Estimating Physical Parameters in a Diffusive Field

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Abstract

In the recent researches some methods have been proposed for interpolating the diffusive field values along space, but they either do not use point-wise sampling or deploy large number of sensors in space to keep themselves near Niquist's sampling rate. In this research we propose some algorithms in which we benefit from temporal-spatial correlation of field values and try to reduce the number of sensors deployed along space. First we will consider the extreme case with only one sensor in space and use the whole time line to estimate the signals and second the discrete time for which we use M sensors in space and K time samples to reconstruct the field. Finally the stability of the proposed methods will be explored and the best will be chosen in the sense of being the most stable.

Chapter 1

Introduction

Sensor networks are typically used to monitor a physical quantity over space and time. A particular challenge of the sampling problem in sensor networks arises from the fact that the considered signals are usually neither bandlimited in space nor in time [1]. Examples are found in diffusion processes, where the initially localized release of a point source is spread (filtered) over space in a Gaussian manner. Practical situations include heat diffusion, gas diffusion, pollutants diffusion in water, etc.

Sampling theory treats a very fundamental problem, with so many practical repercussions, that it lies at the core of signal processing and communications. Sampling is all about representing a continuous-time signal $f(x)$ by a discrete set of values $f[n]$, $n \in \mathbb{Z}$. Often, in practice, instead of sampling the waveform itself, one has access only to its filtered version. If $f(x)$ is the original waveform, its filtered version is given by $g(x) = f(x) * \widetilde{h}(x)$, where $\widetilde{h}(x) = h(-x)$ is the convolution kernel. Then, uniform sampling with a sampling interval X_s yields samples $g(nX_s)$, which can be expressed as

$$g(nX_s) = \langle f(x), h(x - nX_s) \rangle = \int f(x)h(x - nX_s)\mathrm{d}x$$

Now the key question that arises is the following. Under what conditions is the original signal $f(x)$ uniquely defined by its samples $g(nX_s)$? The crucial result was stated by Shannon in 1949, in the form of the following sampling theorem [2]:

Theorem. [*Shannon's Sampling Theorem*] If $f(x)$ is bandlimited to ω_m, that is, $F(\omega) = 0, |\omega| > \omega_m$, then $f(x)$ is uniquely determined by its samples taken at twice ω_m or $f(n\pi/\omega_m)$. The reconstruction formula that complements the sampling theorem is given by

$$f(x) = \sum_{n \in \mathbb{Z}} g(nX_s)\mathrm{sinc}(\frac{x}{X_s} - n)$$

where the uniform samples of $g(nX_s)$ can be interpreted as coefficients of basis functions obtained by appropriate shifting and scaling of the sinc function $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. But a large class of signals exist which are not bandlimited and need very large number of samples to reconstruct, and they have a certain formulation with which it seems to make us able to use fewer number of samples.

1.1 Related works

In a recent work by Vetterli, Marziliano and Blu [3], it was shown that it is possible to develop exact sampling schemes for some classes of signals that are neither bandlimited nor live on shift-invariant spaces, namely, certain signals with finite *rate of innovation*. Examples include streams of Diracs, non-uniform splines and piecewise polynomials. A common feature of such signals is that they have a parametric representation with a finite number of degrees of freedom per unit of time, or finite rate of innovation ρ, and can be perfectly reconstructed from a set of samples taken at a rate $R \geq \rho$, after appropriate smoothing. The key in all constructions is to identify the innovative part of a signal, such as time instants of Diracs, using an annihilating or locator filter, a well-known tool from spectral analysis or error correction coding. This allows for standard computational procedures for solving the sampling problem for a wide class of non-bandlimited signals and leads to some interesting results.

In another work done by Jovanovic, Sbaiz and Vetterli [1], reconstructing a 2-D diffusive field induced by sources localized in space and time is cinsidered and the goal is to compute the positions of the sources, the origin of time and the total amount released. They use a set of 3-D weighted Diracs to model the field and come up with the following formula for the field:

$$g(x, y, t) = \sum_{k=0}^{K-1} \frac{c_k}{4\pi D(t - t_k)} e^{-\frac{(x-x_k)^2+(y-y_k)^2}{4D(t-t_k)}} \tag{1.1}$$

So the job is to estimate parameters $\{c_k, x_k, y_k, t_k\}$. In first step they first assume that the time origins are equal to zero and use tomographic approaches to estimate the parameters $\{c_k, x_k, y_k\}$ and in the second step put the time origins equal but not zero and use effective rank [4] to make a matrix deficient rank and find the the best estimate of t_k.

In a work by Nordio, Chiasserini and Viterbo, [5] they consider a bandlimited real valued signal $p(t)$ written as:

$$p(t) = \sum_{k=-M'}^{M'} a_k e^{2\pi ikt}$$

and estimate

$$\hat{p}(t) = \sum_{k=M}^{M} \hat{a}_k e^{2\pi ikt}$$

They suppose to have r randomly deployed sensors in $(0, 1)$ and translate the problem to the form of $\boldsymbol{T}_w \hat{\boldsymbol{a}} = \boldsymbol{b}$. They define δ to be the maximum distance between two consecutive sensors and show that if $\delta < 1/2M$ then show that the following bound is valid for condition number of matrix $\boldsymbol{T}_w$:

$$\kappa_{(\boldsymbol{T}_w)} \leq (\frac{1 + 2\delta M}{1 - 2\delta M})^2$$

In this research we try to benefit from the time-space correlation of signals to lower the number of sensors used in space. In this work, for simplicity we consider sources that reside in a 1-D diffusive environment, which mainly have an initial function along space having exponentially decreasing sine Fourier transform coefficients. After the activation of a source, the induced field, although non-bandlimited, is completely determined by a set of parameters. Intuitively, only a finite number of samples is required for perfect reconstruction. But the question is how many sensors is needed in space and how many time samples will be enough to guarantee a good reconstruction of signals, in the sense of well-stabilized solution and in the presence of noise. In the sequence we will first introduce some algorithms and then consider different sampling schemes along space and investigate the numerical stability of our algorithm with these schemes.

Chapter 2

Temporal Model, Diffusive Field

As we have seen before, considering the sensor values taken during the time independently to reconstruct the field leads to using of many sensors in space and may in some situations cause bad conditioning, so we propose to use the temporal and spatial correlation of the signals. So we need to introduce a temporal model, and as a temporal model we consider the diffusive field model which is widely used in different areas. As applications of this model we can name; model of temperature along space; flow of pollutants and other chemicals; it is connected with Brownian Motions of particles and it is considered as the price of stock in financial mathematics.

2.1 Problem Statement

in the sequence the following 1-D diffusion model has been considered as the source model along space and during time.

$$\begin{cases} U_t = U_{xx} \\ U(0,t) = U(\pi,t) = 0 \\ U(x,0) = f(x) \end{cases} \tag{2.1}$$

We consider homogenous Dirichlet boundary condition which makes the signal value equal to zero in 0 and π, and the initial function at $t = 0$ is called as $f(x)$. The solution for such a PDE is as follows:

$$U(x,t) = \sum_{k=1}^{\infty} a_k \sin(kx) e^{-k^2 t} \tag{2.2}$$

Where $U(x,0) = f(x) = \sum_{k-1}^{\infty} a_k \sin kx$, so a_k's are the sine Fourier coefficients of initial function $f(x)$.
From this solution it is obvious that finding coefficients a_k is equivalent to finding the whole field along space and time.
If we find the coefficients appropriately, we may find the initial function in each time instance and in this way we can go through time and track continuous

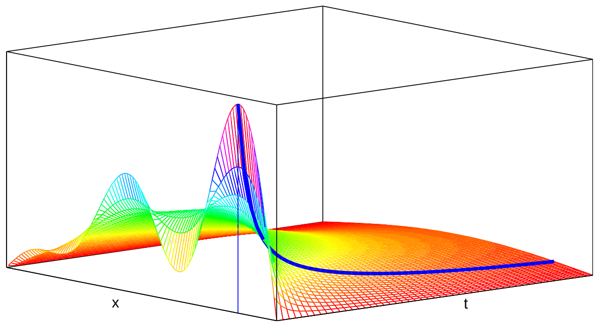


Figure 2.1 – Sampling in space using only one sensor at x_0, we suppose to have the whole time line.

sources along time. So the main job will be a good approximation of the initial function and since finding $f(x)$ and a_ks are equivalent we will focus on estimating a_ks.

