an object of  $\mathbf{C}$  underlies a  $\mathbb{T}$ -algebra, then it is strongly  $\mathbb{T}$ -complete, whence our vision of strongly  $\mathbb{T}$ -complete objects as  $\infty$ - $\mathbb{T}$ -algebras.

## 4.2. Derived cocompletion along a comonad.

*Convention 4.16.* Unless otherwise specified,  $\mathbb{K} = (K, \Delta, \varepsilon)$  denotes a comonad on a simplicial model category  $\mathbf{M}$  such that  $U_{\mathbb{K}} : \mathbf{M}_{\mathbb{K}} \rightarrow \mathbf{M}$  left-induces a model category structure on  $\mathbf{M}_{\mathbb{K}}$  and  $U^{\mathbb{T}_{\mathbb{K}}} : \mathcal{D}^{co}(\mathbb{K}) \rightarrow \mathbf{M}_{\mathbb{K}}$  then right-induces a model category structure on  $\mathcal{D}^{co}(\mathbb{K})$  (cf. appendix C). Assume that  $K$  is a simplicial functor and that  $\mathbf{M}_{\mathbb{K}}$  and  $\mathcal{D}^{co}(\mathbb{K})$  are endowed with the simplicial enrichment of Lemma 3.9.

Note that it follows from the convention above that

$$\mathbf{M}_{\mathbb{K}} \begin{array}{c} \xrightarrow{U_{\mathbb{K}}} \\ \xleftarrow{F_{\mathbb{K}}} \end{array} \mathbf{M}, \quad \mathbf{M}_{\mathbb{K}} \begin{array}{c} \xrightarrow{F^{\mathbb{T}_{\mathbb{K}}}} \\ \xleftarrow{U^{\mathbb{T}_{\mathbb{K}}}} \end{array} \mathcal{D}^{co}(\mathbb{K}), \quad \text{and} \quad \mathcal{D}^{co}(\mathbb{K}) \begin{array}{c} \xrightarrow{Q^{\mathbb{T}_{\mathbb{K}}}} \\ \xleftarrow{\text{Can}^{\mathbb{T}_{\mathbb{K}}}} \end{array} \mathbf{M}_{\mathbb{K}}$$

are all Quillen pairs. Moreover,  $U_{\mathbb{K}}$  and  $U^{\mathbb{T}_{\mathbb{K}}}$  both preserve all weak equivalences.

All of the proofs in this section are formally dual to those in the previous section, so they are omitted.

Dualizing the construction of the previous section, we formulate the following definitions. We begin by enlarging the class of morphisms considered to be equivalences.

**Definition 4.17.** Let  $X$  and  $Y$  be fibrant objects in  $\mathbf{M}$ . A morphism  $f : X \rightarrow Y$  of in  $\mathbf{M}$  is a  $\mathbb{K}$ -equivalence if the induced morphism of simplicial sets

$$\text{Map}_{\mathbf{M}}(U_{\mathbb{K}}C, X) \xrightarrow{f_*} \text{Map}_{\mathbf{M}}(U_{\mathbb{K}}C, Y)$$

is a weak equivalence for all cofibrant  $\mathbb{K}$ -coalgebras  $C$ .

*Remark 4.18.* Arguments strictly dual to those in Remark 4.3 show that, if the additional axiom of Definition B.8 holds for the simplicial enrichment of  $\mathbf{M}_{\mathbb{K}}$ , then any weak equivalence  $f : X \rightarrow Y$  of fibrant objects in  $\mathbf{M}$  is a  $\mathbb{K}$ -equivalence. More generally, if  $f : X \rightarrow Y$  is any morphism of fibrant objects in  $\mathbf{M}$  such that  $Kf$  is a weak equivalence, then  $f$  is a  $\mathbb{K}$ -equivalence.

**Definition 4.19.** A cofibrant object  $Z$  in  $\mathbf{M}$  is  $\mathbb{K}$ -cocomplete if for every  $\mathbb{K}$ -equivalence  $f : X \rightarrow Y$ , the induced morphism of simplicial sets

$$\text{Map}_{\mathbf{M}}(Z, X) \xrightarrow{f_*} \text{Map}_{\mathbf{M}}(Z, Y)$$

is a weak equivalence.

*Remark 4.20.* Obviously, if  $C$  is a cofibrant  $\mathbb{K}$ -algebra, then  $U_{\mathbb{K}}C$  is a  $\mathbb{K}$ -cocomplete object of  $\mathbf{M}$ . In particular, if  $KX$  is a cofibrant object of  $\mathbf{M}$ , then it is  $\mathbb{K}$ -complete, since  $F_{\mathbb{K}}X$  is then cofibrant  $\mathbb{K}$ -coalgebra.

A more interesting class of  $\mathbb{K}$ -cocomplete objects can be formed as follows, using the  $\mathbb{K}$ -bar construction.

**Lemma 4.21.** *Let  $Z$  be an object of  $\mathbf{M}$ . If the  $\mathbb{K}$ -bar construction  $\mathcal{B}_{\bullet}^{\mathbb{K}}Z$  is Reedy cofibrant, then  $|\mathcal{B}_{\bullet}^{\mathbb{K}}Z|$  is  $\mathbb{K}$ -cocomplete.*

Lemma 4.21 justifies the next definition.

**Definition 4.22.** Let  $\tilde{X}$  be a cofibrant replacement of an object  $X$  in  $\mathbf{M}$ . If  $\mathcal{B}_{\bullet}^{\mathbb{K}}\tilde{X}$  is Reedy cofibrant, then  $|\mathcal{B}_{\bullet}^{\mathbb{K}}\tilde{X}|$  is a *model of the derived  $\mathbb{K}$ -cocompletion of  $X$* .

Homotopy invariance of derived  $\mathbb{K}$ -cocompletion is guaranteed under the following conditions.

**Lemma 4.23.** *Let  $X$  and  $Y$  be objects in  $\mathbf{M}$ , and let  $\mathcal{B}_{\bullet}^{\mathbb{K}}\tilde{X}$  and  $\mathcal{B}_{\bullet}^{\mathbb{K}}\tilde{Y}$  be models of their derived  $\mathbb{K}$ -cocompletions. If  $K$  preserves weak equivalences between cofibrant objects, then any weak equivalence  $X \xrightarrow{\sim} Y$  induces a weak equivalence  $|\mathcal{B}_{\bullet}^{\mathbb{K}}\tilde{X}| \xrightarrow{\sim} |\mathcal{B}_{\bullet}^{\mathbb{K}}\tilde{Y}|$ .*

Lemmas 4.21 and 4.23 imply that all models of the derived  $\mathbb{K}$ -cocompletion of an object of  $\mathbf{M}$  are indeed  $\mathbb{K}$ -cocomplete, as well as weakly equivalent to each other if  $K$  preserves weak equivalences between cofibrant objects, and thus motivate the notation introduced below.

*Notation 4.24.* Let  $X$  be an object in  $\mathbf{M}$ . If  $\mathcal{B}_{\bullet}^{\mathbb{K}}\tilde{X}$  is a model of its derived  $\mathbb{K}$ -cocompletion, then, abusing notation slightly, we write

$$X_{\mathbb{K}}^{\vee} := |\mathcal{B}_{\bullet}^{\mathbb{K}}\tilde{X}|.$$

A stronger, purely categorical notion of  $\mathbb{K}$ -cocompleteness, which can be defined for a comonad acting on almost any category, is necessary to our discussion of homotopic codescent.

**Definition 4.25.** Let  $\mathbb{K}$  be a comonad on a category  $\mathbf{D}$  that admits all finite limits and colimits. An object  $Z$  in  $\mathbf{D}$  is *strongly  $\mathbb{K}$ -cocomplete* if the augmentation  $\varepsilon_{\bullet} : \mathcal{B}_{\bullet}^{\mathbb{K}}Z \rightarrow cs_{\bullet}Z$  fits into an external simplicial SDR

$$cs_{\bullet}Z \begin{array}{c} \xrightarrow{\sigma_{\bullet}} \\ \xleftarrow{\varepsilon_{\bullet}} \end{array} \mathcal{B}_{\bullet}^{\mathbb{K}}Z \circlearrowleft h$$

(cf. Definition B.18).

The terminology chosen above is justified: under reasonable hypotheses, strong  $\mathbb{K}$ -cocompleteness implies  $\mathbb{K}$ -cocompleteness for cofibrant objects.

**Lemma 4.26.** *Under the hypotheses of Convention 4.16, let  $Z$  be a cofibrant object in  $\mathbf{M}$  such that  $\mathcal{B}_{\bullet}^{\mathbb{K}}Z$  is Reedy cofibrant. If  $Z$  is strongly  $\mathbb{K}$ -cocomplete, then it is  $\mathbb{K}$ -cocomplete.*

Strong  $\mathbb{K}$ -cocompleteness is also not an overly restrictive condition.

**Lemma 4.27.** *Suppose that  $\mathbb{K}$  is the comonad associated to an adjunction*

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : U.$$

*If  $X$  is any object of  $\mathbf{C}$ , then  $FX$  is strongly  $\mathbb{K}$ -cocomplete.*

*Remark 4.28.* When applied to the adjunction  $U_{\mathbb{K}} : \mathbf{D}_{\mathbb{K}} \rightleftarrows \mathbf{D} : F_{\mathbb{K}}$ , for some comonad  $\mathbb{K}$  on  $\mathbf{D}$ , this lemma implies that  $U_{\mathbb{K}}(C, \delta)$  is strongly  $\mathbb{K}$ -cocomplete for all  $\mathbb{K}$ -coalgebras  $(C, \delta)$ . In particular,  $KY$  is strongly  $\mathbb{K}$ -cocomplete for all objects  $Y$  in  $\mathbf{D}$ , since  $KY = U_{\mathbb{K}}F_{\mathbb{K}}Y$ .

*Remark 4.29.* Let  $\mathbb{K}$  be a comonad on a category  $\mathbf{D}$  admitting all finite limits and colimits. A careful analysis of Definition 4.25 leads one naturally to view strongly  $\mathbb{K}$ -complete objects as “ $\infty$ - $\mathbb{K}$ -coalgebras”. The formulas involved are dual to those in Remark 4.15, and lead easily to the conclusion that if an object of  $\mathbf{D}$  underlies a  $\mathbb{K}$ -coalgebra, then it is strongly  $\mathbb{K}$ -cocomplete, whence our vision of strongly  $\mathbb{K}$ -cocomplete objects as  $\infty$ - $\mathbb{K}$ -coalgebras.

**4.3. Assembly, derived cocompletion and isomorphism conjectures.** In this section, as an indication of the importance of the theory developed above, we describe the intriguing relationship between assembly maps and derived cocompletion.

Let  $\Phi : \mathbf{C} \rightarrow \mathbf{D}$  be a functor between small categories, and let  $\mathbf{M}$  be a simplicial model category that is combinatorial or cofibrantly generated. The functor categories  $\mathbf{M}^{\mathbf{C}}$  and  $\mathbf{M}^{\mathbf{D}}$  are then also simplicial model categories that are combinatorial or cofibrantly generated, where fibrations and weak equivalences are defined objectwise.

The functor  $\Phi$  induces a Quillen pair of adjoint functors

$$\Phi_* : \mathbf{M}^{\mathbf{C}} \rightleftarrows \mathbf{M}^{\mathbf{D}} : \Phi^*,$$

where, if  $X : \mathbf{C} \rightarrow \mathbf{M}$  is any functor, then  $\Phi_*(X) : \mathbf{D} \rightarrow \mathbf{M}$  denotes its left Kan extension. We let  $\mathbb{K}_\Phi = (K_\Phi, \Delta_\Phi, \varepsilon_\Phi)$  denote the comonad associated to this adjunction, i.e.,

$$K_\Phi := \Phi_* \Phi^* : \mathbf{M}^{\mathbf{D}} \rightarrow \mathbf{M}^{\mathbf{D}}.$$

Recall that for all  $X \in \mathbf{M}^{\mathbf{D}}$  and all  $D \in \text{Ob } \mathbf{D}$ ,

$$K_\Phi(X)(D) = \Phi_* \Phi^*(X)(D) = \text{colim}_{\mathbf{Simp}_\Phi(D)} (X \circ \text{dom}),$$

where  $\mathbf{Simp}_\Phi(D)$  is the full subcategory of the overcategory  $\mathbf{D}/D$  with object set

$$\{f : \Phi(C) \rightarrow D \in \text{Mor } \mathbf{D} \mid C \in \text{Ob } \mathbf{C}\},$$

and

$$\text{dom} : \mathbf{Simp}_\Phi(D) \rightarrow \mathbf{D} : (\Phi(C) \xrightarrow{f} D) \mapsto \Phi(C)$$

is the “domain” functor. Since  $\mathbf{M}$  is a model category, it is natural to replace the colimit in the formula above for  $K_\Phi(X)(D)$  by a homotopy colimit, obtaining a homotopy invariant replacement

$$K_\Phi^{ho}(X)(D) = \text{hocolim}_{\mathbf{Simp}_\Phi(D)} (X \circ \text{dom}),$$

of the comonad  $K_\Phi$ , built from what is called the *homotopy left Kan extension* of  $\Phi^*(X)$ . The *assembly map* of the functor  $X : \mathbf{D} \rightarrow \mathbf{M}$  evaluated at an object  $D$  is then the canonical map

$$A_{X,D} : K_\Phi^{ho}(X)(D) \rightarrow X(D).$$

On the other hand, for all  $X \in \mathbf{M}^{\mathbf{D}}$  and all  $D \in \text{Ob } \mathbf{D}$ ,

$$|\mathcal{B}_\bullet^{\mathbb{K}_\Phi}(X)(D)| \in \mathbf{M},$$

which is just  $X_{\mathbb{K}_\Phi}^\vee(D)$  if  $\mathcal{B}_\bullet^{\mathbb{K}_\Phi}(X)$  is Reedy cofibrant, is another natural “homotopy replacement” of  $K_\Phi(X)(D)$ , and the natural map

$$B_{X,D} : |\mathcal{B}_\bullet^{\mathbb{K}_\Phi}(X)(D)| \rightarrow X(D)$$

could also reasonably be termed an “assembly map.” We show below that the two notions of assembly map are equivalent under reasonable cofibrancy conditions.

The article [1] of Balmer and Matthey on model-theoretic formulations of the Baum-Connes and Farrell-Jones conjectures motivates our examination of the relationship between  $A_{X,D}$  and  $B_{X,D}$ . Let  $G$  be a discrete group, and let  $\mathbf{D}$  be the orbit category of  $G$ , while  $\mathbf{C}$  is its full subcategory with as objects those quotients  $G/H$  where  $H$  is virtually cyclic. Let  $\Phi$  denote the inclusion of  $\mathbf{C}$  into  $\mathbf{D}$ . Let  $X_G$  denote a functor from  $\mathbf{D}$  into some nice simplicial model category of spectra  $\mathbf{M}$

such that  $\pi_*(X_G(G/H))$  is canonically isomorphic to  $K_*(H)$  for every subgroup  $H$  of  $G$ . Here,  $K_*(-)$  can denote any of a number of relevant  $K$ - or  $L$ -theories, for which we refer the reader to the extensive literature, but in particular to [1].

The Isomorphism Conjecture for  $G$  at a subgroup  $H$  with respect to the chosen  $K$ - or  $L$ -theory states that

$$A_{X_G, G/H} : K_{\Phi}^{ho}(X_G)(G/H) \rightarrow X_G(G/H)$$

should be a weak equivalence of spectra. If  $A_{X_G, G/H}$  is weakly equivalent to  $B_{X_G, G/H}$  for all  $H \leq G$ , then the Isomorphism Conjecture holds at all  $H \leq G$  if and only if  $X_G$  is derived  $\mathbb{K}_{\Phi}$ -cocomplete. One could apply such an equivalence in two directions: to obtain examples of  $\mathbb{K}_{\Phi}$ -cocomplete objects in those cases where the Isomorphism Conjectures are known to hold and to gain potentially useful insight into the meaning of the Isomorphism Conjectures in still-unresolved cases.

Motivated by this discussion of the Baum-Connes and Farrell-Jones conjectures, we formulate the following definition.

**Definition 4.30.** Let  $\Phi : \mathbf{C} \rightarrow \mathbf{D}$  be a functor between small categories, let  $\mathbf{M}$  be a simplicial model category that is combinatorial or cofibrantly generated, and let  $D$  be an object of  $\mathbf{D}$ . A functor  $X : \mathbf{D} \rightarrow \mathbf{M}$  satisfies the  $\Phi$ -Isomorphism Conjecture at  $D$  if  $A_{X, D} : K_{\Phi}^{ho}(X)(D) \rightarrow X(D)$  is a weak equivalence. If  $A_{X, D}$  is a weak equivalence for all  $D$ , then  $X$  satisfies the  $\Phi$ -Isomorphism Conjecture

The theorem below enables us to characterize those functors that satisfy Isomorphism Conjectures in terms of derived cocompletion.

**Theorem 4.31.** Let  $\Phi : \mathbf{C} \rightarrow \mathbf{D}$  be a functor between small categories, and let  $\mathbf{M}$  be a simplicial model category that is combinatorial or cofibrantly generated.

If  $X : \mathbf{D} \rightarrow \mathbf{M}$  is a functor such that  $\mathcal{B}_{\bullet}^{\mathbb{K}_{\Phi}} X$  is a Reedy cofibrant object in  $(\mathbf{M}^{\mathbf{D}})^{\Delta^{op}}$ , then for all  $D \in \mathbf{D}$  there is a commutative diagram in  $\mathbf{M}$ :

$$\begin{array}{ccc} K_{\Phi}^{ho}(X)(D) & & \\ \sim \uparrow & \searrow^{A_{X, D}} & \\ \bullet & & X(D). \\ \sim \downarrow & \nearrow_{B_{X, D}} & \\ X_{\mathbb{K}_{\Phi}}^{\vee}(D) & & \end{array}$$

In order not to interrupt further the general flow of this article, we refer the reader to appendix A for the proof of this theorem.

The following characterization is an immediate consequence of Theorem 4.31.

**Corollary 4.32.** Let  $\Phi : \mathbf{C} \rightarrow \mathbf{D}$  be a functor between small categories, and let  $\mathbf{M}$  be a simplicial model category that is combinatorial or cofibrantly generated. A functor  $X : \mathbf{D} \rightarrow \mathbf{M}$  satisfies the  $\Phi$ -Isomorphism Conjecture if and only if it is derived  $\mathbb{K}_{\Phi}$ -cocomplete.

It is, of course, possible to dualize all of the discussion above, to the case of (homotopy) right Kan extensions and derived completion.

5. HOMOTOPIC DESCENT AND CODESCENT AND ASSOCIATED SPECTRAL SEQUENCES

We are now ready to introduce our theory of homotopic (co)descent for simplicially enriched categories. Since simplicially enriched categories are a model of  $(\infty, 1)$ -categories (also called  $\infty$ -categories by homotopy theorists), our theory can also be considered to be a model for the theory of  $\infty$ -(co)descent.

