GRAPH-BASED REGULARIZATION FOR SPHERICAL SIGNAL INTERPOLATION

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ABSTRACT

This paper addresses the problem of the interpolation of 2-d spherical signals from non-uniformly sampled and noisy data. We propose a graph-based regularization algorithm to improve the signal reconstructed by local interpolation methods such as nearest neighbour or kernel-based interpolation algorithms. We represent the signal as a function on a graph where weights are adapted to the particular geometry of the sphere. We then solve a total variation (TV) minimization problem with a modified version of Chambolle's algorithm. Experimental results with noisy and uncomplete datasets show that the regularization algorithm is able to improve the result of local interpolation schemes in terms of reconstruction quality.

Index Terms— Spherical Function, Signal Reconstruction, Regularization, Chambolle algorithm

1. INTRODUCTION

Signal processing on non-classical manifolds become increasingly important with the developpement of new types of signals and sensing modes. In particular, omnidirectional audio or visual information, as well as many 2-d signals, can be efficiently mapped on the sphere in order to preserve the intrinsic geometry information. However, the data is generally captured as a irregularly sampled and noisy signal, while efficient processing of such signals on the 2-d sphere is generally performed on uniform or equiangular grids. It is therefore necessary to implement an interpolation step in order to map the function on a regular structure.

Conventional algorithms based on nearest neighbour interpolation generally fail to exploit global signal information. On the other hand, methods based on the Spherical Fourier transform [1] may fail to exploit efficiently the local signal information. Locally weighed linear and nonlinear regression methods [2] overcome these limitations, at the price however of an increased computational complexity.

We propose in this paper a graph-based regularization algorithm that builds on top of local interpolation methods for improved interpolation performance. The graph structure is very appealing for modeling the high dimensional data. It provides a lot of flexibility for the representation of different data structures and fast processing algorithms. Graph-based algorithms have been used in classical image processing problems such as inpainting or denoising, and more recently for optical flow estimation on the sphere [3]. We formulate here a total variation (TV) minimization problem on a graph for the interpolation of non-uniform and noisy samples on the sphere.

We solve the problem with a modified version of Chambolle's algorithm [4] that is adapted to the specific geometry of the sphere. Conducted experiments on geodesic data and omnidirectional image demonstrate that the regularization algorithm is able to improve significantly the performance of local interpolation methods even with noisy input data. We note that this paper does not discuss on selection of parameter values for the algrithm, which can further improve the final result.

Potential applications to this framework are numerous. In particular, many real geodesic signal measurements in astronomy, meteorology, oceanography, etc., as well as omnidirectional audio or visual information are easily mapped and processed on the sphere, due to the signal characteristics.

This paper is organized as follows. Section 2 provides the necessary definitions of discrete operators on graphs. The discrete regularization of 2-d spherical functions is presented in Section 3. Section 4 gives experimental results for the interpolation of noisy and nonuniformly sampled data on the sphere.

2. PRELIMINARIES

2.1. Graph representation

We represent the spherical function as a signal on a graph for efficient processing. A graph $\Gamma = (V, E)$ is composed from the set of vertices V that describe items and a set of edges that describe pairwise connections between pairs of vertices $E \subseteq V \times V$. We focus on undirected graphs, where for each edge $e = (u, v) \in E$ draws $e = (v, u) \in E$. We assume that the graph is connected (for every vertex there exists a path to any other vertex of the graph), and that the graph Γ has no self-loops. A graph is weighted if weights defined as $w : E \to \mathbb{R}^+$ are assigned to the graph edges ($w > 0$ if $(u, v) \in E$), where $w(u, v) = w(v, u)$.

In this work, we denote with $\mathcal{H}(V)$ the Hilbert space of real function on vertices, where $f : V \to \mathbb{R}^+$ assigns a real number to each vertex of the graph. A function on the graph edges is denoted by $h \in \mathcal{H}(E)$, where $h : E \to \mathbb{R}^+$ assigns a real value to each edge. Discrete operators for data processing on graph have been introduced in [5] and [6]. We recall here the main definitions that are necessary in our regularization framework.

- The degree function $d : V \to \mathbb{R}^+$ on the vertices is defined as $d(u) = \sum_{v \sim u} w(u, v)$, where $v \sim u$ denotes all vertices v connected to u by the edges (u, v).