2.2 Solution using one sensor in space, Orthogonal Projection

Suppose we are measuring the field values only in one location using one sensor at x_0. This can be shown in Figure 2.1.
For fixed x_0, we can introduced b_k's to be $b_k = a_k \sin(kx_0)$, and consequently, we will have:

$$U(x_0, t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \tag{2.3}$$

In this section we benefit from properties of orthogonal projection properties to find the solution for both continuous and discrete cases of time.

2.2.1 Continuous case

Let's first introduce some prerequisites.

Lemma 1. *The set $S = \{e^{-k^2 t} : k \in \mathbb{N}\}$ is a linearly independent set.*

Proof. To show this set is linearly independent, we must show each finite subset of it is independent. Consider the finite subset $\{e^{-k^2 t} : k \in \mathbf{I}\}$, We have:

$$\sum_{k \in \mathbf{I}} c_k e^{-k^2 t} = 0$$

So we can rewrite the equation as:

$$\begin{aligned} c_m e^{-m^2 t} &= \sum_{\substack{k\in \mathbf{I} \\ k\neq m}} c_k e^{-k^2 t} \\ c_m &= \sum_{\substack{k\in \mathbf{I} \\ k\neq m}} c_k e^{-(k^2-m^2)t} \end{aligned}$$

Where m is the smallest element of $\mathbf{I}$. Now, taking limit of both sides when $t \to \infty$, we get:

$$c_m = \lim_{t\to\infty} \sum_{\substack{k\in \mathbf{I} \\ k\neq m}} c_k e^{-(k^2-m^2)t} = \sum_{\substack{k\in \mathbf{I} \\ k\neq m}} c_k \lim_{t\to\infty} e^{-(k^2-m^2)t} = 0$$

Consequently using the same procedure for other elements, we will get

$$c_k = 0, \quad \forall k \in \mathbf{I}$$

So each finite subset of S is linearly independent and consequently, it is also an independent set. $\square$

Lemma 2. *If initial function $f(x)$, corresponding to equation (2.3), is finite, we can approximate $\sum_{k=1}^{\infty} b_k e^{-k^2 t}$ by $\sum_{k=1}^{L} b_k e^{-k^2 t}$ and make the error arbitrarily small by choosing appropriate L.*

Proof.

$$\begin{aligned} & U(x_0, 0) = \sum_{k=1}^{\infty} b_k = f(x_0) < \infty \\ & \Rightarrow \lim_{k\to\infty} b_k = 0 \\ & U(x_0, t) = \sum_{k=1}^{L} b_k e^{-k^2 t} + \sum_{k=L+1}^{\infty} b_k e^{-k^2 t} \end{aligned}$$

But

$$|\sum_{k=L+1}^{\infty} b_k e^{-k^2 t}| \leq \sum_{k=L+1}^{\infty} |b_k| \leq \epsilon$$

$\square$

Now we have the tools to find b_k's from the given field values. Since the set $\{e^{-k^2 t} : k = 1...L\}$ is a linearly independent set, we know $e^{-m^2 t} \notin \text{span}\{e^{-k^2 t} : k \neq m\}$. So we find its projection on this span and call it $\boldsymbol{P} e^{-m^2 t}$. To do so we put

$$\boldsymbol{P} e^{-m^2 t} = \sum_{\substack{k=1 \\ k\neq m}}^{L} c_k e^{-k^2 t}$$

We know that the error of projection is orthogonal to all the vectors of the target subspace:

$$\langle e^{-m^2 t} - \boldsymbol{P} e^{-m^2 t}, e^{-i^2 t} \rangle = 0, \quad \forall i \neq m$$

Where

$$\langle f(t), g(t)\rangle = \int_0^\infty f(t)g(t)^*\mathrm{d}t$$

So

$$\langle e^{-m^2t} - \sum_{\substack{k=1\\k\neq m}}^{L} c_k e^{-k^2t}, e^{-i^2t}\rangle = 0, \quad \forall i \neq m$$

$$\Rightarrow \langle e^{-m^2t}, e^{-i^2t}\rangle = \sum_{\substack{k=1\\k\neq m}}^{L} c_k \langle e^{-k^2t}, e^{-i^2t}\rangle, \quad \forall i \neq m$$

and since

$$\langle e^{-m^2t}, e^{-i^2t}\rangle = \int_0^\infty e^{-m^2t}e^{-i^2t}\mathrm{d}t = \frac{1}{m^2+i^2}$$

We will have

$$\frac{1}{m^2+i^2} = \sum_{\substack{k=1\\k\neq m}}^{L} \frac{c_k}{k^2+i^2}, \quad \forall i \neq m$$

So we come up with the following matrix form

$$\underbrace{\begin{bmatrix} \frac{1}{1+m^2} \\ \frac{1}{2^2+m^2} \\ \cdot \\ \cdot \\ \cdot \\ \underline{\frac{1}{m^2+m^2}} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{L^2+m^2} \end{bmatrix}}_{\boldsymbol{M}} = \underbrace{\begin{bmatrix} \frac{1}{1+1} & \frac{1}{1+2^2} & \cdots & \underline{\frac{1}{1+m^2}} & \cdots & \frac{1}{1+L^2} \\ \frac{1}{2^2+1} & \frac{1}{2^2+2^2} & \cdots & \underline{\frac{1}{2^2+m^2}} & \cdots & \frac{1}{2^2+L^2} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \underline{\frac{1}{m^2+1}} & \underline{\frac{1}{m^2+2^2}} & \cdots & \underline{\frac{1}{m^2+m^2}} & \cdots & \underline{\frac{1}{m^2+L^2}} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \frac{1}{L^2+1} & \frac{1}{L^2+2^2} & \cdots & \underline{\frac{1}{L^2+m^2}} & \cdots & \frac{1}{L^2+L^2} \end{bmatrix}}_{\boldsymbol{T}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{c_m} \\ \cdot \\ \cdot \\ \cdot \\ c_L \end{bmatrix}}_{\boldsymbol{C}}$$

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{T}\boldsymbol{C} \\ \boldsymbol{C} &= \boldsymbol{T}^{-1}\boldsymbol{M} \end{aligned} \tag{2.4}$$

Where underlined terms must be omitted from the matrices. Thus up to now we have the projection of e^{-m^2t} onto span$\{e^{-k^2t}, k \neq m\}$. Now consider the following inner product:

$$\langle U(x_0,t), e^{-m^2t} - \boldsymbol{P}e^{-m^2t}\rangle = \langle \sum_{k=1}^{L} b_k e^{-k^2t}, e^{-m^2t} - \boldsymbol{P}e^{-m^2t}\rangle \tag{2.5}$$

Since

$$(e^{-m^2t} - \boldsymbol{P}e^{-m^2t}) \perp e^{-k^2t}, \quad \forall k \neq m$$

