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Counterexamples to a fiber theorem

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ABSTRACT

We exhibit a counterexample to a fiber theorem stated by F. Fumagalli in [Francesco Fumagalli, On the homotopy type of the Quillen complex of finite soluble groups, J. Algebra 283 (2) (2005) 639–654] and show how it affects the rest of Fumagalli's paper. As a consequence, whether the poset $A_p(G)$ is homotopy equivalent to a wedge of spheres for any finite solvable group G seems to remain an open question.

For a finite group G and a prime p, we denote by $S_p(G)$ the poset of non-trivial p-subgroups of G, and by $A_p(G)$ the poset of non-trivial elementary abelian p-subgroups of G. It was proved by Quillen in an influential paper [7], that the two posets $S_p(G)$ and $A_p(G)$ are actually homotopy equivalent. Quillen's proof relies on a general result on topology of posets, often referred to as the "Quillen fiber lemma": Given a poset map $f: P \to Q$, if the fibers $f^{-1}(Q_{\leqslant q})$ are contractible, then f is a homotopy equivalence. Several results similar to the Quillen fiber lemma can be found in the literature and some have proved useful for the study of $A_p(G)$. They are sometimes called "fiber theorems" and have the following general form: Given a poset map $f: P \to Q$, certain properties can be transferred from Q to P, if the fibers $f^{-1}(Q_{\leqslant q})$ are sufficiently well-behaved. A good account of the subject, as well as the following general result subsuming several of the known fiber theorems, can be found in a paper [2] by Björner, Wachs and Welker.

Theorem 1 *(Björner, Wachs, Welker). Let $f: P \to Q$ be a poset map such that for all $q \in Q$, the fiber $f^{-1}(Q_{\leqslant q})$ is non-empty, and for all non-minimal $q \in Q$, the inclusion map $f^{-1}(Q_{<q}) \hookrightarrow f^{-1}(Q_{\leqslant q})$ is homotopic to a constant map. There is then a homotopy equivalence*

$$P \simeq Q \vee \bigvee_{q \in Q} \left(f^{-1}(Q_{\leqslant q}) * Q_{>q}\right).$$

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In [6], Pulkus and Welker proved a fiber theorem of particular interest for the study of the homotopy type of $A_p(G)$, when G is a solvable group. Their results suggest that the homotopy type of $A_p(G)$, for any solvable group G, can be calculated recursively if one knows the structure of upper intervals in the poset $A_p(G)$.

In [4], Fumagalli studies such upper intervals and claims to have proved that if p is an odd prime and G is a solvable group, then the poset $A_p(G)$ is homotopy equivalent to a wedge of spheres [4, Theorem 21]. A crucial step in the proof, namely [4, Lemmas 19 and 20], relies heavily on a particular fiber theorem [4, Corollary 5], which unfortunately turns out to be false. The aim of this note is to provide two counterexamples to [4, Corollary 5]. The second one will also serve as a counterexample to the wedge decomposition formula obtained by Fumagalli in the proof of his Lemma 19. We will show in addition how this affects the rest of Fumagalli's paper.

All posets and groups appearing in this paper are assumed to be finite. To ease notation, we will use the same letter for a poset and for its order complex. Given a poset map $f : P \to Q$ and $q \in Q$, we denote by $f^{-1}_{\leqslant q}$, resp. $f^{-1}_{<q}$, the fiber $f^{-1}(Q_{\leqslant q})$, resp. $f^{-1}(Q_{<q})$. For any group G, we denote by G' its derived subgroup and by $\Phi(G)$ its Frattini subgroup. If G is a p-group, for some prime p, we denote by $\Omega_1(G)$ the subgroup of G generated by all elements of order p in G

In his paper [4], Fumagalli introduces, for $X \in A_p(G) \cup \{1\}$, the poset

$$M_X(G) = \bigl\{U \in S_p(G) \bigm| X < U,\ U = \Omega_1(U),\ \Phi(U) \leqslant X \leqslant Z(U)\bigr\}.$$

During the proof of [4, Lemma 19], he proves the following formula: Let p be an odd prime and let G be a group. Let A be a central elementary abelian p-subgroup of G, let $1 \leqslant X \leqslant Y \leqslant A$ with $|Y : X| = p$, and let $R \leqslant A$ be such that $A = RY$ and $Y \cap R = X$. Fumagalli claims that in this situation, there is a homotopy equivalence

$$M_X(G)_{>A} \simeq M_Y(G)_{>A} \vee \bigvee_{U \in M_Y(G)_{>A}} \bigl(A_p(U/R)_{>A/R} * M_Y(G)_{>U}\bigr). \tag{1}$$

Fumagalli's proof of Formula (1) makes use of Formula (2) below, which is the statement of Corollary 5 in Fumagalli's paper [4].