- The edge derivative of function f along e at the vertex u, $\frac{\partial f}{\partial e}|_u : \mathcal{H}(V) \to \mathbb{R}^+$ is defined by:

$$\frac{\partial f}{\partial e}\big|_u = \sqrt{\frac{w(u,v)}{d(u)}}f(u) - \sqrt{\frac{w(u,v)}{d(v)}}f(v).$$

- The graph gradient is an operator $\nabla : \mathcal{H}(V) \to \mathcal{H}(E)$ defined by:
$$(\nabla f)(u,v) = \sqrt{\frac{w(u,v)}{d(v)}}f(v) - \sqrt{\frac{w(u,v)}{d(u)}}f(u), \forall(u,v) \in E.$$

- The norm of the gradient measures the roughness of a function around a vertex. A local variation of f at each vertex v is defined with the operator $\|\nabla\| : \mathbb{R}^N \to \mathbb{R}^+$ as:
$$\|\nabla_v f\| = \sqrt{\sum_{e \dashv v}(\frac{\partial f}{\partial e}\big|_v)^2}, \text{ where } e \dashv v \text{ denotes the set of}$$
the edges incident with vertex v.

- The graph divergence operator is given as: $div : \mathcal{H}(E) \to \mathcal{H}(V)$ which satisfies the property
$$< \nabla f, h >_{\mathcal{H}(E)} = < f, -div(h) >_{\mathcal{H}(V)}.$$

- The graph Laplacian is the operator $\triangle : \mathcal{H}(V) \to \mathcal{H}(V)$ defined by: $\triangle f = -\frac{1}{2}div(\|\nabla f\|)$. Combined with previous definitions, it can be rewritten as:
$$\triangle f(v) = f(v) - \sum_{u \sim v} \frac{w(u,v)}{\sqrt{d(u)d(v)}}f(u).$$

- The graph curvature $\mathcal{K} : \mathcal{H}(V) \to \mathcal{H}(V)$ is defined by: $\mathcal{K}f = \frac{1}{2}div(\frac{\nabla f}{\|\nabla f\|})$.

2.2. TV regularization on graphs

Equipped with the above definitions, we can now describe the total variation (TV) minimization problem on a graph. Originally, TV-based plane image restoration models were introduced by [7] and written as the following optimization problem

$$\min_{f^*} \int_\Omega |\nabla f^*| df^*, \quad s.t. \quad \|f^* - f\|_2^2 \le \sigma^2, \quad (1)$$

where $|\cdot|$ represents Euclidean distance, Ω denotes image domain, f is the observed image, f^* is estimated function and σ^2 is variance estimate of the noise.

Rather then solving the above constrained minimization problem, it is more convenient to solve the unconstrained problem:

$$\min_{f^*} \int_\Omega |\nabla f^*| df^* + \frac{1}{2\lambda}\|f^* - f\|_2^2, \quad (2)$$

where λ is a positive constant.

The above problem can be formulated as a discrete optimization problem for data on a graph. Given $\{f, f^*\}$ on V and p on E, Chambolle [4] proved that the solution to Eq. (2) is a function f^* given by $f^* = f - \lambda \cdot div(p)$, where the term $\lambda \cdot div(p)$ is computed from a following minimization problem:

$$\min\{\|\lambda \cdot div(p) - f\|^2 \mid p \in E, |p_{(u,v)}|^2 \le 1, \forall u, v \in f\}. \quad (3)$$

The problem of Eq. (3) can be solved by the semi-implicit algorithm, which is giving as a final result a following iterative procedure:

$$p_{(u,v)}^{(n+1)} = \frac{p_{(u,v)}^{(n)} + \tau(\nabla(div(p^{(n)}) - \frac{f}{\lambda}))_{(u,v)}}{1 + |\tau(\nabla(div(p^{(n)}) - \frac{f}{\lambda}))_{(u,v)}|} \quad (4)$$

The function p is initialized as a zero-valued matrix, the parameter $0 < \tau \le 1/(\mathcal{K}f)^2$ dictates the algorithm convergence speed and the parameter n denotes iteration steps of the algorithm. Note that several solutions have been proposed for duality-based algorithms in TV regularized image restoration [8]. In this paper, we use Chambolle's algorithm due to its fast convergence.
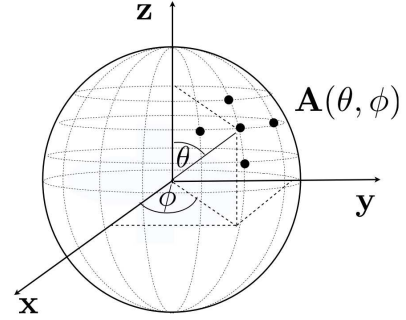
3. GRAPH REGULARIZATION OF SPHERICAL FUNCTIONS

Fig. 1. Illustration of samples of a 2-d spherical function: a grid point A(θ, ϕ) and its neighboring samples.