We will have

$$\begin{aligned}\langle U(x_0,t), e^{-m^2t} - \boldsymbol{P}e^{-m^2t}\rangle &= \langle\sum_{k=1}^{L} b_k e^{-k^2t}, e^{-m^2t} - \boldsymbol{P}e^{-m^2t}\rangle \\ &= \langle b_m e^{-m^2t}, e^{-m^2t} - \boldsymbol{P}e^{-m^2t}\rangle \\ &= b_m\big(\langle e^{-m^2t}, e^{-m^2t}\rangle - \langle e^{-m^2t}, \boldsymbol{P}e^{-m^2t}\rangle\big) \\ &= b_m(\frac{1}{2m^2} - \langle e^{-m^2t}, \boldsymbol{P}e^{-m^2t}\rangle)\end{aligned} \tag{2.6}$$

And finally we will have

$$\begin{aligned}b_m &= \frac{\langle U(x_0,t), e^{-m^2t} - \boldsymbol{P}e^{-m^2t}\rangle}{\frac{1}{2m^2} - \langle e^{-m^2t}, \boldsymbol{P}e^{-m^2t}\rangle} \\ &= \frac{\langle U(x_0,t), e^{-m^2t} - \sum_{\substack{k=1\\k\neq m}}^{L} c_k e^{-k^2t}\rangle}{\frac{1}{2m^2} - \langle e^{-m^2t}, \sum_{\substack{k=1\\k\neq m}}^{L} c_k e^{-k^2t}\rangle} \\ &= \frac{\langle U(x_0,t), \sum_{k=1}^{L} c_k e^{-k^2t}\rangle}{\sum_{k=1}^{L} \frac{c_k}{k^2+m^2}}\end{aligned} \tag{2.7}$$

Where c_k's for $k \neq m$ can be found from equation (2.4) and $c_m = -1$.

2.2.2 Discrete case

Now we just sample everything every T seconds, so the problem is changed as following:
Given $U(x_0, nT)$, find b_m's, knowing that $e^{-m^2nT} \notin \text{span}\{e^{-k^2nT} : k \neq m, k = 1...L\}$ So we find its projection onto this span, $\boldsymbol{P}e^{-m^2nT}$. To do so we put

$$\boldsymbol{P}e^{-m^2nT} = \sum_{\substack{k=1\\k\neq m}}^{L} c_k e^{-k^2nT}$$

and also

$$\langle e^{-m^2nT} - \boldsymbol{P}e^{-m^2nT}, e^{-i^2nT}\rangle = 0, \quad \forall i \neq m$$

Where

$$\langle f_n, g_n\rangle = \sum_{n=0}^{\infty} f(nT)g(nT)^*$$

So

$$\langle e^{-m^2nT} - \sum_{\substack{k=1\\k\neq m}}^{L} c_k e^{-k^2nT}, e^{-i^2nT}\rangle = 0, \quad \forall i \neq m$$

$$\langle e^{-m^2nT}, e^{-i^2nT}\rangle = \sum_{\substack{k=1\\k\neq m}}^{L} c_k \langle e^{-k^2nT}, e^{-i^2nT}\rangle, \quad \forall i \neq m$$

$$\frac{1}{1-e^{-(m^2+i^2)T}} = \sum_{\substack{k=1\\k\neq m}}^{L} \frac{c_k}{1-e^{-(k^2+i^2)T}}, \quad \forall i \neq m$$

Now defining $f(i,j) = \frac{1}{1-e^{-(i^2+j^2)T}}$ we come up with the following matrix form

$$\underbrace{\begin{bmatrix} f(m,1) \\ f(m,2) \\ \cdot \\ \cdot \\ \cdot \\ \underline{f(m,m)} \\ \cdot \\ \cdot \\ \cdot \\ f(m,L) \end{bmatrix}}_{\boldsymbol{M}} = \underbrace{\begin{bmatrix} f(1,1) & f(1,2) & \cdot & \cdot & \cdot & \underline{f(1,m)} & \cdot & \cdot & \cdot & f(1,L) \\ f(2,1) & f(2,2) & \cdot & \cdot & \cdot & \underline{f(2,m)} & \cdot & \cdot & \cdot & f(2,L) \\ \cdot & & & & & & & & & \cdot \\ \cdot & & & & & & & & & \cdot \\ \cdot & & & & & & & & & \cdot \\ \underline{f(m,1)} & \underline{f(m,2)} & \cdot & \cdot & \cdot & \underline{f(m,m)} & \cdot & \cdot & \cdot & \underline{f(m,L)} \\ \cdot & & & & & & & & & \cdot \\ \cdot & & & & & & & & & \cdot \\ \cdot & & & & & & & & & \cdot \\ f(L,1) & f(L,2) & \cdot & \cdot & \cdot & \underline{f(L,m)} & \cdot & \cdot & \cdot & f(L,L) \end{bmatrix}}_{\boldsymbol{T}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{c_m} \\ \cdot \\ \cdot \\ \cdot \\ c_L \end{bmatrix}}_{\boldsymbol{C}}$$

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{TC} \\ \boldsymbol{C} &= \boldsymbol{T}^{-1}\boldsymbol{M} \end{aligned} \tag{2.8}$$

Where underlined terms must be omitted from the matrices. Thus up to now we have the projection of e^{-m^2nT} onto span$\{e^{-k^2nT}, k \neq m\}$. Now consider the following inner product:

$$\langle U(x_0,nT), e^{-m^2nT} - \boldsymbol{P}e^{-m^2nT}\rangle = \langle \sum_{k=1}^{L} b_k e^{-k^2nT}, e^{-m^2nT} - \boldsymbol{P}e^{-m^2nT}\rangle \tag{2.9}$$

and since

$$(e^{-m^2nT} - \boldsymbol{P}e^{-m^2nT}) \perp e^{-k^2nT}, \quad \forall k \neq m$$

We will have

$$\begin{aligned} \langle U(x_0,nT), e^{-m^2nT} - \boldsymbol{P}e^{-m^2nT}\rangle &= \langle \sum_{k=1}^{L} b_k e^{-k^2nT}, e^{-m^2nT} - \boldsymbol{P}e^{-m^2nT}\rangle \\ &= \langle b_m e^{-m^2nT}, e^{-m^2nT} - \boldsymbol{P}e^{-m^2nT}\rangle \\ &= b_m\big(\langle e^{-m^2nT}, e^{-m^2nT}\rangle - \langle e^{-m^2nT}, \boldsymbol{P}e^{-m^2nT}\rangle\big) \\ &= b_m(f(m,m) - \langle e^{-m^2nT}, \boldsymbol{P}e^{-m^2nT}\rangle) \end{aligned} \tag{2.10}$$

And finally we will have

$$b_m = \frac{\langle U(x_0, nT), e^{-m^2nT} - \boldsymbol{P}e^{-m^2nT}\rangle}{f(m,m) - \langle e^{-m^2nT}, \boldsymbol{P}e^{-m^2nT}\rangle}$$

$$= \frac{\langle U(x_0, nT), e^{-m^2nT} - \sum_{\substack{k=1\\k\neq m}}^{L} c_k e^{-k^2nT}\rangle}{f(m,m) - \langle e^{-m^2nT}, \sum_{\substack{k=1\\k\neq m}}^{L} c_k e^{-k^2nT}\rangle} \tag{2.11}$$

$$= \frac{\langle U(x_0, nT), \sum_{k=1}^{L} c_k e^{-k^2nT}\rangle}{\sum_{k=1}^{L} \frac{c_k}{1-e^{-(m^2+k^2)T}}}$$

Where c_k's for $k \neq m$ can be found from equation (2.8) and $c_m = -1$.

