$L_{q,p}$-Cohomology of Riemannian Manifolds and Simplicial Complexes of Bounded Geometry

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Abstract

The $L_{q,p}$-cohomology of a Riemannian manifold (M, g) is defined to be the quotient of closed L_p-forms, modulo the exact forms which are derivatives of L_q-forms, where the measure considered comes from the Riemannian structure.

The $L_{q,p}$-cohomology of a simplicial complex K is defined to be the quotient of p-summable cocycles of K, modulo the coboundaries of q-summable cocycles.

We introduce those two notions together with a variant for coarse cohomology on graphs, and we establish their main properties. We define the categories we work on, i.e. manifolds and simplicial complexes of bounded geometry, and we show how cohomology classes can be represented by smooth forms.

The first result of the thesis is a de Rham type theorem: we prove that for an orientable, complete and (non compact) Riemannian manifold with bounded geometry (M, g) together with a triangulation K with bounded geometry, the $L_{q,p}$-cohomology of the manifold coincides with the $L_{q,p}$-cohomology of the triangulation. This is a generalization of an earlier result from Gol'dshteĭn, Kuz'minov and Shvedov.

The second result is a quasi-isometry invariance one: we prove how this de Rham type isomorphism together with a result in coarse cohomology induces the fact that the $L_{q,p}$-cohomology of a Riemannian manifold depends only on its quasi-invariance class. This result was proved in the $q = p$ case by Elek.

We establish some consequences, such as monocity results for $L_{q,p}$-cohomology, and the quasi-isometry invariance of the existence of Sobolev inequalities.

Keywords : $L_{q,p}$-Cohomology, bounded geometry, quasi-isometry invariance, de Rham theorem, coarse cohomology.

Résumé

La $L_{q,p}$-cohomologie d'une variété Riemannienne (M, g) est le quotient des formes L_p fermées, modulo les formes exactes qui sont dérivée d'une forme L_q, la mesure provenant de la métrique Riemannienne g.

La $L_{q,p}$-cohomologie d'un complexe simplicial K est le quotient des cocycles p-sommables de K, modulo les cobords des cocycles q-sommables.

On introduit ces deux notions avec une variante pour la cohomologie grossière des graphes, et on établit leurs propriétés principales. On définit les categories sur lesquelles on travaille (les variétés et complexes simpliciaux à géométrie bornée), et on montre comment les formes L^p peuvent être représentées par des formes lisses.

Le premier résultat de la thèse est un analogue au théorème de de Rham : on montre que pour une variété Riemannienne complete, orientable, non nécessairement compacte à géometrie bornée (M, g) munie d'une triangulation à géometrie bornée, la $L_{q,p}$-cohomologie de la variété coïncide avec la $L_{q,p}$-cohomologie de la triangulation. Ceci est une généralisation d'un précédent résulat dû à Gol'dshteĭn, Kuz'minov et Shvedov.

Le second résultat est un résulat d'invariance sous quasi-isométries : on montre comment ce théorème de de Rham, avec un résultat en cohomologie $L_{q,p}$ des graphes, induit le fait que la $L_{q,p}$-cohomologie d'une variété Riemannienne ne dépend que de sa classe de cohomologie. Dans le cas $q = p$, ce résultat a été prouvé par G. Elek.

On établit quelques conséquences, comme des résultats de monotonie pour la $L_{q,p}$-cohomologie en volume infini, et l'invariance sous quasi-isométries des inégalités de Sobolev pour les formes différentielles.

Mots-clés : Cohomologie $L_{q,p}$, géométrie bornée, invariance sous quasi-isométries, théorème de de Rham, cohomologie grossière.

Contents

Introduction

For a smooth manifold, the de Rham cohomology provides invariants which carry some information on the topology of the manifold. However it will not give any indication on the metric aspect. It is completely non-sensitive to the Riemannian structure.

In order to provide invariants sensitive to the geometry of a manifold, one has to restrict to certain classes of forms whose definition take metric into account. The resulting theory is the $L_{q,p}$-cohomology.

An introductory problem : Sobolev inequalities for differential forms

This thesis is about $L_{q,p}$-cohomology, however as an introduction we begin by asking a question on Sobolev inequalities for differential forms:

Question: : *Suppose that M and M' are quasi-isometric manifolds, and suppose that M satisfies a Sobolev inequality for differential forms. Is it the case for M as well ?*

Let us state what this question exactly means. First, we formulate a classical result for compact manifolds. Let Z^k be the vector space of closed forms.

Proposition: *Let M be a compact n-manifold, $k = 1, \dots, n$ and $1 < p, q < \infty$. There exists a constant $C > 0$ such that for any differential form ω of degree k with coefficients in L^q,*

$$\inf_{\theta \in Z^k} \|\omega - \theta\|_{L^q} \leq C \|d\omega\|_{L^p}$$

if and only if

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}.$$

We are interested in generalizing this type of inequality to the non-compact setting. Let us give a definition for a form to be L^p on any orientable manifold, provided it is given a Riemannian structure. There is a natural norm $|\cdot|_x$ on the exterior algebra $\Lambda^k T_x^* M$ coming from the Riemannian metric (it will be defined in the first chapter). The set of L^p forms on M is simply the completion of compactly supported smooth forms on M with respect to the norm

$$\|\omega\|_p = \left(\int_M |\omega(x)|_x^p d\,\mathrm{vol}_g(x) \right)^{\frac{1}{p}}.$$

We can now give a formulation for Sobolev inequalities on a Riemannian manifold:

Definition: We say that an Riemannian manifold (M,g) *satisfies a Sobolev inequality* if there exists a constant $0 < C < \infty$ such that for any L^q differential k-form ω with derivative in L^p, one has

$$\inf_{\theta \in Z^k} \|\omega - \theta\|_{L^q} \leq C \|d\omega\|_{L^p}.$$

Let $\mathrm{Sob}^k_{q,p}(M)$ denote the smallest constant C satisfying this inequality.

To understand our question, we also need the notion of *quasi-isometry*: it is a map which preserves distances "at large scales". More precisely, a map $f : X \to Y$ between two metric spaces is a quasi-isometry if

(i) There exists constants $C > 1, L > 0$ such that for any $x, x') \in X$, one has

$$C^{-1} \cdot d(x,x') - L \leq d(f(x), f(x') \leq C \cdot d(x,x') + L.$$

(ii) There exists a constant ε such that any point $y \in Y$ lies in a ε neighborhood of the image $f(X)$.

Whenever a quasi-isometry exists between two spaces, one says that they are *quasi-isometric.*

Finally our question has meaning, and an answer is given by the following theorem:

Theorem: *Let M, M' be two quasi-isometric orientable Riemannian manifolds, uniformly contractible, with bounded geometry. Let $n = \max\{\dim(M), \dim(M')\}$, and let q, p such that one of the following hypothesis holds:*

(A) $1 < q, p \leq \infty$ *and* $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$, *or*

(B) $1 \leq q, p \leq \infty$ *and* $0 \leq \frac{1}{p} - \frac{1}{q} < \frac{1}{n}$

Then for any k, one has

$$\mathrm{Sob}^k_{q,p}(M) > 0 \textit{ if and only if } \mathrm{Sob}^k_{q,p}(N) > 0.$$

Being uniformly contractible and having bounded geometry are two hypothesis on the topology and the geometry of the manifolds. Being of bounded geometry essentially means that there exist uniform bounds on geometric quantities such as the injectivity radius and derivatives of the curvature (natural exemples are given by compact manifolds, their universal coverings, Lie groups or more generally homogeneous spaces). Being uniformly contractible means that you can retract any ball of fixed radius r onto a point, within a ball of radius R depending only on r.

We will define a little below the $L_{q,p}$-cohomology of a Riemannian manifold, and show how it can help to prove this result. Before this, let us give a nice corollary of this theorem. A result due to Federer-Fleming and Maz'ya (see [Kan86]) says that

$$\left(\mathrm{Sob}^0_{q,1}\right)^{-1} = I_{\frac{q}{1-q}}(M)$$

where the isoperimetric constant $I_m(M)$ of a manifold is defined as

$$I_m(M) = \inf_{\Omega} \frac{\mathrm{area}\partial\Omega}{(\mathrm{Vol}(\Omega))^{\frac{m-1}{m}}}$$

m begin an arbitrary constant. The infimum is taken over all bounded domains Ω in M. The classical *isoperimetric inequality* for a manifold is formulated $I_m(M) > 0$, and such an inequality becomes a quasi-isometry invariant under our hypothesis (this is a well known result from Kanai).

Cohomological formulation

To prove the theorem stated above, we shall use a cohomological interpretation of Sobolev inequalities. Let $Z^k_p(M)$ be the Banach space of closed k-forms which are L^p. Let also $B^k_{q,p}(M)$ denote the space of exact L^p k-forms which are derivatives of L^q forms, and let $\overline{B}^k_{q,p}(M)$ denotes its closure. The $L_{q,p}$ *cohomology group of degree* k of M is the vector space

$$H^k_{q,p}(M) = Z^k_p(M)/B^k_{q,p}(M).$$

One also defines the *reduced cohomology* :

$$\overline{H}^k_{q,p}(M) = Z^k_p(M)/\overline{B}^k_{q,p}(M).$$

The *torsion* is the quotient of those two spaces:

$$T^k_{q,p}(M) = H^k_{q,p}(M)/\overline{H}^k_{q,p}(M).$$

In [GT06], Gol'dshteĭn and Troyanov establish the following link between $L_{q,p}$-cohomology and Sobolev inequalities: Let $1 \leq p < \infty, 1 < q < \infty$. Then $T^k_{q,p}(M) = 0$ if, and only if $\mathrm{Sob}^k_{p,q}(M) > 0$.

Consequently, to obtain the quasi-isometry invariance of Sobolev inequalities, we can simply prove that both $L_{q,p}$ cohomology and its reduced counterpart are quasi-isometry invariants.

Quasi-isometry invariance for $L_{q,p}$-cohomology

Two approaches can be found in the literature for the quasi-isometric invariance of L_p-cohomology. The first approach is to be found in a 1995 preprint of P. Pansu revised in 2004 (see [Pan]) and the other one in a 1998 short paper by G. Elek (see [Ele98]). The approach by Pansu is based on an L_p variant of the Alexander-Spanier cohomology adapted to metric measure space. In this thesis we choose to follow Elek's approach which is based on an L_p variant of the John Roe coarse cohomology.

The quasi-isometry invariance for $L_{q,p}$-cohomology will be achieved in four steps:

1. First, we introduce a notion of simplicial complex with bounded geometry, together with a natural notion of simplicial $L_{q,p}$-cohomology (both reduced and non-reduced). We then prove a de Rham isomorphism theorem: if a manifold is triangulated by such a simplicial complex, then the simplicial $L_{q,p}$-cohomology of the triangulation coincides with the $L_{q,p}$-cohomology of the manifold.

2. Next, we introduce a notion of $L_{q,p}$-cohomology for graphs, called the *coarse cohomology* and prove that it is a quasi-isometry invariant.

3. Then we show that the simplicial cohomology of a simplicial complex with bounded geometry which is uniformly contractible coincides with the coarse $L_{q,p}$-cohomology of its 1-skeleton, and this gives us the quasi-isometry invariance for simplicial $L_{q,p}$-cohomology of uniformly contractible graphs with bounded geometry.

4. Finally, we obtain the quasi-isometry invariance for $L_{q,p}$-cohomologies of manifolds: if M and N are quasi-isometric manifolds satisfying the hypothesis, then we triangulate both of them with uniformly contractible simplicial complexes.of bounded geometry. It suffices to use points (1) and (3).

Let us detail those steps.

Step 1: a de Rham theorem for $L_{q,p}$-cohomology

A finite dimensional simplicial complex K, realized in some euclidean space $\mathbb{R}^N$, has *bounded geometry* if it each vertex admits a uniformly bounded number of neighbors, and if the volumes of its faces are uniformly bounded above and below. Let $C_k(K)$ denote the vector space of k-chains, and $C^k(K) = C_k(K)^*$ the vector space of k-cochains. A cochain $c \in C^k(K)$ is L^p if it is p-summable in the following sense:

$$\sum_{\Delta^k \in K} |c(\Delta)|^p < \infty.$$

Here the sum runs through the k-simplices Δ^k of K. In a way similar to what we did for manifolds, we define the simplicial $L_{q,p}$-cohomology as follows : $H^k_{q,p}(K)$ is the quotient of closed L^p-cochains modulo the exact L^p-cochains which are coboundaries of L^q cochains.