We define here a graph-based representation for functions on the 2-d sphere and we adapt the above regularization framework to the specific geometry of the problem. Points on a 2-d sphere (defined as a sphere whose radius $r = 1$) are defined with an azimuth angle $\theta \in [0, \pi]$ and a zenith angle $\phi \in [-\pi, \pi)$. The spherical function can be represented on a graph, where we assign the function sample values f to vertices of the graph. As the graph is constructed on a regular (equiangular) grid, we have to perform an a priori interpolation for samples that are missing in the input dataset. The function f on the graph is therefore defined from the noisy input samples when they are available, or from values that have been estimated by local interpolation methods.

The values of the edge weight function are given from the geodesic distances between samples. We draw edges only between neighbouring samples, where neighbours are defined as pairs of points with a small geodesic distance. In this work, we consider the four nearest samples on the regular grid as neighbors. The weights between function samples are dependent on the desired reconstruction resolution ($2N_\theta \times 2N_\phi$). The samples (θ_i, ϕ_i) on a regular grid are typically chosen from the set of values: $\theta_i \in \{\frac{\pi}{2N_\theta+1}, \frac{2\pi}{2N_\theta+1}, \dots, \frac{2N_\theta\pi}{2N_\theta+1}\}$, while $\phi_i \in \{-\pi, -\pi + \frac{2\pi}{2N_\phi}, \dots, \pi - \frac{2\pi}{2N_\phi}\}$. An interpolation point $A(\theta_i, \phi_i)$ and its neighbors are illustrated on Fig. 1. The geodesic distance among neighbors with the same θ_i value is given by $w^a = \frac{2\pi \sin\theta}{2N_\phi}$, while neighbors with same ϕ_i angle have a geodesic distance of $w^b = \frac{\pi}{2N_\theta+1}$. The degree function on the graph finally reads:

$$d(u) = \sum_{v \sim u} w(u,v) = 2w^a + w_{up}^b + w_{down}^b. \quad (5)$$

The regularization problem then consists in computing a function f^* on the sphere from noisy or a priori interpolated samples $f_i(\theta_i, \phi_i)$, where $i = 1, \dots, q$ with q the total number of samples. In other words, we have to estimate f^* in the presence of noise ϵ_i, i.e.,:

$$f_i = f^*(\theta_i, \phi_i) + \epsilon_i. \quad (6)$$

The objective is to find a set of values f^* on vertices V^* that preserves the edges and that is simultaneously close to the input data samples on the sphere, for each of the local set of neighbors in the

subgraphs. This is formalized as the following optimization problem:

$$\min_{f^*}\{\frac{||f^* - f||^2}{2\lambda} + |\nabla f^*|\}, \tag{7}$$

where f^* is the unknown function, and f is a noisy input set. This problem is a discrete version of problem given with Eq. (2). The first term is measuring a fidelity to initial data, while the second one is preserving the sharp discontinuities (edges) of the spherical function.

In our formulation, we assume that the underlying analog function on a 2-d sphere is differentiable and that the vertex points that are connected are 'close'. The latter condition is fulfilled by enabling only the local connections for each point on the equiangular grid.

Discrete values of the function samples and gradient and divergence operators on a graph defined in the previous section are therefore obtained by analog function sampling and discretization of the corresponding analog operators. In particular, when u is a grid point and $\{(u+1)^a, (u+1)^b, (u-1)^a, (u-1)^b\}$ are its neighboring points in the longitude and latitude direction, the gradient and divergence operators on the spherical grid are given as follows.