Now let's extend the result for when we have only finite number of samples in time, namely $K+1$ samples, then we have:

$$\langle e^{-i^2nT}, e^{-j^2nT}\rangle = \sum_{n=0}^{K} e^{-i^2nT} e^{-j^2nT} = \frac{1 - e^{-(i^2+j^2)T(K+1)}}{1 - e^{-(i^2+j^2)T}}$$

Again defining $f(i,j) = \frac{1-e^{-(i^2+j^2)T(K+1)}}{1-e^{-(i^2+j^2)T}}$, We come up with the following solution:

$$b_m = \frac{\langle U(x_0, nT), \sum_{k=1}^{L} c_k e^{-k^2nT}\rangle}{\sum_{k=1}^{L} c_k \frac{1-e^{-(m^2+k^2)T(K+1)}}{1-e^{-(m^2+k^2)T}}} \tag{2.12}$$

Where c_k's for $k \neq m$ come from equation (2.8) but with new values for $f(i,j)$ and $c_m = -1$ and

$$\langle f, g\rangle = \sum_{n=0}^{K} f(nT) g(nT)^*$$

2.2.3 Experimental Results

In our experiment we suppose the initial condition function is a smooth enough so the coefficients satisfy $a_k < c\,e^{-a\,k}$. We also define a quantity SNR to measure the precision of algorithm in estimating the initial condition:

$$SNR = -10\log_{10} \frac{\text{energy of error}}{\text{energy of original signal}}$$

$$= -10\log_{10} \frac{\sum_{k=1}^{L+1} (f(k) - \widehat{f}(k))^2}{\sum_{k=1}^{L+1} f(k)^2}$$

We put $L = 9$, $T = 0.01$ time unit and $a_k < 20e^{-0.6k}$. The average SNR in 100 experiments with random a_k's for estimating the initial function using 9 coefficients was 75.3677 dB. Figure 2.2 shows the estimation of initial condition function with different number of coefficients. The original number of coefficients is 9.

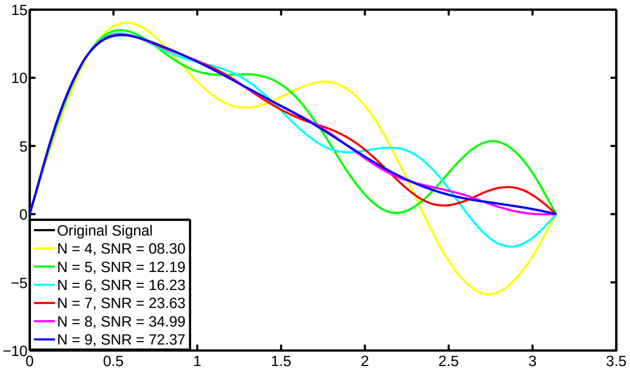


Figure 2.2 – Experimental result for estimating initial function using orthogonal projection.

2.3 Solution using several sensors in space, matrix inversion

Now how can we improve the previously introduced algorithm, namely benefitting from temporal-spatial correlation of signal. Suppose we are approximating the source with finite number of coefficients (N), and consider the case that we are using exactly N sensors in space and we just use only one sample in time (at $t = T$). Then we will have the following matrix format for the problem

$$\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_N \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-T}\sin x_1 & e^{-2^2T}\sin 2x_1 & . \;\; . \;\; . & e^{-N^2T}\sin Nx_1 \\ e^{-T}\sin x_2 & e^{-2^2T}\sin 2x_2 & . \;\; . \;\; . & e^{-N^2T}\sin Nx_2 \\ . & & & . \\ . & & & . \\ . & & & . \\ e^{-T}\sin x_N & e^{-2^2T}\sin 2x_N & . \;\; . \;\; . & e^{-N^2T}\sin Nx_N \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_N \end{bmatrix} \tag{2.13}$$

For finding the coefficients we need to compute the inverse matrix A^{-1}. But we have:

$$A = \begin{bmatrix} \sin x_1 & \sin 2x_1 & \cdots & \sin Nx_1 \\ \sin x_2 & \sin 2x_2 & \cdots & \sin Nx_2 \\ \vdots & \vdots & & \vdots \\ \sin x_N & \sin 2x_N & \cdots & \sin Nx_N \end{bmatrix} \begin{bmatrix} e^{-T} & & & \\ & e^{-2^2T} & & \\ & & \ddots & \\ & & & e^{-N^2T} \end{bmatrix}$$

So

$$A^{-1} = \begin{bmatrix} e^{T} & & & \\ & e^{2^2T} & & \\ & & \ddots & \\ & & & e^{N^2T} \end{bmatrix} \begin{bmatrix} \sin x_1 & \sin 2x_1 & \cdots & \sin Nx_1 \\ \sin x_2 & \sin 2x_2 & \cdots & \sin Nx_2 \\ \vdots & \vdots & & \vdots \\ \sin x_N & \sin 2x_N & \cdots & \sin Nx_N \end{bmatrix}^{-1}$$

Now the question is whether the second matrix is invertible for several $x \in (0, \pi)$ or not.

Lemma 3. *The matrix A_N, as defined below for different $x_i \in (0, \pi)$, is not singular.*

$$A_N = \begin{bmatrix} \sin x_1 & \sin 2x_1 & \cdots & \sin Nx_1 \\ \sin x_2 & \sin 2x_2 & \cdots & \sin Nx_2 \\ \vdots & \vdots & & \vdots \\ \sin x_N & \sin 2x_N & \cdots & \sin Nx_N \end{bmatrix} \tag{2.14}$$

Proof. We are considering the case where $x_i \neq x_j$. For this matrix to be singular we need its columns to be linearly dependant. We can interpret this as follows: We know the set of functions $\{\sin kx\}_{k=1\cdots N}$ are linearly independent in interval $(0, \pi)$. But here for this matrix to be singular we need these functions to be linearly dependant in only N points in $(0, \pi)$. In other words the function

$$f(x) = \sum_{k=1}^{N} c_k \sin kx$$

must have at least N zeros in $(0, \pi)$ for $c_k \neq 0$. But we have:

$$\begin{aligned} h(x) &= \sum_{k=1}^{N} c_k \sin kx \\ &= \sum_{k=1}^{N} c_k \left(\frac{e^{jkx} - e^{-jkx}}{2j}\right) \\ \Rightarrow g(x) = 2je^{jNx}h(x) &= \sum_{k=1}^{N} c_k (e^{j(k+N)x} - e^{-j(k-N)x}) \\ &= \sum_{k=0}^{2N} c'_k e^{jkx} \\ &= \sum_{k=0}^{2N} c'_k z^k \Big|_{z=e^{jkx}, x\in(0,\pi)} \end{aligned}$$

This is a polynomial of degree $2N$, so has at most $2N$ zeros in $[0, \pi]$. We already know that $g(0) = g(\pi) = 0$, So we have found two roots of $g(x)$ and of course these are of no any interest for us from beginning. So there is only $2N - 2$ roots left. But also by looking into the definition of $f(x)$ and consequently $g(x)$, it is obvious that if $f(x_0) = g(x_0) = 0$, then $f(-x_0) = g(-x_0) = 0$, and since we are looking for roots in $(0, \pi)$, there will remain only $N - 1$ roots and we are done. This means there are not different points in $(0, \pi)$ for which the above matrix is singular. $\square$

Thus this is an acceptable job to use several sensors along space and use matrix inversion to get the appropriate result. But this way has a drawback and it is that we need number of sensors to be at least equal to number of coefficients we are estimating.