The reduced simplicial $L_{q,p}$-cohomology of K is the quotient $\overline{H}^k_{q,p}(K)$ of closed L^p-cochains modulo the closure of exact L^p-cochains which are coboundaries of L^q cochains.

We then have the following theorem, which we will prove in chapter 2 (result 2.13):

de Rham isomorphism theorem: *Let (M, g) be a non-compact, orientable, complete and connected Riemannian manifold, and assume that M admits a bounded geometry triangulation $\tau : |K| \to M$. Let q, p such that one of the following hypothesis holds:*

(1) $1 < q, p < \infty$ *and* $0 \le \frac{1}{q} - \frac{1}{p} \le \frac{1}{n}$, *or*

(2) $1 \le q, p < \infty$ *and* $0 \le \frac{1}{q} - \frac{1}{p} < \frac{1}{n}$.

Then for any k there are vector space isomorphisms

$$H^k_{q,p}(M) = H^k_{q,p}(K) \qquad \text{and} \qquad \overline{H}^k_{q,p}(M) = \overline{H}^k_{q,p}(K)$$

and the latter is continuous.

To prove this theorem, we will need an intermediary object: The *Sullivan complex.*

The Sullivan Complex First, let us introduce the notion of *flat forms* on a manifold: it is a form which is L^∞, with exterior derivative in L^∞.

If K is a simplicial complex triangulating a manifold, a *Sullivan k-form* of K is the data, for each simplex $\Delta \in K$, of a flat k-form ω_Δ satisfying the following restriction condition: if Δ' is a face of Δ, then $\omega_\Delta|_{\Delta'} = \omega_{\Delta'}$. The space of such forms is the Sullivan space $S^k(K)$, and with the differential d one has a cochain complex $S^\bullet(K)$.

The Sullivan complex admits a $L_{q,p}$ version $S^\bullet_{q,p}(K)$: it is the set of Sullivan forms for which the following norm is finite:

$$\|\omega\|_{S^\bullet_{q,p}(K)} = \left(\sum_{\Delta \in K} \operatorname{esssup} |\omega_\Delta|^q\right)^{\frac{1}{q}} + \left(\sum_{\Delta \in K} \operatorname{esssup} |d\omega_\Delta|^p\right)^{\frac{1}{p}}.$$

The proof of the de Rham theorem rests on the existence of isomorphisms in cohomology as in the following pattern:

$$H^k_{q,p}(M) \underset{\iota}{\overset{R^M}{\rightleftarrows}} H^k\left(S^\bullet_{q,p}(K)\right) \underset{w}{\overset{I}{\rightleftarrows}} H^k_{q,p}(K) \qquad \overline{H}^k_{q,p}(M) \underset{\iota}{\overset{R^M}{\rightleftarrows}} \overline{H}^k\left(S^\bullet_{q,p}(K)\right) \underset{w}{\overset{I}{\rightleftarrows}} \overline{H}^k_{q,p}(K)$$

Defining those isomorphisms is the object of chapter 2. At a glance:

- R^M is a regularisation operator; as the name suggests, it allows to obtain a smooth form out of a non-smooth one.

- ι is an inclusion operator.
- w is called the *Whitney transformation*. It associates a differential form to a simplicial cochain.
- I is the classical *integration morphism*: essentially, it associate a simplicial cochain to a differential form.

Step 2: coarse $L_{q,p}$-cohomology and quasi-isometry invariance

The second part of the proof relies on a notion of $L_{q,p}$-cohomology for graphs. First, let us define the *penumbra* of a graph G of bounded geometry (i.e. a locally finite graph, whose vertex have a uniformly bounded number of neighors). For $k \in \mathbb{N}$ and $R > 0$, the *penumbra* of radius R and order k of G is the set

$$\mathrm{Pen}(G, R) = \left\{ (x_0, \dots, x_k) \in V_G^{k+1} \;\middle|\; d(x_i, x_j) \leq R \right\}.$$

For, $1 \leq p < \infty$, we define the L^p cochains by

$$CX_p^k(G) = \left\{ \alpha : V_G^{k+1} \to \mathbb{R} \;\middle|\; \sum_{(x_o,\dots,x_k)\in \mathrm{Pen}(G,R)} |\alpha(x_0, \dots, x_k)|^p < \infty \quad \text{for any } R > 0 \right\}.$$

The differential map is defined by

$$d\alpha(x_0, \dots, x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i \alpha(x_0, \dots, \widehat{x_i}, \dots, x_{k+1})$$

and as usual, the $L_{q,p}$-coarse cohomology space of degree k of G is the quotient $HX_{q,p}^k(G)$ of closed L^p cochains modulo the exact L^p cochains which are derivatives of L^q cochains. The $L_{q,p}$-coarse reduced cohomology space of degree k of G is the quotient $\overline{HX}_{q,p}^k(G)$ of closed L^p cochains modulo the closure of exact L^p cochains which are derivatives of L^q cochains

In chapter 3, we will prove the following result (results 3.5 and 3.4):

Proposition : *Let G and G' be two quasi-isometric graphs, and $q \geq p$. Then $HX_{q,p}^k(G) = HX_{q,p}^k(G')$ and $\overline{HX}_{q,p}^k(G) = \overline{HX}_{q,p}^k(G')$*

Step 3 : relating coarse and simplicial cohomology

The work is almost done. In chapter 3, we also prove the following result (results 3.14 and 3.15):

Proposition : *If K is a uniformly contractible bounded geometry simplicial complex, and if G_K is its 1-skeleton, then for any integer k and any pair q, p with $q \geq p$, one has*

$$H_{q,p}^k(K) = HX_{q,p}^k(G_K) \text{ and } \overline{H}_{q,p}^k(K) = \overline{HX}_{q,p}^k(G_K)$$

.

Step 4 : combine the preceeding steps

Those points combined with the $L_{q,p}$-de Rham theorem will achieve the proof of the quasi-isometry invariance for $L_{q,p}$-cohomology (result 3.17):

Proposition : *Let M, N be two uniformly contractible manifolds with bounded geometry, and suppose that M, N are quasi-isometric. Let q, p such that one of the following hypothesis holds:*

(1) $1 < q, p < \infty$ *and* $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$, *or*

(2) $1 \leq q, p < \infty$ *and* $0 \leq \frac{1}{p} - \frac{1}{q} < \frac{1}{n}$.

Then

$$H^k_{q,p}(M) = H^k_{q,p}(N) \qquad \text{and} \qquad \overline{H}^k_{q,p}(M) = \overline{H}^k_{q,p}(N)$$

and the latter is continuous.

A brief historical viewpoint

In the middle 70's, Atiyah (see [Ati76]) and Dodziuk (see [Dod74]) introduced, for manifolds, a variant of the de Rham cohomology, by adding a L^2-condition on considered forms. The result was the L^2-cohomology, together with a link with a combinatorial Hodge theory introduced earlier by Eckman. The L^2-forms of degree k of Dodziuk are the completion of the usual smooth k-forms with respect to the inner product given by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta$$

where $\star$ is the star-Hodge operator. In 1981 (See [Dod81]), Dodziuk generalizes the notion of L^2-forms to manifolds with bounded geometry, and proves that harmonic L^2-forms on such a manifold are exactly the classes of square summable cochains of a good triangulation.

In the late eighties (see [GKS88]), V. Gold'Shtein, V. Kuz'Minov and I. Shvedov considered, for manifolds with bounded geometry, the space of forms which are L^p, namely the completion of compactly supported forms with respect to the norm

$$\|\alpha\|_{L^p} = \left(\int_M |\alpha_x|^p_x d\,\mathrm{vol}_g(x) \right)^{\frac{1}{p}}.$$

They proved that the resulting cohomology classes coincide with the cohomology classes of p-summable simplicial cochains of a good triangulation, which is a de Rham isomorphism theorem. Their methods are based on a different approach from Dodziuk's one.

In 1998 (see [Ele98]), G. Elek defines for a graph a L^p-version of the coarse cohomology introduced by J. Roe in [Roe93]. He shows that it is a quasi-isometry invariant, and shows that the simplicial L_p-cohomology of a bounded geometry simplicial complex equals the L_p-coarse-cohomology of its 0-skeleton. This, together with the isomorphism theorem of V. Gol'dshteĭn, V. Kuz'Minov and I. Shvedov, show that the L^p-cohomology of a Riemannian manifold with bounded geometry and convenient topology is a quasi-isometry invariant.

Here is a list of persons who also discovered nice results in this field: P. Pansu, Xiang Dong Li, S. Zucker, M. Gromov, A. Kopylov.

Chapter 1

Preliminary notions

In this chapter, we introduce the principal objects of the thesis: the $L_{q,p}$ cohomology of a Riemannian manifold, as well as the L_π-cohomology of a simplicial complex. We also prove an extension of the de Rham's regularization theorem, which states in particular that the classes of cohomology can be represented by smooth forms. We then introduce the $L_{q,p}$ and L_π cohomologies of a simplicial complex. We finish by a discussion of manifolds and simplicial complexes of bounded geometry.

The $L_{q,p}$-cohomology of a Riemannian manifold

In this section, we define the $L_{q,p}$-cohomology of a Riemannian manifold. First, we begin by defining a norm for differential forms in each point of the manifold. A form will belong to L^p if its norm is L^p in the usual sense, i.e. as a real valued function.

A norm for differential forms In the sequel and throughout all this text, (M, g) is an orientable, connected and complete Riemannian manifold, x is a point of M, and we denote by $\Lambda^k T_x M$ the vector space of multilinear alternate maps

$$\alpha_x : T_x^* M \times \ldots \times T_x^* M \to \mathbb{R}.$$

Recall that an exterior form of degree k on M is a section of the k-th cotangent bundle

$$\Lambda^k M = \coprod_{x \in M} \Lambda^k T_x^* M \overset{\pi}{\mapsto} M.$$

In practice, for each point $x \in M$, one has a multilinear map $\alpha_x \in \Lambda^k T_x M$. If $(e_1, \ldots, e_n)$ is a basis of $T_x M$ and $(\varepsilon_1, \ldots, \varepsilon_n)$ is a dual basis, one can write

$$\alpha_x = \sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1 \ldots i_k .x} \varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k}$$

where $a_{i_1 \ldots i_k} = \alpha_x(e_{i_1}, \ldots, e_{i_k})$.

In particular, if $x^1, \ldots, x^n$ are local coordinates on a open subset U of M, we have a basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ of T_xM for each $x \in U$, with dual basis $dx^1, \ldots, dx^n$. On the whole open set U, one can write

$$\alpha = \sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

In this formula, $a_{i_1 \ldots i_k}$ is a real-valued function defined by $a_{i_1 \ldots i_k}(x) = a_{i_1 \ldots i_k \cdot x}$.

We will consider the set of *measurable forms*, i.e. the forms for which there exists a coordinate system for which $a_{i_1 \ldots i_k}$ is measurable (and in this case, it is measurable in any coordinate system). We will not require our forms to be smooth.

Since one has a scalar product on T_xM for each $x \in M$ (namely the Riemannian metric on M), we can define a norm for each $\alpha(x) = \alpha_x$.