Let $f : P \to Q$ be a poset map and assume the following:

Assumption 1. The poset Q is a meet-semilattice with least element $\hat{0}$.

Assumption 2. For every $q \in Q_{>\hat{0}}$, $f^{-1}_{\leqslant q} \supsetneq f^{-1}(\hat{0})$.

Assumption 3. For every $q \in Q_{>\hat{0}}$, the complex $f^{-1}_{\leqslant q}$ is either contractible, or a wedge of n_q-dimensional spheres, with $0 \leqslant n_{q'} < n_q$ if $q' < q$ in Q.

Fumagalli claims that under these assumptions, there exists a homotopy equivalence

$$P \simeq \bigl(f^{-1}(\hat{0}) * Q_{>\hat{0}}\bigr) \vee \bigvee_{q \in Q_{>\hat{0}}} \bigl(f^{-1}_{\leqslant q} * Q_{>q}\bigr). \tag{2}$$

This equivalence is given in [4] as a corollary to three standard lemmas, namely the Projection Lemma [4, Lemma 1], the Homotopy Lemma [4, Lemma 2] and the Wedge Lemma [4, Lemma 3], but no detailed proof is given. It is also mentioned as a corollary to [2, Theorem 2.5] (see Theorem 1 above) or [6, Corollary 2.4], but we will see in the following example that these two results actually don't apply in this context.

Example 1. Let $f : P = Q \setminus \{z_1, \hat{0}\} \to Q$ be the inclusion of posets defined by their Hasse diagrams in Fig. 1. The poset Q is a meet-semilattice with least element $\hat{0}$, so that Assumption 1 is satisfied, as

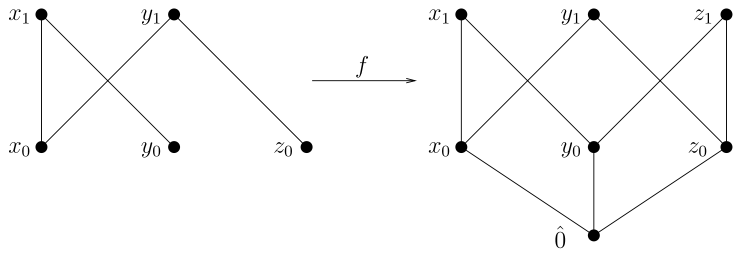

Fig. 1. Hasse diagrams of P and Q.

well as Assumption 2, since $f^{-1}(\hat{0}) = \emptyset$ and for any $q \in Q_{>\hat{0}}$, the fiber $f^{-1}_{\leqslant q}$ is non-empty. The fiber $f^{-1}_{\leqslant q}$ is contractile for every $q \in Q_{>\hat{0}}$, except for $q = z_1$ in which case $f^{-1}_{\leqslant z_1} = \{y_0, z_0\}$ is a sphere of dimension 0, hence Assumption 3 is also satisfied. However, the poset P is contractible, whereas the right-hand side of Formula (2) contains the wedge summand $Q_{>\hat{0}}$ which is homotopy equivalent to a sphere of dimension 1, and thus Formula (2) does not hold.

Note furthermore, that in this example the inclusion map $f^{-1}_{<z_1} \hookrightarrow f^{-1}_{\leqslant z_1}$ is the identity map on $\{y_0, z_0\}$, hence is not homotopic to a constant map. Therefore, classical fiber theorems such as [2, Theorem 2.5] (see Theorem 1 above) or [6, Corollary 2.4] don't apply.