In the case of a (co)monad acting on simplicial model category, we provide criteria for homotopic (co)descent (Theorems 5.3 and 5.8), which are strongly analogous to the Main Theorem in Mandell’s paper [17]. The criteria are formulated in terms of strong (co)completion, providing another important motivation for the study of these notions in the previous section.

In the last part of this section we consider spectral sequences that arise naturally from the action of a (co)monad on a simplicial model category. The notions of both derived (co)completion and homotopic (co)descent play a role in interpreting these spectral sequences.

Many well-known spectral sequences, such as the Adams and Adams-Novikov spectral sequences and various “descent” spectral sequences for bundles, arise as (co)descent spectral sequences in the way we describe here. In the section 6 we treat certain of these examples in detail.

**5.1. Homotopic descent.** The definition of homotopic descent requires only simplicial enrichment and not the full structure of a simplicial model category. We suggest that the reader compare the definition below to Definition 2.11 and Proposition 3.15.

**Definition 5.1.** Let  $\mathbf{S}$  be a simplicially enriched category. Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on  $\mathbf{S}$  such that  $T$  is a simplicial functor, and  $\mu$  and  $\eta$  are simplicial natural transformations.

If each component

$$(\text{Can}_{\mathbb{T}})_{X,Y} : \text{Map}_{\mathbf{S}}(X, Y) \rightarrow \text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{T}}(X), \text{Can}_{\mathbb{T}}(Y))$$

of the simplicial functor  $\text{Can}_{\mathbb{T}} : \mathbf{S} \rightarrow \mathcal{D}(\mathbb{T})$  is a weak equivalence of simplicial sets, then  $\mathbb{T}$  satisfies *homotopic descent*. If, in addition, every descent datum is isomorphic in the path-component category  $\pi_0 \mathcal{D}(\mathbb{T})$  (cf. Definition B.1) to an object of the form  $\text{Can}_{\mathbb{T}}(X)$ , then  $\mathbb{T}$  satisfies *effective homotopic descent*.

Our criterion for homotopic descent does require model category structure and is expressed in terms of the following important subcategory of the underlying model category.

*Notation 5.2.* If  $\mathbf{M}$  is a simplicial model category, and  $\mathbb{T} = (T, \mu, \eta)$  is a monad on  $\mathbf{M}$ , we let  $\mathbf{M}_{\mathbb{T}}^{\wedge}$  denote the full simplicial subcategory of  $\mathbf{M}$  determined by the bifibrant, strongly  $\mathbb{T}$ -complete objects.

The observation in Remark 4.14 implies that if  $T$  preserves bifibrant objects, then  $T$  fixes  $\mathbf{M}_{\mathbb{T}}^{\wedge}$ , i.e.,  $\mathbb{T}$  restricts and corestricts to a monad on  $\mathbf{M}_{\mathbb{T}}^{\wedge}$ , which we denote  $\widehat{\mathbb{T}}$ .

**Theorem 5.3.** *Let  $\mathbf{M}$  be a simplicial model category. Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on  $\mathbf{M}$  such that  $T$  is a simplicial functor preserving bifibrant objects, and  $\mu$  and  $\eta$  are simplicial naturals.*

*If  $\Omega_{\mathbb{T}}^{\bullet}$  preserves fibrant objects, then  $\widehat{\mathbb{T}}$  satisfies homotopic descent.*

Theorem 5.3 is an immediate consequence of the following proposition, which generalizes Proposition 3.15, in the case where the underlying category is a simplicial model category.

**Proposition 5.4.** *Let  $\mathbf{M}$  be a simplicial model category. Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on  $\mathbf{M}$  such that  $T$  is a simplicial functor, and  $\mu$  and  $\eta$  are simplicial natural transformations.*

*If  $Y$  is fibrant and strongly  $\mathbb{T}$ -complete, and  $\Omega_{\mathbb{T}}^{\bullet} Y$  is Reedy fibrant, then*

$$(\text{Can}_{\mathbb{K}^{\mathbb{T}}})_{X,Y} : \text{Map}_{\mathbf{M}}(X, Y) \rightarrow \text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \text{Can}_{\mathbb{K}^{\mathbb{T}}}(Y))$$

*is a weak equivalence of simplicial sets, for all cofibrant objects  $X$ .*

*Proof.* Since  $Y$  is strongly  $\mathbb{T}$ -complete, there is an external cosimplicial SDR

$$cc^{\bullet} Y \begin{array}{c} \xrightarrow{\eta^{\bullet}} \\ \xleftarrow{\rho^{\bullet}} \end{array} \Omega_{\mathbb{T}}^{\bullet}(Y) \circlearrowleft h.$$

Applying  $\text{Can}_{\mathbb{K}^{\mathbb{T}}}$  levelwise, we obtain a new SDR

$$cc^{\bullet} \text{Can}_{\mathbb{K}^{\mathbb{T}}}(Y) \begin{array}{c} \xrightarrow{\text{Can}_{\mathbb{K}^{\mathbb{T}}} \eta^{\bullet}} \\ \xleftarrow{\text{Can}_{\mathbb{K}^{\mathbb{T}}} \rho^{\bullet}} \end{array} \text{Can}_{\mathbb{K}^{\mathbb{T}}}^{\Delta} \Omega_{\mathbb{T}}^{\bullet}(Y) \circlearrowleft h.$$

By Lemma 3.3  $\text{Can}_{\mathbb{K}^{\mathbb{T}}}^{\Delta} \Omega_{\mathbb{T}}^{\bullet}(Y)$  is Reedy fibrant, which implies that  $cc^{\bullet} \text{Can}_{\mathbb{K}^{\mathbb{T}}}(Y)$  is as well, since a retract of a fibrant object is always fibrant, in any model category.

Consider the following commutative diagram of maps of fibrant cosimplicial simplicial sets, in which the righthand vertical arrow is an isomorphism by Proposition 3.15.

$$\begin{array}{ccc} \text{Map}_{\mathbf{M}}(X, cc^{\bullet} Y) & \begin{array}{c} \xrightarrow{\eta^{\bullet}} \\ \xleftarrow{\rho^{\bullet}} \end{array} & \text{Map}_{\mathbf{M}}(X, \Omega_{\mathbb{T}}^{\bullet}(Y)) \circlearrowleft h \\ \text{(Can}_{\mathbb{K}^{\mathbb{T}}})_{X, cc^{\bullet} Y} \downarrow & & \cong \downarrow \text{(Can}_{\mathbb{K}^{\mathbb{T}}})_{X, \Omega_{\mathbb{T}}^{\bullet} Y} \\ \text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), cc^{\bullet} \text{Can}_{\mathbb{K}^{\mathbb{T}}}(Y)) & \begin{array}{c} \xrightarrow{\text{Can}_{\mathbb{K}^{\mathbb{T}}} \eta^{\bullet}} \\ \xleftarrow{\text{Can}_{\mathbb{K}^{\mathbb{T}}} \rho^{\bullet}} \end{array} & \text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \text{Can}_{\mathbb{K}^{\mathbb{T}}}^{\Delta} \Omega_{\mathbb{T}}^{\bullet}(Y)) \circlearrowleft h \end{array}$$

The top and the bottom of this diagram consist of external cosimplicial SDR's of simplicial sets.

By Corollary B.15, applying  $\text{Tot}$  to the diagram above leads to the commutative diagram of fibrant simplicial sets

$$\begin{array}{ccc} \text{Tot Map}_{\mathbf{M}}(X, cc^{\bullet} Y) & \begin{array}{c} \xrightarrow{\sim} \\ \xleftarrow{\sim} \end{array} & \text{Tot Map}_{\mathbf{M}}(X, \Omega_{\mathbb{T}}^{\bullet}(Y)) \\ \text{Tot}(\text{Can}_{\mathbb{K}^{\mathbb{T}}})_{X, cc^{\bullet} Y} \downarrow & & \cong \downarrow \text{Tot}(\text{Can}_{\mathbb{K}^{\mathbb{T}}})_{X, \Omega_{\mathbb{T}}^{\bullet} Y} \\ \text{Tot Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), cc^{\bullet} \text{Can}_{\mathbb{K}^{\mathbb{T}}}(Y)) & \begin{array}{c} \xrightarrow{\sim} \\ \xleftarrow{\sim} \end{array} & \text{Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \text{Can}_{\mathbb{K}^{\mathbb{T}}}^{\Delta} \Omega_{\mathbb{T}}^{\bullet}(Y)), \end{array}$$

whence

$$\text{Tot}(\text{Can}_{\mathbb{K}^{\mathbb{T}}})_{X, cc^{\bullet} Y} : \text{Tot Map}_{\mathbf{M}}(X, cc^{\bullet} Y) \rightarrow \text{Tot Map}_{\mathcal{D}(\mathbb{T})}(\text{Can}_{\mathbb{K}^{\mathbb{T}}}(X), cc^{\bullet} \text{Can}_{\mathbb{K}^{\mathbb{T}}}(Y))$$



is a weak equivalence. Chasing through the commutative diagram

$$\begin{array}{ccc}
 \mathrm{Tot} \mathrm{Map}_{\mathbf{M}}(X, cc^\bullet Y) & \xrightarrow[\sim]{\mathrm{Tot}(\mathrm{Can}_{\mathbb{K}^\tau})_{X, cc^\bullet Y}} & \mathrm{Tot} \mathrm{Map}_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}_{\mathbb{K}^\tau}(X), cc^\bullet \mathrm{Can}_{\mathbb{K}^\tau}(Y)) \\
 \downarrow \sim & & \downarrow \sim \\
 \mathrm{holim} \mathrm{Map}_{\mathbf{M}}(X, cc^\bullet Y) & \xrightarrow{\mathrm{holim}(\mathrm{Can}_{\mathbb{K}^\tau})_{X, cc^\bullet Y}} & \mathrm{holim} \mathrm{Map}_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}_{\mathbb{K}^\tau}(X), cc^\bullet \mathrm{Can}_{\mathbb{K}^\tau}(Y)) \\
 \uparrow \sim & & \uparrow \sim \\
 \mathrm{lim} \mathrm{Map}_{\mathbf{M}}(X, cc^\bullet Y) & \xrightarrow{\mathrm{lim}(\mathrm{Can}_{\mathbb{K}^\tau})_{X, cc^\bullet Y}} & \mathrm{lim} \mathrm{Map}_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}_{\mathbb{K}^\tau}(X), cc^\bullet \mathrm{Can}_{\mathbb{K}^\tau}(Y)) \\
 \uparrow \cong & & \uparrow \cong \\
 \mathrm{Map}_{\mathbf{M}}(X, Y) & \xrightarrow{(\mathrm{Can}_{\mathbb{K}^\tau})_{X, Y}} & \mathrm{Map}_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}_{\mathbb{K}^\tau}(X), \mathrm{Can}_{\mathbb{K}^\tau}(Y)),
 \end{array}$$

in which the vertical arrows are weak equivalences by Theorem 18.7.4 (2), Example 18.3.8 and Theorem 18.4.16 (2) in [11], we conclude that

$$(\mathrm{Can}_{\mathbb{K}^\tau})_{X, Y} : \mathrm{Map}_{\mathbf{M}}(X, Y) \rightarrow \mathrm{Map}_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}_{\mathbb{K}^\tau}(X), \mathrm{Can}_{\mathbb{K}^\tau}(Y))$$

is a weak equivalence as desired.  $\square$

*Remark 5.5.* Let  $\mathbf{M}$  be a simplicial model category. Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on  $\mathbf{M}$  such that  $T$  is a simplicial functor, and  $\mu$  and  $\eta$  are simplicial natural transformations. Let  $\mathbb{DT}$  denote the monad on  $\mathcal{D}(\mathbb{T})$  with underlying endofunctor  $F_{\mathbb{K}^\tau} U_{\mathbb{K}^\tau}$ .

According to Remark 2.6

$$\mathrm{Can}_{\mathbb{K}^\tau} T = \mathrm{Can}_{\mathbb{K}^\tau}(U^\mathbb{T} F^\mathbb{T}) = F_{\mathbb{K}^\tau} F^\mathbb{T} = (F_{\mathbb{K}^\tau} U_{\mathbb{K}^\tau}) \mathrm{Can}_{\mathbb{K}^\tau},$$

whence, for all objects  $Y$  in  $\mathbf{M}$ ,

$$\mathrm{Can}_{\mathbb{K}^\tau}^\Delta \Omega_{\mathbb{T}}^\bullet(Y) = \Omega_{\mathbb{DT}}^\bullet(\mathrm{Can}_{\mathbb{K}^\tau}(Y)).$$

Thus, if  $Y$  is strongly  $\mathbb{T}$ -complete, i.e., there is an external cosimplicial SDR

$$cc^\bullet Y \begin{array}{c} \xrightarrow{\eta^\bullet} \\ \xleftarrow{\rho^\bullet} \end{array} \Omega_{\mathbb{T}}^\bullet(Y) \circlearrowleft h,$$

then  $\mathrm{Can}_{\mathbb{K}^\tau}(Y)$  is strongly  $\mathbb{DT}$ -complete, as there is an external cosimplicial SDR

$$cc^\bullet \mathrm{Can}_{\mathbb{K}^\tau}(Y) \begin{array}{c} \xrightarrow{\mathrm{Can}_{\mathbb{K}^\tau} \eta^\bullet} \\ \xleftarrow{\mathrm{Can}_{\mathbb{K}^\tau} \rho^\bullet} \end{array} \Omega_{\mathbb{DT}}^\bullet(\mathrm{Can}_{\mathbb{K}^\tau}(Y)) \circlearrowleft h.$$

The image of the restriction of  $\mathrm{Can}_{\mathbb{K}^\tau}$  to  $\mathbf{M}_{\mathbb{T}}^\Delta$  therefore consists of strongly  $\mathbb{DT}$ -complete descent data.

It is natural to wonder under what conditions any strongly  $\mathbb{DT}$ -complete descent datum is weakly equivalent to the canonical descent datum of a strongly  $\mathbb{T}$ -complete object of  $\mathbf{M}$ . We intend to investigate this question in future work, i.e., to determine when the homotopic descent of Theorem 5.3 is effective.

**5.2. Homotopic codescent.** We now dualize the definitions and results of the previous section. Since the proofs dualize as well, we omit them.

We suggest this time that the reader compare the definition below to Definition 2.14 and Proposition 3.15.

**Definition 5.6.** Let  $\mathbf{S}$  be a simplicially enriched category. Let  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a comonad on  $\mathbf{S}$  such that  $K$  is a simplicial functor, and  $\Delta$  and  $\varepsilon$  are simplicial natural transformations.

If each component

$$(\text{Can}^{\mathbb{K}})_{X,Y} : \text{Map}_{\mathbf{S}}(X, Y) \rightarrow \text{Map}_{\mathcal{D}^{co}(\mathbb{K})}(\text{Can}^{\mathbb{K}}(X), \text{Can}^{\mathbb{K}}(Y))$$

of the simplicial functor  $\text{Can}^{\mathbb{K}} : \mathbf{S} \rightarrow \mathcal{D}^{co}(\mathbb{K})$  is a weak equivalence of simplicial sets, then  $\mathbb{K}$  satisfies *homotopic codescent*. If, in addition, every codescent datum is isomorphic in the path-component category  $\pi_0 \mathcal{D}^{co}(\mathbb{K})$  to an object of the form  $\text{Can}^{\mathbb{K}}(X)$ , then  $\mathbb{K}$  satisfies *effective homotopic descent*.

The criterion for homotopic codescent for comonads on simplicial model categories is formulated in terms of the following subcategory of the underlying model category.

*Notation 5.7.* If  $\mathbf{M}$  is a simplicial model category, and  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a comonad on  $\mathbf{M}$ , we let  $\mathbf{M}_{\mathbb{K}}^{\vee}$  denote the full simplicial subcategory of  $\mathbf{M}$  determined by the bifibrant, strongly  $\mathbb{K}$ -cocomplete objects.

The observation in Remark 4.28 implies that if  $K$  preserves bifibrant objects, then  $K$  fixes  $\mathbf{M}_{\mathbb{K}}^{\vee}$ , i.e.,  $\mathbb{K}$  restricts and corestricts to a comonad on  $\mathbf{M}_{\mathbb{K}}^{\vee}$ , which we denote  $\check{\mathbb{K}}$ .

**Theorem 5.8.** *Let  $\mathbf{M}$  be a simplicial model category. Let  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a comonad on  $\mathbf{M}$  such that  $K$  is a simplicial functor, and  $\Delta$  and  $\varepsilon$  are simplicial natural transformations.*

*If  $\mathcal{B}_{\bullet}^{\mathbb{K}}$  preserves cofibrant objects, then  $\check{\mathbb{K}}$  satisfies homotopic codescent.*

Theorem 5.8 is an immediate consequence of the following proposition, which generalizes Proposition 3.15, in the case where the underlying category is a simplicial model category.

**Proposition 5.9.** *Let  $\mathbf{M}$  be a simplicial model category. Let  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a comonad on  $\mathbf{S}$  such that  $K$  is a simplicial functor, and  $\Delta$  and  $\varepsilon$  are simplicial natural transformations.*

*If  $X$  is cofibrant and strongly  $\mathbb{K}$ -cocomplete, and  $\mathcal{B}_{\bullet}^{\mathbb{K}}X$  is Reedy cofibrant, then*

$$(\text{Can}^{\mathbb{K}})_{X,Y} : \text{Map}_{\mathbf{S}}(X, Y) \rightarrow \text{Map}_{\mathcal{D}^{co}(\mathbb{K})}(\text{Can}^{\mathbb{K}}(X), \text{Can}^{\mathbb{K}}(Y))$$

*is a weak equivalence of simplicial sets, for all fibrant objects  $Y$ .*

*Remark 5.10.* Let  $\mathbf{M}$  be a simplicial model category. Let  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a comonad on  $\mathbf{S}$  such that  $K$  is a simplicial functor, and  $\Delta$  and  $\varepsilon$  are simplicial natural transformations. Let  $\mathbb{D}\mathbb{K}$  denote the comonad on  $\mathcal{D}^{co}(\mathbb{K})$  with underlying endofunctor  $F^{\mathbb{K}}U^{\mathbb{K}}$ .

An analysis dual to that in Remark 5.5 shows that the image of the restriction of  $\text{Can}^{\mathbb{K}}$  to  $\mathbf{M}_{\mathbb{K}}^{\vee}$  consists of strongly  $\mathbb{D}\mathbb{K}$ -cocomplete codescent data. We intend to determine in future work under what conditions any strongly  $\mathbb{D}\mathbb{K}$ -cocomplete codescent datum is weakly equivalent to the canonical codescent datum of a strongly

$\mathbb{K}$ -cocomplete object of  $\mathbf{M}$ , i.e., to determine when the homotopic codescent of Theorem 5.8 is effective.

### 5.3. Descent and codescent spectral sequences.

5.3.1. *The descent spectral sequence.* Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on a simplicial model category  $\mathbf{M}$  such that Convention 4.1 holds. We require in addition that  $\mu$  and  $\eta$  be simplicial natural transformations.