Let us denote by $\langle u, v \rangle_x = g_x(u, v)$ the scalar product on T_xM, and let us chose an orthonormal basis $(e_1, \ldots, e_n)$ of T_xM with dual basis $\varepsilon_1, \ldots, \varepsilon_n$. Such a basis always exists: one simply has to apply Gram-Schmidt's process to a basis given by local coordinates. Let us define a map $G : \Lambda^k T_xM \times \Lambda^k T_xM \to \mathbb{R}$ by the following formula: for $\alpha_x = \sum a_{i_1 \ldots i_k, x} \varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k}$ and $\beta_x = \sum a_{j_1 \ldots j_k, x} \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k} \in \Lambda_k T_xM$, we set

$$G(\alpha_x, \beta_x) = \sum_{i_1 \ldots i_k} \alpha_{i_1 \ldots i_k, x} \beta_{i_1 \ldots, k, x}$$

The verification of the following lemma is straightforward:

Lemma 1.1 *The map $G : \Lambda_k T_xM \times \Lambda_k T_xM \to \mathbb{R}$ is symmetric and positive-definite, and hence is a scalar product. It does not depend on the choice of the particular basis $(e_1, \ldots, e_n)$ among the orthonormal ones, and the basis $(\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k})$ is orthonormal for G.*

□

One can give intrinsic definitions as well: for $\theta^1 \wedge \ldots \wedge \theta^k, \eta^1 \wedge \ldots \wedge \eta^k \in \Lambda_k T_x^*M$, we have

$$G(\theta^1 \wedge \ldots \wedge \theta^k, \eta^1 \wedge \ldots \wedge \eta^k) = \det \begin{pmatrix} g(\theta^1, \eta^1) & \ldots & g(\theta^1, \eta^n) \\ \vdots & \ddots & \vdots \\ g(\theta^n, \eta^1) & \ldots & g(\theta^n, \eta^n) \end{pmatrix}.$$

Similarly, G can be defined by the relation

$$\alpha \wedge \star\beta = G(\alpha, \beta) \star 1$$

where $\star$ is the star-Hodge operator.

Let $|\cdot|_x$ denote the norm induced by G on $\Lambda_k T_xM$, and $d\,\mathrm{vol}_g(x)$ denote the Riemannian measure on M. Now that we have a norm for each α_x, we can ask the function $x \mapsto |\alpha(x)|_x$

to be integrable: this will be our notion for L^p-forms. Let $L^1_{\mathrm{loc}}(M,\Lambda^k)$ be the set of forms on M with *locally integrable norm*: for any compact $K \subset M$, one has

$$\int_K |\alpha(x)|_x d\,\mathrm{vol}_g(x) < \infty.$$

Definition (L^p forms) : Let (M,g) be a Riemannian manifold of dimension n, and $1 \le p < \infty$. A form $\alpha \in L^1_{\mathrm{loc}}(M,\Lambda^k)$ is said to be L^p if the function $x \mapsto |\alpha(x)|_x$ belongs to $L^p(M)$ in the usual sense, i.e.

$$\int_M |\alpha(x)|_x^p \, d\,\mathrm{vol}_g(x) < \infty.$$

Let $L^p(M,\Lambda^k)$ be the Banach space of L^p-forms on M, together with the norm

$$\|\alpha\|_p = \left(\int_M |\alpha(x)|_x^p \, d\,\mathrm{vol}_g(x)\right)^{\frac{1}{p}}.$$

Let us also define

$$L^\infty(M,\Lambda^k) = \left\{\alpha \in L^1_{\mathrm{loc}}(M,\Lambda^k) \middle| \operatorname{esssup} \|\alpha\|_x < \infty\right\}.$$

Let us also introduce two more notations:

- $C^\infty(M,\Lambda^k) \subset L^1_{\mathrm{loc}}(M,\Lambda^k)$ is the space of smooth forms of degree k on M;
- $C^\infty_c(M,\Lambda^k) \subset C^\infty(M,\Lambda^k)$ is the space of compactly supported smooth forms of degree k on M.

Since the forms that we consider are not smooth, their exterior derivative in the usual sense does not necessarily exist. However, there still is a weak sense (the sense of currents):

Definition (Weak derivative) : Let $\alpha \in L^1_{\mathrm{loc}}(M,\Lambda^k)$ be a locally integrable form on an orientable Riemannian manifold (M,g) without boundary. We say that $\theta \in L^1_{loc}(M,\Lambda^{k+1})$ is a *weak derivative* of α if for any compactly supported smooth $(n-k-1)$-form $\omega \in C^\infty_c(M,\Lambda^{n-k-1})$, the following identity holds:

$$\int_M \theta \wedge \omega = (-1)^{k+1} \int_M \alpha \wedge d\omega.$$

Lemma 1.2 *Let $\alpha \in C^\infty_c(M,\Lambda^k)$. Then the usual derivative $d\alpha \in C^\infty(M,\Lambda^{k+1})$ of ω is a weak derivative.*

Proof : Let us recall Stoke's theorem: for any smooth form η of degree $(n-1)$ with compact support on a n-manifold M, one has

$$\int_M d\eta = \int_{\partial M} \eta.$$

In the case where M has no boundary, one has

$$\int_M d\eta = 0.$$

Let us apply this result to the particular form $\alpha \wedge \omega$, where ω is an arbitrary smooth form of degree $(n-k-1)$ with compact support on M. One has in this case

$$\int_M d(\alpha \wedge \omega) = 0.$$

By Leibniz's formula, one has $d(\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^k \alpha \wedge d\omega$. This means that

$$\int_M \alpha \wedge d\omega = (-1)^{k+1} \int_M d\alpha \wedge \omega.$$

That is, $d\alpha$ is a weak derivative of α. □

Remark 1.1 If $\alpha \in L^1_{\text{loc}}(M, \Lambda^k)$ admits a weak derivative, it is unique up to a set of measure zero.

Proof : Suppose that θ and ζ are weak derivatives of a form α of degree k. Then for any smooth form ω of degree $(n-k-1)$ with compact support, one has

$$\int_M (\theta - \zeta) \wedge \omega = 0.$$

But any continuous form (i.e. a form with continuous coefficients) ϕ with compact support can be approximated by a smooth form ω of same degree, with compact support arbitrarily close to the support of ϕ. In particular, this means that for any continuous form ϕ of degree $(n-k-1)$ with compact support, one has

$$\int_M (\theta - \zeta) \wedge \phi = 0.$$

Moreover, any measurable form is the limit (for almost everywhere convergence) of a sequence of continuous forms. One can therefore conclude that for any measurable and bounded form with compact support χ

$$\int_M (\theta - \zeta) \wedge \chi = 0.$$

Let $\chi_a = \star \left(f_a \cdot \frac{\theta - \zeta}{|\theta - \zeta|_x} \right)$ where f_a is the indicator function of a ball $B(a)$ of radius a. The form χ_a is measurable and bounded, and therefore the preceeding result can be applied:

$$\int_{B(a)} |\theta - \zeta|_x d\,\text{vol}_g(x) = \int_M (\theta - \zeta) \wedge \chi_a = 0.$$

One concludes that $|\theta - \zeta|_x = 0$ almost everywhere since the integral of its norm is zero on any ball of radius a. □

Notation: We denote by $d\alpha$ the weak derivative of a locally integrable form when it exists.

The square-cancelation property $d \circ d = 0$ of the usual exterior derivative still holds, as well as Hölder's inequality :

Lemma 1.3 $d \circ d = 0$.

Proof : Let $\gamma = d\beta$, where $\beta = d\alpha$, both derivatives being in the weak sense. For any compactly supported smooth form ω, one has

$$\begin{aligned} \int_M \gamma \wedge \omega &= \int_M d\beta \wedge \omega \\ &= \pm \int_M \beta \wedge d\omega \\ &= \pm \int_M d\alpha \wedge d\omega \\ &= \pm \int_M \alpha \wedge (d \circ d\omega) \\ &= \pm \int_M \alpha \wedge 0 \\ &= 0. \end{aligned}$$

Hence $\gamma = 0$ almost everywhere.

□

Proposition 1.4 (Hölder's inequality) *Let* $1 \leq q, p < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. If* $\alpha \in L^p(M, \Lambda^k)$ *and* $\beta \in L^q(M, \Lambda^\ell)$*, then* $\alpha \wedge \beta \in L^1(M, \Lambda^{k+\ell})$ *and*

$$\|\alpha \wedge \beta\|_1 \leq \|\alpha\|_p \cdot \|\beta\|_q .$$

Proof : For functions $f \in L^p(M)$ and $g \in L^q(M)$, the usual Hölder's inequality tells us that $f \cdot g \in L^1(M)$ and

$$\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q .$$

Now let $\alpha \in L^p(M, \Lambda^k)$ and $\beta \in L^q(M, \Lambda^\ell)$. Let us choose an orthonormal basis $\varepsilon^1, \ldots, \varepsilon^n$ of $T^*_x M$, and let us write

$$\alpha = \sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1 \ldots i_k} \varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k}$$

$$\beta = \sum_{1 \leq j_1 < \ldots < j_\ell \leq n} b_{j_1 \ldots j_\ell} \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_\ell}$$

Let $|\alpha(x)|_x$ denote the norm of $\alpha(x) \in \Lambda^k T_x^* M$, and $|\beta(x)|$ the norm of $\beta(x) \in \Lambda^\ell T_x^* M$. One has

$$|\alpha(x)|_x^2 = \sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1 \ldots i_k}^2$$

$$|\beta(x)|_x^2 = \sum_{1 \leq j_1 < \ldots < j_\ell \leq n} b_{j_1 \ldots j_\ell}^2$$

Now

$$\alpha \wedge \beta(x) = \sum_{\substack{1 \leq i_1 < \ldots < i_k \leq n \\ 1 \leq j_1 < \ldots < j_\ell \leq n}} a_{i_1 \ldots i_k} b_{j_1 \ldots j_\ell} \varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k} \wedge \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_\ell}$$

Some of the terms of the type $\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k} \wedge \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_\ell}$ may be zero: it is the case when $i_r = j_s$ for some s, r (it is of course the case for all of them if $k + \ell > n$). The non-zero elements are of the form $\pm \varepsilon^{\mu_1} \wedge \ldots \wedge \varepsilon^{\mu_{k+l}}$, and thus

$$|\alpha \wedge \beta(x)|_x^2 \leq \sum_{\substack{1 \leq i_1 < \ldots < i_k \leq n \\ 1 \leq j'_1 < \ldots < j'_\ell \leq n}} a_{i_1 \ldots i_k}^2 b_{j_1 \ldots j_\ell}^2$$

But

$$\begin{aligned} |\alpha(x)|_x^2 \cdot |\beta(x)|_x^2 &= \left(\sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1 \ldots i_k}^2 \right) \cdot \left(\sum_{1 \leq j_1 < \ldots < j_\ell \leq n} b_{j_1 \ldots j_\ell}^2 \right) \\ &= \sum_{\substack{1 \leq i_1 < \ldots < i_k \leq n \\ 1 \leq j_1 < \ldots < j_\ell \leq n}} a_{i_1 \ldots i_k}^2 b_{j_1 \ldots j_\ell}^2 \\ &\geq |\alpha \wedge \beta(x)|_x^2 \end{aligned}$$

Thus $|\alpha(x)|_x \cdot |\beta(x)|_x \geq |\alpha \wedge \beta(x)|_x$. Since $\alpha \in L^p(M, \Lambda^k)$ and $\beta \in L^q(M, \Lambda^\ell)$, the functions $x \mapsto |\alpha(x)|_x$ and $x \mapsto |\beta(x)|_x$ are in $L^p(M)$ and $L^q(M)$ respectively. By Hölder's inequality for functions, one has

$$\begin{aligned} \int_M |\alpha(x)|_x \cdot |\beta(x)|_x d\,\mathrm{vol}_g(x) &\leq \left(\int_M |\alpha(x)|_x^p d\,\mathrm{vol}_g(x) \right)^{\frac{1}{p}} \cdot \left(\int_M |\beta(x)|_x^q d\,\mathrm{vol}_g(x) \right)^{\frac{1}{q}} \\ &= \|\alpha\|_p \cdot \|\beta\|_q \end{aligned}$$

Since $|\alpha \wedge \beta(x)|_x \leq |\alpha(x)|_x \cdot |\beta(x)|_x$, one has

$$\int_M |\alpha \wedge \beta(x)|_x d\,\mathrm{vol}_g(x) \qquad \leq \qquad \int_M |\alpha(x)|_x |\beta(x)|_x d\,\mathrm{vol}_g(x)$$
$$\leq \|\alpha\|_p \cdot \|\beta\|_q$$

This is exactly the inequality $\|\alpha \wedge \beta\|_1 \leq \|\alpha\|_p \cdot \|\beta\|_q$ and thus finishes our proof.

□

Notations: Let $1 \leq q, p \leq \infty$. We introduce the following notations:

- $Z_p^k(M) = \left\{ \alpha \in L^p(M, \Lambda^k) \middle| \, d\alpha = 0 \right\} = L^p(M, \Lambda^k) \cap \ker d$;
- $B_{q,p}^k(M) = dL^q(M, \Lambda^{k-1}) \cap L^p(M, \Lambda^k)$;
- $\overline{B}_{q,p}^k(M) = \overline{B_{q,p}^k(M)}^{L^p(M,\Lambda^k)}$.

