

The primitive idempotents of the p -permutation ring

Serge Bouc and Jacques Thévenaz

Abstract : Let G be a finite group, let p be a prime number, and let K be a field of characteristic 0 and k be a field of characteristic p , both large enough. In this note we state explicit formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} pp_k(G)$, where $pp_k(G)$ is the ring of p -permutation kG -modules.

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1. Introduction

Let G be a finite group, let p be a prime number, and let K be a field of characteristic 0 and k be a field of characteristic p , both large enough. In this note we state explicit formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} pp_k(G)$, where $pp_k(G)$ is the ring of p -permutation kG -modules (also called the trivial source ring).

To obtain these formulae, we first use induction and restriction to reduce to the case where G is cyclic modulo p , i.e. G has a normal Sylow p -subgroup with cyclic quotient. Then we solve the easy and well known case where G is a cyclic p' -group. Finally we conclude by using the natural ring homomorphism from the Burnside ring $B(G)$ of G to $pp_k(G)$, and the classical formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} B(G)$.

Our formulae are an essential tool in [2], where Cartan matrices of Mackey algebras are considered, and some invariants of these matrices (determinant, rank) are explicitly computed.

2. p -permutation modules

2.1. Notation.

- Throughout the paper, G will be a fixed finite group and p a fixed prime number. We consider a field k of characteristic p and we denote by kG the group algebra of G over k . We assume that k is large enough in the sense that it is a splitting field for every group algebra $k(N_G(P)/P)$, where P runs through the set of all p -subgroups of G .

- We let K be a field of characteristic 0 and we assume that K is large enough in the sense that it contains the values of all the Brauer characters of the groups $N_G(P)/P$, where P runs through the set of all p -subgroups of G .

We recall quickly how Brauer characters are defined. We let \bar{k} be an extension of k containing all the n -th roots of unity, where n is the p' -part of the exponent of G . We choose an isomorphism $\theta : \mu_n(\bar{k}) \rightarrow \mu_n(\mathbb{C})$ from the group of n -th roots of unity in \bar{k} and the corresponding group in \mathbb{C} . If V is an r -dimensional kH -module for the group $H = N_G(P)/P$ and if s is an element of the set $H_{p'}$ of all p' -elements of H , the matrix of the action of s on V has eigenvalues $(\lambda_1, \dots, \lambda_r)$ in the group $\mu_n(\bar{k})$. The *Brauer character* ϕ_V of V is the central function defined on $H_{p'}$, with values in the field $\mathbb{Q}[\mu_n(\mathbb{C})]$, sending s to $\sum_{i=1}^r \theta(\lambda_i)$. The actual values of Brauer characters may lie in a subfield of $\mathbb{Q}[\mu_n(\mathbb{C})]$ and we simply require that K contains all these values.

2.2. Remark : Let V be as above and let W be a t -dimensional kH -module. If s has eigenvalues (μ_1, \dots, μ_t) on W , its eigenvalues for the diagonal action of H on $V \otimes_k W$ are $(\lambda_i \mu_j)_{1 \leq i \leq r, 1 \leq j \leq t}$. It follows that $\phi_{V \otimes_k W}(s) = \sum_{i=1}^r \sum_{j=1}^t \theta(\lambda_i \mu_j) = \phi_V(s) \phi_W(s)$.

- When H is a subgroup of G , and M is a kG -module, we denote by $\text{Res}_H^G M$ the kH -module obtained by restricting the action of G to H . When L is a kH -module, we denote by $\text{Ind}_H^G L$ the induced kG -module.

- When M is a kG -module, and P is a subgroup of G , the k -vector space of fixed points of P on M is denoted by M^P . When $Q \leq P$ are subgroups of G , the *relative trace* is the map $\text{tr}_Q^P : M^Q \rightarrow M^P$ defined by $\text{tr}_Q^P(m) = \sum_{x \in [P/Q]} x \cdot m$.

- When M is a kG -module, the *Brauer quotient* of M at P is the k -vector space

$$M[P] = M^P / \sum_{Q < P} \text{tr}_Q^P M^Q .$$

This k -vector space has a natural structure of $k\bar{N}_G(P)$ -module, where as usual $\bar{N}_G(P) = N_G(P)/P$. It is equal to zero if P is not a p -group.

- If P is a normal p -subgroup of G and M is a $k(G/P)$ -module, denote by $\text{Inf}_{G/P}^G M$ the kG -module obtained from M by inflation to G . Then there is an isomorphism

$$(\text{Inf}_{G/P}^G M)[P] \cong M$$

of $k(G/P)$ -modules.

- When G acts on a set X (on the left), and $x, y \in X$, we write $x =_G y$ if x and y are in the same G -orbit. We denote by $[G \setminus X]$ a set of representatives

of G -orbits on X , and by X^G the set of fixed points of G on X . For $x \in X$, we denote by G_x its stabilizer in G .