The gradient is defined as a 2-d vector $(\nabla f)(u) = (\nabla f_{(u)}^a, \nabla f_{(u)}^b)$, where:

$$\nabla f_{(u)}^b = \begin{cases} 0, \text{ if } \theta_i = \theta_{max} = \frac{2N_\theta \pi}{2N_\theta+1}, \\ \sqrt{\frac{w^b(u,u+1)}{d(u)}}f(u) - \sqrt{\frac{w^b(u,u+1)}{d(u+1)}}f^b(u+1), \text{ otherwise.} \end{cases}$$

$$\nabla f_{(u)}^a = \sqrt{\frac{w^a(u,u+1)}{d(u)}}f(u) - \sqrt{\frac{w^a(u,u+1)}{d(u+1)}}f^a(u+1) \tag{8}$$

The computation of the divergence relies on a function h on the graph edges that satisfy the property: $< \nabla f, h >_{\mathcal{H}(E)} = < f, -div(h) >_{\mathcal{H}(V)}$. We choose a Laplacian function $\triangle f(u) = -div(\nabla f(u))$, since it fulfills this property. It is defined by:

$$\triangle f_{(u)}^b = \begin{cases} f(u), \text{ if } \theta = \theta_{min} = \frac{\pi}{2N_\theta+1}, \\ f(u) - \frac{w^b(u,u-1)}{\sqrt{d(u)d(u-1)}}f^b(u-1), \text{ otherwise.} \end{cases}$$

$$\triangle f_{(u)}^a = f(u) - \frac{w^a(u,u-1)}{\sqrt{d(u)d(u-1)}}f^a(u-1). \tag{9}$$

Equipped with these operators, we can solve the TV-minimization problem on the graph with Chambolle's algorithm [4]. The iterative procedure of Eq. (4) is used to compute the values $p_{(u,v)}^{(n)}$, where the initial value $p_{(u,v)}^{(0)} = 0, \forall (u,v) \in V$. The resulting solution is $f^* = f - \lambda div(p^{(iter)})$, where $iter$ is a parameter that denotes a total number of iterative steps.

Critical point of the regularization algorithm is the estimation of parameter values. Though there are several known methods for their optimization [CITE], in this work we simply use a trial-error method to set the parameters and we keep it fixed throughout the experiments.

4. EXPERIMENTAL RESULTS

We compare the reconstruction results of a graph-based regularization method with the local interpolation methods whose parameters are chosen in order to minimize the reconstruction error. The nearest neighbour (NN) interpolation method and a kernel regression method (KerInt) described in [2] are the local interpolation methods used in our experiments. The latter approach approximates the function on a 2-d sphere with a Taylor series expansion. We use the Gaussian kernel function projected on the 2-d sphere by inverse stereographic projection. The variance of the Gaussian function represents a geodesic distance on the sphere. For each interpolation point, the set of neighboring points are the points whose geodesic distance to the interpolation point is smaller than the kernel variance σ. The samples are weighted locally, according to their distance to the interpolation point.

Experiments are conducted for spherical functions whose value points lie on the uniform grid (ground-truth values). We use a dataset representing the topography of the Earth[1] and a synthetic omnidirectional image. The datasets are normalized to values in the range $[0, 1]$ and down-sampled to a size 256×256.

We add an Gaussian noise to the sample values. Then we remove randomly a certain percentage k of pixels from the original image and estimate missing sample values of the spherical function from the remaining samples. This permits to analyze the performance of the proposed interpolation algorithm for uncomplete and noisy datasets on the sphere. We measure the performance with respect to the original data in means of the Peak-Signal-to-Noise-Ratio (PSNR), defined with $PSNR(dB) = 10log_{10}(\frac{MAX^2}{MSE})$ (MAX is maximum pixel intensity value), the $MSE = \frac{1}{10}\sum_{i=1}^{10} MSE_{rd}(i)$ and the Mean-Square-Error of random matrix interpolation errors is $MSE_{rd} = \frac{1}{2N_\theta \times 2N_\phi}\sum_{u=1}^{2N_\theta}\sum_{v=1}^{2N_\phi}(O_{u,v} - I_{u,v})^2$ (O is original data matrix and I is the interpolation result).