Lemma 1.5 *$Z_p^k(M)$ is a closed subspace of $L^p(M, \Lambda^k)$ (and therefore it is a Banach space).*

Proof : Let $z \in L^p(M, \Lambda^k)$, and let $(z_i) \subset Z_p^k(M)$ be a sequence converging in the L^p norm to z. We need to prove that the weak exterior differential of z satisfies $dz = 0$.
By hypothesis, one has $dz_i = 0$ for any $i \in \mathbb{N}$. Using the definition of the weak derivative, this can be also written

$$\int_M z_i \wedge d\omega = 0 \qquad \text{for any } \omega \in C_c^\infty(M, \Lambda^{n-k-1}).$$

Using Hölder's inequality, for q such that $\frac{1}{p} + \frac{1}{q} = 1$ one has

$$\int_M |(z - z_i) \wedge d\omega|_x \, d\,\mathrm{vol}_g(x) \leq \|z - z_i\|_p \cdot \|d\omega\|_q$$

Since ω has compact support, its L^q norm is finite, and therefore $\|z - z_i\|_p \cdot \|d\omega\|_q \to 0$. Hence

$$\int_M |(z - z_i) \wedge d\omega|_x \, d\,\mathrm{vol}_g(x) \to 0.$$

But

$$\left| \int_M (z - z_i) \wedge d\omega \right| \leq \int_M |(z - z_i) \wedge d\omega|_x \, d\,\mathrm{vol}_g(x)$$

and thus

$$\int_M (z - z_i) \wedge d\omega \to 0$$

Since $\int_M z_i \wedge d\omega = 0$, this means that

$$\int_M z \wedge d\omega = 0.$$

□

Remark 1.2 Since $d \circ d = 0$, one has $B^k_{q,p}(M) \subset Z^k_p(M)$. Since $Z^k_p(M)$ is closed, one also has $\overline{B}^k_{q,p}(M) \subset Z^k_p(M)$.

Definition **($L_{q,p}$-cohomologies of a Riemannian manifold)** Let (M, g) be an orientable, connected and complete Riemannian manifold of dimension n, and $1 \leq q, p \leq \infty$. The *$L_{q,p}$-cohomology space of degree k of M* is the quotient

$$H^k_{q,p}(M) = Z^k_p(M)/B^k_{q,p}(M).$$

The *reduced $L_{q,p}$-cohomology space of degree k of M* is the quotient

$$\overline{H}^k_{q,p}(M) = Z^k_p(M)/\overline{B}^k_{q,p}(M).$$

The reduced cohomology space is always a Banach space.

Notation: If needed, we specify the metric in the notation and write $H^k_{q,p}(M, g)$ and $\overline{H}^k_{q,p}(M, g)$.

The L_π-cohomology of a Riemannian manifold

The algebraic machinery of Banach complexes can be particularly useful in the study of $L_{q,p}$ cohomology. In the aim of using it, we now introduce a way to see the (reduced)-$L_{q,p}$-cohomology space of degree k of a manifold as the (reduced) cohomology space of degree k of a particular Banach complex.
Let $1 \leq p, q \leq \infty$ be real numbers, and let us denote by

$$\Omega^k_{q,p}(M) = \left\{ \alpha \in L^q(M, \Lambda^k) \mid d\alpha \in L^p(M, \Lambda^{k+1}) \right\}$$

Equipped with the graph norm $\|\alpha\|_{q,p} = \|\alpha\|_q + \|d\alpha\|_p$, it is a Banach space.

Let $\pi = (p_0, p_1, \ldots)$ be a sequence of real numbers $1 \leq p_k \leq \infty$, and let us denote

$$\Omega^k_\pi(M) = \Omega^k_{p_k p_{k+1}}(M), \quad \|\alpha\|_{\Omega^k_\pi(M)} = \|\alpha\|_{p_k p_{k+1}} \text{ for } \alpha \in \Omega^k_\pi(M)$$

The differential $d : \Omega^k_\pi(M) \to \Omega^{k+1}_\pi(M)$ is a bounded operator, and thus we have a Banach complex:

$$\ldots \to \Omega^{k-1}_\pi(M) \to \Omega^k_\pi(M) \to \Omega^{k+1}_\pi(M) \to \ldots$$

Definition **(L_π-cohomology of a Riemannian manifold)** The cohomology of the Banach complex $(\Omega^*_\pi(M), d)$ is called the *(de Rham) L_π-cohomology of the manifold* (M, g):

$$H^k_\pi(M) = Z^k_{p_k}(M)/d\Omega^{k-1}_\pi(M).$$

The reduced cohomology of the Banach complex $(\Omega^*_\pi(M), d)$ is called the *(de Rham) reduced L_π-cohomology of the manifold* (M, g):

$$\overline{H}^k_\pi(M) = Z^k_{p_k}(M)/\overline{d\Omega^{k-1}_\pi(M)}.$$

We also define the *torsion* of M to be the torsion of that complex, i.e.

$$T_\pi^k(M) = H_\pi^k(M)/\overline{H}_\pi^k(M).$$

Remark 1.3 Let π be a sequence of real numbers $1 \leq p_k \leq \infty$ with $p_{k-1} = q$ and $p_k = p$. Then

$$H_{q,p}^k(M) = H_\pi^k(M) \text{ and } \overline{H}_{q,p}^k(M) = \overline{H}_\pi^k(M).$$

We thus have realized the $L_{q,p}$-cohomology spaces as spaces of cohomology of Banach complexes.

There exists a regularization theorem (see [GKS88], [GKS84] and [GT06]):

Proposition 1.6 (L_π regularization) *Let M be a Riemannian manifold, and suppose that M admits an atlas whose maps changes are uniformly bilipschitz. Let π be a sequence of real numbers $1 \leq p_j < \infty$.*
There exists a sequence of regularization operators $R_\varepsilon^M : L^1_{\text{loc}}(M, \Lambda^k) \to L^1_{\text{loc}}(M, \Lambda^k)$ and a sequence homotopy operators $A_\varepsilon^M : L^1_{\text{loc}}(M, \Lambda^k) \to L^1_{\text{loc}}(M, \Lambda^{k-1})$ such that

(1) For any $\omega \in L^1_{loc}(M, \Lambda^k)$, the form $R_\varepsilon^M \omega$ is smooth on M.

(2) For any $\omega \in \Omega_\pi^k(M)$, we have $dR_\varepsilon^M \omega = R_\varepsilon^M d\omega$;

(3) For any $\varepsilon > 0$, the operator $R_\varepsilon^M : \Omega_\pi^k(M) \to \Omega_\pi^k(M)$ is bounded and satisfies $\lim_{\varepsilon \to 0} \left\| R_\varepsilon^M \right\|_\pi = 1$;

(4) For any $\omega \in \Omega_\pi^k(M)$, we have $\lim_{\varepsilon \to 0} \left\| R_\varepsilon^M \omega - \omega \right\|_\pi = 0$;

(5) The operator $A_\varepsilon^M : \Omega_\pi^k(M) \to \Omega_\pi^{k-1}(M)$ is bounded in the following cases:

(i) $1 \leq p_j \leq \infty$ and $\frac{1}{p_k} - \frac{1}{p_{k-1}} < \frac{1}{n}$ or

(ii) $1 < p_j \leq \infty$ and $\frac{1}{p_k} - \frac{1}{p_{k-1}} \leq \frac{1}{n}$.

(6) We have the homotopy formula

$$\omega - R_\varepsilon^M \omega = dA_\varepsilon^M \omega + A_\varepsilon^M d\omega.$$

This theorem has a number of important corollaries. Let us see two of them.

Theorem 1.7 *Let (M, g) be a Riemannian manifold. For any choice of π, the space $C^\infty \Omega_\pi^k(M) = C^\infty(M, \Lambda) \cap \Omega_\pi^k(M, \Lambda^k)$ of smooth forms in L^{p_k} with derivative in $L^{p_{k+1}}$ is dense in $\Omega_\pi^k(M)$.*

Proof : Let $\omega \in \Omega_\pi^k(M)$. By property (4), $\lim_{\varepsilon \to 0} \left\| R_\varepsilon^M \omega - \omega \right\|_\pi = 0$, hence the sequence $(R_\varepsilon^M \omega)$ converges to ω. □

The L_π cohomology can be represented by smooth forms: let us denote by $C^\infty H_\pi^k(M)$ the cohomology space in degree k of the complex $C^\infty \Omega_\pi^*(M)$.

Theorem 1.8 *Let (M,g) be a Riemannian manifold. For any sequence of real numbers π such that*

(i) $\frac{1}{p_k} - \frac{1}{p_{k-1}} < \frac{1}{n}$ *or*

(ii) $1 < p_k \leq \infty$ *and* $\frac{1}{p_k} - \frac{1}{p_{k-1}} \leq \frac{1}{n}$.

there is a vector space isomorphism

$$C^\infty H^k_\pi(M) = H^k_\pi(M).$$

Proof : By (6), the regularization operator $R^M_\varepsilon : \Omega^k_\pi(M) \to C^\infty\Omega^k_\pi(M)$ is homotopic to the identity operator $I : C^\infty\Omega^k_\pi(M) \to C^\infty\Omega^k_\pi(M)$. Proposition A.2 allows to conclude.

□

Corollary 1.9 *If M is a compact manifold, $H^k_\pi(M) = H^k_{dR}(M)$, where $H^k_{dR}(M)$ denotes the de Rham cohomology group of degree k of M in the usual sense.*

Proof : Since M is compact, every smooth form is L^p. Hence,

$$C^\infty\Omega^k_\pi(M) = C^\infty(M,\Lambda) \cap \Omega^k_\pi(M) = C^\infty(M,\Lambda^k).$$

□

Before proving this regularization theorem, we need some auxilliary results. We begin by a result from Iwaniec and Lutoborski (see [IL93]):

Proposition 1.10 *Let U be a bounded and convex open subset of R^n, and $k = 1,\dots,n$. There exists an operator $T : L^1_{\text{loc}}(U,\Lambda^k) \to L^1_{\text{loc}}(U,\Lambda^{k-1})$ such that*

(i) $dT\omega + Td\omega = \omega$,

(ii) $|T\omega(x)| \leq C \int_U \frac{|\omega(y)|}{|y-x|^{n-1}} dy$.

Proof : We first prove the results for smooth forms. For $y \in U$, let $K_y : \Omega^k(U) \to \Omega^{k-1}(U)$ be defined by

$$(K_y\omega)(x, v_1, \dots, v_{k-1}) = \int_0^1 t^{k-1}\omega(tx + y - ty, x - y, v_1, \dots, v_{k-1}).$$

We begin by proving that K_y satisfies the following homotopy formula:

$$\omega = dK_y\omega + K_y d\omega.$$

We will then average K_y on $y \in U$ and we'll prove that it satisfies the desired estimate.