2.3. Review of p -permutation modules. We begin by recalling some definitions and basic results. We refer to [3], and to [1] Sections 3.11 and 5.5 for details :

2.4. Definition. A permutation kG -module is a kG -module admitting a G -invariant k -basis. A p -permutation kG -module M is a kG -module such that $\text{Res}_S^G M$ is a permutation kS -module, where S is a Sylow p -subgroup of G .

The p -permutation kG -modules are also called *trivial source modules*, because the indecomposable ones coincide with the indecomposable modules having a trivial source (see [3] 0.4). Moreover, the p -permutation modules also coincide with the direct summands of permutation modules (see [1], Lemma 3.11.2).

2.5. Proposition.

1. If H is a subgroup of G , and M is a p -permutation kG -module, then the restriction $\text{Res}_H^G M$ of M to H is a p -permutation kH -module.
2. If H is a subgroup of G , and L is a p -permutation kH -module, then the induced module $\text{Ind}_H^G L$ is a p -permutation kG -module.
3. If N is a normal subgroup of G , and L is a p -permutation $k(G/N)$ -module, the inflated module $\text{Inf}_{G/N}^G L$ is a p -permutation kG -module.
4. If P is a p -group, and M is a permutation kP -module with P -invariant basis X , then the image of the set X^P in $M[P]$ is a k -basis of $M[P]$.
5. If P is a p -subgroup of G , and M is a p -permutation kG -module, then the Brauer quotient $M[P]$ is a p -permutation $k\overline{N}_G(P)$ -module.
6. If M and N are p -permutation kG -modules, then their tensor product $M \otimes_k N$ is again a p -permutation kG -module.

Proof : Assertions 1,2,3, and 6 are straightforward consequences of the same assertions for *permutation* modules. For Assertion 4, see [3] 1.1.(3). Assertion 5 follows easily from Assertion 4 (see also [3] 3.1). \square

This leads to the following definition :

2.6. Definition. The p -permutation ring $pp_k(G)$ is the Grothendieck group of the category of p -permutation kG -modules, with relations corresponding to direct sum decompositions, i.e. $[M] + [N] = [M \oplus N]$. The ring structure

on $pp_k(G)$ is induced by the tensor product of modules over k . The identity element of $pp_k(G)$ is the class of the trivial kG -module k .

As the Krull-Schmidt theorem holds for kG -modules, the additive group $pp_k(G)$ is a free (abelian) group on the set of isomorphism classes of indecomposable p -permutation kG -modules. These modules have the following properties :

2.7. Theorem. [[3] Theorem 3.2]

1. The vertices of an indecomposable p -permutation kG -module M are the maximal p -subgroups P of G such that $M[P] \neq \{0\}$.
2. An indecomposable p -permutation kG -module has vertex P if and only if $M[P]$ is a non-zero projective $k\overline{N}_G(P)$ -module.
3. The correspondence $M \mapsto M[P]$ induces a bijection between the isomorphism classes of indecomposable p -permutation kG -modules with vertex P and the isomorphism classes of indecomposable projective $k\overline{N}_G(P)$ -modules.

2.8. Notation. Let $\mathcal{P}_{G,p}$ denote the set of pairs (P, E) , where P is a p -subgroup of G , and E is an indecomposable projective $k\overline{N}_G(P)$ -module. The group G acts on $\mathcal{P}_{G,p}$ by conjugation, and we denote by $[\mathcal{P}_{G,p}]$ a set of representatives of G -orbits on $\mathcal{P}_{G,p}$.

For $(P, E) \in \mathcal{P}_{G,p}$, let $M_{P,E}$ denote the (unique up to isomorphism) indecomposable p -permutation kG -module such that $M_{P,E}[P] \cong E$.

2.9. Corollary. The classes of the modules $M_{P,E}$, for $(P, E) \in [\mathcal{P}_{G,p}]$ form a \mathbb{Z} -basis of $pp_k(G)$.

2.10. Notation. The operations Res_H^G , Ind_H^G , $\text{Inf}_{G/N}^G$ extend linearly to maps between the corresponding p -permutations rings, denoted with the same symbol.

The maps Res_H^G and $\text{Inf}_{G/N}^G$ are ring homomorphisms, whereas Ind_H^G is not in general. Similarly :

2.11. Proposition. Let P be a p -subgroup of G . Then the correspondence $M \mapsto M[P]$ induces a ring homomorphism $\text{Br}_P^G : pp_k(G) \rightarrow pp_k(\overline{N}_G(P))$.