**Definition 5.11.** Let  $X, Y \in \mathbf{M}$ , and let  $f \in \mathbf{M}(X, Y)$ . Let

$$p : \tilde{X} \xrightarrow{\sim} X$$

be a cofibrant replacement of  $X$ , and let

$$j^\bullet : \Omega_{\mathbb{T}}^\bullet Y \xrightarrow{\sim} \widehat{\Omega}_{\mathbb{T}}^\bullet Y$$

denote a fibrant replacement of  $\Omega_{\mathbb{T}}^\bullet Y$  in  $\mathbf{M}^\Delta$ .

The extended homotopy spectral sequence [4, X.6] of the fibrant cosimplicial simplicial set

$$\mathrm{Map}_{\mathbf{M}}(\tilde{X}, \widehat{\Omega}_{\mathbb{T}}^\bullet Y),$$

with basepoint

$$j^n \circ \eta_{T^n Y} \circ \cdots \circ \eta_{TY} \circ \eta_Y \circ f \circ p : \tilde{X} \rightarrow \widehat{\Omega}_{\mathbb{T}}^n Y$$

in level  $n$ , denoted  $E_f^{\mathbb{T}}$ , is the  $\mathbb{T}$ -descent spectral sequence for the pair  $(X, Y)$  with respect to  $f$ .

*Remark 5.12.* The  $\mathbb{T}$ -descent spectral sequence is clearly independent, up to isomorphism, of the choices of cofibrant and fibrant replacements made in the definition above, which justifies our use of the definite article in the definition.

*Remark 5.13.* If  $\mathbb{R}$  is the monad on  $\mathbf{sSet}$  of [4, I.2], then the  $\mathbb{R}$ -descent spectral sequence is exactly the unstable Adams spectral sequence studied there.

*Remark 5.14.* The analysis in sections X.6 and X.7 of [4] shows that for all  $t \geq s \geq 0$ ,

$$(E_f^{\mathbb{T}})_2^{s,t} = \pi^s \pi_t \mathrm{Map}_{\mathbf{M}}(\tilde{X}, \widehat{\Omega}_{\mathbb{T}}^\bullet Y),$$

where, for  $t \geq 2$ , the cosimplicial cohomotopy  $\pi^*(\pi_t \mathrm{Map}(\tilde{X}, \widehat{\Omega}_{\mathbb{T}}^\bullet Y))$  is the usual cohomology of the normalized cochain complex of the cosimplicial abelian group  $\pi_t \mathrm{Map}_{\mathbf{M}}(\tilde{X}, \widehat{\Omega}_{\mathbb{T}}^\bullet Y)$ .

Moreover, under conditions described in section IX.5 of [4], the spectral sequence  $E_f^{\mathbb{T}}$  abuts to  $\pi_* \mathrm{Tot} \mathrm{Map}_{\mathbf{M}}(\tilde{X}, \widehat{\Omega}_{\mathbb{T}}^\bullet Y)$ . The convergence issue is quite subtle, especially since the spectral sequence is fringed, and we do not claim to say anything new about convergence here.

We can now easily identify the term to which a  $\mathbb{T}$ -descent spectral sequence abuts, at least if  $\Omega_{\mathbb{T}}^\bullet$  preserves fibrant objects.

**Lemma 5.15.** *Suppose that  $\Omega_{\mathbb{T}}^\bullet$  preserves fibrant objects. If  $X$  is cofibrant and  $Y$  is fibrant, then the  $\mathbb{T}$ -descent spectral sequence for  $(X, Y)$  abuts to*

$$\pi_* \mathrm{Map}_{\mathbf{M}}(X, Y_{\mathbb{T}}^\wedge),$$

for any choice of basepoint  $f : X \rightarrow Y$ .

*Proof.* Since  $\Omega_{\mathbb{T}}^{\bullet}Y$  is Reedy fibrant by hypothesis,

$$\mathrm{Tot} \mathrm{Map}_{\mathbf{M}}(X, \Omega_{\mathbb{T}}^{\bullet}Y) \cong \mathrm{Map}_{\mathbf{M}}(X, \mathrm{Tot} \Omega_{\mathbb{T}}^{\bullet}\widehat{Y}) = \mathrm{Map}_{\mathbf{M}}(X, Y_{\mathbb{T}}^{\wedge}).$$

□

Under reasonable conditions, the  $E_2$ -term of the  $\mathbb{T}$ -descent spectral sequence admits an interesting interpretation. We refer the reader to Definition B.20 for the meaning of  $\mathrm{Ext}$  in simplicial model categories and to Remark 5.5 for the definition of the monad  $\mathbb{D}\mathbb{T}$ .

**Theorem 5.16.** *Let  $X$  and  $Y$  be a cofibrant object and a fibrant object in  $\mathbf{M}$ , respectively. If  $\Omega_{\mathbb{T}}^{\bullet}Y$  is Reedy fibrant and  $\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(Y)$  is strongly  $\mathbb{D}\mathbb{T}$ -complete, then*

$$(E_f^{\mathbb{T}})_2^{s,t} \cong \mathrm{Ext}_{\mathcal{D}(\mathbb{T})}^{s,t}(\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(Y))$$

for any choice of basepoint  $f$ .

*Remark 5.17.* Though it is not immediate that  $\mathcal{D}(\mathbb{T})$  satisfies the extra axiom of Definition B.8, and therefore that  $\mathrm{Map}_{\mathcal{D}(\mathbb{T})}(H, H')$  is fibrant if  $H$  and  $H'$  is cofibrant, in this case it follows from the proof that the mapping spaces in the definition of  $\mathrm{Ext}$  are indeed fibrant, so that their homotopy groups are well defined.

*Remark 5.18.* We can interpret Theorem 5.16 to mean that the  $\mathbb{T}$ -descent spectral sequence interpolates from  $\mathrm{Map}_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(Y))$  to  $\mathrm{Map}_{\mathbf{M}}(X, Y)$ , when  $\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(Y)$  is strongly  $\mathbb{D}\mathbb{T}$ -complete.

When  $Y$  is strongly  $\mathbb{T}$ -complete, then  $\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(Y)$  is strongly  $\mathbb{D}\mathbb{T}$ -complete, but also  $\mathrm{Map}_{\mathbf{M}}(X, Y)$  and  $\mathrm{Map}_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(Y))$  are weakly equivalent by Theorem 5.3. The spectral sequence probably collapses in this case.

*Proof.* By Proposition 3.15, the simplicial functor  $\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}$  induces an isomorphism

$$\mathrm{Map}_{\mathbf{M}}(X, \Omega_{\mathbb{T}}^n Y) \cong \mathrm{Map}_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(\Omega_{\mathbb{T}}^n Y))$$

for all  $n \geq 0$ . There is therefore an isomorphism of cosimplicial simplicial sets

$$\mathrm{Map}_{\mathbf{M}}(X, \Omega_{\mathbb{T}}^{\bullet}Y) \cong \mathrm{Map}_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(X), \mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}^{\Delta}(\Omega_{\mathbb{T}}^{\bullet}Y)),$$

where the functor  $\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}$  is applied levelwise in the second component on the right-hand side.

To conclude, we show that  $\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}^{\Delta}(\Omega_{\mathbb{T}}^{\bullet}Y)$  is a fibrant cosimplicial resolution of  $\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(Y)$  in  $\mathcal{D}(\mathbb{T})^{\Delta}$ . Since  $\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}(Y)$  is strongly  $\mathbb{D}\mathbb{T}$ -complete, there is an external cosimplicial SDR in  $\mathcal{D}(\mathbb{T})$ :

$$cc^{\bullet} \mathrm{Can}_{\mathbb{K}^{\mathbb{T}}} Y \begin{array}{c} \xrightarrow{\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}^{\Delta} \eta^{\bullet}} \\ \xleftarrow{\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}^{\Delta} \rho^{\bullet}} \end{array} \mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}^{\Delta} \Omega_{\mathbb{T}}^{\bullet}(Y) \circlearrowleft h.$$

Since Lemma 3.3 implies that  $\mathrm{Can}_{\mathbb{K}^{\mathbb{T}}}^{\Delta} \Omega_{\mathbb{T}}^{\bullet}(Y)$  is Reedy fibrant, we can conclude. □

5.3.2. *The codescent spectral sequence.* Let  $\mathbb{K} = (K, \Delta, \varepsilon)$  be a comonad on a simplicial model category  $\mathbf{M}$  such that Convention 4.16 holds and such that  $\Delta$  and  $\varepsilon$  are simplicial natural transformations.

**Definition 5.19.** Let  $X, Y \in \mathbf{M}$ , and let  $f \in \mathbf{M}(X, Y)$ . Let

$$j : Y \xrightarrow{\sim} \widehat{Y}$$

be a fibrant replacement of  $Y$ , and let

$$p_\bullet : \widetilde{\mathcal{B}}_\bullet^{\mathbb{K}} X \xrightarrow{\sim} \mathcal{B}_\bullet^{\mathbb{K}} X$$

denote a cofibrant replacement of  $\mathcal{B}_\bullet^{\mathbb{K}} X$  in  $\mathbf{M}^{\Delta^{op}}$ .

The extended homotopy spectral sequence [4, X.6] of the fibrant cosimplicial simplicial set

$$\mathrm{Map}_{\mathbf{M}}(\widetilde{\mathcal{B}}_\bullet^{\mathbb{K}} X, \widehat{Y}),$$

with basepoint

$$j \circ f \circ \varepsilon_X \circ \cdots \circ \varepsilon_{K^n X} \circ p_n : \widetilde{\mathcal{B}}_n^{\mathbb{K}} X \rightarrow \widehat{Y}$$

in level  $n$ , denoted  $E_f^{\mathbb{K}}$ , is the  $\mathbb{K}$ -codescent spectral sequence for the pair  $(X, Y)$  with respect to  $f$ .

*Remark 5.20.* The  $\mathbb{K}$ -codescent spectral sequence is clearly independent, up to isomorphism, of the choices of cofibrant and fibrant replacements made in the definition above.

*Remark 5.21.* The analysis in sections X.6 and X.7 of [4] shows that for all  $t \geq s \geq 0$ ,

$$(E_f^{\mathbb{K}})_2^{s,t} = \pi^s \pi_t \mathrm{Map}_{\mathbf{M}}(\widetilde{\mathcal{B}}_\bullet^{\mathbb{K}} X, \widehat{Y}),$$

where, for  $t \geq 2$ , the cosimplicial cohomotopy  $\pi^*(\pi_t \mathrm{Map}(\widetilde{\mathcal{B}}_\bullet^{\mathbb{K}} X, \widehat{Y}))$  is the usual cohomology of the normalized cochain complex of the cosimplicial abelian group  $\pi_t \mathrm{Map}_{\mathbf{M}}(\widetilde{\mathcal{B}}_\bullet^{\mathbb{K}} X, \widehat{Y})$ .

Moreover, under conditions described in section IX.5 of [4], the spectral sequence  $E_f^{\mathbb{K}}$  abuts to  $\pi_* \mathrm{Tot} \mathrm{Map}_{\mathbf{M}}(\widetilde{\mathcal{B}}_\bullet^{\mathbb{K}} X, \widehat{Y})$ . Again, we do not claim to say anything new about convergence here.

In the next two results, we identify both the term to which a  $\mathbb{K}$ -codescent spectral sequence abuts and its  $E_2$ -term, under reasonable cofibrancy hypotheses. The proofs are dual to those of Lemma 5.15 and Theorem 5.16.

**Lemma 5.22.** *Suppose that  $\mathcal{B}_\bullet^{\mathbb{K}}$  preserves cofibrant objects. If  $X$  is cofibrant and  $Y$  is fibrant, then the  $\mathbb{K}$ -descent spectral sequence for  $(X, Y)$  abuts to*

$$\pi_* \mathrm{Map}_{\mathbf{M}}(X_{\mathbb{K}}^{\vee}, Y),$$

for any choice of basepoint  $f : X \rightarrow Y$ .

We refer the reader to Remark 5.10 for the definition of the comonad  $\mathbb{D}\mathbb{K}$  used in the next statement.

**Theorem 5.23.** *Let  $X$  and  $Y$  be a cofibrant object and a fibrant object in  $\mathbf{M}$ , respectively. If  $\mathcal{B}_\bullet^{\mathbb{K}} X$  is Reedy cofibrant and  $\mathrm{Can}^{\mathbb{T}\mathbb{K}}(X)$  is strongly  $\mathbb{D}\mathbb{K}$ -cocomplete, then*

$$(E_f^{\mathbb{T}})_2^{s,t} \cong \mathrm{Ext}_{\mathcal{D}^{co}(\mathbb{K})}^{s,t}(\mathrm{Can}^{\mathbb{T}\mathbb{K}}(X), \mathrm{Can}^{\mathbb{T}\mathbb{K}}(Y))$$

for any choice of basepoint  $f$ .

*Remark 5.24.* We can interpret Theorem 5.23 to mean that the  $\mathbb{K}$ -codescent spectral sequence interpolates from  $\mathrm{Map}_{\mathcal{D}^{co}(\mathbb{K})}(\mathrm{Can}^{\mathbb{T}\mathbb{K}}(X), \mathrm{Can}^{\mathbb{T}\mathbb{K}}(Y))$  to  $\mathrm{Map}_{\mathbf{M}}(X, Y)$ , when  $\mathrm{Can}^{\mathbb{T}\mathbb{K}}(X)$  is strongly  $\mathbb{D}\mathbb{K}$ -cocomplete.

When  $X$  is strongly  $\mathbb{K}$ -cocomplete, then  $\mathrm{Can}^{\mathbb{T}\mathbb{K}}(X)$  is strongly  $\mathbb{D}\mathbb{K}$ -cocomplete, but also  $\mathrm{Map}_{\mathbf{M}}(X, Y)$  and  $\mathrm{Map}_{\mathcal{D}^{co}(\mathbb{K})}(\mathrm{Can}^{\mathbb{T}\mathbb{K}}(X), \mathrm{Can}^{\mathbb{T}\mathbb{K}}(Y))$  are weakly equivalent by Theorem 5.8. The spectral sequence probably collapses in this case.

## 6. HOMOTOPIC GROTHENDIECK DESCENT AND ITS DUAL

We return in this section to the monads and comonads introduced in sections 2.2.1 and 2.2.2 and apply to them the theories of derived (co)completion and of homotopic (co)descent. We also examine the associated (co)descent spectral sequences and see that certain of them are already very familiar.

**6.1. The Grothendieck framework.** Throughout this section, let

$$\varphi : B \rightarrow A$$

be a monoid morphism in a simplicial model category  $\mathbf{M}$  that is also endowed with a monoidal structure  $(\wedge, I)$ . We study here the theories of derived completion and of homotopic descent for the monad  $\mathbb{T}_\varphi$  associated to the adjunction

$$- \wedge_B A : \mathbf{Mod}_B \rightleftarrows \mathbf{Mod}_A : \varphi^*.$$

We first need to determine conditions under which  $\mathbf{Mod}_B$  and  $\mathbf{Mod}_A$  are equipped with a simplicial model category structure.

**Definition 6.1.** A simplicial model category  $\mathbf{M}$ , which is also endowed with a monoidal structure  $(\wedge, I)$ , is *monoidally simplicial* if

- (1) the simplicial-tensoring functor  $- \otimes - : \mathbf{M} \times \mathbf{sSet} \rightarrow \mathbf{M}$  is monoidal and op-monoidal, i.e., there are appropriately “associative” natural transformations

$$\tau : (X \otimes L) \wedge (X' \otimes L') \rightarrow (X \wedge X') \otimes (L \times L')$$

and

$$v : (X \wedge X') \otimes (L \times L') \rightarrow (X \otimes L) \wedge (X' \otimes L'),$$

and

- (2) the monoidal product in  $\mathbf{M}$  is itself a simplicial functor, i.e., the mapping space functor  $\mathrm{Map}_{\mathbf{M}}(-, -) : \mathbf{M}^{op} \times \mathbf{M} \rightarrow \mathbf{sSet}$  is monoidal, and the associated natural transformation

$$\xi : \mathrm{Map}_{\mathbf{M}}(X, Y) \times \mathrm{Map}_{\mathbf{M}}(X', Y') \rightarrow \mathrm{Map}_{\mathbf{M}}(X \wedge X', Y \wedge Y')$$

is appropriately compatible with the composition

*Remark 6.2.* If  $\mathbf{M}$  is monoidally simplicial, then for all objects  $X$  and  $X'$  in  $\mathbf{M}$  and all simplicial sets  $L$  and  $L'$ , there is morphism in  $\mathbf{M}$

$$(X^L \wedge (X')^{L'}) \otimes (L \times L') \xrightarrow{v} (X^L \otimes L) \wedge ((X')^{L'} \otimes L') \xrightarrow{ev_X \wedge ev_{Y'}} X \wedge X',$$

where  $ev$  is the counit of the tensoring/cotensoring-adjunction, i.e., the “evaluation” map. Taking the transpose of this composite, we obtain a natural map

$$\zeta : X^L \otimes (X')^{L'} \rightarrow (X \wedge X')^{L \times L'}.$$

*Example 6.3.* Let  $\mathbf{M}$  be a simplicial model category that is also endowed with a monoidal structure. It is not difficult to show that if the simplicial structure on  $\mathbf{M}$  is induced by a strong symmetric monoidal functor from simplicial sets to  $\mathbf{M}$ , then  $\mathbf{M}$  is monoidally simplicial. For example, the geometric realization functor gives rise to the simplicial structure on the category of topological spaces, while the simplicial suspension spectrum functor induces the simplicial structure on the category of symmetric spectra.

*Remark 6.4.* If  $A$  is any monoid in  $\mathbf{M}$ , then there is a monad  $\mathbb{T}_A$  on  $\mathbf{M}$  with underlying endofunctor  $- \wedge A$ , which is a simplicial functor if  $\mathbf{M}$  is monoidally simplicial. Applying Lemmas 3.9 and 3.11, we see that  $\mathbf{M}^{\mathbb{T}_A}$ , which is isomorphic to  $\mathbf{Mod}_A$ , is simplicially enriched and censored over  $\mathbf{sSet}$ . As stated in the next lemma, we can actually complete this structure on  $\mathbf{Mod}_A$  to that of a simplicial model category.