If $\omega \in \Omega^k(U)$, the exterior derivative of ω can be written

$$d\omega(x, v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^k \left[D\omega(x)v_i\right](v_0, \ldots, \widehat{v_i}, \ldots, v_k)$$

where $D\omega(x)$ is the Frechet derivative of the map $\omega(x) : \mathbb{R}^n \to \left(\Lambda^k \mathbb{R}^n\right)^*$. For any $x \in U, v_0, \ldots, v_k \in \mathbb{R}^n$, one thus has

$$d\omega(x, v_0, \ldots, v_k) = D\omega(x)(v_0) + \sum_{i=1}^{k} (-1)^k \left[D\omega(x)v_i\right](v_1, \ldots, \widehat{v_i}, \ldots, v_k)$$

Hence

$$\begin{aligned} d\omega(tx + y - ty, x - y, v_1, \ldots, v_k) &= [D\omega(tx + y - ty)(x - y)](v_1, \ldots, v_k) \\ &+ \sum_{i=1}^{k} (-1)^i [D\omega(tx + y - ty)(v_i)](x - y, v_1, \ldots, \widehat{v_i}, \ldots, v_k). \end{aligned}$$

Applying this formula to the k-form $K_y d\omega$ yields

$$\begin{aligned} (K_y d\omega)(x, v_1, \ldots, v_k) &= \int_0^1 t^k [D\omega(tx + y - ty)(x - y)](v_0, \ldots, v_k) dt \\ &+ \sum_{i=0}^{k} (-1)^i \int_0^1 t^k [D\omega(tx + y - ty)v_i](x - y, v_1, \ldots, \widehat{v_i}, \ldots, v_k) dt \end{aligned}$$

Similarly:

$$\begin{aligned} (dK_y\omega)(x, v_1, \ldots, v_k) &= \sum_{i=1}^{k} (-1)^{i-1} \int_0^1 t^k [D\omega(tx + y - ty)v_i](x - y, v_1, \ldots, \widehat{v_i}, \ldots, v_k) dt \\ &+ \sum_{i=1}^{k} (-1)^{i-1} \int_0^1 t^{k-1} \omega(tx + y - ty, v_i, v_1, \ldots, \widehat{v_i}, \ldots, v_k) dt. \end{aligned}$$

From these two results, we compute

$$\begin{aligned} (dK_y\omega + K_y d\omega)(x, v_1, \ldots, v_k) &= \int_0^1 t^k [D\omega(tx + y - ty)(x - y)](v_1, \ldots, v_k) dt \\ &+ k \int_0^1 t^{k-1} \omega(tx + y - ty, v_1, \ldots, v_k) dt \\ &= \int_0^1 \frac{d}{dt} \left[t^k \omega(tx + y - ty, v_1, \ldots, v_k) \right] dt \\ &= \omega(x, v_1, \ldots, v_k) \end{aligned}$$

Hence K_y satisfies the homotopy formula. Now let $T : \Omega^k(U) \to \Omega^{k-1}(U)$ be defined by

$$T\omega = \int_U \phi(y) K_y \omega dy$$

where ϕ is a compactly supported smooth function on U such that $\displaystyle\int_U \phi(y)dy = 1$.

It is clear that the operator T satisfies the same homotopy formula:

$$\omega = dT\omega + Td\omega.$$

Let $v = (v_0, \ldots, v_{k-1})$. By Fubini's theorem, one has for any multivector v

$$T\omega(x, v) = \int_0^1 t^{k-1} \int_U \phi(y)\omega(tx + y - ty, x - y, v)dydt.$$

Let $z = tx + y - ty$ and $t = \frac{s}{1+s}$. By change of coordinates, one obtains

$$T\omega(x, v) = \int_U \omega(z, \zeta(z, x - z), v)dz$$

where

$$\begin{aligned} \zeta(z, h) &= \int_0^\infty s^{k-1}(1 + s)^{n-k}\phi(z - sh) \cdot hds \\ &= h \sum_{l=k}^n \binom{n-k}{l-k} \int_0^\infty s^{l-1}\phi(z - sh) \cdot hds \\ &= \sum_{l=k}^n \binom{n-k}{l-k} \frac{h}{|h|^l} \int_0^\infty s^{l-1}\phi\left(z - s\frac{h}{|h|}\right) ds. \end{aligned}$$

Whenever $s > \operatorname{diam}(U)$, one has $\phi\left(z - s\dfrac{h}{|h|}\right) = 0$. Hence the integration is over the interval $0 \leq s \leq \operatorname{diam}(U)$. We thus obtain the following estimate for $|h| \leq \operatorname{diam}(U)$:

$$|\zeta(z, h)| \leq \frac{2^{n-k}(\operatorname{diam} U)^n \|\phi\|_\infty}{k|h|^{n-1}} = \text{const}.$$

From the estimate above, we find the following estimate: for any convex $F \subset U$,

$$|T\omega(x)| \leq 2^n \mu(U) \int_F \frac{|\omega(y)|}{|x - y|^{n-1}} dy.$$

This results extends to $L^1_{\text{loc}}(U, \Lambda^k)$ by approximation.

□

We will need two corollaries of this proposition.

Corollary 1.11 *Suppose that $1 \leq q, p \leq \infty$ satisfies the following*

$$\frac{1}{p} - \frac{1}{q} < \frac{1}{n}.$$

Then T maps $L^p(U, \Lambda^k)$ continuously on $L^q(U, \Lambda^{k-1})$.

Proof :

- Let us first see that for $s = \frac{q.p}{p+pq-q}$, we have $s(1-n) > -n$. One has

$$\begin{aligned} \frac{1}{s} &= \frac{p+pq-q}{pq} \\ &= \frac{1}{q} - \frac{1}{p} + 1 \\ &> 1 - \frac{1}{n} \\ &= \frac{n-1}{n}. \end{aligned}$$

- By the first point, the function $g(x) = |x|^{1-n}$ belongs to $L^s(U)$. Let $\omega \in L^p(U, \Lambda^k)$, and $f = |\omega|$. The map f is in $L^p(U)$ by hypothesis.

- Let $t = q, r = p$. One has $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{t}$. Indeed,

$$\begin{aligned} \frac{1}{r} + \frac{1}{s} &= \frac{1}{p} + \frac{p+pq-q}{pq} \\ &= \frac{p+pq}{pq} \\ &= \frac{1}{q} + 1 \\ &= \frac{1}{t} + 1. \end{aligned}$$

- Since r, s and t satisfy $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{t}$, one can use Young's inequality on convolutions (see A.11): if $f \in L^r(U), g \in L^s(U)$, then the convolution product $f \star g$ is in $L^t(U)$, and moreoever

$$\|f \star g\|_{L^t(U)} \leq \|f\|_{L^r(U)} \cdot \|g\|_{L^s(U)}\,.$$

- By proposition 1.10, one has

$$|T\omega(x)| \leq C \int_U \frac{|\omega(y)|}{|y-x|^{n-1}} dy = C|f \star g(x)|.$$

  Hence,

$$\begin{aligned}\|T\omega\|_q &\leq C\,\|f\star g\|_{L^q(U)}\\ &= C\,\|f\star g\|_{L^t(U)}\\ &\leq C\,\|f\|_{L^r(U)}\cdot\|g\|_{L^s(U)}\\ &= C\,\|f\|_{L^p(U)}\cdot\|g\|_{L^s(U)}\\ &= C\,\|\omega\|_q\,\|g\|_{L^s(U)}\end{aligned}$$

Since U is bounded, $\|g\|_{L^s(U)}$ is finite and depends only on U. Therefore, $T\omega \in L^q(U,\Lambda^k)$, and T is bounded. □

Proposition 1.10 admits another corollary, which is similar but supposes slightly different assumptions.

Corollary 1.12 *If $1 < p, q \leq \infty$ satisfy $\frac{1}{p}-\frac{1}{q} \leq \frac{1}{n}$, the conclusion of the previous corollary remains true.*

Proof : If $\frac{1}{p}-\frac{1}{q} < \frac{1}{n}$, the previous corollary can be applied. Hence, we can suppose that $\frac{1}{p}-\frac{1}{q} = \frac{1}{n}$. In this case, we can use the Hardy-Littlewood-Sobolev inequality (see A.12): if $f \in L^p(U)$ and $g(x) = \int_U f(y)(y-x)^{1-n}dy$, then $g \in L^q(U)$ and

$$\|g\|_{L^q(U)} \leq C\|f\|_{L^p(U)}.$$

Let $\omega \in L^p(U,\Lambda^k)$, and let $f = |\omega| \in L^p(U)$. If $g(x) = \displaystyle\int_U \frac{|\omega(y)|dy}{|y-x|^{n-1}}$, one has $|T\omega(x)| \leq C|g(x)|$, hence

$$\begin{aligned}\|T\omega(x)\|_{L^q} &\leq C\|g\|_{L^q(U)}\\ &\leq C\|f\|_{L^p(U)}\\ &= C\|\omega\|_{L^p}\end{aligned}$$

□

As a consequence of Proposition 1.10 and Corollaries 1.11 and 1.12, one has the following result:

Proposition 1.13 *Suppose that one of the two following hypothesis holds:*

(i) $1 \leq p, q \leq \infty$, $\frac{1}{p}-\frac{1}{q} < \frac{1}{n}$ *and* $\frac{1}{r}-\frac{1}{p} < \frac{1}{n}$ *or*

(ii) $1 < p, q \leq \infty$, $\frac{1}{p}-\frac{1}{q} \leq \frac{1}{n}$ *and* $\frac{1}{r}-\frac{1}{p} \leq \frac{1}{n}$.

Then the two following consequences hold:

(a) $dT\omega + Td\omega = \omega$ *for any* $\omega \in \Omega^k_{p,r}(U)$ *and*

(b) T *sends* $\Omega^k_{p,r}(U)$ *on* $\Omega^{k-1}_{q,p}(U)$ *continuously.*

Proof :

(a) One has $dT\omega + Td\omega = \omega$ for any $\omega \in L^1_{\mathrm{loc}}(U)$. Hence it remains true for any $\omega \in \Omega^k_{q,p}(U)$.

(b) Let $\omega \in \Omega^k_{p,r}(U)$. One thus has $\omega \in L^p(U, \Lambda^k)$ and $d\omega \in L^r(U, \Lambda^{k+1})$, and $\|\omega\|_{p,r} = \|\omega\|_p + \|d\omega\|_r$. By corollaries 1.11 and 1.12, there exists a constant $C > 0$ such that $\|T\omega\|_{L^q} \le C\|\omega\|_{L^p}$ and $\|Td\omega\|_{L^p} \le C\|d\omega\|_{L^r}$. Hence,

$$\|T\omega\|_q + \|Td\omega\|_p \le C\left(\|\omega\|_p + \|d\omega\|_r\right).$$

However,

$$\begin{aligned} \|T\omega\|_{q,p} &= \|T\omega\|_q + \|dT\omega\|_p \\ &= \|T\omega\|_q + \|Td\omega - \omega\|_p \\ &\le \|T\omega\|_q + \|Td\omega\|_p + \|\omega\|_p \end{aligned}$$

Finally, one has $\|T\omega\|_{q,p} \le (C+1)\|\omega\|_{p,r}$. Hence, T sends continuously $\Omega^k_{p,r}(U)$ to $\Omega^{k-1}_{q,p}(U)$, with norm at most $C+1$.

□

We are now going to show how we can regularize a locally integrable form, by convolution against a smooth mollifier. Let $f : (0,1) \to \mathbb{R}$ be a smooth function such that

(i) $f'(r) > 0$ for any $r \in (0,1)$,

(ii) $f(r) = \begin{cases} r & \text{if } 0 < r < \frac{1}{3} \\ e^{(r-1)^{-2}} & \text{if } \frac{2}{3} < r < 1. \end{cases}$

Let g be the inverse function of f, and $h : \mathbb{R}^n \to \mathbb{R}^n$ be the function defined by

$$h(\xi) = \begin{cases} 0 & \text{if } \xi = 0 \\ \frac{\xi}{\|\xi\|} g\left(\|\xi\|\right) & \text{if } \xi \neq 0. \end{cases}$$

The function h is a C^∞ homeomorphism from $\mathbb{R}^n$ to $\mathbb{B}^n$. For any $v \in \mathbb{R}^n$, let

$$s_v(x) = \begin{cases} h\left(h^{-1}(x) + v\right) & \text{if } \|x\| < 1, \\ x & \text{if } \|x\| \ge 1. \end{cases}$$

Lemma 1.14 *For any* $v \in \mathbb{R}^n$*, the map* $s_v : \mathbb{R}^n \to \mathbb{R}^n$ *is smooth and equal to* Id *outside* $\overline{\mathbb{B}^n}$.