Remark. Let $f : P \to Q$ be a poset map and let $q \in Q$. If $f^{-1}_{\leqslant q}$ is r-connected and $f^{-1}_{<q}$ is at most r-dimensional, then a standard topological argument gives that the inclusion map $f^{-1}_{<q} \hookrightarrow f^{-1}_{\leqslant q}$ is homotopic to a constant map. This situation happens in particular if $f^{-1}_{\leqslant q'}$ is at most r-dimensional for all $q' < q$, since

$$f^{-1}_{<q} = \bigcup_{q'<q} f^{-1}_{\leqslant q'},$$

and the union of spaces of dimension at most r is also at most r-dimensional. This fact was used in particular by Pulkus and Welker (see [6, Lemma 3.2]), and it is probably a similar argument that Fumagalli had in mind. There is unfortunately no such argument under the conditions of Assumption 3, since there is no condition on the dimensions and the union of k-connected spaces does not necessarily remain k-connected.

The next example gives another counterexample to Formula (2) and shows furthermore that Formula (1) is not true in general.

Example 2. Let p be an odd prime. We denote by L the p-group on the generators x_1, y_1, x_2, y_2, v, w of order p and with commuting relations $[x_1, y_1] = [x_2, y_2] = w$, $[x_1, x_2] = v$ and all other commutators between generators trivial. Let $V = \langle v\rangle$ and $W = \langle w\rangle$. The group L has the following properties:

(i) $Z(L) = \langle v, w\rangle = V \times W$ is elementary abelian of rank 2;
(ii) $L' = \Phi(L) = Z(L)$;
(iii) L has exponent p;
(iv) L/V is extraspecial of order p^5 and exponent p with $Z(L/V) = Z(L)/V$.

For any subgroup H of L containing V, we denote by $\bar{H}$ the quotient H/V. Let P be the poset $\mathcal{A}_p(L)_{>Z(L)}$ and Q the poset $\mathcal{A}_p(\bar{L})_{\geqslant Z(\bar{L})}$ (note that Q is a meet-semilattice with $\hat{0} = Z(\bar{L})$) and consider the poset map $f : P \to Q$ given by $f(A) = \bar{A}$. For $q = \bar{B} \in Q_{>\hat{0}}$, we have $f^{-1}_{\leqslant q} = \mathcal{A}_p(B)_{>Z(L)}$ and either B is elementary abelian, or $B = X \times W$ with X extraspecial of order p^3 and exponent p with

$Z(X) = V$. In the first case, $f^{-1}_{\leqslant q}$ is contractible and in the second case $f^{-1}_{\leqslant q}$ is a wedge of p spheres of dimension 0. All of the assumptions are satisfied and Formula (2) can be rewritten as the following homotopy equivalence

$$A_p(L)_{>Z(L)} \simeq A_p(\bar{L})_{>Z(\bar{L})} \vee \bigvee_{\bar{B} \in A_p(\bar{L})_{>Z(\bar{L})}} \big(A_p(B)_{>Z(L)} * A_p(\bar{L})_{>\bar{B}}\big). \tag{3}$$

Since $\bar{L}$ is extraspecial, we know from [7, Example 10.4] that the poset $A_p(\bar{L})_{>Z(\bar{L})}$ is homotopy equivalent to the building of a symplectic group. It follows from the Solomon–Tits theorem that $A_p(\bar{L})_{>Z(\bar{L})}$ is a wedge of p^4 spheres of dimension 1, so that the right-hand side of Eq. (3) is in particular not contractible. It turns out, however, that the left-hand side of Eq. (3), namely $A_p(L)_{>Z(L)}$, is contractible. To see this, we use the following wedge decomposition, which is slightly modified from [3].

Lemma 2. *Let A be a central elementary abelian p-subgroup of the p-group G and suppose that G contains a normal subgroup $E_0 \in A_p(G)_{>A}$ such that $|E_0 : A| = p$ and $M := C_P(E_0)$ is of index p in G. There is then a homotopy equivalence*

$$A_p(G)_{>A} \simeq \bigvee_{F \in \mathcal{F}} A_p\big(C_M(F)\big)_{>A}, \tag{4}$$

where $\mathcal{F} = \{F \in A_p(G)_{>A} \mid F \cap M = A,\ |F : A| = p\}$.