**Lemma 6.5.** *If  $\mathbf{M}$  is a monoidally simplicial model category and  $A$  is any monoid in  $\mathbf{M}$ , then  $\mathbf{Mod}_A$  inherits a simplicial model category structure from  $\mathbf{M}$ . In particular,*

- (1)  $\text{Map}_A : \mathbf{Mod}_A^{op} \times \mathbf{Mod}_A \rightarrow \mathbf{sSet}$  is defined for any right  $A$ -modules  $(M, \rho_M)$  and  $(N, \rho_N)$  by

$$\text{Map}_A(M, N) = \text{equal} \left( \text{Map}_{\mathbf{M}}(M, N) \begin{array}{c} \xrightarrow{(\rho_M)^*} \\ \xrightarrow{(\rho_N)^* \circ (- \wedge A)} \end{array} \text{Map}_{\mathbf{M}}(M \wedge A, N) \right);$$

- (2)  $- \otimes^A - : \mathbf{Mod}_A \times \mathbf{sSet} \rightarrow \mathbf{Mod}_A$  is defined for all  $A$ -modules  $(M, \rho)$  and simplicial sets  $K$  by

$$M \otimes^A K = (M \otimes K, \rho'),$$

where  $\rho'$  is the composite

$$(M \otimes K) \wedge A \cong (M \otimes K) \wedge (A \otimes \Delta[0]) \xrightarrow{\tau} (M \wedge A) \otimes (K \times \Delta[0]) \cong (M \wedge A) \otimes K \xrightarrow{\rho \otimes K} M \otimes K;$$

and

- (3)  $(-)_A^{(-)} : \mathbf{Mod}_A \times \mathbf{sSet}^{op} \rightarrow \mathbf{Mod}_A$  is defined for all  $A$ -modules  $(M, \rho)$  and simplicial sets  $K$  by

$$M_A^K = (M^K, \rho''),$$

where  $\rho''$  is the composite

$$M^K \wedge A \cong M^K \wedge A^{\Delta[0]} \xrightarrow{\zeta} (M \wedge A)^{K \times \Delta[0]} \cong (M \wedge A)^K \xrightarrow{\rho^K} M^K.$$

Recall from section 2.2.1 that if  $A$  is augmented, then  $(\mathbf{Mod}_A)^{\mathbb{T}_\varphi} \simeq \mathbf{Mod}_B$  and that  $\mathcal{D}(\mathbb{T}_\varphi) \cong \mathbf{M}_A^{W_\varphi}$ , the category of comodules over the descent co-ring  $W_\varphi$  in the category of right  $A$ -modules. Finally,

$$\text{Can}_\varphi := \text{Can}_{\mathbb{K}_\varphi} : \mathbf{Mod}_B \rightarrow \mathbf{M}_A^{W_\varphi} : M \mapsto (M \wedge_B A, M \wedge_B \varphi \wedge_B A).$$

We write  $\text{Map}_{W_\varphi, A}(-, -)$  for the induced mapping space functor on  $(\mathbf{M}_A^{W_\varphi})^{op} \times \mathbf{M}_A^{W_\varphi}$ .

To apply the theory of derived completion from section 4.1, the following conditions must be satisfied.

- (1) The forgetful functor  $U_{\mathbb{K}_\varphi} : \mathbf{M}_A^{W_\varphi} \rightarrow \mathbf{Mod}_B$  left-induces a model category structure on  $\mathbf{M}_A^{W_\varphi}$  (cf. appendix C).
- (2) The functor  $\varphi^*(- \wedge_B A) : \mathbf{Mod}_B \rightarrow \mathbf{Mod}_B$  is simplicial.

Condition (1) holds if fibrations of  $B$ -modules admit nice enough descriptions as limits of Postnikov-type towers, which seems to be a reasonable condition (cf. Proposition C.5). We refer the reader to [10] for examples of categories in which such axioms are verified. Furthermore, if the monoidal structure is actually cartesian, then comodule categories are simply overcategories, and therefore inherit left-

and right-induced model structures automatically. It is clear that condition (2) always holds, since  $\varphi^*(-\wedge_B A)$  is a composite of simplicial functors, by definition of the mapping spaces in the module categories. Note that it is also evident that the multiplication  $\mu_\varphi$  and the unit  $\eta_\varphi$  of the monad  $\mathbb{T}_\varphi$  are simplicial natural transformations.

In this Grothendieck framework, Definition 4.7 and Notation 4.9 translate into the following definition.

**Definition 6.6.** Let  $M$  be a  $B$ -module. Let  $\widehat{M}$  be a fibrant replacement of  $M$  such that  $\Omega_{\mathbb{T}_\varphi}^\bullet \widehat{M}$  is Reedy fibrant. The *derived completion of  $M$  along  $\varphi$*  is

$$M_\varphi^\wedge := M_{\mathbb{T}_\varphi}^\wedge = \text{Tot } \Omega_{\mathbb{T}_\varphi}^\bullet \widehat{M}.$$

When  $\mathbf{M}$  is a monoidal model category of spectra, the definition above agrees with Carlsson's definition of derived completion of module spectra [5], when  $A$  is cofibrant as a  $B$ -algebra.

*Remark 6.7.* Recall from Remark 4.15 that a strongly  $\mathbb{T}_\varphi$ -complete  $B$ -module  $M$  can be seen as an  $\infty$ - $\mathbb{T}_\varphi$ -algebra. In other words,  $M$  is a sort of  $\infty$ - $A$ -module, with respect to the monoidal product  $-\wedge_B -$ .

Specializing Definition 5.1 to the Grothendieck framework, we obtain the following notion of homotopic descent. Note that this definition makes sense even if the module categories admit only a simplicial enrichment, rather than a full simplicial model category structure.

**Definition 6.8.** The monad  $\mathbb{T}_\varphi$  satisfies *homotopic descent* if each component

$$(\text{Can}_\varphi)_{M, M'} : \text{Map}_B(M, M') \rightarrow \text{Map}_{W_\varphi, A}(\text{Can}_\varphi(M), \text{Can}_\varphi(M'))$$

of the simplicial functor  $\text{Can}_\varphi$  is a weak equivalence of simplicial sets.

The criterion for homotopic descent in the Grothendieck framework is a special case of Theorem 5.3.

**Theorem 6.9.** Let  $(\mathbf{Mod}_B)_{\mathbb{T}_\varphi}^\wedge$  denote the full simplicial subcategory of  $\mathbf{Mod}_B$  determined by the bifibrant, strongly  $\mathbb{T}_\varphi$ -complete objects. If  $\varphi^*(-\wedge_B A) : \mathbf{Mod}_B \rightarrow \mathbf{Mod}_B$  preserves bifibrant objects and  $\Omega_{\mathbb{T}_\varphi}^\bullet$  preserves fibrant objects, then  $\mathbb{T}_\varphi$  restricts and corestricts to a monad  $\widehat{\mathbb{T}}_\varphi$  on  $(\mathbf{Mod}_B)_{\mathbb{T}_\varphi}^\wedge$  that satisfies homotopic descent.

The relevant spectral sequence in this case is defined as follows. The choice of terminology is justified by the examples sketched below.

**Definition 6.10.** Let  $M$  and  $N$  be right  $B$ -modules, and let  $f : M \rightarrow N$  be a morphism of  $B$ -modules. The  $\varphi$ -Adams spectral sequence for  $(M, N)$  with respect to  $f$ , denoted  $E_f^\varphi$ , is the  $\mathbb{T}_\varphi$ -descent spectral sequence for  $(M, N)$  with respect to  $f$ .

The next result is an immediate specialization of Lemma 5.15 and Theorem 5.16.

**Proposition 6.11.** Let  $M$  and  $N$  be a cofibrant and a fibrant right  $B$ -module, respectively.

- (1) If  $\Omega_{\mathbb{T}_\varphi}^\bullet$  preserves fibrant objects, then the  $\varphi$ -Adams spectral sequence for  $(M, N)$  with respect to  $f$  abuts to  $\pi_* \text{Map}_B(M, N_\varphi^\wedge)$ .



(2) If  $\Omega_{\mathbb{T}_\varphi}^\bullet N$  is Reedy fibrant and  $\text{Can}_\varphi(N)$  is strongly  $\mathbb{D}\mathbb{T}_\varphi$ -complete, then

$$(E_f^\varphi)_2^{s,t} \cong \text{Ext}_{W_{\varphi,A}}^{s,t}(\text{Can}_\varphi(M), \text{Can}_\varphi(N))$$

for any choice of basepoint  $f$ .

In a future article we will treat in detail the following special, very important cases of  $\varphi$ -Adams spectral sequences. Here we sketch only the proposed applications.

6.1.1. *The stable Adams spectral sequence.* Let  $H\mathbb{F}_p$  denote the Eilenberg-MacLane spectrum of  $\mathbb{F}_p$ , in some appropriate monoidal model category  $\mathbf{M}$  of spectra, and let  $\eta_{H\mathbb{F}_p} : S \rightarrow H\mathbb{F}_p$  denote its unit. As we sketch below, it is possible to show that the  $\eta_{H\mathbb{F}_p}$ -Adams spectral sequence is the usual stable Adams spectral sequence.

Note that in this case

$$W_{\eta_{H\mathbb{F}_p}} = H\mathbb{F}_p \wedge H\mathbb{F}_p,$$

which is actually a comonoid in the category of commutative  $S$ -algebras and not just an  $S$ -co-ring. The stable homotopy groups of  $W_{\eta_{H\mathbb{F}_p}}$  form the dual mod  $p$  Steenrod algebra  $\mathcal{A}_p^\sharp$ , while for any spectrum  $X$ , the stable homotopy groups of  $\text{Can}_{\eta_{H\mathbb{F}_p}}(X) = X \wedge H\mathbb{F}_p$  are its singular homology groups with coefficients in  $\mathbb{F}_p$ .

Applying Proposition 5.4, we see that homotopic descent is satisfied if  $Y$  is fibrant and strongly  $\mathbb{T}_{\eta_{H\mathbb{F}_p}}$ -complete,  $\Omega_{\mathbb{T}_{\eta_{H\mathbb{F}_p}}}^\bullet(Y)$  is Reedy fibrant, and  $X$  is cofibrant: the simplicial map

$$(\text{Can}_{\eta_{H\mathbb{F}_p}})_{X,Y} : \text{Map}_{\mathbf{M}}(X, Y) \rightarrow \text{Map}_{W_{\eta_{H\mathbb{F}_p}}, H\mathbb{F}_p}(X \wedge H\mathbb{F}_p, Y \wedge H\mathbb{F}_p)$$

is then a weak equivalence, where we have suppressed the coactions from the notation for  $\text{Can}_{\eta_{H\mathbb{F}_p}}(X)$  and  $\text{Can}_{\eta_{H\mathbb{F}_p}}(Y)$ .

6.1.2. *The Adams-Novikov spectral sequence.* Let  $MU$  denote the complex cobordism spectrum, in some appropriate monoidal model category  $\mathbf{M}$  of spectra, and let  $\eta_{MU} : S \rightarrow MU$  denote its unit. We again sketch the argument to show that the  $\eta_{MU}$ -Adams spectral sequence is the Adams-Novikov spectral sequence.

Note that in this case

$$W_{\eta_{MU}} = MU \wedge MU,$$

which again is actually a comonoid in the category of commutative  $S$ -algebras and not just an  $S$ -co-ring. The stable homotopy groups of  $W_{\eta_{MU}}$  form the Hopf algebra of cooperations on  $MU_*$ -homology, while for any spectrum  $X$ , the stable homotopy groups of  $\text{Can}_{\eta_{MU}}(X) = X \wedge MU$  are its  $MU_*$ -homology groups.

As in the previous case, Proposition 5.4 tells us that homotopic descent is satisfied if  $Y$  is fibrant and strongly  $\mathbb{T}_{\eta_{MU}}$ -complete,  $\Omega_{\mathbb{T}_{\eta_{MU}}}^\bullet(Y)$  is Reedy fibrant, and  $X$  is cofibrant: the simplicial map

$$(\text{Can}_{\eta_{MU}})_{X,Y} : \text{Map}_{\mathbf{M}}(X, Y) \rightarrow \text{Map}_{W_{\eta_{MU}}, MU}(X \wedge MU, Y \wedge MU)$$

is then a weak equivalence, where we have suppressed the coactions from the notation for  $\text{Can}_{\eta_{MU}}(X)$  and  $\text{Can}_{\eta_{MU}}(Y)$ .

6.1.3. *Algebraic vs. étale K-theory.* Let  $R = \mathbb{Z}[\frac{1}{\ell}]$ , where  $\ell$  is a fixed prime, and let  $A$  be a noetherian  $R$ -algebra. In [8] Dwyer and Friedlander constructed a natural map of ring spectra

$$\varphi_A^{DF} : K_A \rightarrow \widehat{K}_A^{\text{ét}},$$

where  $K_A$  is the algebraic K-theory spectrum of  $A$ , and  $\widehat{K}_A^{\text{ét}}$  is its étale K-theory spectrum, i.e.,

$$\pi_* K_A \cong K_*(A) \quad \text{and} \quad \pi_* \widehat{K}_A^{\text{ét}} \cong \widehat{K}_*(A).$$

More generally, if  $v \geq 1$  and  $M(v)$  denotes the mod  $\ell^v$ -Moore spectrum with bottom cell in stable dimension 0, then

$$\pi_*(M(v) \wedge K_A) \cong K_*(A; \mathbb{Z}/\ell^v \mathbb{Z}) \quad \text{and} \quad \pi_*(M(v) \wedge \widehat{K}_A^{\text{ét}}) \cong \widehat{K}_*(A; \mathbb{Z}/\ell^v \mathbb{Z}).$$

The famous Lichtenbaum-Quillen conjecture lists conditions under which the homomorphism

$$(\varphi_A^{DF})_* : K_*(A; \mathbb{Z}/\ell^v \mathbb{Z}) \rightarrow \widehat{K}_*(A; \mathbb{Z}/\ell^v \mathbb{Z})$$

induced by the Dwyer-Friedlander map should be an isomorphism. We refer the reader to the extensive literature on this conjecture for cases in which it is known to hold.

Note that the associated canonical descent co-ring is the  $\widehat{K}_A^{\text{ét}}$ -bimodule

$$W_{\varphi_A^{DF}} = \widehat{K}_A^{\text{ét}} \wedge_{K_A} \widehat{K}_A^{\text{ét}},$$

endowed with the usual canonical comultiplication and counit.

Suppose that we can choose a model for the Dwyer-Friedlander map such that the necessary conditions (e.g., Convention 4.1) hold and so that  $\Omega_{\mathbb{T}_{\varphi_A^{DF}}}^{\bullet}$  preserves fibrant objects. The  $\varphi_A^{DF}$ -Adams spectral sequence for a cofibrant  $K_A$ -module  $M$  and a fibrant  $K_A$ -module  $N$  such that  $N \wedge_{K_A} \widehat{K}_A^{\text{ét}}$  is strongly  $\mathbb{D}\mathbb{T}_{\varphi_A^{DF}}$ -complete then satisfies

$$(E_f^{\varphi_A^{DF}})_2^{s,t} \cong \text{Ext}_{W_{\varphi_A^{DF}}, \widehat{K}_A^{\text{ét}}}^{s,t} (M \wedge_{K_A} \widehat{K}_A^{\text{ét}}, N \wedge_{K_A} \widehat{K}_A^{\text{ét}})$$

and abuts to

$$\pi_* \text{Map}_{K_A} (M, N \wedge_{\varphi_A^{DF}})$$

for any choice of basepoint  $f : M \rightarrow N$ . Moreover, Proposition 5.4 implies that the simplicial map

$$(\text{Can}_{\varphi_A^{DF}})_{M,N} : \text{Map}_{K_A} (M, N) \rightarrow \text{Map}_{W_{\varphi_A^{DF}}, \widehat{K}_A^{\text{ét}}} (M \wedge_{K_A} \widehat{K}_A^{\text{ét}}, N \wedge_{K_A} \widehat{K}_A^{\text{ét}})$$

is a weak equivalence if  $N$  is actually strongly  $\mathbb{T}_{\varphi_A^{DF}}$ -complete, where we have suppressed the coactions from the notation for  $\text{Can}_{\varphi_A^{DF}}(M)$  and  $\text{Can}_{\varphi_A^{DF}}(N)$ .

Let  $S$  denote the sphere spectrum, i.e., the unit object in  $\mathbf{M}$ . In the case where  $M = K_A$  and  $N = M(v) \wedge K_A$ , and  $f$  is determined by the inclusion of the basepoint into  $M(v)$ ,

$$\text{Map}_{K_A} (M, N) \cong \text{Map}_{\mathbf{M}} (S, M(v) \wedge K_A),$$

while

$$\text{Map}_{W_{\varphi_A^{DF}}, \widehat{K}_A^{\text{ét}}} (M \wedge_{K_A} \widehat{K}_A^{\text{ét}}, N \wedge_{K_A} \widehat{K}_A^{\text{ét}}) \cong \text{Map}_{W_{\varphi_A^{DF}}, \widehat{K}_A^{\text{ét}}} (\widehat{K}_A^{\text{ét}}, M(v) \wedge \widehat{K}_A^{\text{ét}}),$$

which is a simplicial subset of

$$\text{Map}_{\widehat{K}_A^{\text{ét}}} (\widehat{K}_A^{\text{ét}}, \widehat{K}_A^{\text{ét}}) \cong \text{Map}_{\mathbf{M}} (S, M(v) \wedge \widehat{K}_A^{\text{ét}}).$$

Let  $\Psi_{A,v} = \pi_*(\text{Can}_{\varphi_A^{DF}})_{K_A, M(v) \wedge K_A}$ . It follows from the analysis above that there is a commutative diagram of group homomorphisms

$$\begin{array}{ccc} K_*(A; \mathbb{Z}/\ell^v\mathbb{Z}) & \xrightarrow{(\varphi_A^{DF})_*} & \widehat{K}_*^{\text{ét}}(A; \mathbb{Z}/\ell^v\mathbb{Z}), \\ & \searrow \Psi_{A,v} & \nearrow \iota_* \\ & \pi_* \text{Map}_{W_{\varphi_A^{DF}}, \widehat{K}_A^{\text{ét}}}(\widehat{K}_A^{\text{ét}}, \widehat{K}_A^{\text{ét}}) & \end{array}$$

where  $\iota_*$  is induced by the inclusion. There is thus a close relationship, mediated by  $\iota_*$ , between the Lichtenbaum-Quillen conjecture for  $A$  and the question of when  $\mathbb{T}_{\varphi_A^{DF}}$  satisfies homotopic descent. In particular, if  $M(v) \wedge K_A$  is bifibrant as a  $K_A$ -module and strongly  $\mathbb{T}_{\varphi_A^{DF}}$ -complete, then  $\Psi_{A,v}$  is an isomorphism, so that the Lichtenbaum-Quillen conjecture holds for  $A$  and  $v$  if and only if  $\iota_*$  is an isomorphism.