Proof : Let M and N be p -permutation kG -modules. The canonical bilinear map $M \times N \rightarrow M \otimes_k N$ is G -equivariant, hence it induces a bilinear map $\beta_P : M[P] \times N[P] \rightarrow (M \otimes_k N)[P]$ (see [3] 1.2), which is $\overline{N}_G(P)$ -equivariant. Now if X is a P -invariant k -basis of M , and Y a P -invariant k -basis of N ,

then $X \times Y$ is a P -invariant basis of $M \otimes_k N$. The images of the sets X^P , Y^P , and $(X \times Y)^P$ are bases of $M[P]$, $N[P]$, and $(M \otimes_k N)[P]$, respectively, and the restriction of β_P to these bases is the canonical bijection $X^P \times Y^P \rightarrow (X \times Y)^P$. It follows that β_P induces an isomorphism $M[P] \otimes_k N[P] \rightarrow (M \otimes_k N)[P]$ of $k\overline{N}_G(P)$ -modules. Proposition 2.11 follows. \square

2.12. Notation. Let $\mathcal{Q}_{G,p}$ denote the set of pairs (P, s) , where P is a p -subgroup of G , and s is a p' -element of $\overline{N}_G(P)$. The group G acts on $\mathcal{Q}_{G,p}$, and we denote by $[\mathcal{Q}_{G,p}]$ a set of representatives of G -orbits on $\mathcal{Q}_{G,p}$.

If $(P, s) \in \mathcal{Q}_{G,p}$, we denote by $N_G(P, s)$ the stabilizer of (P, s) in G , and by $\langle Ps \rangle$ the subgroup of $N_G(P)$ generated by Ps (i.e. the inverse image in $N_G(P)$ of the cyclic group $\langle s \rangle$ of $\overline{N}_G(P)$).

2.13. Remarks :

- When H is a subgroup of G , there is a natural inclusion of $\mathcal{Q}_{H,p}$ into $\mathcal{Q}_{G,p}$, as $\overline{N}_H(P) \leq \overline{N}_G(P)$ for any p -subgroup P of H . We will consider $\mathcal{Q}_{H,p}$ as a subset of $\mathcal{Q}_{G,p}$.
- When $(P, s) \in \mathcal{Q}_{G,p}$, the group $N_G(P, s)$ is the set of elements g in $N_G(P)$ whose image in $\overline{N}_G(P)$ centralizes s . In other words, there is a short exact sequence of groups

$$(2.14) \quad \mathbf{1} \rightarrow P \rightarrow N_G(P, s) \rightarrow C_{\overline{N}_G(P)}(s) \rightarrow \mathbf{1} .$$

In particular $N_G(P, s)$ is a subgroup of $N_G(\langle Ps \rangle)$.

2.15. Notation. Let $(P, s) \in \mathcal{Q}_{G,p}$. Let $\tau_{P,s}^G$ denote the additive map from $pp_k(G)$ to K sending the class of a p -permutation kG -module M to the value at s of the Brauer character of the $\overline{N}_G(P)$ -module $M[P]$.

2.16. Remarks :

- It is clear that $\tau_{P,s}^G(M)$ only depends on the restriction of M to the group $\langle Ps \rangle$. In other words

$$\tau_{P,s}^G = \tau_{P,s}^{\langle Ps \rangle} \circ \text{Res}_{\langle Ps \rangle}^G .$$

Furthermore, it is clear from the definition that

$$(2.17) \quad \tau_{P,s}^G = \tau_{\mathbf{1},s}^{\langle Ps \rangle / P} \circ \text{Br}_P^{\langle Ps \rangle} \circ \text{Res}_{\langle Ps \rangle}^G .$$

- It is easy to see that $\tau_{P,s}^G$ only depends on the G -orbit of (P, s) , that is, $\tau_{P^g, s^g}^G = \tau_{P,s}^G$ for every $g \in G$.

The following proposition is Corollary 5.5.5 in [1], but our construction of the species is slightly different (but equivalent, of course). For this reason, we sketch an independent proof :

2.18. Proposition.

1. The map $\tau_{P,s}^G$ is a ring homomorphism $pp_k(G) \rightarrow K$ and extends to a K -algebra homomorphism (a species) $\tau_{P,s}^G : K \otimes_{\mathbb{Z}} pp_k(G) \rightarrow K$.
2. The set $\{\tau_{P,s}^G \mid (P, s) \in [\mathcal{Q}_{G,p}]\}$ is the set of all distinct species from $K \otimes_{\mathbb{Z}} pp_k(G)$ to K . These species induce a K -algebra isomorphism

$$T = \prod_{(P,s) \in [\mathcal{Q}_{G,p}]} \tau_{P,s}^G : K \otimes_{\mathbb{Z}} pp_k(G) \rightarrow \prod_{(P,s) \in [\mathcal{Q}_{G,p}]} K .$$

Proof : By 2.17, to prove Assertion 1, it suffices to prove that $\tau_{\mathbf{1},s}^{\langle Ps \rangle / P}$ is a ring homomorphism, since both $\text{Res}_{\langle Ps \rangle}^G$ and $\text{Br}_P^{\langle Ps \rangle}$ are ring homomorphisms. In other words, we can assume that $P = \mathbf{1}$. Now the value of $\tau_{\mathbf{1},s}^G$ on the class of a kG -module M is the value $\phi_M(s)$ of the Brauer character of M at s , so Assertion 1 follows from Remark 2.2.