So if we use M sensors in space and get K time samples in time we will have the following matrix format for the problem

$$
\begin{bmatrix} y_{1,1} \\ y_{1,2} \\ \vdots \\ y_{1,K} \\ \cdot \\ \cdot \\ \cdot \\ y_{M,1} \\ y_{M,2} \\ \vdots \\ y_{M,K} \end{bmatrix}
=
\underbrace{\begin{bmatrix}
e^{-T}\sin x_1 & e^{-2^2T}\sin 2x_1 & \cdots & e^{-N^2T}\sin Nx_1 \\
e^{-2T}\sin x_1 & e^{-2\times 2^2T}\sin 2x_1 & \cdots & e^{-2\times N^2T}\sin Nx_1 \\
\vdots & \vdots & & \vdots \\
e^{-KT}\sin x_1 & e^{-K\times 2^2T}\sin 2x_1 & \cdots & e^{-K\times N^2T}\sin Nx_1 \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
e^{-T}\sin x_M & e^{-2^2T}\sin 2x_M & \cdots & e^{-N^2T}\sin Nx_M \\
e^{-2T}\sin x_M & e^{-2\times 2^2T}\sin 2x_M & \cdots & e^{-2\times N^2T}\sin Nx_M \\
\vdots & \vdots & & \vdots \\
e^{-KT}\sin x_M & e^{-K\times 2^2T}\sin 2x_M & \cdots & e^{-K\times N^2T}\sin Nx_M
\end{bmatrix}}_{\boldsymbol{A}}
\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_N \end{bmatrix}
\tag{2.15}
$$

Now the question comes to mind if this matrix is again non-singular for different values of $x_i \in (0, \pi)$ or not.

Lemma 4. *In the matrix A defined as in* (2.15)*, if $M = N$, the matrix A is non-singular.*

Proof. rewrite the matrix as follows:

$$
A = \begin{bmatrix}
e^{-T}\sin x_1 & e^{-2^2T}\sin 2x_1 & \cdots & e^{-N^2T}\sin Nx_1 \\
e^{-T}\sin x_2 & e^{-2^2T}\sin 2x_2 & \cdots & e^{-N^2T}\sin Nx_2 \\
\vdots & \vdots & & \vdots \\
e^{-T}\sin x_N & e^{-2^2T}\sin 2x_N & \cdots & e^{-N^2T}\sin Nx_N \\
e^{-2T}\sin x_1 & e^{-2\times 2^2T}\sin 2x_1 & \cdots & e^{-2\times N^2T}\sin Nx_1 \\
e^{-2T}\sin x_2 & e^{-2\times 2^2T}\sin 2x_2 & \cdots & e^{-2\times N^2T}\sin Nx_2 \\
\vdots & \vdots & & \vdots \\
e^{-2T}\sin x_N & e^{-2\times 2^2T}\sin 2x_N & \cdots & e^{-2\times N^2T}\sin Nx_N \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
e^{-KT}\sin x_1 & e^{-K\times 2^2T}\sin 2x_1 & \cdots & e^{-K\times N^2T}\sin Nx_1 \\
e^{-KT}\sin x_2 & e^{-K\times 2^2T}\sin 2x_2 & \cdots & e^{-K\times N^2T}\sin Nx_2 \\
\vdots & \vdots & & \vdots \\
e^{-KT}\sin x_N & e^{-K\times 2^2T}\sin 2x_N & \cdots & e^{-K\times N^2T}\sin Nx_N
\end{bmatrix}
\tag{2.16}
$$

$$
A = \begin{bmatrix} B_{N\times N} \\ C_{N(K-1)\times N} \end{bmatrix}
$$

From Lemma 3 we know that B is non-singular, so A is also non-singular and we are done. ∎

Now let's consider the situation where $M \neq N$:

Proof For $M \neq N$, First Approach

Let's review some prerequisites.

Implicit Function Theorem: Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function and let $\mathbb{R}^{n+m}$ have coordinates $(\boldsymbol{x}, \boldsymbol{y})$. Fix a point $(a_1, \ldots, a_n, b_1, \ldots, b_m) = (\boldsymbol{a}, \boldsymbol{b})$ with $f(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{c}$, where $\boldsymbol{c} \in \mathbb{R}^m$. If the matrix $[(\partial f_i/\partial y_j)(\boldsymbol{a}, \boldsymbol{b})]$ is invertible, then there exists an open set U containing $\boldsymbol{a}$, an open set V containing $\boldsymbol{b}$, and a unique continuously differentiable function $g : U \to V$ such that

$$\{(\boldsymbol{x}, g(\boldsymbol{x}))\} = \{(\boldsymbol{x}, \boldsymbol{y}) | f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{c}\} \cap (U \times V).$$

Preposition 1. *Consider the polynomial function $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$. The set of its roots has measure zero in $\mathbb{R}^3$*

Proof. For the proof, we use induction on degree of z. Call degree of z equal to N. For $N = 0$, there is nothing to prove. Suppose we know the set of results for $deg(z) = N$ is of measure zero. At stage $N + 1$, If we have $\partial f/\partial z \neq 0$, we use the implicit function theorem, for $m = 1$, $n = 2$ and $f(\underbrace{x, y}_{\boldsymbol{x}}, \underbrace{z}_{\boldsymbol{y}}) : \mathbb{R}^3 \to \mathbb{R}$.

For any fixed (x_0, y_0, z_0) where $f(x_0, y_0, z_0) = 0$, there exists V containing (x_0, y_0) and U containing z_0 and a unique function $g(x_0, y_0) : V \to U$ for which $f\big(x, y, g(x, y)\big) = 0, \quad \forall x, y \in V$.
Since $g(x, y)$ is a 2-dimensional surface defined on an open set V it has measure zero in $\mathbb{R}^3$ (look at Figure 2.3). Let's define

$$A_{x,y} = \{\big(x, y, g(x, y)\big) | f\big(x, y, g(x, y)\big) = 0\}$$

as the set of answers for point$(x, y) \in \mathbb{R}^2$. So all along $\mathbb{R}^3$ we have

$$A = \bigcup_{x,y \in \mathbb{R}^2} A_{x,y}.$$

We know that

$$\lambda(\bigcup_{i \in \mathbb{N}} A_i) \leq \bigcup_{i \in \mathbb{N}} \lambda(A_i)$$

where λ is the usual Lebesgue measure.
So if we show that $A = \bigcup_{x,y \in \mathbb{R}^2} A_{x,y} = \bigcup_{i,j \in \mathbb{N}} A_{x_i, y_j}$, we are done, because then

$$\lambda_3(A) \leq \bigcup_{i,j \in \mathbb{N}} \lambda_3(A_{x_i, y_j}) = 0$$

Where λ_3 is 3-dimensional Lebesgue measure.
To show this it is enough to show that $\mathbb{R}^2$ can be covered by a countable number of open sets. Consider the following partitioning of $\mathbb{R}^2$. We have:

$$\mathbb{R}^2 = \bigcup_{x,y \in \mathbb{R}} O_{x,y}$$

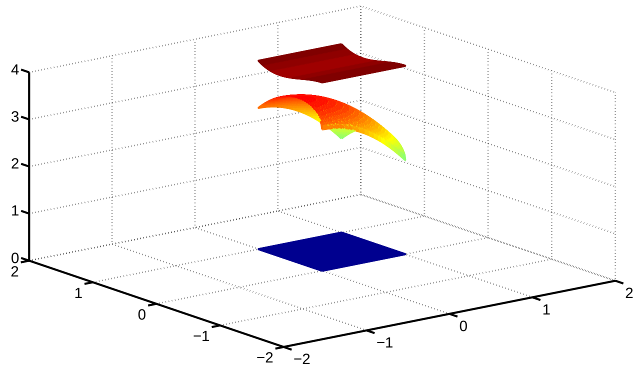
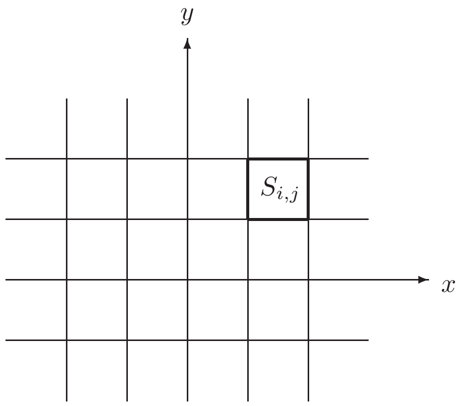