6.1.4. *Quillen homology.* Let  $\mathbf{M}$  be a particularly nice, simplicially monoidal model category. Let  $A$  be a monoid in  $\mathbf{M}$  that is endowed with an augmentation  $\varepsilon : A \rightarrow I$ . Let

$$A \xrightarrow{\varphi} Q \xrightarrow{\sim} I$$

be a factorization of  $\varepsilon$  in the category of  $A$ -algebras (i.e., monoids in the category of  $A$ -bimodules) into a cofibration followed by an acyclic fibration. The descent co-ring associated to the cofibration  $\varphi$  is then

$$W_\varphi = Q \wedge_A Q.$$

If  $M$  is a cofibrant  $A$ -module and  $N$  is a fibrant  $A$ -module, then the  $\varphi$ -Adams spectral sequence abuts to

$$\pi_* \text{Map}_A(M, N^\wedge).$$

If, in addition,  $\text{Can}_\varphi(N)$  is strongly  $\mathbb{D}\mathbb{T}_\varphi$ -complete, then the  $E_2$ -term is isomorphic to

$$E_2^{s,t} = \text{Ext}_{A, W_\varphi}(\text{Can}_\varphi(M), \text{Can}_\varphi(N)).$$

When, as is often the case, the forgetful functor  $A\text{-Alg} \rightarrow {}_A\mathbf{Mod}$  preserves cofibrations, and  $A$  is cofibrant in  ${}_A\mathbf{Mod}$ , then  $Q$  is a cofibrant replacement of  $I$  in the category left  $A$ -modules. Under reasonable conditions, therefore, the  $A$ -module underlying  $\text{Can}_\varphi(M)$  is a model of the homotopy orbits (or homotopy indecomposables) of the  $A$ -action on  $M$ :

$$\text{Can}_\varphi(M) = M \wedge_A Q = M_{hA}.$$

Thus, the  $\varphi$ -Adams spectral sequence in this case can be seen as converging from Quillen homology to homotopy.

6.2. **The dual Grothendieck framework.** We now specialize our definitions of derived cocompletion and of the codescent spectral sequence to the dual Grothendieck framework (cf. section 2.2.2). Though establishing the theoretical foundations of this special form of codescent consists essentially of a formal game of dualizing the previous case, it is a game worth playing as the spectral sequences we obtain are both important and of a rather different nature from those we studied in the previous section. A helpful reference for the material in this section is [14].

To simplify the discussion somewhat, particularly in terms of the model category structures involved, we consider only the cartesian case here: we fix a simplicial model category  $\mathbf{M}$ , endowed with the monoidal structure given by the categorical product  $\times$ . Recall from section 2.2.2 that any object  $X$  of  $\mathbf{M}$  is a comonoid, where the comultiplication is the canonical diagonal map  $\delta_X : X \rightarrow X \times X$  and the counit is the unique map  $\epsilon_X : X \rightarrow e$  to the terminal object. Moreover, the category of right  $(X, \delta_X, \epsilon_X)$ -comodules is isomorphic to the overcategory  $\mathbf{M}/X$ .

For any object  $X$ , the category  $\mathbf{M}/X$  is endowed with a simplicial model category structure inherited from  $\mathbf{M}$  [21, Proposition II.2.6]. The cofibrant objects in  $\mathbf{M}/X$  are those morphisms  $f : X \rightarrow B$  in  $\mathbf{M}$  such that  $X$  is cofibrant, while the fibrant objects are the fibrations  $p : X \twoheadrightarrow B$ . If  $\mathbf{M}$  is cellular ([11, Definition 12.1.1]) and proper ([11, Definition 13.1.1]), then  $\mathbf{M}/X$  is also cellular and proper, for all objects  $X$  [11, Proposition 12.1.6]. Examples of proper, cellular cartesian categories include  $\mathbf{sSet}$  and  $\mathbf{Top}$ , the category of topological spaces.

Throughout this section, let

$$\varphi : E \rightarrow B$$

be any morphism in the simplicial model category  $\mathbf{M}$ , which we view as a morphism of comonoids. We study here the theories of derived cocompletion and of homotopic codescent for the monad  $\mathbb{K}_\varphi$  associated to the adjunction

$$\varphi_! : \mathbf{M}/E \rightleftarrows \mathbf{M}/B : \varphi^*,$$

where  $\varphi_!$  is given by postcomposing with  $\varphi$ , while  $\varphi^*$  is given by pullback. In other words, we develop the theories of derived cocompletion and of homotopic codescent along a morphism in  $\mathbf{M}$  for bundles in  $\mathbf{M}$ .

Let  $\mathbb{T}_\varphi$  denote the monad associated to the  $(\varphi_!, \varphi^*)$ -adjunction. Recall from section 2.2.2 that  $\mathbf{M}/E \simeq (\mathbf{M}/B)_{\mathbb{K}_\varphi}$  and that

$$\mathcal{D}^{co}(\mathbb{K}_\varphi) \cong (\mathbf{M}/E)^{\mathbb{T}_\varphi} \cong \mathbf{M}_{V_\varphi}^E,$$

the category of modules in  $\mathbf{M}/E$  over the codescent ring

$$V_\varphi = (E \times_B E, \gamma_{can}, \eta_{can}),$$

where  $E \times_B E \rightarrow E$  is given by projection onto the second factor, and  $\gamma_{can}$  is the composite

$$(E \times_B E) \times_E (E \times_B E) \cong E \times_B E \times_B E \xrightarrow{E \times_B \varphi \times_B E} E \times_B B \times_B E \cong E \times_B E.$$

Furthermore,

$$\text{Can}^\varphi := \text{Can}^{\mathbb{T}_\varphi} : \mathbf{M}/B \rightarrow \mathbf{M}_{V_\varphi}^E : (X \xrightarrow{f} B) \mapsto (\varphi^*(f), X \times_B \varphi \times_B E),$$

where

$$\begin{array}{ccc} X \times_B E & \longrightarrow & X \\ \varphi^*(f) \downarrow & & \downarrow f \\ E & \xrightarrow{\varphi} & B \end{array}$$

is a pullback diagram. Finally

$$F^{\mathbb{T}} = \text{Can}^\varphi \circ \varphi_! : \mathbf{M}/E \rightarrow \mathbf{M}_{V_\varphi}^E.$$

We let  $\text{Map}_{V^\varphi, E}(-, -)$  denote the induced mapping space functor on  $(\mathbf{M}_{V^\varphi}^E)^{op} \times \mathbf{M}_{V^\varphi}^E$ .

*Remark 6.12.* A careful analysis of the definition of  $\mathbb{T}_\varphi$  shows that the existence of a  $\mathbb{T}_\varphi$ -algebra structure on a map  $f : X \rightarrow E$  is equivalent to the existence of a map  $\tau : X \times_B E \rightarrow X \times_B E$  satisfying both a cocycle condition (corresponding to the associativity of the multiplication) and a normalization condition (corresponding to the unitality of the multiplication). If  $\text{Can}^\varphi$  is an equivalence, i.e., if  $\mathbb{K}_\varphi$  satisfies effective codescent, then any bundle over  $E$  endowed with such a map  $\tau$  can be pushed forward to a bundle over  $B$ , and every bundle over  $B$  arises in this way, up to isomorphism.

To apply the theory of derived cocompletion from section 4.2, the following conditions must be satisfied.

- (1) The forgetful functor  $U^{\mathbb{T}_\varphi} : \mathbf{M}_{V^\varphi}^E \rightarrow \mathbf{M}/E$  right-induces a model category structure on  $\mathbf{M}_{V^\varphi}^E$  (cf. appendix C).
- (2) The functor  $\varphi_! \varphi^* : \mathbf{M}/B \rightarrow \mathbf{M}/B$  is simplicial.

Condition (1) holds if, for example, the model category structure on  $\mathbf{M}$  is cellular (and therefore cofibrantly generated), all objects are small and the so-called monoid axiom is satisfied [23, Theorem 4.1(1)]. It follows that condition (1) is satisfied if  $\mathbf{M} = \mathbf{sSet}$ . By Proposition C.2, condition (1) also holds if  $\mathbf{M}$  is cellular and proper and  $\varphi$  is a fibration, since  $F^{\mathbb{T}_\varphi}$  will then preserve all weak equivalences.

It is clear that condition (2) always holds, since  $\varphi_! \varphi^*$  is a composite of simplicial functors, by definition of the mapping spaces in the overcategories. Note that it is also evident that the comultiplication  $\Delta_\varphi$  and the counit  $\varepsilon_\varphi$  of the comonad  $\mathbb{K}_\varphi$  are simplicial natural transformations.

In this dual Grothendieck framework, Definition 4.22 and Notation 4.24 translate into the following definition.

**Definition 6.13.** Let  $f : X \rightarrow B$  be a morphism in  $\mathbf{M}$ . Let  $p : \tilde{X} \xrightarrow{\sim} X$  be a cofibrant replacement of  $X$  such that  $\mathcal{B}_{\bullet}^{\mathbb{K}_\varphi}(fp)$  is Reedy cofibrant. The *derived cocompletion of  $f$  along  $\varphi$*  is

$$(f)_\varphi^\vee := (f)_{\mathbb{K}_\varphi}^\vee = |\mathcal{B}_{\bullet}^{\mathbb{K}_\varphi}(fp)|.$$

*Remark 6.14.* If  $\mathbf{M} = \mathbf{sSet}$ , then all objects in  $\mathbf{M}$  are cofibrant, whence the same is true of  $\mathbf{M}/X$ , for all  $X$ . Moreover,  $\mathcal{B}_{\bullet}^{\mathbb{K}_\varphi}(f)$  is cofibrant for all  $f : X \rightarrow B$ , since all objects in  $\mathbf{sSet}^{\Delta^{op}}$  are Reedy cofibrant. It makes sense therefore to write

$$(f)_\varphi^\vee = |\mathcal{B}_{\bullet}^{\mathbb{K}_\varphi}(f)|$$

for all simplicial maps  $f : X \rightarrow B$ .

*Remark 6.15.* Expanding the formula for the functor  $\mathcal{B}_{\bullet}^{\mathbb{K}_\varphi}$ , we note that if  $f : X \rightarrow B$  is a morphism in  $\mathbf{M}$ , then

$$\mathcal{B}_{\bullet}^{\mathbb{K}_\varphi}(f) = \left( \begin{array}{c} X \times_B E \rightleftharpoons X \times_B E \times_B E \rightleftharpoons X \times_B E \times_B E \times_B E \cdots \\ \searrow \varphi_! \varphi^*(f) \quad \downarrow (\varphi_! \varphi^*)^2(f) \quad \swarrow (\varphi_! \varphi^*)^3(f) \\ B \end{array} \right).$$

In particular, if  $\mathbf{M} = \mathbf{sSet}$ , then  $\mathcal{B}_{\bullet}^{\mathbb{K}_{\varphi}}(\varphi)$  is the bisimplicial set underlying the “cohomological descent spectral sequence of a map” (cf. e.g., [20] and [7, Section 5.3]).

*Remark 6.16.* Examining the formula for  $\mathbb{K}_{\varphi}$ , one sees that a  $\mathbb{K}_{\varphi}$ -coalgebra structure on  $f : X \rightarrow B$  is equivalent to a section

$$\sigma : X \rightarrow X \times_B E : x \mapsto (x, e(x))$$

of the projection  $\varepsilon_f : X \times_B E \rightarrow X$ , which is counit of the adjunction, such that  $e(x) \in \varphi^{-1}(f(x))$ , since any such section automatically satisfies the coassociativity and counit conditions of a coalgebra.

It follows that the strongly  $\mathbb{K}_{\varphi}$ -cocomplete objects in  $\mathbf{M}/B$  are exactly the  $\mathbb{K}_{\varphi}$ -coalgebras: for all  $f : X \rightarrow B$ , a simplicial section  $\sigma_{\bullet} : cs_{\bullet}(f) \rightarrow \mathcal{B}_{\bullet}^{\mathbb{K}_{\varphi}}(f)$  of the simplicial augmentation  $\varepsilon_{\bullet}$  begins with a section  $\sigma_0 : X \rightarrow X \times_B E$  of  $\varepsilon_f$  as maps over  $B$ . This bottom stage of the simplicial section therefore defines a  $\mathbb{K}_{\varphi}$ -coalgebra structure on  $f$ .

Specializing Definition 5.6 to the dual Grothendieck framework, we obtain the following notion of homotopic codescent. Note that this definition makes sense even if the module categories admit only a simplicial enrichment, rather than a full simplicial model category structure.

**Definition 6.17.** The comonad  $\mathbb{K}_{\varphi}$  satisfies *homotopic codescent* if each component

$$(\text{Can}^{\varphi})_{f,g} : \text{Map}_{\mathbf{M}/B}(f, g) \rightarrow \text{Map}_{V^{\varphi}, E}(\text{Can}^{\varphi}(f), \text{Can}^{\varphi}(g))$$

of the simplicial functor  $\text{Can}^{\varphi}$  is a weak equivalence of simplicial sets.

The criterion for homotopic codescent in the dual Grothendieck framework is a special case of Theorem 5.8, where Remark 6.16 comes clearly into play.

**Theorem 6.18.** Let  $(\mathbf{M}/B)_{\mathbb{K}_{\varphi}}^{\vee}$  denote the full simplicial subcategory of  $\mathbf{M}/B$  determined by the bifibrant  $\mathbb{K}_{\varphi}$ -coalgebras. If  $\varphi_! \varphi^* : \mathbf{M}/B \rightarrow \mathbf{M}/B$  preserves bifibrant objects, and  $\mathcal{B}_{\bullet}^{\mathbb{K}_{\varphi}}$  preserves cofibrant objects, then  $\mathbb{K}_{\varphi}$  restricts and corestricts to a comonad  $\check{\mathbb{K}}_{\varphi}$  on  $(\mathbf{M}/B)_{\mathbb{K}_{\varphi}}^{\vee}$  that satisfies homotopic codescent.

*Remark 6.19.* If  $\varphi : E \rightarrow B$  is a fibration, then  $\varphi_! \varphi^* : \mathbf{M}/B \rightarrow \mathbf{M}/B$  preserves fibrant objects. Thus, if  $\mathbf{M} = \mathbf{sSet}$ , and  $\varphi$  is a fibration, then  $\varphi_! \varphi^* : \mathbf{M}/B \rightarrow \mathbf{M}/B$  preserves bifibrant objects. The criterion for homotopic codescent therefore applies to all Kan fibrations.

If a Kan fibration  $\varphi : E \rightarrow B$  admits a section, then for all  $f : X \rightarrow B$ , the counit map  $\varepsilon_f : X \times_B E \rightarrow X$  also admits a section, i.e., every  $f$  is a  $\mathbb{K}_{\varphi}$ -coalgebra. It follows that  $\mathbb{K}_{\varphi}$  itself then satisfies homotopic codescent.

The relevant spectral sequence in this case is defined as follows.

**Definition 6.20.** Let  $i : X \rightarrow B$  and  $p : Y \rightarrow B$  be morphisms in  $\mathbf{M}$ , and let  $f : X \rightarrow Y$  be a morphism in  $\mathbf{M}$  satisfying  $pf = i$ . The  $\varphi$ -bundle spectral sequence for  $(i, p)$  with respect to  $f$ , denoted  $E_f^{\varphi}$ , is the  $\mathbb{K}_{\varphi}$ -codescent spectral sequence for  $(i, p)$  with respect to  $f$ .

The next result is an immediate specialization of Lemma 5.22 and Theorem 5.23.

**Proposition 6.21.** *Let  $i : X \rightarrow B$  and  $p : Y \rightarrow B$  be morphisms in  $\mathbf{M}$  such that  $X$  is cofibrant and  $p$  is a fibration.*

- (1) *If  $\mathcal{B}_\bullet^{\mathbb{K}_\varphi}$  preserves cofibrant objects in  $\mathbf{M}/B$ , then the  $\varphi$ -bundle spectral sequence for  $(i, p)$  with respect to  $f$  abuts to  $\pi_* \text{Map}_{\mathbf{M}/B}(i_\varphi^\vee, p)$ .*
- (2) *If  $\mathcal{B}_\bullet^{\mathbb{K}_\varphi}(i)$  is Reedy cofibrant and  $\text{Can}^\varphi(i)$  is strongly  $\mathbb{D}\mathbb{K}_\varphi$ -cocomplete, then*

$$(E_f^\varphi)_2^{s,t} \cong \text{Ext}_{V_\varphi, E}^{s,t}(\text{Can}^\varphi(i), \text{Can}^\varphi(p))$$

for any choice of basepoint  $f$ .

As a concrete example of the theory developed here, we present the following example.

*Example 6.22.* Let  $B$  be any reduced simplicial set, and let  $\varphi : PB \rightarrow B$  be the Kan fibration with  $PB = B \times_{\tau} GB$ , the twisted cartesian product of  $B$  and its Kan loop group  $GB$  over the universal twisting function  $\tau : B \rightarrow GB$  [18]. The associated codescent ring is

$$V^\varphi = PB \times_B PB,$$

which is multiplicatively homotopy equivalent to  $GB$ , via the simplicial map

$$PB \times_B PB \rightarrow GB : ((\lambda_1, b), (\lambda_2, b)) \mapsto \lambda_1 \lambda_2^{-1}.$$

Moreover, for all  $f : X \rightarrow B$ , the canonical codescent datum  $\text{Can}^\varphi(f)$  is the natural projection from homotopy fiber of  $f$  down to  $PB$ , endowed with the usual action of  $GB$ .

In this case, since  $\varphi$  itself admits a section, the counit (projection)  $\varepsilon_f : X \times_B E \rightarrow X$  admits a canonical section for all  $f : X \rightarrow B$ . In other words, by Remark 6.16, every object in  $\mathbf{sSet}/B$  is strongly  $\mathbb{K}_\varphi$ -cocomplete and therefore  $\mathbb{K}_\varphi$ -cocomplete, since all objects are cofibrant.

Abusing notation a little, we see that for all simplicial maps  $f : X \rightarrow B$  and all Kan fibrations  $p : Y \rightarrow B$ , Proposition 5.9 implies that

$$\text{Can}_{f,p}^\varphi : \text{Map}_{\mathbf{M}/B}(f, p) \rightarrow \text{Map}_{PB, GB}(\text{hfib}(f), \text{hfib}(p))$$

is a weak equivalence.

#### APPENDIX A. THE PROOF OF THEOREM 4.31

Since the proof of the theorem relating assembly to cocompletion is of a rather different nature from the rest of the paper, we have reserved it for this appendix.

*Proof of Theorem 4.31.* We begin by establishing, via a dual version of the argument in [4, XI.10], a formula for  $|\mathcal{B}_\bullet^{\mathbb{K}_\Phi}(X)(D)|$  that is analogous to the definition of  $K_\Phi^{ho}(X)(D)$ . Let  $\mathbf{Simp}_{K_\Phi}(X)$  be the full subcategory of the overcategory  $(\mathbf{M}^{\mathbf{D}})/X$  with object set

$$\{\tau : K_\Phi(Y) \rightarrow X \in \text{Mor } \mathbf{M}^{\mathbf{D}} \mid Y : \mathbf{D} \rightarrow \mathbf{M} \text{ such that } K_\Phi(Y) \text{ is cofibrant} \},$$

and let

$$\text{Dom} : \mathbf{Simp}_{K_\Phi}(X) \rightarrow \mathbf{M}^{\mathbf{D}} : (\tau : K_\Phi(Y) \rightarrow X) \mapsto K_\Phi(Y)$$

be the “domain” functor. Note that  $(\varepsilon_\Phi)_X : K_\Phi(X) \rightarrow X$  is an object of  $\mathbf{Simp}_{K_\Phi}(X)$ .