For Assertion 2, it suffices to prove that T is an isomorphism. Since $[\mathcal{P}_{G,p}]$ and $[\mathcal{Q}_{G,p}]$ have the same cardinality, the matrix \mathcal{M} of T is a square matrix. Let $(P, E) \in \mathcal{P}_{G,p}$, and $(Q, s) \in \mathcal{Q}_{G,p}$. Then $\tau_{Q,s}(M_{P,E})$ is equal to zero if Q is not contained in P up to G -conjugation, because in this case $M_{P,E}(Q) = \{0\}$ by Theorem 2.7. It follows that \mathcal{M} is block triangular. As moreover $M_{P,E}[P] \cong E$, we have that $\tau_{P,s}(M_{P,E}) = \phi_E(s)$. This means that the diagonal block of \mathcal{M} corresponding to P is the matrix of Brauer characters of projective $k\overline{N}_G(P)$ -modules, and these are linearly independent by Lemma 5.3.1 of [1]. It follows that all the diagonal blocks of \mathcal{M} are non singular, so \mathcal{M} is invertible, and T is an isomorphism. \square

2.19. Corollary. *The algebra $K \otimes_{\mathbb{Z}} pp_k(G)$ is a split semisimple commutative K -algebra. Its primitive idempotents $F_{P,s}^G$ are indexed by $[\mathcal{Q}_{G,p}]$, and the idempotent $F_{P,s}^G$ is characterized by*

$$\forall (R, u) \in \mathcal{Q}_{G,p}, \quad \tau_{R,u}^G(F_{P,s}^G) = \begin{cases} 1 & \text{if } (R, u) =_G (P, s) \\ 0 & \text{otherwise.} \end{cases}$$

3. Restriction and induction

3.1. Proposition. *Let $H \leq G$, and $(P, s) \in \mathcal{Q}_{G,p}$. Then*

$$\text{Res}_H^G F_{P,s}^G = \sum_{(Q,t)} F_{Q,t}^H ,$$

where (Q, t) runs through a set of representatives of H -conjugacy classes of G -conjugates of (P, s) contained in H .

Proof : Indeed, as Res_H^G is an algebra homomorphism, the element $\text{Res}_H^G F_{P,s}^G$ is an idempotent of $K \otimes_{\mathbb{Z}} pp_k(H)$, hence it is equal to a sum of some distinct primitive idempotents $F_{Q,t}^H$. The idempotent $F_{Q,t}^H$ appears in this decomposition if and only if $\tau_{Q,t}^H(\text{Res}_H^G F_{P,s}^G) = 1$. By Remark 2.16

$$\begin{aligned} \tau_{Q,t}^H(\text{Res}_H^G F_{P,s}^G) &= \tau_{Q,t}^{\langle Qt \rangle}(\text{Res}_{\langle Qt \rangle}^H \text{Res}_H^G F_{P,s}^G) \\ &= \tau_{Q,t}^{\langle Qt \rangle}(\text{Res}_{\langle Qt \rangle}^G F_{P,s}^G) \\ &= \tau_{Q,t}^G(F_{P,s}^G) . \end{aligned}$$

Now $\tau_{Q,t}^G(F_{P,s}^G)$ is equal to 1 if and only if (Q, t) and (P, s) are G -conjugate. This completes the proof. \square

3.2. Proposition. *Let $H \leq G$, and $(Q, t) \in \mathcal{Q}_{H,p}$. Then*

$$\text{Ind}_H^G F_{Q,t}^H = |N_G(Q, t) : N_H(Q, t)| F_{Q,t}^G .$$

Proof : Since $K \otimes_{\mathbb{Z}} pp_k(G)$ is a split semisimple commutative K -algebra, any element X in $K \otimes_{\mathbb{Z}} pp_k(G)$ can be written

$$(3.3) \quad X = \sum_{(P,s) \in [\mathcal{Q}_{G,p}]} \tau_{P,s}^G(X) F_{P,s}^G ,$$

and moreover for any $(P, s) \in \mathcal{Q}_{G,p}$

$$\tau_{P,s}^G(X) F_{P,s}^G = X \cdot F_{P,s}^G .$$

Setting $X = \text{Ind}_H^G F_{Q,t}^H$ in this equation gives

$$\begin{aligned} \tau_{P,s}^G(\text{Ind}_H^G F_{Q,t}^H) F_{P,s}^G &= (\text{Ind}_H^G F_{Q,t}^H) \cdot F_{P,s}^G \\ &= \text{Ind}_H^G(F_{Q,t}^H \cdot \text{Res}_H^G F_{P,s}^G) . \end{aligned}$$

By Proposition 3.1, the element $\text{Res}_H^G F_{P,s}^G$ is equal to the sum of the distinct idempotents F_{P^y, s^y}^H associated to elements y of G such that $\langle Ps \rangle^y \leq H$. The product $F_{Q,t}^H \cdot F_{P^y, s^y}^H$ is equal to zero, unless (Q, t) is H -conjugate to (P^y, s^y) , which implies that (Q, t) and (P, s) are G -conjugate. It follows that the only non zero term in the right hand side of Equation 3.3 is the term corresponding to (Q, t) . Hence

$$\text{Ind}_H^G F_{Q,t}^H = \tau_{Q,t}^G(\text{Ind}_H^G F_{Q,t}^H) F_{Q,t}^G .$$