Proof. Let P be the poset $A_p(G) \backslash \mathcal{F}$ and let $B \in P$. Since $|G : M| = p$ and $B \notin \mathcal{F}$, we have $B \cap M > A$ and the following sequence of inequalities in P shows that the poset P is indeed contractible:

$$B \geqslant B \cap M \leqslant (B \cap M)E_0 \geqslant E_0.$$

The set $\mathcal{F}$ consists of minimal elements of $A_p(G)_{>A}$ and the result follows from the homotopy complementation formula (see for example [1, Corollary 2.3]) and the observation that $\mathrm{lk}_P(F) = A_p(G)_{>F} \simeq A_p(C_M(F))_{>A}$, where the homotopy equivalence comes from the contraction $B \mapsto B \cap M$ with homotopy inverse $E \mapsto EF$. □

We apply Lemma 2 to our group L defined above, with $A = Z(L)$ and $E_0 = \langle y_1, v, w\rangle$. In this situation, $M = C_L(E_0) = \langle y_1, x_2, y_2, v.w\rangle$ and the elements of $\mathcal{F}$ are of the form $F = \langle x_1 x_2^r y_1^{s_1} y_2^{s_2}, v, w\rangle$. For such an element F, we have $C_M(F) = \langle y_1^{-r} y_2, v, w\rangle$ is elementary abelian, so that $A_p(C_M(F))_{>A}$ is always contractible, showing that $A_p(L)_{>Z(L)}$ is contractible.

It follows that Eq. (3) does not hold and we show now how this implies that Formula (1) does not hold in general. For this, we set

$$A = Z(L), \qquad X = \{1\}, \qquad Y = V \quad \text{and} \quad R = W.$$

The following identifications follow easily:

$$M_X(L)_{>A} = A_p(L)_{>Z(L)}, \quad \text{and} \quad M_Y(L)_{>A} \cong A_p(\bar{L})_{>Z(\bar{L})}.$$

Furthermore, for $B \in M_Y(L)_{>A}$, it is not difficult to see that $A_p(B/R)_{>A/R}$ is isomorphic to $A_p(B)_{>Z(L)}$. We see thus that Formulas (1) and (3) are identical in this situation, showing that Formula (1) does not hold in general.

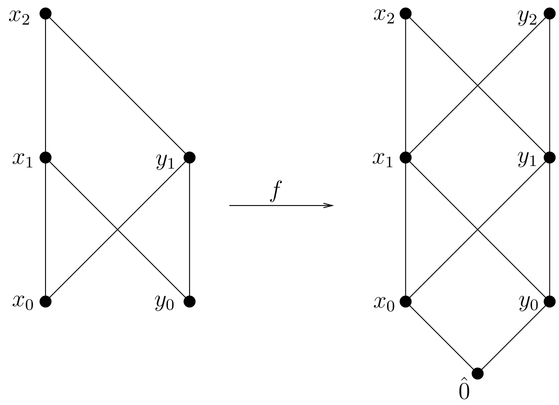

Fig. 2. Fibers of dimension 1.

Remark. (a) It should be noted that our second example does not contradict the statement of Fumagalli's Lemma 19, but only Formula (1) appearing in its proof.

(b) In our second example, Formula (3) is true if we choose W instead of V. But there is no good choice in general as can be seen with the 3-group L of order 3^6 numbered 469 in the GAP library [5]. The center of L is elementary abelian of rank 2 and every quotient of L by a non-trivial proper subgroup of its center is extraspecial of order 3^5 and exponent p, so that the right-hand side of Formula (3) contains spheres of dimension 1. It turns out however that the left-hand side, namely the poset $A_3(L)_{>Z(L)}$, is a wedge of 0-dimensional spheres.

(c) Bouc and Thévenaz applied Lemma 2 successfully to show that in the case G is a finite p-group and $|A| = p$, then $A_p(G)_{>A}$ has the homotopy type of a wedge of spheres. To our knowledge it is not known when $|A| > p$. Whether $A_p(G)$ is homotopy equivalent to a wedge of spheres for any solvable group G seems then to remain an open question.

(d) In both of our examples, spheres only appear in dimension 0. However, Fumagalli's fiber theorem [4, Corollary 5] is unlikely to be true, even if Assumption 3 is strengthened by requiring $n_q > 0$. In the inclusion of posets shown in Fig. 2, all the fibers are contractible, except $f^{-1}_{\leqslant y_2}$ which is a sphere of dimension 1. The poset P is contractible, whereas $Q_{>\hat{0}}$ is homotopy equivalent to a 2-dimensional sphere. Note that this is not strictly speaking a counterexample, since Q is not a meet-semilattice.

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