Since  $\mathcal{B}_{\bullet}^{\mathbb{K}_{\Phi}} X$  comes equipped with a coaugmentation down to  $X$ , via the counit of the comonad  $\mathbb{K}_{\Phi}$ , the functor  $\mathcal{B}_{\bullet}^{\mathbb{K}_{\Phi}} X : \mathbf{\Delta}^{op} \rightarrow \mathbf{M}^{\mathbf{D}}$  factors through  $\mathbf{Simp}_{K_{\Phi}}(X)$ , as

$$\mathcal{B}_{\bullet}^{\mathbb{K}_{\Phi}} X = \text{Dom} \circ \widetilde{\mathcal{B}}_{\bullet}^{\mathbb{K}_{\Phi}} X.$$

The dual of Proposition XI.9.3 in [4] implies that  $\widetilde{\mathcal{B}}_{\bullet}^{\mathbb{K}_{\Phi}} X$  is homotopy right cofinal, as

- (1)  $\mathbf{\Delta}^{op}$  is obviously right filtered,
- (2) for all objects  $\tau : K_{\Phi}(Y) \rightarrow X$  in  $\mathbf{Simp}_{K_{\Phi}}(X)$

$$\begin{array}{ccc} K_{\Phi}(Y) & \xrightarrow{K_{\Phi}(\tau) \circ (\Delta_{\Phi})_Y} & K_{\Phi}(X) \\ & \searrow \tau & \swarrow (\varepsilon_{\Phi})_X \\ & & X \end{array}$$

commutes, and

- (3) for every commuting diagram

$$\begin{array}{ccc} K_{\Phi}(Y) & \xrightarrow{\sigma} & K_{\Phi}(X) \\ & \searrow \tau & \swarrow (\varepsilon_{\Phi})_X \\ & & X \end{array}$$

the morphism  $(\Delta_{\Phi})_X : K_{\Phi}(X) \rightarrow K_{\Phi}^2(X)$  coequalizes  $\sigma$  and  $K_{\Phi}(\tau) \circ (\Delta_{\Phi})_Y$ :

$$\begin{aligned} (\Delta_{\Phi})_X K_{\Phi}(\tau) (\Delta_{\Phi})_Y &= (\Delta_{\Phi})_X K_{\Phi}((\varepsilon_{\Phi})_X) K_{\Phi}(\sigma) (\Delta_{\Phi})_Y \\ &= (\Delta_{\Phi})_X K_{\Phi}((\varepsilon_{\Phi})_X) (\Delta_{\Phi})_X \sigma \\ &= (\Delta_{\Phi})_X \sigma. \end{aligned}$$

Theorem 19.6.7 in [11] then implies that the natural map

$$\text{hocolim}_{\mathbf{\Delta}^{op}} \mathcal{B}_{\bullet}^{\mathbb{K}_{\Phi}} X \rightarrow \text{hocolim}_{\mathbf{Simp}_{K_{\Phi}}(X)} \text{Dom}$$

is a weak equivalence, whence

$$|\mathcal{B}_{\bullet}^{\mathbb{K}_{\Phi}} X| \sim \text{hocolim}_{\mathbf{Simp}_{K_{\Phi}}(X)} \text{Dom},$$

since the Reedy cofibrancy of  $\mathcal{B}_{\bullet}^{\mathbb{K}_{\Phi}} X$  implies that the Bousfield-Kan map

$$\text{hocolim}_{\mathbf{\Delta}^{op}} \mathcal{B}_{\bullet}^{\mathbb{K}_{\Phi}} X \rightarrow |\mathcal{B}_{\bullet}^{\mathbb{K}_{\Phi}} X|$$

is a weak equivalence [11, Theorem 18.7.4]. The derived  $\mathbb{K}_{\Phi}$ -cocompletion of  $X$  therefore admits a description highly analogous to that of  $K_{\Phi}^{ho}(X)(D)$ .

The next step in the proof consists in showing that for all  $X \in \text{Ob } \mathbf{M}^{\mathbf{D}}$  and all  $D \in \text{Ob } \mathbf{D}$ ,

$$(A.1) \quad (\text{colim}_{\mathbf{Simp}_{K_{\Phi}}(X)} \text{Dom})(D) \cong K_{\Phi}(X)(D),$$

by expressing both sides, up to isomorphism, as the same colimit in two variables. Let

$$E_{X,D} : \mathbf{Simp}_{K_{\Phi}}(X) \times \mathbf{Simp}_{\Phi}(D) \rightarrow \mathbf{M}$$

denote the functor specified on objects by

$$E_{X,D}(\tau : K_{\Phi}(Y) \rightarrow X, \alpha : \Phi(C) \rightarrow D) = K_{\Phi}(Y)(\Phi(C)).$$



Note that for all  $\tau : K_\Phi(Y) \rightarrow X$ ,

$$(A.2) \quad K_\Phi(Y)(\Phi(C)) = \operatorname{colim}_{\mathbf{Simp}_\Phi(\Phi(C))} (Y \circ \operatorname{dom}) = Y(\Phi(C)),$$

since the identity map on  $\Phi(C)$  is a terminal object in  $\mathbf{Simp}_\Phi(\Phi(C))$ . Thus,

$$(\varepsilon_\Phi)_{X, \Phi(C)} = \operatorname{Id}_{X(\Phi(C))},$$

which implies that if

$$\begin{array}{ccc} K_\Phi(Y) & \xrightarrow{\sigma} & K_\Phi(X) \\ & \searrow \tau & \swarrow (\varepsilon_\Phi)_X \\ & X & \end{array}$$

is a morphism in  $\mathbf{Simp}_{K_\Phi}(X)$ , then  $\tau_{\Phi(C)} = \sigma_{\Phi(C)}$  for all objects  $C$  in  $\mathbf{C}$ . In particular, observation (2) above implies that

$$(A.3) \quad \tau_{\Phi(C)} = (K_\Phi(\tau) \circ (\Delta_\Phi)_Y)_{\Phi(C)},$$

for all  $\tau : K_\Phi(Y) \rightarrow X$ .

The key to establishing (A.1) is the fact that for all objects  $C$  in  $\mathbf{C}$ ,

$$(A.4) \quad \operatorname{colim}_{\mathbf{Simp}_{K_\Phi}(X)} Y(\Phi(C)) \cong X(\Phi(C)),$$

which we prove as follows. If  $M$  is an object in  $\mathbf{M}$ , let  $\operatorname{Cst}_M : \mathbf{Simp}_{K_\Phi}(X) \rightarrow \mathbf{M}$  denote the constant functor at  $M$ .

If

$$\begin{array}{ccc} K_\Phi(Y) & \xrightarrow{\sigma} & K_\Phi(Y') \\ & \searrow \tau & \swarrow \tau' \\ & X & \end{array}$$

is any morphism in  $\mathbf{Simp}_{K_\Phi}(X)$ , then evaluating at  $\Phi(C)$  gives rise, by (A.2), to a commutative diagram in  $\mathbf{M}$ :

$$\begin{array}{ccc} Y(\Phi(C)) & \xrightarrow{\sigma_{\Phi(C)}} & Y'(\Phi(C)) \\ & \searrow \tau_{\Phi(C)} & \swarrow \tau'_{\Phi(C)} \\ & X(\Phi(C)) & \end{array}$$

There is therefore a natural transformation

$$\Gamma : (-)(\Phi(C)) \rightarrow \operatorname{Cst}_{X(\Phi(C))} : \mathbf{Simp}_{K_\Phi}(X) \rightarrow \mathbf{M},$$

where  $\Gamma_\tau = \tau_{\Phi(C)} : Y(\Phi(C)) \rightarrow X(\Phi(C))$  for all  $\tau : K_\Phi(Y) \rightarrow X$ . Moreover, if  $M$  is any object in  $\mathbf{M}$ , and

$$\Psi : (-)(\Phi(C)) \rightarrow \operatorname{Cst}_M : \mathbf{Simp}_{K_\Phi}(X) \rightarrow \mathbf{M}$$

is any natural transformation, then

$$\Psi_{(\varepsilon_\Phi)_X} : X(\Phi(C)) \rightarrow M,$$

and for all objects  $\tau : K_\Phi(Y) \rightarrow X$  in  $\mathbf{Simp}_{K_\Phi}(X)$

$$\Psi_\tau = \Psi_{(\varepsilon_\Phi)_X} \circ \Gamma_\tau : Y(\Phi(C)) \rightarrow M.$$

To prove this last equality, we observe that since  $\Psi$  is a natural transformation, and  $K_\Phi(\tau) \circ (\Delta_\Phi)_Y$  is a morphism from  $\tau$  to  $(\varepsilon_\Phi)_X$  for all  $\tau : K_\Phi(Y) \rightarrow X$ , we can deduce from (A.3) that

$$\Psi_\tau = \Psi_{(\varepsilon_\Phi)_X} \circ (K_\Phi(\tau) \circ (\Delta_\Phi)_Y)_{\Phi(C)} = \Psi_{(\varepsilon_\Phi)_X} \circ \tau_{\Phi(C)}.$$

It follows from equation (A.4) that

$$\begin{aligned} K_\Phi(X)(D) &= \operatorname{colim}_{\mathbf{Simp}_\Phi(D)} (X \circ \operatorname{dom}) \\ &= \operatorname{colim}_{\mathbf{Simp}_\Phi(D)} X(\Phi(C)) \\ &\cong \operatorname{colim}_{\mathbf{Simp}_\Phi(D)} \operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)}} Y(\Phi(C)) \\ &\cong \operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)} \times \mathbf{Simp}_\Phi(D)} E_{X,D}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)}} \operatorname{Dom})(D) &= \operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)}} (K_\Phi(Y)(D)) \\ &= \operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)}} \operatorname{colim}_{\mathbf{Simp}_\Phi(D)} (Y \circ \operatorname{dom}) \\ &= \operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)}} \operatorname{colim}_{\mathbf{Simp}_\Phi(D)} Y(\Phi(C)) \\ &\cong \operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)} \times \mathbf{Simp}_\Phi(D)} E_{X,D}, \end{aligned}$$

since colimits in  $\mathbf{M}^{\mathbf{D}}$  are calculated objectwise. We can conclude that

$$(\operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)}} \operatorname{Dom})(D) \cong \operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)} \times \mathbf{Simp}_\Phi(D)} E_{X,D} \cong K_\Phi(X)(D).$$

To complete the proof of the theorem, we exhibit weak equivalences from

$$\operatorname{hocolim}_{\mathbf{Simp}_{K_\Phi(X)} \times \mathbf{Simp}_\Phi(D)} E_{X,D}$$

to  $(\operatorname{hocolim}_{\mathbf{Simp}_{K_\Phi(X)}} \operatorname{Dom})(D)$  and to  $K_\Phi^{ho}(X)(D)$ . Since we have already shown that  $|\mathcal{B}_{\bullet}^{K_\Phi} X|$  is weakly equivalent to  $\operatorname{hocolim}_{\mathbf{Simp}_{K_\Phi(X)}} \operatorname{Dom}$ , we can then conclude.

By definition  $K_\Phi(Y)$  is cofibrant in  $\mathbf{M}^{\mathbf{D}}$  for all  $\tau : K_\Phi(Y) \rightarrow X$  in  $\mathbf{Simp}_{K_\Phi(X)}$ , which implies that  $K_\Phi(Y)(D)$  is cofibrant in  $\mathbf{M}$  for all  $D \in \operatorname{Ob} \mathbf{D}$ . It follows that  $E_{X,D}$  is an objectwise cofibrant functor and therefore that we can apply Lemma B.13 to conclude that there are weak equivalences

$$\begin{array}{ccc} \operatorname{hocolim}_{\mathbf{Simp}_{K_\Phi(X)}} \operatorname{colim}_{\mathbf{Simp}_\Phi(D)} E_{X,D} & \xleftarrow{\sim} & \operatorname{hocolim}_{\mathbf{Simp}_{K_\Phi(X)} \times \mathbf{Simp}_\Phi(D)} E_{X,D} \\ & & \downarrow \sim \\ & & \operatorname{hocolim}_{\mathbf{Simp}_\Phi(D)} \operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)}} E_{X,D}. \end{array}$$

Because

$$\operatorname{colim}_{\mathbf{Simp}_\Phi(D)} E_{X,D}(\tau, -) = \operatorname{colim}_{\mathbf{Simp}_\Phi(D)} Y(\Phi(C)) = K_\Phi(Y)(D)$$

for all  $\tau : K_\Phi(Y) \rightarrow X$  by definition of  $K_\Phi$ , and

$$\operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)}} E_{X,D}(-, \alpha) = \operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)}} Y(\Phi(X)) = X(\Phi(C))$$

for all  $\alpha : \Phi(C) \rightarrow D$  by (A.4), it follows that

$$\operatorname{hocolim}_{\mathbf{Simp}_{K_\Phi(X)}} \operatorname{colim}_{\mathbf{Simp}_\Phi(D)} E_{X,D} \cong (\operatorname{hocolim}_{\mathbf{Simp}_{K_\Phi(X)}} \operatorname{Dom})(D)$$

and

$$\operatorname{hocolim}_{\mathbf{Simp}_\Phi(D)} \operatorname{colim}_{\mathbf{Simp}_{K_\Phi(X)}} E_{X,D} \cong K_\Phi^{ho}(X)(D).$$

The proof is thus complete, since it is clear from the definitions of the weak equivalences in the proof of Lemma B.13 that the following diagram commutes, where the maps from homotopy colimits to colimits are the natural ones.

$$\begin{array}{ccc}
 \text{hocolim}_{\mathbf{Simp}_{\Phi}(D)} \text{colim}_{\mathbf{Simp}_{K_{\Phi}}(X)} E_{X,D} & & \\
 \uparrow \sim & \searrow^{A_{X,D}} & \\
 \text{hocolim}_{\mathbf{Simp}_{\Phi}(D) \times \mathbf{Simp}_{K_{\Phi}}(X)} E_{X,D} & \longrightarrow & \text{colim}_{\mathbf{Simp}_{\Phi}(D) \times \mathbf{Simp}_{K_{\Phi}}(X)} E_{X,D} \longrightarrow X(D) \\
 \downarrow \sim & \nearrow_{B_{X,D}} & \\
 \text{hocolim}_{\mathbf{Simp}_{K_{\Phi}}(X)} \text{colim}_{\mathbf{Simp}_{\Phi}(D)} E_{X,D} & & 
 \end{array}$$

□

## APPENDIX B. USEFUL SIMPLICIAL STRUCTURES

In this appendix we recall various simplicial structures that are necessary to this paper, in order to fix notation and terminology and to make this paper as self-contained as possible and reasonable.

**B.1. Reedy model category structure.** Let  $\mathbf{M}$  be any model category. We describe here the fibrant objects in the Reedy model category structure on  $\mathbf{M}^{\Delta}$  and the cofibrant objects in the Reedy model category structure on  $\mathbf{M}^{\Delta^{op}}$ . We refer the reader to Chapter 15 of [11] for a complete description of these model category structures.

By Corollary 15.10.5 in [11], if  $X$  is a fibrant object of  $\mathbf{M}$ , then  $cc^{\bullet}X$  is Reedy fibrant in  $\mathbf{M}^{\Delta}$ . Dually, if  $X$  is cofibrant, then  $cs_{\bullet}X$  is Reedy cofibrant in  $\mathbf{M}^{\Delta^{op}}$ .

More generally, if  $X^{\bullet}$  is any cosimplicial object in  $\mathbf{M}$ , then the  $n^{\text{th}}$ -matching object of  $X^{\bullet}$  is

$$M_n X^{\bullet} = \text{equal} \left( \prod_{0 \leq i \leq n} X^n \begin{array}{c} \xrightarrow{\psi_1} \\ \xrightarrow{\psi_2} \end{array} \prod_{0 \leq i < j \leq n-1} X^{n-1} \right),$$

where, if  $p_i : \prod_{0 \leq i \leq n} X^n \rightarrow X^n$  and  $p_{i,j} : \prod_{0 \leq i < j \leq n-1} X^{n-1} \rightarrow X^{n-1}$  are the obvious projections, then

$$p_{i,j} \psi_1 = s^i p_j \quad \text{and} \quad p_{i,j} \psi_2 = s^{j-1} p_i.$$

The cosimplicial object  $X^{\bullet}$  is Reedy fibrant if the natural map

$$\sigma_n : X^n \rightarrow M_n X^{\bullet},$$

which is induced by the codegeneracies  $s^0, \dots, s^{n-1} : X^n \rightarrow X^{n-1}$ , is a fibration for all  $n$ .

Dually, if  $X_{\bullet}$  is any simplicial object in  $\mathbf{M}$ , then the  $n^{\text{th}}$ -latching object of  $X_{\bullet}$  is

$$L^n X_{\bullet} = \text{coequal} \left( \prod_{0 \leq i < j \leq n-1} X_{n-1} \begin{array}{c} \xrightarrow{\psi_1} \\ \xrightarrow{\psi_2} \end{array} \prod_{0 \leq i \leq n} X_n \right),$$

where, if  $\iota_i : X_n \rightarrow \prod_{0 \leq i \leq n} X_n$  and  $\iota_{i,j} : X_{n-1} \rightarrow \prod_{0 \leq i < j \leq n-1} X_{n-1}$  are the obvious summand “inclusions,” then

$$\psi_1 \iota_{i,j} = \iota_j s_i \quad \text{and} \quad \psi_2 \iota_{i,j} = \iota_i s_{j-1}.$$

The simplicial object  $X_\bullet$  is Reedy cofibrant if the natural map

$$\sigma^n : L^n X_\bullet \rightarrow X_n,$$

which is induced by the degeneracies  $s_0, \dots, s_{n-1} : X_{n-1} \rightarrow X_n$ , is a cofibration for all  $n$ .

**B.2. Simplicial enrichments, tensoring and cotensoring.** For a detailed introduction to the theory of enriched categories, we refer the reader to [16]. Here we recall only those elements of the theory of enrichments over  $\mathbf{sSet}$  that we use in this paper, from the point of view usual to homotopy theorists: we think of simplicial categories as ordinary categories with extra structure (cf. [11, Ch 9]).

**Definition B.1.** Let  $\mathbf{S}$  be a category. A *simplicial enrichment* of  $\mathbf{S}$  consists of a function

$$\mathrm{Map}_{\mathbf{S}} : \mathrm{Ob} \mathbf{S} \times \mathrm{Ob} \mathbf{S} \rightarrow \mathrm{ObsSet}$$

together with two collections of set maps

$$\{c_{X,Y,Z} : \mathrm{Map}_{\mathbf{S}}(Y, Z) \times \mathrm{Map}_{\mathbf{S}}(X, Y) \rightarrow \mathrm{Map}_{\mathbf{S}}(X, Z) \mid X, Y, Z \in \mathrm{Ob} \mathbf{S}\}$$

called *simplicial composition*, and

$$\{i_X : * \rightarrow \mathrm{Map}_{\mathbf{S}}(X, X) \mid X \in \mathrm{Ob} \mathbf{S}\},$$

called *units*, such that simplicial composition is associative and unital, in the obvious sense. Moreover, for all  $X, Y \in \mathrm{Ob} \mathbf{S}$ , there is a bijection

$$\mathrm{Map}_{\mathbf{S}}(X, Y)_0 \cong \mathbf{S}(X, Y)$$

that is compatible with composition.