Now by Remark 2.16 and the Mackey formula

$$\begin{aligned}\tau_{Q,t}^G(\text{Ind}_H^G F_{Q,t}^H) &= \tau_{Q,t}^{\langle Qt \rangle}(\text{Res}_{\langle Qt \rangle}^G \text{Ind}_H^G F_{Q,t}^H) \\ &= \tau_{Q,t}^{\langle Qt \rangle} \left(\sum_{x \in \langle Qt \rangle \backslash G/H} \text{Ind}_{\langle Qt \rangle \cap xH}^{\langle Qt \rangle} {}^x \text{Res}_{\langle Qt \rangle \cap xH}^H F_{Q,t}^H \right).\end{aligned}$$

By Proposition 3.1, the element $\text{Res}_{\langle Qt \rangle \cap xH}^H F_{Q,t}^H$ is equal to the sum of the distinct idempotents $F_{Q^y, ty}^{\langle Qt \rangle \cap xH}$ corresponding to elements $y \in H$ such that $\langle Qt \rangle^y \leq \langle Qt \rangle \cap xH$. This implies $\langle Qt \rangle^y = \langle Qt \rangle^x$, i.e. $y \in N_G(\langle Qt \rangle)x$, thus $x \in N_G(\langle Qt \rangle) \cdot H$. This gives

$$\begin{aligned}\tau_{Q,t}^G(\text{Ind}_H^G F_{Q,t}^H) &= \tau_{Q,t}^{\langle Qt \rangle} \left(\sum_{\substack{x \in N_G(\langle Qt \rangle)H/H \\ y \in N_H(Q,t) \backslash N_G(\langle Qt \rangle)x}} {}^x F_{Q^y, ty}^{\langle Qt \rangle} \right) \\ &= \sum_{\substack{x \in N_G(\langle Qt \rangle)/N_H(\langle Qt \rangle) \\ y \in N_H(Q,t) \backslash N_G(\langle Qt \rangle)x}} \tau_{Q,t}^{\langle Qt \rangle} (F_{Q^{yx^{-1}}, tyx^{-1}}^{\langle Qt \rangle}) \\ &= \sum_{z \in N_H(Q,t) \backslash N_G(\langle Qt \rangle)} \tau_{Q,t}^{\langle Qt \rangle} (F_{Q^z, tz}^{\langle Qt \rangle}),\end{aligned}$$

where $z = yx^{-1}$. Finally $\tau_{Q,t}^{\langle Qt \rangle} (F_{Q^z, tz}^{\langle Qt \rangle})$ is equal to 1 if (Q^z, t^z) is conjugate to (Q, t) in $\langle Qt \rangle$, and to zero otherwise.

If $u \in \langle Qt \rangle$ is such that $(Q^z, t^z)^u = (Q, t)$, then $zu \in N_G(Q, t)$. But since $[\langle Qt \rangle, t] \leq Q$, we have $\langle Qt \rangle \leq N_G(Q, t)$, so $u \in N_G(Q, t)$, hence $z \in N_G(Q, t)$, and $(Q^z, t^z) = (Q, t)$. It follows that

$$\tau_{Q,t}^G(\text{Ind}_H^G F_{Q,t}^H) = |N_G(Q, t) : N_H(Q, t)|,$$

which completes the proof of the proposition. \square

3.4. Corollary. *Let $(P, s) \in \mathcal{Q}_{G,p}$. Then*

$$F_{P,s}^G = \frac{|s|}{|C_{\overline{N}_G(P)}(s)|} \text{Ind}_{\langle Ps \rangle}^G F_{P,s}^{\langle Ps \rangle}.$$

Proof : Apply Proposition 3.2 with $(Q, t) = (P, s)$ and $H = \langle Ps \rangle$. Then $N_H(Q, t) = \langle Ps \rangle$, thus by Exact sequence 2.14

$$|N_G(Q, t) : N_H(Q, t)| = \frac{|P| |C_{\overline{N}_G(P)}(s)|}{|P| |s|} = \frac{|C_{\overline{N}_G(P)}(s)|}{|s|},$$

and the corollary follows. \square

4. Idempotents

It follows from Corollary 3.4 that, in order to find formulae for the primitive idempotents $F_{P,s}^G$ of $K \otimes_{\mathbb{Z}} pp_k(G)$, it suffices to find the formula expressing the idempotent $F_{P,s}^{\langle Ps \rangle}$. In other words, we can assume that $G = \langle Ps \rangle$, i.e. that G has a normal Sylow p -subgroup P with cyclic quotient generated by s .