Proof : For any $x \in \mathbb{R}^n$ and $\|s_v(x) - x\|$ small enough, one has $s_v(x) - x \sim \tilde{V}(x)$, where $\tilde{V}$ is a vector field equal to zero outside $\overline{\mathbb{B}^n}$, and of the form h_*V in $\mathbb{B}^n$, where V is the constant vector field equal to x. Indeed, this is evident for $\|x\| \geq 1$. For any $x \in \mathbb{B}^n$, one has

$$\begin{aligned} s_v(x) - x &= h\left(h^{-1}(x) + v\right) - x \\ &= h\left(h^{-1}(x) + V(x)\right) - x \\ &\sim h\left(h^{-1}(x)\right) + h_*V(x) - x \\ &\sim h_*V(x) \end{aligned}$$

Hence, the transformations s_v form a group of transformations of $\mathbb{R}^n$, whose infinitesimal transformations are given by vector fields which are zero outside of the unit ball, and to a transformation of a constant vector field by h inside the unit ball. We simply need to show that theses vector fields are smooth. It is clear inside and outside $\mathbb{B}^n$. Hence, one simply needs to show this for points on the border of $\mathbb{B}^n$.
Let $x = h(\xi) = \frac{\xi}{\|\xi\|} g(\|\xi\|)$ and $\rho = \|\xi\|$. Let also $r = g(\rho)$, or equivalently $\rho = f(r)$. One has $x_i = \frac{\xi_i}{\rho} g(\rho)$. On the other hand,

$$\frac{\partial g(\rho)}{\partial \xi_j} = \frac{\partial \rho}{\partial \xi_j} \frac{1}{f'(r)} = \frac{\xi_j}{f(r)} \frac{1}{f'(r)}.$$

Hence,

$$\begin{aligned} \frac{\partial x_i}{\partial \xi_j} &= \left(\frac{\delta_i^j}{\rho} - \frac{\xi_i \xi_j}{\rho^3}\right) g(\rho) + \frac{\xi_i \xi_j}{\rho^2} \frac{1}{f'(r)} \\ &= \frac{\delta_i^j r}{f(r)} - \frac{x_i x_j}{r^2} \frac{g(\rho)}{\rho} + \frac{x_i x_j}{r^2} \frac{1}{f'(r)} \\ &= \frac{\delta_i^j r}{f(r)} - \frac{x_i x_j}{r f(r)} + \frac{x_i x_j}{r^2 f'(r)} \end{aligned}$$

When $r \to 1$, the functions $\frac{1}{f(r)}$ and $\frac{1}{f'(r)}$ converge to 0, as well as all their derivatives. Hence, the expression equal to $\frac{\partial x_i}{\partial \xi_j}$ inside $\mathbb{B}^n$ and to 0 outside is smooth. □

We can now prove the regularization theorem. Let $\varepsilon > 0$, and let $\rho_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, with compact support contained in the ball $B_0(\varepsilon)$ of center 0 and radius ε. Let us moreover chose ρ_ε in such a way that $\int_{\mathbb{R}^n} \rho_\varepsilon(v) dv = 1$.
Let U be a bounded and convex open subset of $\mathbb{R}^n$, containing the unit ball, and let us set for any $\omega \in L^1_{\text{loc}}(U, \Lambda^k)$:

$$R_\varepsilon \omega = \int_{\mathbb{R}^n} s_v^* \omega \rho_\varepsilon(v) dv.$$

The following result is due to G. de Rham (see Proposition 1, paragraph 15 of [dR73]).

Proposition 1.15 *(i) $R_\varepsilon\omega$ is smooth inside $\mathbb{B}^n$, and equal to ω outside of $\overline{\mathbb{B}^n}$.*

(ii) If ω has continuous coefficients, then $R_\varepsilon\omega$ converges uniformly to ω.

Moreover, R_ε acts in a natural way:

Lemma 1.16 *For any $\omega \in L^1_{\text{loc}}(U, \Lambda^k)$, one has $dR_\varepsilon\omega = R_\varepsilon d\omega$.*

Proof : One has $ds_v^*\omega = s_v^* d\omega$. Hence, we obtain

$$\int_{\mathbb{R}^n} ds_v^*\omega\rho_\varepsilon(v)dv = \int_{\mathbb{R}^n} s_v^* d\omega\rho_\varepsilon(v)dv.$$

This equality can be rewritten coefficient by coefficient. The differential is a operation of partial differentiation, and we can thereafter employ the dominated convergence theorem, as ρ_ε has compact support. We thus have

$$d\int_{\mathbb{R}^n} s_v^*\omega\rho_\varepsilon(v)dv = \int_{\mathbb{R}^n} s_v^* d\omega\rho_\varepsilon(v)dv.$$

This is exactly $dR_\varepsilon\omega = R_\varepsilon d\omega$. □

Proposition 1.17 *The operator R_ε sends continuously $\Omega^k_{q,p}(U)$ to itself, and is bounded. Moreover,*

$$\lim_{\varepsilon\to 0} \|R_\varepsilon\|_{q,p} \leq 1.$$

Proof : First we quote a result of Gol'dshteĭn, Kuz'Minov and Shvedov (this is lemma 2 of [GKS84]): the operator R_ε sends $L^p(U, \Lambda^k)$ to $L^p(U, \Lambda^k)$ and moreover it satisfies the estimate

$$\|R_\varepsilon\|_p \leq C(\varepsilon)$$

where $C(\varepsilon) \to 1$ as $\varepsilon \to 0$.
Hence, if $\omega \in \Omega^k_{q,p}(U)$, one has $\omega \in L^q(U, \Lambda^k)$ and $d\omega \in L^p(U, \Lambda^{k+1})$. We thus have $R_\varepsilon\omega \in L^q(U, \Lambda^k)$ as well as $dR_\varepsilon\omega = R_\varepsilon d\omega \in L^p(U, \Lambda^k)$. Hence, $R_\varepsilon\omega \in \Omega^k_{q,p}(U)$. Moreover,

$$
\begin{aligned}
\|R_\varepsilon\|_{q,p} &= \sup_{\omega\neq 0}\frac{\|R_\varepsilon\omega\|_{q,p}}{\|\omega\|_{q,p}} \\
&= \sup_{\omega\neq 0}\frac{\|R_\varepsilon\omega\|_q+\|dR_\varepsilon\omega\|_p}{\|\omega\|_{q,p}} \\
&= \sup_{\omega\neq 0}\frac{\|R_\varepsilon\omega\|_q+\|R_\varepsilon d\omega\|_p}{\|\omega\|_{q,p}} \\
&\leq \sup_{\omega\neq 0}\frac{\|R_\varepsilon\|_q\|\omega\|_q+\|R_\varepsilon\|_p\|\omega\|_p}{\|\omega\|_{q,p}} \\
&\leq \sup_{\omega\neq 0}\frac{C(\varepsilon)\|\omega\|_q+C(\varepsilon)\|d\omega\|_p}{\|\omega\|_{q,p}} \\
&= \sup_{\omega\neq 0}\frac{C(\varepsilon)(\|\omega\|_q+\|d\omega\|_p)}{\|\omega\|_{q,p}} \\
&= \sup_{\omega\neq 0}\frac{C(\varepsilon)\|\omega\|_{q,p}}{\|\omega\|_{q,p}} \\
&= C(\varepsilon).
\end{aligned}
$$

□

Proposition 1.18 *If $\omega \in L^p(M,\Lambda^k)$, then $\|R_\varepsilon\omega-\omega\|_p \stackrel{\varepsilon\to 0}{\longrightarrow} 0$.*

Proof : Let $\omega\in L^p(U,\Lambda^k)$. One has $\|R_\varepsilon\omega\|_p\leq C(\varepsilon)\|\omega\|_p$. Now let ξ a form of degree k with continuous coefficients such that for any fixed $\delta>0$, one has

$$\|\omega-\xi\|_p\leq\delta.$$

Since $R_\varepsilon\xi$ converges uniformly to ξ, one has $\|R_\varepsilon\xi-\xi\|_p\to 0$. Hence, $\|R_\varepsilon\xi-\xi\|_p<\delta$ for $\varepsilon>0$ sufficently small. We thus have, for $\varepsilon>0$ sufficently small,

$$
\begin{aligned}
\|\omega-R_\varepsilon\omega\|_p &\leq \|\omega-\xi\|_p+\|\xi-R_\varepsilon\xi\|_p+\|R_\varepsilon\xi-R_\varepsilon\omega\|_p \\
&\leq \|\omega-\xi\|_p+\|\xi-R_\varepsilon\xi\|_p+\|R_\varepsilon\|_p\|\xi-R_\varepsilon\omega\|_p \\
&\leq \delta+\|R_\varepsilon\|_p\delta+\delta \\
&= \delta(\|R_\varepsilon\|_p+2)
\end{aligned}
$$

Since this inequality is true for any $\delta>0$, one has $\|R_\varepsilon\omega-\omega\|_p\stackrel{\varepsilon\to 0}{\longrightarrow}0$. □

We now construct a homotopy between R_ε and the identity operator of the convex set U. Let $\varepsilon>0$, and $A_\varepsilon=(I-R_\varepsilon)\circ T$. We have the following result:

Lemma 1.19 *For any $\omega\in L^1_{\text{loc}}(U,\Lambda^k)$, one has*

$$\omega-R_\varepsilon\omega=dA_\varepsilon\omega+A_\varepsilon d\omega.$$

Proof : It's a simple computation:

$$\begin{aligned} dA_\varepsilon\omega + A_\varepsilon d\omega &= d(I - R_\varepsilon)T\omega + (I - R_\varepsilon) \circ Td\omega \\ &= dT\omega - dR_\varepsilon T\omega + Td\omega - R_\varepsilon Td\omega \\ &= dT\omega + Td\omega - R_\varepsilon(dT\omega + Td\omega) \\ &= \omega - R_\varepsilon\omega \end{aligned}$$

□

Proposition 1.20 *Suppose that one of the following hypothesis is satisfied:*

(i) $1 < p, q, r \leq \infty$ *and* $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}, \frac{1}{r} - \frac{1}{p} \leq \frac{1}{n}$;

(ii) $1 \leq p, q, r \leq \infty$ *and* $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}, \frac{1}{r} - \frac{1}{p} < \frac{1}{n}$.

Then A_ε *sends* $\Omega^k_{p,r}(U)$ *onto* $\Omega^{k-1}_{q,p}(U)$ *continuously, and moreover one has*

$$dA_\varepsilon\omega + A_\varepsilon d\omega = \omega - R_\varepsilon\omega \ \forall\omega \in \Omega^k_{p,r}(U).$$

Proof : It is immediate with Corollaries 1.11 and 1.12 and the previous lemma. □

Remark 1.4 Since $I = R_\varepsilon$ outside $\overline{\mathbb{B}^n}$, one has $A_\varepsilon = 0$ on this set.

We now generalize this construction on smooth manifolds instead of $\mathbb{R}^n$. Let (M, g) be a Riemannian manifold, and assume that one has a countable atlas $\mathcal{A} = (\phi_i : V_i \to U_i \subset \mathbb{R}^n)$. Let us moreover assume that the atlas $\mathcal{A}$ satisfies the following properties:

1. $\mathcal{A}$ is locally finite.
2. U_i is a convex open set containing the unit ball $\mathbb{B}^n$.
3. If $B_i = \phi_i^{-1}(\mathbb{B}^n)$, the B_i cover M.

Remark that since $B_i \subset V_i$, the covering (B_i) is locally finite as well. Let $\varepsilon > 0$, and let

$$\begin{cases} R_{i,\varepsilon} = \phi_i^* \circ R_\varepsilon \circ (\phi_i^{-1})^* \\ R_\varepsilon^{(m)} = R_{1,\varepsilon} \circ R_{2,\varepsilon} \circ \ldots \circ R_{m,\varepsilon} \\ R_\varepsilon^M = \lim_{m\to\infty} R_\varepsilon^{(m)} = \Pi_{i=1}^\infty R_{i,\varepsilon} \end{cases} \qquad \begin{cases} A_{i,\varepsilon} = \phi_i^* \circ A_\varepsilon \circ (\phi_i^{-1})^* \\ A_\varepsilon^{(m)} = R_{1,\varepsilon} \circ R_{2,\varepsilon} \circ \ldots \circ R_{m-1,\varepsilon} \circ A_{m,\varepsilon} \\ A_\varepsilon^M = \sum_{m=1}^\infty A_\varepsilon^{(m)} \end{cases}$$

Remark 1.5 (1) Each operator $R_{\varepsilon,i}, A_{\varepsilon,i}$ is defined for forms on V_i. However, R_ε equals the identical operator and A_ε equals zero for forms outside of U_i. Hence, both $R_{\varepsilon,i}$ and $A_{\varepsilon,i}$ can be extended to forms defined on M.

(2) Since (B_i) is locally finite, the operators R_ε^M and A_ε^M are defined for any form with compact support, and consequently for locally integrable forms.