If  $\mathbf{S}$  is a simplicially enriched category, then its *path-component category*  $\pi_0 \mathbf{S}$  is the ordinary category with  $\mathrm{Ob} \pi_0 \mathbf{S} = \mathrm{Ob} \mathbf{S}$  and  $\pi_0 \mathbf{S}(X, Y) = \pi_0 \mathrm{Map}_{\mathbf{S}}(X, Y)$ .

We usually drop the subscripts on simplicial composition, in the interest of simplifying notation.

*Remark B.2.* Suppose that  $\mathbf{S}$  is simplicially enriched. A morphism  $f : Y \rightarrow Y'$  in  $\mathbf{S}$  induces morphisms of simplicial sets

$$f_* : \mathrm{Map}_{\mathbf{S}}(X, Y) \rightarrow \mathrm{Map}_{\mathbf{S}}(X, Y') \quad \text{and} \quad f^* : \mathrm{Map}_{\mathbf{S}}(Y', X) \rightarrow \mathrm{Map}_{\mathbf{S}}(Y, X)$$

for all objects  $X$  in  $\mathbf{S}$ , which are defined as follows. If  $g \in \mathrm{Map}_{\mathbf{S}}(X, Y)_n$ , then

$$f_*(g) = c(s_0^n f, g),$$

which makes sense since we can view  $f$  as a 0-simplex of  $\mathrm{Map}_{\mathbf{S}}(Y, Y')$ . Similarly, if  $g \in \mathrm{Map}_{\mathbf{S}}(Y', X)_n$ , then

$$f^*(g) = c(g, s_0^n f).$$

Recall that associativity of composition in a category implies that if both small squares of a diagram

$$\begin{array}{ccccc} X & \xrightarrow{a} & Y & \xrightarrow{b} & Z \\ f \downarrow & & g \downarrow & & h \downarrow \\ X' & \xrightarrow{a'} & Y' & \xrightarrow{b'} & Z' \end{array}$$

commute, then so does the outer rectangle. The next lemma, which is a simplicial analog of this fact that proves useful in section 3.2, is an easy consequence of the associativity of the simplicial composition maps.

**Lemma B.3.** *Let  $\mathbf{S}$  be a simplicially enriched category. Let  $f : X \rightarrow X'$ ,  $g : Y \rightarrow Y'$  and  $h : Z \rightarrow Z'$  be morphisms in  $\mathbf{S}$ . Let  $a \in \text{Map}_{\mathbf{S}}(X, Y)_n$ ,  $b \in \text{Map}_{\mathbf{S}}(Y, Z)_n$ ,  $a' \in \text{Map}_{\mathbf{S}}(X', Y')_n$ , and  $b' \in \text{Map}_{\mathbf{S}}(Y', Z')_n$ .*

*If  $g_*(a) = f^*(a')$  and  $h_*(b) = g^*(b')$ , then  $h_*(c(b, a)) = f^*(c(b', a'))$ .*

**Definition B.4.** Let  $\mathbf{S}$  and  $\mathbf{S}'$  be simplicially enriched categories. A *simplicial functor*  $F$  from  $\mathbf{S}$  to  $\mathbf{S}'$  consists of

- (1) a function  $F : \text{Ob } \mathbf{S} \rightarrow \text{Ob } \mathbf{S}'$ , and
- (2) a collection of simplicial maps

$$\{F_{X,Y} : \text{Map}_{\mathbf{S}}(X, Y) \rightarrow \text{Map}_{\mathbf{S}'}(FX, FY) \mid X, Y \in \text{Ob } \mathbf{S}\},$$

called the *components* of the simplicial functor,

such that

$$c \circ (F_{Y,Z} \times F_{X,Y}) = F_{X,Z} \circ c \quad \text{and} \quad F_{X,X} \circ i_X = i_{FX}$$

for all  $X, Y, Z \in \text{Ob } \mathbf{S}$ .

**Definition B.5.** Let  $F, G : \mathbf{S} \rightarrow \mathbf{S}'$  be two simplicial functors. A *simplicial natural transformation*  $\tau$  from  $F$  to  $G$  consists of a collection

$$\{\tau_X : * \rightarrow \text{Map}_{\mathbf{S}'}(FX, GX) \mid X \in \text{Ob } \mathbf{S}\}$$

of simplicial maps such that

$$(\tau_Y)_* \circ F_{X,Y} = (\tau_X)^* \circ G_{X,Y} : \text{Map}_{\mathbf{S}}(X, Y) \rightarrow \text{Map}_{\mathbf{S}'}(FX, GY)$$

for all  $X, Y \in \text{Ob } \mathbf{S}$ .

**Definition B.6.** Let  $\text{Map}_{\mathbf{S}\mathbf{sSet}}$  denote the usual mapping space functor in  $\mathbf{sSet}$ . A simplicially enriched category  $\mathbf{S}$  is *tensoried over  $\mathbf{sSet}$*  if there is a functor

$$- \otimes - : \mathbf{S} \times \mathbf{sSet} \rightarrow \mathbf{S}$$

together with natural isomorphisms

$$\text{Map}_{\mathbf{S}}(X \otimes L, Y) \cong \text{Map}_{\mathbf{sSet}}(L, \text{Map}_{\mathbf{S}}(X, Y))$$

for all  $X, Y \in \text{Ob } \mathbf{S}$  and  $L \in \text{Ob } \mathbf{sSet}$ . Dually,  $\mathbf{S}$  is *cotensoried over  $\mathbf{sSet}$*  if there is a functor

$$(-)^{(-)} : \mathbf{S} \times \mathbf{sSet}^{op} \rightarrow \mathbf{S},$$

together with natural isomorphisms

$$\text{Map}_{\mathbf{S}}(X, Y^L) \cong \text{Map}_{\mathbf{sSet}}(L, \text{Map}_{\mathbf{S}}(X, Y))$$

for all  $X, Y \in \text{Ob } \mathbf{S}$  and  $L \in \text{Ob } \mathbf{sSet}$ .

*Remark B.7.* If  $\mathbf{S}$  is tensoried and cotensoried over  $\mathbf{sSet}$ , then, for all simplicial sets  $L$ , the functors  $- \otimes L$  and  $(-)^L$  form an adjoint pair, with counit  $ev : X^L \otimes L \rightarrow X$  and unit  $coev : X \rightarrow (X \otimes L)^L$ . It follows as well that

$$\text{Map}_{\mathbf{S}}(X, Y)_n \cong \mathbf{S}(X \otimes \Delta[n], Y) \cong \mathbf{S}(X, Y^{\Delta[n]})$$

for all  $X, Y \in \text{Ob } \mathbf{S}$  and  $n \geq 0$ .

**B.3. Simplicial model categories.** We sketch here those elements of the theory of simplicial model categories that we need in this paper. We refer the reader to, e.g., [11, Ch 9], for a thorough exposition of the subject, including a detailed discussion of homotopy (co)limits, as well as the Tot and realization functors.

**Definition B.8.** A model category that is simplicially enriched, as well as tensored and cotensored over  $\mathbf{sSet}$ , is a *simplicial model category* if for every cofibration  $i : X \twoheadrightarrow Y$  and every fibration  $p : E \twoheadrightarrow B$  in  $\mathbf{M}$ , the induced map of simplicial sets

$$\mathrm{Map}_{\mathbf{M}}(Y, E) \xrightarrow{i^* \times p_*} \mathrm{Map}_{\mathbf{M}}(Y, B) \times_{\mathrm{Map}_{\mathbf{M}}(X, B)} \mathrm{Map}_{\mathbf{M}}(X, E)$$

is a fibration that is acyclic if  $i$  or  $p$  is a weak equivalence.

**Theorem B.9.** [11, Theorem 9.8.5] *Let  $\mathbf{M}$  and  $\mathbf{M}'$  be simplicial model categories. A functor  $F : \mathbf{M} \rightarrow \mathbf{M}'$  can be extended to a simplicial functor if and only if there is a natural transformation*

$$\theta : F(-) \otimes - \Rightarrow F(- \otimes -)$$

of functors from  $\mathbf{M} \times \mathbf{sSet}$  into  $\mathbf{M}'$  such that

- (1) *a unit condition holds:  $\theta_{X, \Delta[0]} : F(X) \otimes \Delta[0] \rightarrow F(X \otimes \Delta[0])$  is an isomorphism for all  $X$ , compatible with the natural isomorphisms  $X \otimes \Delta[0] \cong X$  and  $F(X) \otimes \Delta[0] \cong F(X)$ , and*
- (2) *an associativity condition holds: the two possible natural composites*

$$F(X) \otimes (L \times L') \rightarrow F((X \otimes L) \otimes L')$$

*built from  $\theta$  and the tensoring isomorphisms are equal, for all  $X, L$  and  $L'$ .*

*Remark B.10.* The functor  $F : \mathbf{M} \rightarrow \mathbf{M}'$  of the previous theorem is simplicial if and only if there is a natural transformation

$$\tau : F((-)^{(-)}) \Rightarrow F(-)^{(-)}$$

of functors from  $\mathbf{M} \times \mathbf{sSet}^{op}$  into  $\mathbf{M}'$ , which is associative and unital in a sense similar to that above. Indeed, given  $\theta$ , let  $\tau_{X,L} : F(X^L) \rightarrow F(X)^L$  be the transpose, with respect to the  $(- \otimes L, (-)^L)$ -adjunction, of the composite

$$F(X^L) \otimes L \xrightarrow{\theta_{X^L, L}} F(X^L \otimes L) \xrightarrow{F(ev)} F(X).$$

Similarly, given  $\tau$ , let  $\theta_{X,L}$  be the transpose, with respect to the  $(- \otimes L, (-)^L)$ -adjunction, of the composite

$$F(X) \xrightarrow{F(coev)} F((X \otimes L)^L) \xrightarrow{\tau_{X \otimes L, L}} F(X \otimes L)^L.$$

**Definition B.11.** Let  $\mathbf{M}$  be a simplicial model category. The *totalization functor*

$$\mathrm{Tot} : \mathbf{M}^{\Delta} \rightarrow \mathbf{M}$$

is defined using the cotensoring on  $\mathbf{M}$  to be

$$\mathrm{Tot} X^{\bullet} = \mathrm{equal} \left( \prod_{n \geq 0} (X^n)^{\Delta[n]} \rightrightarrows \prod_{\Delta(n,k); k, n \geq 0} (X^k)^{\Delta[n]} \right),$$

where the maps are given in the obvious way by projection onto different factors, either in the simplicial component or in the  $\mathbf{M}$ -component. The *realization functor*

$$|-| : \mathbf{M}^{\Delta^{op}} \rightarrow \mathbf{M}$$

is defined using the tensoring on  $\mathbf{M}$  to be

$$|X_\bullet| = \text{coeq} \left( \coprod_{\Delta(n,k); k, n \geq 0} X_n \otimes \Delta[k] \rightrightarrows \coprod_{n \geq 0} X_n \otimes \Delta[n] \right),$$

where the maps are given in the obvious way by injection into different summands, either in the simplicial component or in the  $\mathbf{M}$ -component.

*Remark B.12.* We recall that if  $X^\bullet$  is Reedy fibrant, then  $\text{Tot } X^\bullet$  is fibrant in  $\mathbf{M}$ . If  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  is a levelwise weak equivalence between Reedy fibrant objects, then  $\text{Tot } f^\bullet$  is a weak equivalence in  $\mathbf{M}$ .

Dually, if  $X_\bullet$  is Reedy cofibrant, then  $|X_\bullet|$  is cofibrant in  $\mathbf{M}$ . If  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  is a levelwise weak equivalence between Reedy cofibrant objects, then  $|f_\bullet|$  is a weak equivalence in  $\mathbf{M}$ .

In appendix A we need the following comparison result for homotopy colimits.

**Lemma B.13.** *Let  $\mathbf{M}$  be a simplicial model category, and let  $\mathbf{C}$  and  $\mathbf{D}$  be small categories. If  $F : \mathbf{C} \times \mathbf{D} \rightarrow \mathbf{M}$  is a functor such that  $F(C, D)$  is cofibrant for all objects  $C$  and  $D$ , then there are weak equivalences*

$$\text{hocolim}_{\mathbf{D}} (\text{colim}_{\mathbf{C}} F(-, D)) \xleftarrow{\sim} \text{hocolim}_{\mathbf{C} \times \mathbf{D}} F \xrightarrow{\sim} \text{hocolim}_{\mathbf{C}} (\text{colim}_{\mathbf{D}} F(C, -)).$$

*Proof.* Let  $\mathcal{N} : \mathbf{Cat} \rightarrow \mathbf{sSet}$  denote the nerve (or classifying space) functor from the category of small categories to the category of simplicial sets. According to Corollary 14.8.8 in [11], the functor

$$\mathcal{N}(-/\mathbf{E})^{op} : \mathbf{E}^{op} \rightarrow \mathbf{sSet}$$

is cofibrant in the model category structure on  $\mathbf{sSet}^{\mathbf{E}^{op}}$  in which weak equivalences and fibrations are defined objectwise. Moreover  $\mathcal{N}(E/\mathbf{E}) \rightarrow *$  is a weak equivalence for all objects  $E$ , since  $E/\mathbf{E}$  admits a terminal object,  $Id_E : E \rightarrow E$ .

Recall that if  $G : \mathbf{E} \rightarrow \mathbf{M}$  is any functor, then its homotopy colimit can be constructed as a coend over  $\mathbf{E}$ :

$$\text{hocolim}_{\mathbf{E}} G = G \otimes_{\mathbf{E}} \mathcal{N}(-/\mathbf{E})^{op}.$$

It follows that

$$\text{hocolim}_{\mathbf{C} \times \mathbf{D}} F = F \otimes_{\mathbf{C} \times \mathbf{D}} \mathcal{N}((-)/(\mathbf{C} \times \mathbf{D}))^{op},$$

while

$$\text{hocolim}_{\mathbf{D}} (\text{colim}_{\mathbf{C}} F(-, D)) \cong F \otimes_{\mathbf{C} \times \mathbf{D}} (\mathcal{N}(-/\mathbf{D})^{op} \circ P^{op}),$$

where  $P : \mathbf{C} \times \mathbf{D} \rightarrow \mathbf{D}$  is the projection functor.

We can define a natural transformation

$$\pi : \mathcal{N}((-)/(\mathbf{C} \times \mathbf{D}))^{op} \rightarrow \mathcal{N}(-/\mathbf{D})^{op} \circ P^{op},$$

by specifying that  $\pi_{C,D}$  sends an  $n$ -simplex

$$\begin{array}{ccccc} & (C, D) & & & \\ & \swarrow (f_0, g_0) & \downarrow (f_1, g_1) & \searrow (f_n, g_n) & \\ (C_0, D_0) & \xleftarrow{(c_1, d_1)} & (C_1, D_1) & \xleftarrow{\dots} & \xleftarrow{(c_n, d_n)} (C_n, D_n) \end{array}$$

to the  $n$ -simplex

$$\begin{array}{ccccc}
 & & D & & \\
 & g_0 \swarrow & \downarrow g_1 & \searrow g_n & \\
 D_0 & \xleftarrow{d_1} & D_1 & \xleftarrow{\dots} & D_n
 \end{array}$$

for every pair of objects  $(C, D)$  in  $\mathbf{C}^{op} \times \mathbf{D}^{op}$ .

Since both of the functors  $\mathcal{N}((-,-)/(\mathbf{C} \times \mathbf{D}))^{op}$  and  $\mathcal{N}(-/\mathbf{D})^{op} \circ P^{op}$  are objectwise contractible,  $\pi$  is a weak equivalence with cofibrant source and target in  $\mathbf{sSet}^{\mathbf{C}^{op} \times \mathbf{D}^{op}}$ . Corollary 18.4.5 in [11] therefore implies that the induced map

$$Id_F \otimes_{\mathbf{C} \times \mathbf{D}} \pi : \text{hocolim}_{\mathbf{C} \times \mathbf{D}} F \rightarrow \text{hocolim}_{\mathbf{D}} (\text{colim}_{\mathbf{C}} F(-, D))$$

is a weak equivalence of cofibrant objects in  $\mathbf{M}$ , since  $F$  is objectwise cofibrant.  $\square$

**B.4. External (co)simplicial structure.** Let  $\mathbf{C}$  be a category admitting all finite limits and colimits. Quillen showed in [21, §II.1] (cf. also [3, §2.10]) that  $\mathbf{C}^{\Delta^{op}}$  and  $\mathbf{C}^{\Delta}$  both admit a simplicial enrichment that is tensored and cotensored over  $\mathbf{sSet}$ , called the *external simplicial structure*.

Bousfield proved the cosimplicial half of the following useful result in [3, §2.12]. To prove the simplicial half, one can simply dualize his proof.

**Proposition B.14.** [3] *Let  $\mathbf{M}$  be a simplicial model category.*

- (1) *Let  $f^{\bullet}, g^{\bullet} : X^{\bullet} \rightarrow Y^{\bullet}$  be morphisms of Reedy fibrant cosimplicial objects in  $\mathbf{M}^{\Delta}$ . If  $f^{\bullet}$  and  $g^{\bullet}$  are externally homotopic, then  $\text{Tot } f^{\bullet}$  and  $\text{Tot } g^{\bullet}$  are simplicially homotopic.*
- (2) *Let  $f_{\bullet}, g_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$  be morphisms of Reedy cofibrant simplicial objects in  $\mathbf{M}^{\Delta^{op}}$ . If  $f_{\bullet}$  and  $g_{\bullet}$  are externally homotopic, then  $|f_{\bullet}|$  and  $|g_{\bullet}|$  are simplicially homotopic.*

In this paper, we apply the following consequence of Bousfield's result.

**Corollary B.15.** *Let  $\mathbf{M}$  be a simplicial model category.*

- (1) *If  $f^{\bullet} : X^{\bullet} \rightarrow Y^{\bullet}$  is an external homotopy equivalence of Reedy fibrant objects in  $\mathbf{M}^{\Delta}$ , then  $\text{Tot } f^{\bullet} : \text{Tot } X^{\bullet} \rightarrow \text{Tot } Y^{\bullet}$  is a weak equivalence in  $\mathbf{M}$ .*
- (2) *If  $f_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$  is an external homotopy equivalence of Reedy cofibrant objects in  $\mathbf{M}^{\Delta^{op}}$ , then  $|f_{\bullet}| : |X_{\bullet}| \rightarrow |Y_{\bullet}|$  is a weak equivalence in  $\mathbf{M}$ .*

*Proof.* We do the cosimplicial case and leave the strictly dual proof of the simplicial case to the reader.