Figure 2.3 − x,y and $g(x, y)$ for different z_0s



where $O_{x,y}$ is an open interval around (x, y). We also know that

$$S_{i,j} \subset \bigcup_{x,y\in\mathbb{R}} O_{x,y}$$

is compact, so it can be covered by a countable (finite) number of open sets.

$$S_{i,j} \subset \bigcup_{i,j\in\mathbb{N}}^{N_{ij}} O_{x_i,y_j}$$

and at end:

$$\mathbb{R}^2 = \bigcup_{i,j\in\mathbb{N}} S_{i,j} = \bigcup_{i,j\in\mathbb{N}} O_{x_i,y_j}$$

Now suppose the case at stage $N+1$ where we have $\partial f/\partial z = 0$ for roots, but $\partial f/\partial z$ is a polynomial of degree N and according to induction has set of roots of measure zero. $\square$

Lemma 5. *The matrix A defined in (2.15), with $K \times M = N$, is non-singular almost everywhere. In other words the set of values for x_n and T for A to be singular is of measure zero.*

Proof. Again for the proof we use induction. As the base of induction suppose 2×2 matrix

$$\begin{bmatrix} e^{-KT} \sin x_{M-1} & e^{-K2^2T} \sin 2x_{M-1} \\ e^{-KT} \sin x_M & e^{-K2^2T} \sin 2x_M \end{bmatrix}$$

We already have shown this is non-singular. Now suppose we are at stage N and put $u = e^{-T}$ and $v = e^{jx}$. We have the following matrix format:

$$A = \begin{bmatrix} u(v - \frac{1}{v}) & u^4(v^2 - \frac{1}{v^2}) & \cdots & u^{N^2}(v^N - \frac{1}{v^N}) \\ & B & & \vdots \end{bmatrix} \tag{2.17}$$

And from previous stage, we know that matrix B is non-singular almost everywhere. Writing the determinant with respect to the first line, gives us:

$$\det A = 0$$

$$\Rightarrow \sum_{k=1}^{N} c_k u^{k^2} (v^k - \frac{1}{v^k}) = 0$$

Where $u \in [0, 1]$ and $v \in \mathbb{C}$. So

$$f(v, u) = \sum_{k=1}^{N} c_k u^{k^2} (v^{N+k} - v^{N-k})$$

If we put $v = \alpha + j\beta$ we will have:

$$f(v, u) = f_1(\alpha, \beta, u) + jf_2(\alpha, \beta, u) = 0$$

Where $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ are polynomials. And we have shown that roots of polynomials $f : \mathbb{R}^3 \to \mathbb{R}$ have measure zero in $\mathbb{R}^3$.
Now the only work to do is to show that actually the set of roots has measure zero on $S^1 \times [0, 1]$, where S^1 is unit circuit. So we have to intersect these curves with the cylinder of diameter 2 between 0 and 1. But this can be shown easily that the intersections are finite number of lines and in that case we are done. $\square$

Proof For $M \neq N$, Second Approach

Here I propose another approach for the proof of Lemma 5.
First recall Tonelli's Theorem.

Tonelli's Theorem: Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $g \in L^+(X \times Y)$, then the functions $x \mapsto \int_Y g(x, y) d\nu(y)$ and $y \mapsto \int_X g(x, y) d\mu(x)$ are in $L^+(X)$ and $L^+(Y)$ respectively, and furthermore if we denote by $\mu \times \nu$ the product measure, then

$$\int_{X \times Y} g \, d(\mu \times \nu) = \int_X \left[\int_Y g(x, y) d\nu(y) \right] d\mu(x) = \int_Y \left[\int_X g(x, y) \, d\mu(x) \right] d\nu(y).$$

Now what we have is a function $f(v,u) : S^1 \times [0,1] \to \mathbb{C}$,where S^1 is the unit circle. Let's define

$$S = \{(v,u) \in S^1 \times [0,1] | f(v,u) = 0\}$$

and put $g := \mathbb{1}_{(S)}$, so we satisfy Tonelli's theorem conditions.
Then we will have:

$$m(S) = \int_{S^1 \times [0,1]} \mathbb{1}_{(S)} = \int_{S^1} dv \int_{[0,1]} \mathbb{1}_{(S)} du$$

But for a fixed v_0 the number of zeros of function $f(v_0,u)$ is finite, so we have $\int_{[0,1]} \mathbb{1}_{(S)} du = 0$ and consequently

$$m(S) = 0$$

Now we can extend our proof for the most general case:

Lemma 6. *The matrix A defined in* (2.21)*, with $K \times M \geq N$ is non-singular almost every $x_i \in (0,\pi)$ and every $T \in \mathbb{R}^+$.*

Proof. To prove this we just pick first N rows of A and apply Lemma 5, and we are done. ∎

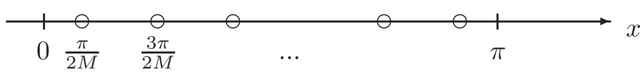
So as can be seen from above discussions, we are allowed to use the matrix format and use oversampling in time and instead reduce the number of sensors in space.

2.3.1 Uniform sampling along space

As we saw in the previous section we can associate almost any sensor locations between 0 and π, and still have a non-singular matrix for A. But actually non-singularity is not enough, we also need numerical stability of our solution. To investigate this we define the condition number of our matrix to be :

$$\kappa_A = \|A^{-1}\| \cdot \|A\| = \frac{\sigma_{max}}{\sigma_{min}} \tag{2.18}$$

Where σ_{max} and σ_{min} are maximum and minimum singular values of matrix A. We believe that different spatial allocation of sensors will cause different condition numbers. So let's try uniform allocation of sensors in space. Suppose the sensors are deployed in space as follows



Let's investigate the properties of the matrix A, introduced in (2.15).

Fix Condition Number for $M > N$

Let's first consider the simple matrix for $K = 1$ and $T = 0$. So the matrix format will be :

$$A = \begin{bmatrix} \sin x_1 & \sin 2x_1 & \cdots & \sin Nx_1 \\ \sin x_2 & \sin 2x_2 & \cdots & \sin Nx_2 \\ \vdots & \vdots & & \vdots \\ \sin x_M & \sin 2x_M & \cdots & \sin Nx_M \end{bmatrix} \tag{2.19}$$

Since we have $\kappa_{(A)} = \sqrt{\kappa_{(A^*A)}}$, where A^* is the conjugate transpose of A, we can investigate the condition number of A^*A.

$$A^*A = \begin{bmatrix} \sum_{i=1}^{M}(\sin x_i)^2 & \sum_{i=1}^{M}\sin x_i \sin 2x_i & \cdots & \sum_{i=1}^{N}\sin x_i \sin Nx_i \\ \sum_{i=1}^{M}\sin 2x_i \sin x_i & \sum_{i=1}^{M}(\sin 2x_i)^2 & \cdots & \sum_{i=1}^{M}\sin 2x_i \sin Nx_i \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{M}\sin Nx_i \sin x_i & \sum_{i=1}^{M}\sin Nx_i \sin 2x_i & \cdots & \sum_{i=1}^{M}(\sin Nx_i)^2 \end{bmatrix}$$

Let's first investigate the diagonal elements.