Lemma 1.21 *For any $\omega \in L^1_{\text{loc}}(M, \Lambda^k)$, one has $dR_\varepsilon^M \omega = R_\varepsilon^M d\omega$.*

Proof : Since d commutes with $\phi_i^*, (\phi_i^{-1})^*$ and R_ε, it also commutes with $R_{i,\varepsilon}$, and hence with $R_\varepsilon^{(m)}$. Moreover, the differential is locally defined. Hence, d commutes with R_ε^M. □

Lemma 1.22 *For any $\omega \in L^1_{\text{loc}}(M, \Lambda^k)$, the form $R_\varepsilon^M \omega$ is smooth.*

Proof : Since ϕ_i is a diffeormorphism, the form $R_{i,\varepsilon}\omega$ is smooth on B_i, and equal to ω outside of $\overline{B_i}$. Hence, the form $R_\varepsilon^M \omega$ is smooth on $\bigcup_{i=1}^{\infty} B_i = M$. □

The following lemma is a direct corollary of proposition 1.17:

Lemma 1.23 *The operator R_ε^M maps $\Omega^k_{q,p}(M)$ continuously onto itself. Moreover, one has $\lim_{\varepsilon \to 0} \|R_\varepsilon^M\|_{q,p} \leq 1$.*

□

Similarly, the following lemma is a corollary of proposition 1.18:

Lemma 1.24 *For $\omega \in L^p(M, \Lambda^k)$, one has $\|R_\varepsilon^M \omega - \omega\|_p \stackrel{\varepsilon \to 0}{\longrightarrow} 0$.*

□

From proposition 1.20, one obtains the following result:

Lemma 1.25 *Suppose that one of the following hypothesis is satisfied:*

(i) $1 < p, q, r \leq \infty$ *and* $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}, \frac{1}{r} - \frac{1}{p} \leq \frac{1}{n}$;

(ii) $1 \leq p, q, r \leq \infty$ *and* $\frac{1}{p} - \frac{1}{1} < \frac{1}{n}, \frac{1}{r} - \frac{1}{p} < \frac{1}{n}$.

Then A_ε^M maps $\Omega^k_{q,p}(M)$ to $\Omega^{k-1}_{pr}(M)$ continuously.

□

Finally we also have a homotopy formula:

Lemma 1.26 *For any $\omega \in L^1_{\text{loc}}(M, \Lambda^k)$, one has*

$$dA_\varepsilon^M \omega + A_\varepsilon^M d\omega = I - R_\varepsilon^M \omega.$$

Proof : For any m, one has

$$\begin{aligned} \omega - R_{m,\varepsilon}\omega &= \omega - \phi_m^* \circ R_\varepsilon \circ (\phi_m^{-1})^*\omega \\ &= \phi_m^* \circ (I - R_\varepsilon) \circ (\phi_m^{-1})^*\omega \\ &= \phi_m^* \circ (A_\varepsilon d + dA_\varepsilon) \circ (\phi_m^{-1})^*\omega \\ &= dA_{m,\varepsilon}\omega + A_{m,\varepsilon}d\omega \end{aligned}$$

Let us compose on the left with $R_\varepsilon^{(m-1)}$:

$$R_\varepsilon^{(m-1)}\omega - R_\varepsilon^{(m)}\omega = dA_\varepsilon^{(m)}\omega + A_\varepsilon^{(m)}d\omega.$$

If m varies, we obtain a telescopic sequence:

$$\begin{aligned} \omega - R_\varepsilon^{(1)}\omega &= dA_\varepsilon^{(1)}\omega + A_\varepsilon^{(1)}d\omega. \\ R_\varepsilon^{(1)}\omega - R_\varepsilon^{(2)}\omega &= dA_\varepsilon^{(2)}\omega + A_\varepsilon^{(2)}d\omega. \\ \ldots &= \ldots \\ R_\varepsilon^{(m-1)}\omega - R_\varepsilon^{(m)}\omega &= dA_\varepsilon^{(m)}\omega + A_\varepsilon^{(m)}d\omega. \end{aligned}$$

If we sum for $m = 1$ to ∞, we obtain

$$dA_\varepsilon^M\omega + A_\varepsilon^M d\omega = I - R_\varepsilon^M\omega.$$

□

Finally, lemmas 1.21 to 1.26 constitute the regularization theorem, which we quote once again:

Proposition 1.27 (L_π regularization) *Let π be a sequence of real numbers $1 \leq p_j < \infty$.*
For any Riemannian manifold M, there exists a sequence of regularization operators $R_\varepsilon^M : L^1_{\text{loc}}(M, \Lambda^k) \to L^1_{\text{loc}}(M, \Lambda^k)$ and a sequence homotopy operators $A_\varepsilon^M : L^1_{\text{loc}}(M, \Lambda^k) \to L^1_{\text{loc}}(M, \Lambda^{k-1})$ such that

(1) For any $\omega \in L^1_{loc}(M, \Lambda^k)$, the form $R_\varepsilon^M\omega$ is smooth on M.

(2) For any $\omega \in \Omega^k_\pi(M)$, we have $dR_\varepsilon^M\omega = R_\varepsilon^M d\omega$;

(3) For any $\varepsilon > 0$, the operator $R_\varepsilon^M : \Omega^k_\pi(M) \to \Omega^k_\pi(M)$ is bounded and satisfies $\lim_{\varepsilon\to 0} \left\| R_\varepsilon^M \right\|_\pi = 1$;

(4) For any $\omega \in \Omega^k_\pi(M)$, we have $\lim_{\varepsilon\to 0} \left\| R_\varepsilon^M\omega - \omega \right\|_\pi = 0$;

(5) The operator $A_\varepsilon^M : \Omega^k_\pi(M) \to \Omega^{k-1}_\pi(M)$ is bounded in the following cases:

(i) $1 < p_j \leq \infty$ and $\frac{1}{p_k} - \frac{1}{p_{k-1}} < \frac{1}{n}$ or

(ii) $1 < p_j \leq \infty$ *and* $\frac{1}{p_k} - \frac{1}{p_{k-1}} \leq \frac{1}{n}$.

(6) We have the homotopy formula

$$\omega - R_\varepsilon^M \omega = dA_\varepsilon^M \omega + A_\varepsilon^M d\omega.$$

Remark 1.6 The regularization theorem does not hold for L^∞ forms (even if one adds a condition on the derivative - i.e. Sobolev injections won't help).

A few examples

This thesis is not about computing integrable cohomology spaces, which is not an easy task and moreover uses completely different techniques than the one we use. However we give here a couple examples. First, we have a Poincaré lemma:

Example (The $L_{q,p}$-cohomology of the ball) Let U be a convex and bounded subset of $\mathbb{R}^n$, and suppose that p, q satisfy one of the following:

(i) $1 \leq p, q \leq \infty$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$ or

(ii) $1 < p, q \leq \infty$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$.

Then for any $k = 1, \ldots, n$, one has $H_{q,p}^k(U) = 0$. Indeed, from proposition 1.13, we know that $\omega = dT\omega + Td\omega$, and T maps $\Omega_{q,p}^k(U)$ to $\Omega_{q,q}^{k-1}(U) = L^q(U, \Lambda^{k-1})$ continuously. In particular, any closed form ω in $\Omega_{q,p}^k(U)$ admits $T\omega \in L^q(U, \Lambda^{k-1})$ as a primitive.

Example (The $L_{q,p}$-cohomology of the Poincaré space) Let us consider the hyperbolic space $\mathbb{H}_1^m$ with curvature -1, seen as the unit ball $B_1(0) \subset \mathbb{R}^m$ together with the Riemannian metric

$$h = \frac{4 \sum dx^i \otimes dx^i}{(1 - |x|^2)^2}.$$

We begin by the case $q = p$ (one speaks of L_p-cohomology), using a method of Gromov (see [PRS08]). We have

$$\overline{H}_p^k(\mathbb{H}^m) \neq 0.$$

In fact, we begin by proving that for such a choice of k and p, the space of L^p-forms of any Riemannian manifold (M, g) of dimension m is a conformal invariant (we will see a generalization of this fact below, see 1.29). So let (M, g) be a Riemannian manifold, and h be a metric on M conformally equivalent to g, i.e. $h = \lambda^2 g$, where λ is a smooth positive function. Then for any form ω of degree k, one has

$$|\omega_x|_{x,h} = \lambda^{-k} |\omega_x|_{x,g}.$$

Here, $|\omega_x|_{x,h}$ denotes the norm of the multilinear map ω_x with respect to the metric h. Moreover,

$$d\,\mathrm{vol}_h = \lambda^m d\,\mathrm{vol}_g\,.$$

Hence, for any form ω of degree k, one has

$$\int_M |\omega_x|^p_{x,h} d\,\mathrm{vol}_h(x) = \int_M |\omega_x|^p_{x,g} d\,\mathrm{vol}_g(x).$$

This proves that $L^p((M,g),\Lambda^k) = L^p((M,h),\Lambda^k)$.

Now, remark that on the ball $B_1(0)$, the hyperbolic metric is conformally equivalent to the Euclidean metric. Moreover, the ball $B_1(0)$ has finite euclidean volume, hence for any choice of k and p, the inclusion induces a pullback bounded linear map

$$i^* : \Lambda^k(\mathbb{R}^m) \to L^p((B_1(0),g),\Lambda^k)$$

where g is the euclidean metric on $B_1(0)$. By conformal invariance, for $kp = m$, one thus has a bounded linear map

$$i^* : \Lambda^k(\mathbb{R}^m) \to L^p(\mathbb{H}^m,\Lambda^k).$$

Let $p = \dfrac{m}{k}, q = \dfrac{m}{m-k}$ and $j^* : \Lambda^{m-k}(\mathbb{R}^m) \to L^q(\mathbb{H}^m),\Lambda^{m-k})$ the equivalent operator.
Let $\omega_1 = i^*(dx^1 \wedge \ldots \wedge dx^k)$, and $\omega_2 = j^*(dx^{k+1} \wedge \ldots \wedge dx^m)$. One has

$$\omega_1 \in Z^k_p(\mathbb{H}^m) \text{ and } \omega_2 \in Z^{m-k}_q(\mathbb{H}^m).$$

We claim that ω_1 represents a non-zero class of reduced cohomology. Suppose by contradiction that $\omega_1 \in \overline{B}^k_p(\mathbb{H}^m)$. Then there exists a sequence (τ_n) of differential forms of degree $k-1$ with $d\tau_n \to \omega_1$ in L^p-norm:

$$\|d\tau_n - \omega_1\|_{L^p} \to 0.$$

Since compactly supported smooth forms $C^\infty_c(M,\Lambda^{k-1})$ are dense in $L^p(M,\Lambda^{k-1})$, we can suppose that each τ_n is a compactly supported smooth form.
Using Hölder's inequality, one has

$$\begin{aligned}\left|\int_{\mathbb{H}^m} d\tau_n \wedge \omega_2 - \int_{\mathbb{H}^m} \omega_1 \wedge \omega_2\right| &\leq \int_{\mathbb{H}^m} |(d\tau_n - \omega_1)\wedge\omega_2| \\ &\leq \|d\tau_n - \omega_1\|_{L^p}\cdot\|\omega_2\|_{L^q} \to 0.\end{aligned}$$

On the other hand, By Stokes,

$$0 = \lim_{n\to\infty}\int_{\mathbb{H}^m} d\tau_n \wedge \omega_2 = \int_{\mathbb{H}^m} \omega_1 \wedge \omega_2.$$

Hence, $\int_{\mathbb{H}^m} \omega_1 \wedge \omega_2 = 0$. But the left-hand side is the euclidean volume of $B_1(0)$, which is certainly non-zero. Therefore, there is a contradiction and ω_1 must represent a non-trivial cohomology class. Hence

$$H_p^k(\mathbb{H}^m) \neq 0.$$

Using monotonicity results that we will prove in Chapter 2(more precisely, Lemmas 2.14 and 2.15), we can now deduce some non-vanishing results on the $L_{q,p}$ cohomology of the hyperbolic space :

1. For $p \geq q$, one has $H_p^k(\mathbb{H}^n) \subset H_{q,p}^k(\mathbb{H}^n)$. In particular, since $H_p^k(\mathbb{H}^n) \neq 0$ for $p = \frac{n}{k}$, one has
$$H_{q,p}^k(\mathbb{H}^n) \neq 0 \text{ for any } q \leq p = \frac{n}{k}.$$
2. This reasoning is still true for reduced cohomology, hence:
$$\overline{H}_{q,p}^k(\mathbb{H}^n) \neq 0 \text{ for any } q \leq p = \frac{n}{k}.$$

Those two results about the cohomology of $\mathbb{H}^n$ have been generalized by Troyanov and Gol'dshteĭn in [GT09], removing the condition $p \geq q$:

Theorem 1.28 *Let (M, g) be a n-dimensional Cartan-Hadamard manifold with sectional curvature $K \leq -1$ and Ricci curvature $Ric \geq -(1+\varepsilon)^2(n-1)$.*

1. *Assume that*
$$\frac{1+\varepsilon}{p} < \frac{k}{n-1} \text{ and } \frac{k-1}{n-1} + \varepsilon < \frac{1+\varepsilon}{q},$$
then $H_{q,p}^k(M) \neq 0$.
2. *Assume furthermore that*
$$\frac{1+\varepsilon}{p} < \frac{k}{n-1} \text{ and } \frac{k-1}{n-1} + \varepsilon < \min\left\{\frac{1+\varepsilon}{q}, \frac{1+\varepsilon}{p}\right\},$$
Then $\overline{H}_{q,p}^k(M) \neq 0$.