Fig. 2 illustrates the performance of the different interpolation methods for the Earth topology data when $k \in \{5, 10, \ldots, 55\}$ per cents of the original image samples are missing and in the presence of additive noise with Signal-to-Noise-Ratio $SNR(dB) \in \{25, 30\}$. To achieve statistically significant results we form a set of 10 randomly chosen interpolation points for each choice of k. We clearly see that the proposed regularization method permits to improve the performance of local interpolation methods. We see that the regularization peforms better when combined with the kernel-based approach (GrKerInt) rather than the NN-based interpolation method (GrNNInt), especially for a smaller number of missing points, since its input data set is more accurate.

Fig. 3 illustrates the results of the kernel-based method and the graph-based method combined with kernel-based interpolation for the Earth topology data when the added noise has a $SNR = 30dB$. Fig. 4 represents the same experiments with the synthetic image. Both figures confirm the benefits of the proposed regularization methods for different kinds of data.

5. CONCLUSIONS

This paper proposes a graph regularization-based method for interpolation of a spherical function from non-uniform and noisy samples. Our method achieves better performance in terms of PSNR than baseline nearest neighbour or kernel-based methods, especially for a small percentage of missing data. In addition, the proposed solution based on an adaptation of Chambolle's algorithm results in a fast algorithm whose complexity is similar to baseline methods. The framework proposed in this paper can further be extended to signal manifolds different from the sphere, when the graph representation is properly adapted to the corresponding geometry.

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[1]Available from the National Geophysical Data Center, NOAA US Department of Commerce under data announcement 88-MGG-02.
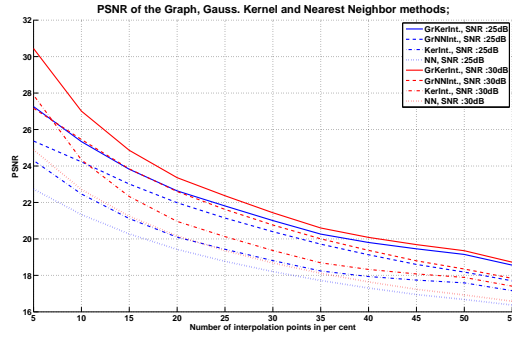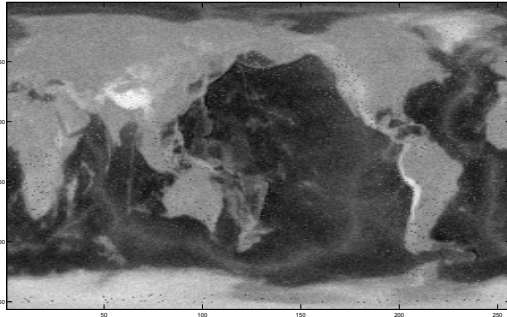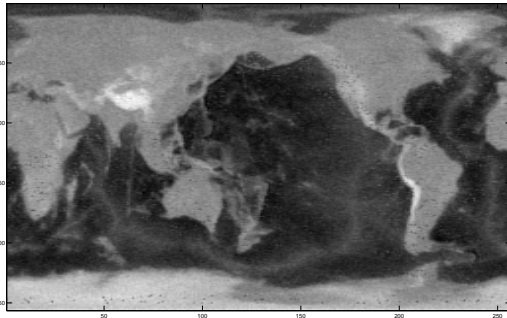
Fig. 2. PSNR values reconstruction with the proposed (GrNNInt, GrKerInt) and baseline methods (NN, KerInt) for different number of input point and $SNR \in \{25, 30\}$ dB. The values are averaged over 10 randomly chosen interpolation sets.

(a)

(b)

Fig. 3. Reconstruction results, $SNR = 30dB$, zero order Taylor expansion and $k = 5\%$ of missing data: (a) Kernel interpolation, $\sigma = 0.02, PSNR = 27.94dB$; (b) Graph regul. with Kernel interp. data, $(\sigma, iter., \tau, \lambda) = (0.02, 150, 25e^{-4}, 1e^{-5}), PSNR = 30.52dB$

6. ACKNOWLEDGEMENTS

(a)

(b)

Fig. 4. Reconstruction results, $SNR = 30dB$, zero order Taylor expansion and $k = 5\%$ of missing data: (a) Kernel interpolation, $\sigma = 0.02, PSNR = 27.95dB$; (b) Graph regul. with Kernel interp. data, $(\sigma, iter., \tau, \lambda) = (0.02, 200, 25e^{-4}, 1e^{-5}), PSNR = 28.55dB$

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