Let  $g^{\bullet} : Y^{\bullet} \rightarrow X^{\bullet}$  be the external homotopy inverse to  $f^{\bullet}$ . By Proposition B.14 (1),  $\text{Tot}(g^{\bullet})\text{Tot}(f^{\bullet}) = \text{Tot}(g^{\bullet}f^{\bullet})$  is simplicially homotopic to the identity morphism of  $\text{Tot } X^{\bullet}$  and  $\text{Tot}(f^{\bullet})\text{Tot}(g^{\bullet}) = \text{Tot}(f^{\bullet}g^{\bullet})$  is simplicially homotopic to the identity morphism of  $\text{Tot } Y^{\bullet}$ , i.e.,  $\text{Tot}(f^{\bullet})$  and  $\text{Tot}(g^{\bullet})$  are simplicial homotopy equivalences. By Proposition 9.5.16 in [11], we can conclude that  $\text{Tot}(f^{\bullet})$  and  $\text{Tot}(g^{\bullet})$  are weak equivalences in  $\mathbf{M}$ .  $\square$

We are particularly interested here in the following excellent, classical source of external homotopy equivalences of cosimplicial (respectively, simplicial) objects.

**Definition B.16.** Let  $\mathbf{C}$  be any category.



- (1) A *contractible cosimplicial object* in  $\mathbf{C}$  consists of an object  $Y^\bullet$  in  $\mathbf{C}^\Delta$  and an object  $X$  in  $\mathbf{C}$ , together with morphisms in  $\mathbf{C}$

$$\eta : X \rightarrow Y^0 \quad \text{and} \quad s^n : Y^n \rightarrow Y^{n-1}, \forall n \geq 0,$$

where  $Y^{-1} := X$ , satisfying

$$d^0 \eta = d^1 \eta : X \rightarrow Y^1,$$

$$s^0 \eta = Id_X \quad \text{and} \quad s^n d^n = Id_{Y^{n-1}} \quad \forall n \geq 0,$$

and

$$s^1 d^0 = \eta s^0 \quad \text{and} \quad s^n d^i = d^i s^{n-1} \quad \forall 0 \leq i \leq n-1.$$

- (2) A *contractible simplicial object* in  $\mathbf{C}$  consists of an object  $Y_\bullet$  in  $\mathbf{C}^{\Delta^{op}}$  and an object  $X$  in  $\mathbf{C}$ , together with morphisms in  $\mathbf{C}$

$$\varepsilon : Y_0 \rightarrow X \quad \text{and} \quad s_n : Y_{n-1} \rightarrow Y_n, \forall n \geq 0,$$

where  $Y_{-1} := X$ , satisfying

$$\varepsilon d_0 = \varepsilon d_1,$$

$$\varepsilon s_0 = Id_X \quad \text{and} \quad d_n s_n = Id_{Y_{n-1}} \quad \forall n \geq 0,$$

and

$$d_0 s_1 = s_0 \varepsilon \quad \text{and} \quad d_i s_n = s_{n-1} d_i \quad \forall 0 \leq i \leq n-1.$$

As proved by Barr in [2], contractible (co)simplicial objects give rise to external homotopy equivalences of the following special sort.

**Proposition B.17.** [2, Prop. 3.3] *Let  $\mathbf{C}$  be a category admitting all finite limits and colimits.*

- (1) *If  $X \xrightarrow{\eta} Y^\bullet$  is a contractible cosimplicial object in  $\mathbf{C}$ , then there are cosimplicial maps*

$$\eta^\bullet : cc^\bullet X \rightarrow Y^\bullet \quad \text{and} \quad \rho^\bullet : Y^\bullet \rightarrow cc^\bullet X$$

*such that  $\rho^\bullet \eta^\bullet = Id_{cc^\bullet X}$  and  $\eta^\bullet \rho^\bullet$  is externally homotopic to  $Id_{Y^\bullet}$ .*

- (2) *If  $Y_\bullet \xrightarrow{\varepsilon} X$  is a contractible simplicial object in  $\mathbf{C}$ , then there are simplicial maps*

$$\varepsilon_\bullet : Y_\bullet \rightarrow cs_\bullet X \quad \text{and} \quad \sigma_\bullet : cs_\bullet X \rightarrow Y_\bullet$$

*such that  $\varepsilon_\bullet \sigma_\bullet = Id_{cs_\bullet X}$  and  $\sigma_\bullet \varepsilon_\bullet$  is externally homotopic to  $Id_{Y_\bullet}$ .*

Motivated by the proposition above, we formulate the following definition.

**Definition B.18.** Let  $\mathbf{C}$  be a category admitting all finite limits and colimits.

- (1) An *external cosimplicial strong deformation retract (SDR)* in  $\mathbf{C}$  consists of a pair of morphisms in  $\mathbf{C}^\Delta$

$$X^\bullet \begin{array}{c} \xrightarrow{\eta^\bullet} \\ \xleftarrow{\rho^\bullet} \end{array} Y^\bullet$$

*such that  $\rho^\bullet \eta^\bullet = Id_{cc^\bullet X}$  and  $\eta^\bullet \rho^\bullet$  is externally homotopic to  $Id_{Y^\bullet}$ .*

- (2) An *external simplicial strong deformation retract (SDR)* in  $\mathbf{C}$  consists of a pair of morphisms in  $\mathbf{C}^{\Delta^{op}}$

$$X_\bullet \begin{array}{c} \xrightarrow{\sigma_\bullet} \\ \xleftarrow{\varepsilon_\bullet} \end{array} Y_\bullet$$

*such that  $\varepsilon_\bullet \sigma_\bullet = Id_{cs_\bullet X}$  and  $\sigma_\bullet \varepsilon_\bullet$  is externally homotopic to  $Id_{Y_\bullet}$ .*

*Notation B.19.* An external cosimplicial SDR is denoted

$$X^\bullet \begin{array}{c} \xrightarrow{\eta^\bullet} \\ \xleftarrow{\rho^\bullet} \end{array} Y^\bullet \circlearrowleft h ,$$

while an external simplicial SDR is denoted

$$X_\bullet \begin{array}{c} \xrightarrow{\sigma_\bullet} \\ \xleftarrow{\varepsilon_\bullet} \end{array} Y_\bullet \circlearrowleft h .$$

The next definition is motivated by the fact that, if  $R$  is a ring and  $M$  and  $N$  are right  $R$ -modules, then the elements of  $\text{Ext}_R^n(M, N)$  are in bijective correspondence with chain homotopy classes of chain maps of degree  $-n$  from a resolution of  $M$  to an injective resolution of  $N$ , as well as with chain homotopy classes of maps of degree  $-n$  from a projective resolution of  $M$  to a resolution of  $N$ .

**Definition B.20.** Let  $\mathbf{S}$  be a simplicially enriched model category that satisfies the extra axiom of Definition B.8, but that is not necessarily tensored or cotensored over  $\mathbf{sSet}$ . If  $X, Y \in \text{Ob } \mathbf{M}$ , then for all  $s, t \geq 0$ , the group  $\text{Ext}_{\mathbf{S}}^{s,t}(X, Y)$  is defined to be the common value of

$$\pi^s \pi_t \text{Map}_{\mathbf{S}}(X, \widehat{Y}^\bullet),$$

where

$$cc^\bullet Y \begin{array}{c} \xrightarrow{\eta^\bullet} \\ \xleftarrow{\rho^\bullet} \end{array} \widehat{Y}^\bullet \circlearrowleft h ,$$

is a *fibrant cosimplicial resolution* of  $Y$ , i.e., an external cosimplicial SDR where  $\widehat{Y}^\bullet$  is Reedy fibrant, and of

$$\pi^s \pi_t \text{Map}_{\mathbf{S}}(\widetilde{X}_\bullet, Y),$$

where

$$cs_\bullet X \begin{array}{c} \xrightarrow{\sigma_\bullet} \\ \xleftarrow{\varepsilon_\bullet} \end{array} \widetilde{X}_\bullet \circlearrowleft h .$$

is a *cofibrant simplicial resolution* of  $X$ , i.e., an external simplicial SDR where  $\widetilde{X}_\bullet$  is Reedy cofibrant.

*Remark B.21.* It follows from the proof of [11, Proposition 9.3.1] that the hypotheses on  $\mathbf{S}$  in the definition above suffice to prove that  $\text{Map}_{\mathbf{S}}(Q, R)$  is a fibrant simplicial set for all cofibrant  $Q$  and fibrant  $R$ . The homotopy groups in the definition are thus well defined.

*Remark B.22.* It is not hard to show that the value of  $\text{Ext}_{\mathbf{S}}^{s,t}(X, Y)$  is independent of which component is resolved and which resolution is taken. The key to the proof is Bousfield's observation in section 2.12 of [3] that if  $f^\bullet : A^\bullet \rightarrow B^\bullet$  is an external homotopy equivalence of cosimplicial abelian groups (respectively, groups or sets), then it induces an isomorphism  $\pi^s f^\bullet : \pi^s A^\bullet \xrightarrow{\cong} \pi^s B^\bullet$  for all  $s \geq 0$  (respectively,  $s = 0, 1$  or  $s = 0$ ). It follows that if  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  is an external homotopy equivalence of Reedy fibrant cosimplicial simplicial sets, then

$$\pi^s \pi_t f_\bullet : \pi^s \pi_t X_\bullet \rightarrow \pi^s \pi_t Y_\bullet$$

is an isomorphism for all  $t \geq s \geq 0$ .

Checking independence of resolution in the definition of  $\text{Ext}_{\mathbf{S}}^{s,t}(X, Y)$ , we suppose that

$$cc^\bullet Y \begin{array}{c} \xrightarrow{\eta_1^\bullet} \\ \xleftarrow{\rho_1^\bullet} \end{array} \widehat{Y}_1^\bullet \circlearrowleft h \quad \text{and} \quad cc^\bullet Y \begin{array}{c} \xrightarrow{\eta_2^\bullet} \\ \xleftarrow{\rho_2^\bullet} \end{array} \widehat{Y}_2^\bullet \circlearrowleft h$$

are fibrant cosimplicial resolutions of  $Y$ . Applying  $\text{Map}_{\mathbf{S}}(X, -)$  to both resolutions, we obtain two external cosimplicial SDR's of simplicial sets

$$\text{Map}_{\mathbf{S}}(X, cc^\bullet Y) \begin{array}{c} \xrightarrow{(\eta_1^\bullet)_*} \\ \xleftarrow{(\rho_1^\bullet)_*} \end{array} \text{Map}_{\mathbf{S}}(X, \widehat{Y}_1^\bullet) \circlearrowleft h,$$

and

$$\text{Map}_{\mathbf{S}}(X, cc^\bullet Y) \begin{array}{c} \xrightarrow{(\eta_2^\bullet)_*} \\ \xleftarrow{(\rho_2^\bullet)_*} \end{array} \text{Map}_{\mathbf{S}}(X, \widehat{Y}_2^\bullet) \circlearrowleft h,$$

which we can compose to form the diagram of cosimplicial simplicial sets

$$\text{Map}_{\mathbf{S}}(X, \widehat{Y}_1^\bullet) \begin{array}{c} \xrightarrow{(\eta_2^\bullet)_*(\rho_1^\bullet)_*} \\ \xleftarrow{(\eta_1^\bullet)_*(\rho_2^\bullet)_*} \end{array} \text{Map}_{\mathbf{S}}(X, \widehat{Y}_2^\bullet).$$

Since

$$(\eta_1^\bullet)_*(\rho_2^\bullet)_*(\eta_2^\bullet)_*(\rho_1^\bullet)_* = (\eta_1^\bullet)_*(\rho_1^\bullet)_*,$$

which is externally homotopic to the identity, as is the composite in the other direction, both  $(\eta_1^\bullet)_*(\rho_2^\bullet)_*$  and  $(\eta_2^\bullet)_*(\rho_1^\bullet)_*$  are external homotopy equivalences and therefore induce isomorphisms

$$\pi^s \pi_t \text{Map}_{\mathbf{S}}(X, \widehat{Y}_1^\bullet) \cong \pi^s \pi_t \text{Map}_{\mathbf{S}}(X, \widehat{Y}_2^\bullet)$$

for all  $t \geq s \geq 0$ .

To prove independence of resolution in the first variable, we can easily dualize the proof above. Moreover the proof that we can choose which component to resolve without affecting the result is of a very similar nature, passing through the bicosimplicial simplicial set given by resolving both components.

#### APPENDIX C. MODEL CATEGORIES OF EILENBERG-MOORE (CO)ALGEBRAS

In order to define model category structures on categories of Eilenberg-Moore (co)algebras, which we need for our discussion of both (co)completion and homotopic (co)descent, we work with model category structures that are induced across adjunctions, of either of the types specified below.

**Definition C.1.** Let  $U : \mathbf{C} \rightarrow \mathbf{M}$  be a functor, where  $\mathbf{M}$  is a model category. A model structure on  $\mathbf{C}$  is *right-induced* from  $\mathbf{M}$  if  $\text{WE}_{\mathbf{C}} = U^{-1}(\text{WE}_{\mathbf{M}})$  and  $\text{Fib}_{\mathbf{C}} = U^{-1}(\text{Fib}_{\mathbf{M}})$ .

Let  $F : \mathbf{C} \rightarrow \mathbf{M}$  be a functor, where  $\mathbf{M}$  is a model category. A model structure on  $\mathbf{C}$  is *left-induced* from  $\mathbf{M}$  if  $\text{WE}_{\mathbf{C}} = F^{-1}(\text{WE}_{\mathbf{M}})$  and  $\text{Cof}_{\mathbf{C}} = F^{-1}(\text{Cof}_{\mathbf{M}})$ .

Schwede and Shipley provided a careful treatment of the case of algebras over a monad in [23], to which we refer the reader for details of definitions and terminology, as well as a slightly more general version of the next result (Lemma 2.3 in [23]).

**Proposition C.2** ([23]). *Let  $\mathbf{M}$  be a cofibrantly generated model category, with  $\mathcal{I}$  a set of generating cofibrations and  $\mathcal{J}$  a set of generating acyclic cofibrations, in which all objects are small. Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad on  $\mathbf{M}$  such that  $T$  commutes with filtered direct limits. Let  $\mathcal{I}^{\mathbb{T}}$  and  $\mathcal{J}^{\mathbb{T}}$  denote the images of  $\mathcal{I}$  and  $\mathcal{J}$  under the free  $\mathbb{T}$ -algebra functor  $F^{\mathbb{T}}$ .*

*The forgetful functor  $U^{\mathbb{T}} : \mathbf{M}^{\mathbb{T}} \rightarrow \mathbf{M}$  right-induces a cofibrantly generated model category structure on  $\mathbf{M}^{\mathbb{T}}$ , with  $\mathcal{I}^{\mathbb{T}}$  as generating cofibrations and  $\mathcal{J}^{\mathbb{T}}$  as generating acyclic cofibrations, if*

- (a) *every regular  $\mathcal{J}^{\mathbb{T}}$ -cofibration is a weak equivalence, or*
- (b) *every object of  $\mathbf{M}$  is fibrant and every  $\mathbb{T}$ -algebra admits a path object.*

The next proposition, which follows from Corollary 5.15 in [10], provides one example of conditions under which there exist model category structures in the coalgebra case. Other sets of sufficient conditions in particular cases were introduced by Hovey in [12] and by Stanculescu in [24].

Before stating the proposition, we recall a bit of helpful notation and a key definition from [10].

*Notation C.3.* Let  $\mathcal{X}$  be any subset of morphisms in a category  $\mathbf{C}$ . The closure of  $\mathcal{X}$  under formation of retracts is denoted  $\widehat{\mathcal{X}}$ , i.e.,

$$f \in \widehat{\mathcal{X}} \iff \exists g \in \mathcal{X} \text{ such that } f \text{ is a retract of } g.$$

**Definition C.4.** Let  $\mathcal{X}$  be a set of morphisms in a category  $\mathbf{C}$  that is closed under pullbacks. If  $\lambda$  is an ordinal and  $Y : \lambda^{op} \rightarrow \mathbf{C}$  is a functor such that for all  $\beta < \lambda$ , the morphism  $Y_{\beta+1} \rightarrow Y_{\beta}$  fits into a pullback

$$\begin{array}{ccc} Y_{\beta+1} & \longrightarrow & X_{\beta+1} \\ \downarrow & & \downarrow x_{\beta+1} \\ Y_{\beta} & \xrightarrow{k_{\beta}} & X_{\beta} \end{array}$$

for some  $x_{\beta+1} : X_{\beta+1} \rightarrow X_{\beta}$  in  $\mathcal{X}$  and  $k_{\beta} : Y_{\beta} \rightarrow X_{\beta}$  in  $\mathbf{C}$ , while  $Y_{\gamma} := \lim_{\beta < \gamma} Y_{\beta}$  for all limit ordinals  $\gamma < \lambda$ , then the composition of the tower  $Y$

$$\lim_{\lambda^{op}} Y_{\beta} \rightarrow Y_0,$$

if it exists, is an  $\mathcal{X}$ -Postnikov tower. The set of all  $\mathcal{X}$ -Postnikov towers is denoted  $\text{Post}_{\mathcal{X}}$ .

A *Postnikov presentation* of a model category  $\mathbf{M}$  is a pair of sets of morphisms  $\mathcal{X}$  and  $\mathcal{Z}$  satisfying

$$\text{Fib}_{\mathbf{M}} = \widehat{\text{Post}_{\mathcal{X}}} \quad \text{and} \quad \text{Fib}_{\mathbf{M}} \cap \text{WE}_{\mathbf{M}} = \widehat{\text{Post}_{\mathcal{Z}}}$$

and such that for all  $f \in \text{Mor } \mathbf{M}$ , there exist

- (a)  $i \in \text{Cof}$  and  $p \in \text{Post}_{\mathcal{Z}}$  such that  $f = pi$ ;
- (b)  $j \in \text{Cof} \cap \text{WE}$  and  $q \in \text{Post}_{\mathcal{X}}$  such that  $f = qj$ .

**Proposition C.5.** [10] *Let  $\mathbf{M}$  be a model category admitting a Postnikov presentation  $(\mathcal{X}, \mathcal{Z})$ , and let  $\mathbb{K}$  be a comonad on  $\mathbf{M}$  such that  $\mathbf{M}_{\mathbb{K}}$  is finitely bicomplete. Let*

$$\mathbf{W} = (U_{\mathbb{K}})^{-1}(\text{WE}_{\mathbf{M}}) \quad \text{and} \quad \mathbf{C} = (U_{\mathbb{K}})^{-1}(\text{Cof}_{\mathbf{M}}).$$

The forgetful functor  $U_{\mathbb{K}} : \mathbf{M}_{\mathbb{K}} \rightarrow \mathbf{M}$  left-induces a model category structure on  $\mathbf{M}_{\mathbb{K}}$ , where the set of fibrations is  $\widehat{\text{Post}}_{F_{\mathbb{K}}(X)}$ , if  $\text{Post}_{F_{\mathbb{K}}(Z)} \subset \mathbf{W}$  and for all  $f \in \text{Mor } \mathbf{C}$  there exist

- (a)  $i \in \mathbf{C}$  and  $p \in \text{Post}_{F_{\mathbb{K}}(Z)}$  such that  $f = pi$ ;
- (b)  $j \in \mathbf{C} \cap \mathbf{W}$  and  $q \in \text{Post}_{F_{\mathbb{K}}(X)}$  such that  $f = qj$ .

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