4.1. A morphism from the Burnside ring. When G is an arbitrary finite group, there is an obvious ring homomorphism \mathcal{L}_G from the Burnside ring $B(G)$ to $pp_k(G)$, induced by the *linearization* operation, sending a finite G -set X to the permutation module kX , which is obviously a p -permutation module. This morphism also commutes with restriction and induction : if $H \leq G$, then

$$(4.2) \quad \mathcal{L}_H \circ \text{Res}_H^G = \text{Res}_H^G \circ \mathcal{L}_G, \quad \mathcal{L}_G \circ \text{Ind}_H^G = \text{Ind}_H^G \circ \mathcal{L}_H.$$

Indeed, for any G -set X , the kH -modules $k\text{Res}_H^G X$ and $\text{Res}_H^G(kX)$ are isomorphic, and for any H -set Y , the kG -modules $k\text{Ind}_H^G Y$ and $\text{Ind}_H^G(kY)$ are isomorphic.

Similarly, when P is a p -subgroup of G , the ring homomorphism $\Phi_P : B(G) \rightarrow B(\overline{N}_G(P))$ induced by the operation $X \mapsto X^P$ on G -sets, is compatible with the Brauer morphism $\text{Br}_P^G : pp_k(G) \rightarrow pp_k(\overline{N}_G(P))$:

$$(4.3) \quad \mathcal{L}_{\overline{N}_G(P)} \circ \Phi_P = \text{Br}_P^G \circ \mathcal{L}_G.$$

This is because for any G -set X , the $k\overline{N}_G(P)$ -modules $k(X^P)$ and $(kX)[P]$ are isomorphic.

The ring homomorphism \mathcal{L}_G extends linearly to a K -algebra homomorphism $K \otimes_{\mathbb{Z}} B(G) \rightarrow K \otimes_{\mathbb{Z}} pp_k(G)$, still denoted by \mathcal{L}_G . The algebra $K \otimes_{\mathbb{Z}} B(G)$ is also a split semisimple commutative K -algebra. Its species are the K -algebra maps

$$K \otimes_{\mathbb{Z}} B(G) \rightarrow K, \quad X \mapsto |X^H|,$$

where H runs through the set of all subgroups of G up to conjugation. Here we denote by $|X^H|$ the number of H -fixed points of a G -set X and this notation is then extended K -linearly to any $X \in K \otimes_{\mathbb{Z}} B(G)$. The primitive idempotents e_H^G of $K \otimes_{\mathbb{Z}} B(G)$ are indexed by subgroups H of G , up to conjugation. They are given by the following formulae, found by Gluck ([4]) and later independently by Yoshida ([5]) :

$$(4.4) \quad e_H^G = \frac{1}{|N_G(H)|} \sum_{L \leq H} |L| \mu(L, H) G/L,$$

where μ denotes the Möbius function of the poset of subgroups of G . The idempotent e_H^G is characterized by the fact that for any $X \in K \otimes_{\mathbb{Z}} B(G)$

$$X \cdot e_H^G = |X^H| e_H^G .$$

4.5. Remark : Since $|X^H|$ only depends on $\text{Res}_H^G X$, it follows in particular that X is a scalar multiple of the “top” idempotent e_G^G if and only if $\text{Res}_H^G X = 0$ for any proper subgroup H of G . In particular, if N is a normal subgroup of G , then

$$(4.6) \quad (e_G^G)^N = e_{G/N}^{G/N} .$$

This is because for any proper subgroup H/N of G/N

$$\text{Res}_{H/N}^{G/N} (e_G^G)^N = (\text{Res}_H^G e_G^G)^N = 0 .$$

So $(e_G^G)^N$ is a scalar multiple of $e_{G/N}^{G/N}$. As it is also an idempotent, it is equal to 0 or $e_{G/N}^{G/N}$. Finally

$$|((e_G^G)^N)^{G/N}| = |(e_G^G)^G| = 1 ,$$

so $(e_G^G)^N$ is non zero.

4.7. The case of a cyclic p' -group. Suppose that G is a cyclic p' -group, of order n , generated by an element s . In this case, there are exactly n group homomorphisms from G to the multiplicative group k^\times of k . For each of these group homomorphisms φ , let k_φ denote the kG -module k on which the generator s acts by multiplication by $\varphi(s)$. As G is a p' -group, this module is simple and projective. The (classes of the) modules k_φ , for $\varphi \in \widehat{G} = \text{Hom}(G, k^\times)$, form a basis of $pp_k(G)$.

Since moreover for $\varphi, \psi \in \widehat{G}$, the modules $k_\varphi \otimes_k k_\psi$ and $k_{\varphi\psi}$ are isomorphic, the algebra $K \otimes_{\mathbb{Z}} pp_k(G)$ is isomorphic to the group algebra of the group \widehat{G} . This leads to the following classical formula :

4.8. Lemma. *Let G be a cyclic p' -group. Then for any $t \in G$,*

$$F_{1,t}^G = \frac{1}{n} \sum_{\varphi \in \widehat{G}} \tilde{\varphi}(t^{-1}) k_\varphi ,$$

where $\tilde{\varphi}$ is the Brauer character of k_φ .