Lemma 7. *In uniform spatial sampling with the sensor arrangement $x_i = \frac{\pi}{2M} + \frac{\pi}{M}(i-1)$, we have:*

$$\sum_{i=1}^{M}(\sin kx_i)^2 = \begin{cases} \frac{M}{2} & M \nmid k \\ M & k = (2l+1)M \\ 0 & k = 2lM \end{cases} \tag{2.20}$$

Where we mean by $M \nmid k$ that M does not divide k.

Proof.

$$\begin{aligned} &\sum_{i=1}^{M}(\sin kx_i)^2 \\ =&\frac{-1}{4}\sum_{i=1}^{M}[e^{jk(\frac{\pi}{2M}+\frac{\pi}{M}(i-1))} - e^{-jk(\frac{\pi}{2M}+\frac{\pi}{M}(i-1))}]^2 \\ =&\frac{-1}{4}[-2M + \underbrace{\sum_{i=1}^{M} e^{jk(\frac{\pi}{M}+\frac{2\pi}{M}(i-1))}}_{I_1} + \underbrace{\sum_{i=1}^{M} e^{-jk(\frac{\pi}{M}+\frac{2\pi}{M}(i-1))}}_{I_2}] \end{aligned}$$

It can easily be shown that:

$$I_1 = \begin{cases} M(-1)^{\frac{k}{M}} & M|k \\ 0 & Otherwise \end{cases}$$

And also

$$I_2 = \begin{cases} M(-1)^{\frac{k}{M}} & M|k \\ 0 & Otherwise \end{cases}$$

So finally considering the above facts we will have the claim. $\square$

Now let's consider the non-diagonal elements:

$$\sum_{i=1}^{M} \sin(kx_i)\sin(lx_i)$$
$$=\frac{-1}{4}[\underbrace{\sum_{i=1}^{M} e^{j(k+l)x_i}}_{I_1} + \underbrace{\sum_{i=1}^{M} e^{-j(k+l)x_i}}_{I_2} - \underbrace{\sum_{i=1}^{M} e^{j(k-l)x_i}}_{I_3} - \underbrace{\sum_{i=1}^{M} e^{-j(k-l)x_i}}_{I_4}]$$

If $M \nmid (k+l)$ we have:

$$\begin{aligned} I_1 &= \sum_{i=1}^{M} e^{j(k+l)[\frac{\pi}{2M}+\frac{\pi}{M}(i-1)]} \\ &= \frac{1-(-1)^{(k+l)}}{e^{-j(k+l)\frac{\pi}{2M}} - e^{j(k+l)\frac{\pi}{2M}}} \end{aligned}$$

and also

$$I_2 = \frac{-(1-(-1)^{(k+l)})}{e^{-j(k+l)\frac{\pi}{2M}} - e^{j(k+l)\frac{\pi}{2M}}}$$

Which means $I_1 + I_2 = 0$.
Similarly, if $M \nmid (k-l)$, then $I_3 + I_4 = 0$

It can also be shown that if $k + l = uM$, then

$$I_1 + I_2 = \begin{cases} 2M & 4|u \\ -2M & 2|u, 4 \nmid u \\ 0 & Otherwise \end{cases}$$

And similarly if $k - l = sM$, then

$$I_3 + I_4 = \begin{cases} 2M & 4|s \\ -2M & 2|s, 4 \nmid s \\ 0 & Otherwise \end{cases}$$

So in general, the result of $\sum_{i=1}^{M} \sin(kx_i)\sin(lx_i)$ will be

The Result of $\sum_{i=1}^{M} \sin(kx_i)\sin(lx_i)$				
	$M \nmid k+l$	$4\|u$	$2\|u, 4 \nmid u$	u:odd
$M \nmid k-l$	0	$\frac{-M}{2}$	$\frac{-M}{2}$	0
$4\|s$	$\frac{M}{2}$	0	M	$\frac{M}{2}$
$2\|s, 4 \nmid s$	$\frac{-M}{2}$	$-M$	0	$\frac{-M}{2}$
s:odd	0	$\frac{-M}{2}$	$\frac{M}{2}$	0

Lemma 8. *For A defined as in (2.19), if $M > N$, the matrix A^*A will be diagonal with diagonal elements equal to $M/2$.*

Proof. for the diagonal elements, we have $\sum_{i=1}^{M}(\sin kx_i)^2$, where $k < M$, so it will be always equal to $M/2$(first case in Lemma 7).

For non-diagonal elements, $k + l < 2M$, so we are either in the case where $M \nmid (k+l)$ or in the case where u : odd. and $k - l < M$, so we are in the case where $M \nmid (k-l)$ or $k - l = 0$. Summing up gives us that the non-diagonal elements are zero for $M > N$. □

Lemma 8 shows that when $K = 1$ and $T = 0$, the condition number will not change for $M > N$.
For general case when $K > 1$ and $t \neq 0$, we have:

$$A = \begin{bmatrix} B\Lambda \\ B\Lambda^2 \\ \vdots \\ B\Lambda^K \end{bmatrix}$$

Where

$$B = \begin{bmatrix} \sin x_1 & \sin 2x_1 & \cdots & \sin Nx_1 \\ \sin x_2 & \sin 2x_2 & \cdots & \sin Nx_2 \\ \vdots & \vdots & & \vdots \\ \sin x_M & \sin 2x_M & \cdots & \sin Nx_M \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} e^{-T} & & & \\ & e^{-2^2 T} & & \\ & & \ddots & \\ & & & e^{-N^2 T} \end{bmatrix}$$

It is obvious that Λ is independent of M. So we will have:

$$A^* A = \sum_{i=1}^{K} \Lambda^i B^* B \Lambda^i$$

But we have seen that $B^* B$ is diagonal and we will have

$$A^* A = (B^* B) \sum_{i=1}^{K} \Lambda^{2i} = \frac{M}{2} I \sum_{i=1}^{K} \Lambda^{2i}$$

and finally

$$\kappa_{(A)} = \sqrt{\kappa_{(A^* A)}} = \sqrt{\frac{\frac{M}{2} \sum_{i=1}^{K} e^{-2iT}}{\frac{M}{2} \sum_{i=1}^{K} e^{-2iN^2 T}}} = \sqrt{\frac{\sum_{i=1}^{K} e^{-2iT}}{\sum_{i=1}^{K} e^{-2iN^2 T}}}$$

Which is independent of M and is an increasing function of K which can be seen in Figure 2.4. This also verifies the results we get in our simulations and the 3-D plot provided (see Figure 2.5).

But when $M \leq N$, the summation effects computed above show up in the matrix $A^* A$ and the condition number is not constant anymore.

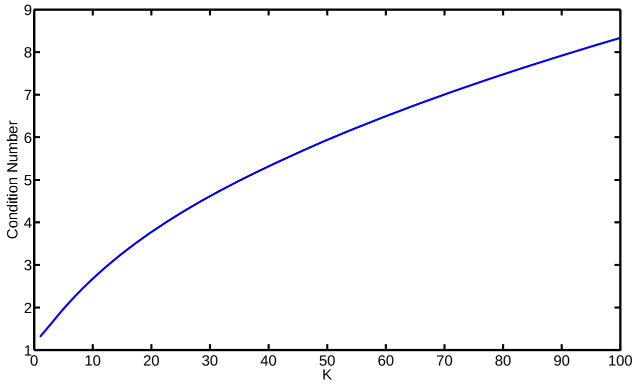


Figure 2.4 – Theoretical value of condition number for different K's for $M > N$

Sudden Jump in $\frac{N}{2}$

In the experiments we saw that with uniform sensor locations, in the plot of condition number with respect to number of sensors, when $M > \frac{N}{2}$, there is a sudden jump from very large values ($1e20$)to reasonable values such as $10 \sim 100$. Here we investigate the reason.