Proof : Indeed for $s, t \in G$

$$\tau_{\mathbf{1},t}^G \left(\frac{1}{n} \sum_{\varphi \in \widehat{G}} \tilde{\varphi}(s^{-1}) k_{\varphi} \right) = \frac{1}{n} \sum_{\varphi \in \widehat{G}} \tilde{\varphi}(s^{-1}) \tilde{\varphi}(t) = \delta_{s,t} ,$$

where $\delta_{s,t}$ is the Kronecker symbol. □

4.9. The case $G = \langle Ps \rangle$. Suppose now more generally that $G = \langle Ps \rangle$, where P is a normal Sylow p -subgroup of G and s is a p' -element. In this case, by Proposition 3.1, the restriction of $F_{P,s}^G$ to any proper subgroup of G is equal to zero. Moreover, since $N_G(P, t) = G$ for any $t \in G/P$, the conjugacy class of the pair (P, t) reduces to $\{(P, t)\}$.

4.10. Lemma. *Suppose $G = \langle Ps \rangle$, and set $E_G^G = \mathcal{L}_G(e_G^G)$. Then*

$$E_G^G = \sum_{\langle t \rangle = \langle s \rangle} F_{P,t}^G .$$

Proof : By 4.2 and by Remark 4.5, the restriction of E_G^G to any proper subgroup of G is equal to zero. Let $(Q, t) \in \mathcal{Q}_{G,p}$, such that the group $L = \langle Qt \rangle$ is a proper subgroup of G . By Proposition 3.2, there is a rational number r such that

$$F_{Q,t}^G = r \operatorname{Ind}_L^G F_{Q,t}^L .$$

It follows that

$$E_G^G \cdot F_{Q,t}^G = r \operatorname{Ind}_L^G ((\operatorname{Res}_L^G E_G^G) \cdot F_{Q,t}^L) = 0 .$$

Now E_G^G is an idempotent of $K \otimes_{\mathbb{Z}} pp_k(G)$, hence is it a sum of some of the primitive idempotents $F_{Q,t}^G$ associated to pairs (Q, t) for which $\langle Qt \rangle = G$. This condition is equivalent to $Q = P$ and $\langle t \rangle = \langle s \rangle$.

It remains to show that all these idempotents $F_{P,t}^G$ appear in the decomposition of E_G^G , i.e. equivalently that $\tau_{P,t}^G(E_G^G) = 1$ for any generator t of $\langle s \rangle$. Now by 4.6 and Remark 2.16

$$\tau_{P,t}^G(E_G^G) = \tau_{\mathbf{1},t}^{G/P}(\operatorname{Br}_P^G(E_G^G)) = \tau_{\mathbf{1},t}^{\langle s \rangle}(E_{\langle s \rangle}^{\langle s \rangle}) .$$

Now the value at t of the Brauer character of a permutation module kX is equal to the number of fixed points of t on X . By K -linearity, this gives

$$\tau_{\mathbf{1},t}^{\langle s \rangle}(E_{\langle s \rangle}^{\langle s \rangle}) = |(e_{\langle s \rangle}^{\langle s \rangle})^t| ,$$

and this is equal to 1 if t generates $\langle s \rangle$, and to 0 otherwise, as was to be shown. □

4.11. Proposition. *Let $(P, s) \in \mathcal{Q}_{G,p}$, and suppose that $G = \langle Ps \rangle$. Then*

$$F_{P,s}^G = E_G^G \cdot \text{Inf}_{G/P}^G F_{1,s}^{G/P} .$$

Proof : Set $E_s = E_G^G \cdot \text{Inf}_{G/P}^G F_{1,s}^{G/P}$. Then E_s is an idempotent of $K \otimes_{\mathbb{Z}} \text{pp}_k(G)$, as it is the product of two (commuting) idempotents. Let $(Q, t) \in \mathcal{Q}_{G,p}$. If $\langle Qt \rangle \neq G$, then $\tau_{Q,t}^G(E_G^G) = 0$ by Lemma 4.10, thus $\tau_{Q,t}^G(E_s) = 0$. And if $\langle Qt \rangle = G$, then $Q = P$ and $\langle t \rangle = \langle s \rangle$. In this case

$$\tau_{Q,t}^G(E_s) = \tau_{P,t}^G(E_G^G) \cdot \tau_{P,t}^G(\text{Inf}_{G/P}^G F_{1,s}^{G/P}) .$$

By Lemma 4.10, the first factor in the right hand side is equal to 1. The second factor is equal to

$$\begin{aligned} \tau_{P,t}^G(\text{Inf}_{G/P}^G F_{1,s}^{G/P}) &= \tau_{1,t}^{G/P} \text{Br}_P^G(\text{Inf}_{G/P}^G F_{1,s}^{G/P}) \\ &= \tau_{1,t}^{G/P}(F_{1,s}^{G/P}) = \delta_{t,s} , \end{aligned}$$

where $\delta_{t,s}$ is the Kronecker symbol. Hence $\tau_{P,t}^G(E_s) = \delta_{t,s}$, and this completes the proof. \square

4.12. Theorem. *Let G be a finite group, and let $(P, s) \in \mathcal{Q}_{G,p}$. Then the primitive idempotent $F_{P,s}^G$ of the p -permutation algebra $K \otimes_{\mathbb{Z}} \text{pp}_k(G)$ is given by the following formula :*

$$F_{P,s}^G = \frac{1}{|P||s||C_{\overline{N}_G(P)}(s)|} \sum_{\substack{\varphi \in \langle \overline{s} \rangle \\ L \leq \langle Ps \rangle \\ PL = \langle Ps \rangle}} \tilde{\varphi}(s^{-1}) |L| \mu(L, \langle Ps \rangle) \text{Ind}_L^G k_{L,\varphi}^{\langle Ps \rangle} ,$$

where $k_{L,\varphi}^{\langle Ps \rangle} = \text{Res}_L^{\langle Ps \rangle} \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} k_{\varphi}$.

Proof : By Corollary 3.4, and Proposition 4.11

$$F_{P,s}^G = \frac{|s|}{|C_{\overline{N}_G(P)}(s)|} \text{Ind}_{\langle Ps \rangle}^G (E_{\langle Ps \rangle}^{\langle Ps \rangle} \cdot \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} F_{1,s}^{\langle s \rangle}) .$$

By Equation 4.4, this gives

$$F_{P,s}^G = \frac{|s|}{|C_{\overline{N}_G(P)}(s)|} \text{Ind}_{\langle Ps \rangle}^G \frac{1}{|P||s|} \sum_{L \leq \langle Ps \rangle} |L| \mu(L, \langle Ps \rangle) \text{Ind}_L^{\langle Ps \rangle} k \cdot \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} F_{1,s}^{\langle s \rangle} .$$

Moreover for each $L \leq \langle Ps \rangle$

$$\begin{aligned} \text{Ind}_L^{\langle Ps \rangle} k \cdot \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} F_{1,s}^{\langle s \rangle} &\cong \text{Ind}_L^{\langle Ps \rangle} (\text{Res}_L^{\langle Ps \rangle} \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} F_{1,s}^{\langle s \rangle}) \\ &\cong \text{Ind}_L^{\langle Ps \rangle} \text{Inf}_{L/L \cap P}^L \text{Iso}_{LP/P}^{L/L \cap P} \text{Res}_{LP/P}^{\langle s \rangle} F_{1,s}^{\langle s \rangle} . \end{aligned}$$

Here we have used the fact that if L and P are subgroups of a group H , with $P \trianglelefteq H$, then there is an isomorphism of functors

$$\text{Res}_L^H \circ \text{Inf}_{H/P}^H \cong \text{Inf}_{L/L \cap P}^L \circ \text{Iso}_{LP/P}^{L/L \cap P} \circ \text{Res}_{LP/P}^{H/P} ,$$

which follows from the isomorphism of $(L, H/P)$ -bisets

$$H \times_H (H/P) \cong {}_L(H/P)_{H/P} \cong (L/L \cap P) \times_{L/L \cap P} (LP/P) \times_{LP/P} (H/P) .$$

Now Proposition 3.1 implies that $\text{Res}_{LP/P}^{\langle s \rangle} F_{1,s}^{\langle s \rangle} = 0$ if $LP/P \neq \langle s \rangle$, i.e. equivalently if $PL \neq \langle Ps \rangle$. It follows that

$$F_{P,s}^G = \frac{1}{|P| |C_{\overline{N}_G(P)}(s)|} \sum_{\substack{L \leq \langle Ps \rangle \\ PL = \langle Ps \rangle}} |L| \mu(L, \langle Ps \rangle) \text{Ind}_L^G (\text{Res}_L^{\langle Ps \rangle} \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} F_{1,s}^{\langle s \rangle}) .$$

By Lemma 4.8, this gives

$$F_{P,s}^G = \frac{1}{|P| |s| |C_{\overline{N}_G(P)}(s)|} \sum_{\substack{\varphi \in \widehat{\langle s \rangle} \\ L \leq \langle Ps \rangle \\ PL = \langle Ps \rangle}} \tilde{\varphi}(s^{-1}) |L| \mu(L, \langle Ps \rangle) \text{Ind}_L^G k_{L,\varphi}^{\langle Ps \rangle} ,$$

where $k_{L,\varphi}^{\langle Ps \rangle} = \text{Res}_L^{\langle Ps \rangle} \text{Inf}_{\langle s \rangle}^{\langle Ps \rangle} k_\varphi$, as was to be shown. \square

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Serge Bouc, CNRS-LAMFA, Université de Picardie - Jules Verne,
33, rue St Leu, F-80039 Amiens Cedex 1, France.
`serge.bouc@u-picardie.fr`

Jacques Thévenaz, Institut de Géométrie, Algèbre et Topologie,
EPFL, Bâtiment BCH, CH-1015 Lausanne, Switzerland.
`Jacques.Thevenaz@epfl.ch`