Considering again the matrix

$$A = \begin{bmatrix} e^{-T}\sin x_1 & e^{-2^2T}\sin 2x_1 & \cdots & e^{-N^2T}\sin Nx_1 \\ e^{-2T}\sin x_1 & e^{-2\times 2^2T}\sin 2x_1 & \cdots & e^{-2\times N^2T}\sin Nx_1 \\ \vdots & \vdots & & \vdots \\ e^{-KT}\sin x_1 & e^{-K\times 2^2T}\sin 2x_1 & \cdots & e^{-K\times N^2T}\sin Nx_1 \\ . & & & . \\ . & & & . \\ . & & & . \\ e^{-T}\sin x_M & e^{-2^2T}\sin 2x_M & \cdots & e^{-N^2T}\sin Nx_M \\ e^{-2T}\sin x_M & e^{-2\times 2^2T}\sin 2x_M & \cdots & e^{-2\times N^2T}\sin Nx_M \\ \vdots & \vdots & & \vdots \\ e^{-KT}\sin x_M & e^{-K\times 2^2T}\sin 2x_M & \cdots & e^{-K\times N^2T}\sin Nx_M \end{bmatrix} \tag{2.21}$$

We have the sensor locations as:

$$X = \begin{bmatrix} \frac{\pi}{2M} & \frac{3\pi}{2M} & \cdots & \frac{\pi(2M-1)}{2M} \end{bmatrix}$$

When we have $M \leq N/2$, this simply means $N \geq 2M$, so one column of the matrix A, will be all zero, resulting in infinite condition number. The large values for κ in the simulations were just the computation errors done by MATLAB in computing the locations and their sinusoids. So for $M \leq N/2$, we have $\kappa_{(A)} = \infty$, but when $M > N/2$, the all zero column disappears from A, and this is why we have less condition numbers. The sudden jump and also the fixed condition number can be seen in Figure 2.5.

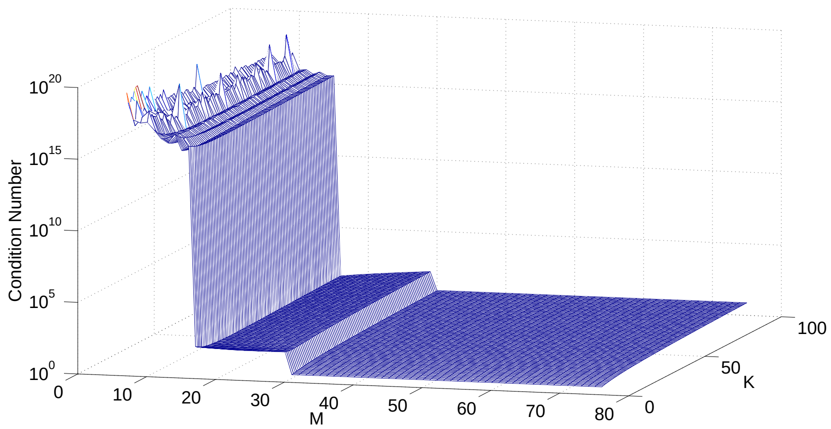
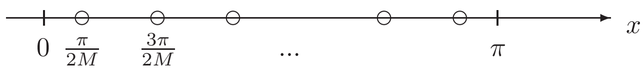


Figure 2.5 – Practical values of condition number for different K's and different M's

2.3.2 Uniform, Shifted Uniform and Periodic Non-Uniform sampling in space

As we saw in the previous section, in spite of good reasons for using uniform sampling, such as easy analysis, easy deployment and lower condition number in large values of M, it has serious drawbacks which makes it almost unusable for small values of M, namely it has infinite condition number for $M \leq \frac{N}{2}$. So we are about to find some other means of sampling. Here we investigate between three different schemes, uniform sampling



shifted uniform sampling

and periodic non-uniform sampling

In our experiments, we fix value of $N = 20$, and $K = 4$. We find the condition number of matrix A for different values of M, T, δ and τ compute the minimum values of condition number and find the corresponding τ, T and δ for each M. You can see the result in Figure 2.6. As it can be seen for $M < N$ best

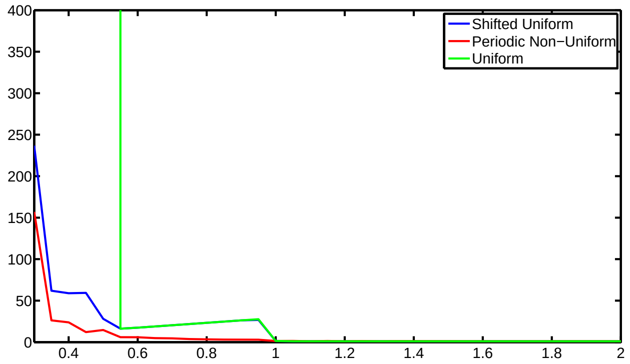
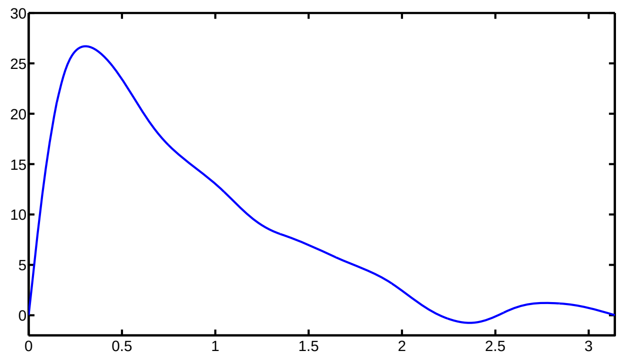


Figure 2.6 – Best condition number for three different schemes, Uniform sampling, Shifted uniform sampling and Periodic non-uniform sampling. As it can be seen for $M < N$ best scheme between these three is shifted uniform sampling and for $M \geq N$ we are better to use Uniform sampling in space.

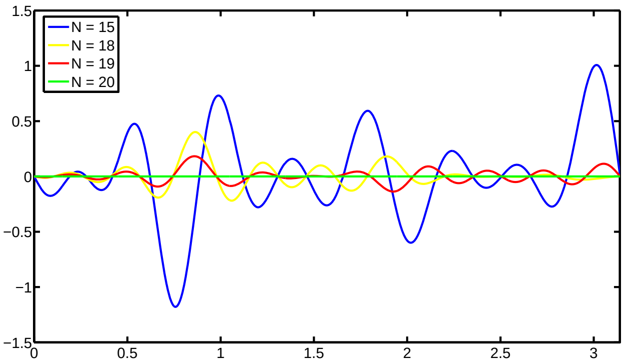
scheme between these three is shifted uniform sampling and for $M \geq N$ we are better to use Uniform sampling in space and minimum value of κ is achieved by taking τ equal to zero and δ equal to $\pi/2M$, so that we get uniform sampling for $M \geq N$.

2.4 Reconstruction

In this regard we did some experiments to see the effect of using more time samples in contrast to more spatial samples. In the Figure 2.7 you can see the results of a simulation for $M = 6$, $N = 20$, $K = 4$ and $T = 3.7e - 4$ time units. Figure 2.7(b) suggests that the condition number of our matrix A is very good, because in the case of aliasing due to large number of coefficients we still get good results in the order of 30 dB.



(a)



(b)

Figure 2.7 – In (a) you can see the original signal and in (b) the error of estimation is shown for different number of coefficients used to estimate signal

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