These two examples show in particular that diffeomorphic manifolds may have different $L_{q,p}$-cohomology spaces, as it was guessed above.

The following theorem allows to compute the $L_{q,p}$ cohomology in degree greater than one for spaces conformally equivalent to some simpler space.

A result of conformal invariance: In the computation of the L^p-cohomology of the hyperbolic plane, we have seen that the space of L^p-forms of degree k is a conformal invariant if $p \cdot k = m$, where m is the dimension of the manifold. If we set $p = \dfrac{m}{k}$ and $q = \frac{m}{k-1}$, the spaces $B_{q,p}^k(M)$ and $Z_p^k(M)$ are thus conformal invariants. Hence, the following result, due to M. Troyanov and V. Gol'dshteĭn, is now evident:

Theorem 1.29 *Let (M, g) be a Riemannian manifold, and h be a Riemannian metric on M conformaly equivalent to g, i.e. there exists a smooth function $\lambda : M \to \mathbb{R}^+$ such that $h = \lambda g$. One has*

$$H^k_{\frac{n}{k-1}, \frac{n}{k}}(M, g) = H^k_{\frac{n}{k-1}, \frac{n}{k}}(M, h).$$

□

The L_π-cohomology of a n-manifold, with the particular choice $p_k = \frac{n}{k}$, is called the *conformal cohomology*, denoted $H^\bullet_{\text{conf}}(M)$. The result above of conformal invariance has been extended by M. Troyanov and V. Gol'dshtein to invariance under quasi-conformal maps. In fact, they proved the following: let $\Omega^k_{\text{conf}}(M) = \Omega^k_{\frac{k}{n}\frac{k+1}{n}}(M)$. Then we have the

Theorem 1.30 *Let (M, g) and (N, g) be Riemannian manifolds, and a homeomorphism $f : M \to N$. Then f is a quasiconformal map if and only if its pullback f^* defines an isomorphism of Banach differential algebras $f^* : \Omega^\bullet_{\text{conf}}(N) \to \Omega^\bullet_{\text{conf}}(M)$.*

In particular, such an isomorphism $f^* : \Omega^\bullet_{\text{conf}}(N) \to \Omega^\bullet_{\text{conf}}(M)$ gives rise to an isomorphism

$$f^* : H^\bullet_{\text{conf}}(N) \to H^\bullet_{\text{conf}}(M)$$

in conformal cohomology.

We end this introduction to $L_{q,p}$-cohomology by quoting three results which relate the $L_{q,p}$-cohomology and Sobolev inequalities. These three propositions are due to V. Gol'dshteĭn and M. Troyanov.

Proposition 1.31 *The following assertions are equivalent:*

(i.) $\dim T^k_{q,p}(M) < \infty$*;*

(ii.) $T^k_{q,p}(M) = 0$*;*

(iii.) $H^k_{q,p}(M)$ *is a Banach space;*

(iv.) $d : \Omega^{k-1}_{q,p}(M) \longrightarrow \Omega^k_{q,p}(M)$ *is a closed operator.*

Proposition 1.32 *The following assertions are equivalent:*

(i.) $H^k_{q,p}(M) = 0$*;*

(ii.) $d : \Omega^{k-1}_{q,p}(M)/Z^{k-1}_q(M) \longrightarrow Z^k_p(M)$ *admits a bounded inverse;*

(iii.) There exists a constant $C > 0$ such that for any closed form $\phi \in Z^k$ of degree k, there exists a form $\psi \in \Omega^{k-1}$ such that $d\psi = \phi$ and $\|\psi\| \leq C_k \|\phi\|$.

Proposition 1.33 *(1) If $T^k_{q,p}(M) = 0$, then there exists a constant C' such that for any differential form $\theta \in \Omega^{k-1}_{q,p}(M)$, there exists a closed form $\zeta \in Z^{k-1}_q(M)$ such that*

$$\|\theta - \zeta\|_q \leq C' \|d\theta\|_p.$$

(2) The converse is true when $1 < q < \infty$.

Those three propositions are direct corollaries of propositions A.4, A.5 and A.6 of chapter 4.

L_π-cohomology of a simplicial complex

In the sequel and throughout all this thesis, K is a locally finite simplicial complex of finite dimension n. We assume that it is realized in some euclidean space $\mathbb{R}^N$. Each face is endowed with the euclidean metric, and the realization itself is given the resulting length metric. Some basic knowledge about simplicial complexes is assumed (see e.g. [Mat06] for an introduction to the subject).

Let K be a locally finite simplicial complex. We use the following notations:

- $\Delta^k = (e_{i_0}, \ldots, e_{i_k})$ denotes the oriented simplex with vertices $e_{i_0}, \ldots, e_{i_k}$, and $-(e_{i_0}, \ldots, e_{i_k})$ denotes the same simplex with opposite orientation.
- $C_k(K)$ denotes the space of real chains of degree k of K, that is the formal vector space with the set of k-simplices of K.
- $C^k(K)$ denotes the space of real cochains of degree k of K, that is the algebraic dual space of $C_k(K)$.
- If $\Delta^k = (e_{i_0}, \ldots, e_{i_k})$ denotes an oriented k-simplex of K, its *boundary* is the $(k-1)$-chain
$$\partial\Delta^k = \sum_{j=0}^{k} (-1)^j (e_{i_0}, \ldots, \widehat{e_{i_j}}, \ldots, e_{i_k})$$
where the symbol $\widehat{e_{i_j}}$ means that the corresponding term is omitted. Extending this formula by linearity defines an operator $\partial : C_k(K) \to C_{k-1}(K)$.
- The *coboundary operator* $\delta : C^k(K) \to C^{k+1}(K)$ is dual to the boundary operator, i.e. it is defined on simplexes by
$$\delta c(\Delta^{k+1}) = c\left(\partial\Delta^{k+1}\right).$$

For $1 \le p < \infty$, let

$$C_p^k(K) = \left\{ c \in C^k(K) \;\middle|\; \sum_{\Delta^k \in K} \left| c\left(\Delta^k\right)\right|^p < \infty \right\}.$$

Together with the norm

$$\|c\|_p = \left(\sum_{\Delta^k \in K} \left| c(\Delta^k)\right|^p \right)^{\frac{1}{p}}$$

it is a Banach space homeomorphic to $l^p(\mathbb{Z})$.

Let

$$Z_p^k(K) = \left\{c \in C_p^k(K) \mid \delta c = 0\right\} \text{ and } B_{q,p}^k(K) = \delta C_q^{k-1}(K) \cap C_p^k(K) \subset Z_p^k(K).$$

We also denote by $\overline{B}_{q,p}^k(K)$ the closure of $B_{q,p}^k(K)$ in $C_p^k(K)$.

Let $C_{q,p}^k(K) = \left\{c \in C_q^k(K) \,\middle|\, \delta c \in C_p^{k+1}(K)\right\}$, together with the operator norm of δ:

$$\|c\|_{q,p} = \|c\|_q + \|\delta c\|_p.$$

In a way similar to what we did for L_π cohomology of a Riemannian manifold, we associate a Banach complex to K: if $\pi = (p_k)$ is a sequence of real numbers $1 \leq p_k < \infty$, let us note $C_\pi^k(K) = C_{p_k p_{k+1}}^k(K)$ and $\|c\|_{C_\pi^k(K)} = \|c\|_{p_k p_{k+1}}$. We thus have have a Banach complex

$$\ldots \xrightarrow{d} C_\pi^{k-1}(K) \xrightarrow{\delta} C_\pi^k(K) \xrightarrow{\delta} C_\pi^{k+1}(K) \xrightarrow{\delta} \ldots$$

Definition (Simplicial L_π-cohomology) The *L_π-cohomology of the simplicial complex K* is the cohomology of the Banach complex $(C_\pi^\bullet(K), \delta)$.

A remark on the notations: We use $\Omega_{q,p}^k(M)$ and $\Omega_\pi^k(M)$ to designate spaces of forms, whereas we use $C_{q,p}^k(K)$ and $C_\pi^k(K)$ to designate spaces of cochains. During the sequel, letters M, N will always designate (Riemannian) manifolds, and K, L will always designate (euclidean) simplicial complexes.
We use $\|\cdot\|_{q,p}$ to designate the norm in $\Omega_{q,p}^k(K)$ and in $\Omega_{q,p}^k(M)$. To make the distinction clear from the context, we will use small latin letters such has c, d for cochains, and small greek letters such has $\zeta, \eta, \theta, \omega$ for forms.

We have two monotonicity results in simplicial cohomology:

Lemma 1.34 *If $q_2 \geq q_1$, then*

$$H_{q_2 p}^k(K) \subset H_{q_1 p}^k(K)$$

and

$$\overline{H}_{q_2 p}^k(K) \subset \overline{H}_{q_1 p}^k(K).$$

Proof : Since $q_2 \geq q_1$, then for any q_1-summable cochains c, one has $\|c\|_{q_1} \leq \|c\|_{q_2}$. Consequently, $C_{q_1}^k(K) \subset C_{q_2}^k(K)$ and therefore $B_{q_1 p}^k(K) \subset B_{q_2 p}^k(K)$. Hence $H_{q_2 p}^k(K) \subset H_{q_1 p}^k(K)$. Moreover, since $B_{q_1 p}^k(K) \subset B_{q_2 p}^k(K)$, one has $\overline{B}_{q_1 p}^k(K) \subset \overline{B}_{q_2 p}^k(K)$, hence $\overline{H}_{q_2 p}^k(K) \subset \overline{H}_{q_1 p}^k(K)$. □

Lemma 1.35 *If $p_2 \leq p_1$, then $H_{q,p_1}^k(K) = 0 \Rightarrow H_{q,p_2}^k(K) = 0$.*

Proof : Suppose that $H^k_{q,p_1}(K) = 0$, and let $[c] \in H^k_{q,p_2}(K)$. In particular, $\delta c = 0$ and $\|c\|_{p_2} < \infty$. Since $p_2 \leq p_1$, one has $\|c\|_{p_1} < \infty$ as well. This shows that $c \in Z^k_{p_1}(K)$. Moreover, since $H^k_{q,p_1}(K) = 0$, the cochain c belongs to $B^k_{q,p_1}(K)$ and thus there exists $b \in C^{k-1}_q(K)$ such that $db = c$. This shows that $c \in B^k_{q,p_2}(K)$ as well, and thus $[c] = 0$ in $H^k_{q,p_2}(K) = 0$.

□

Manifolds and simplicial complexes of bounded geometry

Manifolds of bounded geometry are manifolds whose geometric behavior is uniform in some sense: their Riemannian geometric invariants are locally controled in a uniform way. For instance, they have bounds on injectivity radius as well as on their curvature. They generalize compact manifolds, and coverings of compact manifolds. Another example is given by Lie groups with left-invariant metrics, and more generally by homogeneous spaces.

We give two definitions of manifolds with bounded geometry. The first definition involves normal coordinates. The second one is expressed in terms of curvature bounds, while both of them require a control on the injectivity radius. We will see further that they have a characterization in terms of existence of a nice triangulation.

Let us proceed more formally. In the sequel, (M^n, g) is a Riemannian manifold of dimension n, without boundary, and T_xM is the tangent space of M at the point $x \in M$. The zero-centered open ball of radius i in T_xM is denoted by $\tilde{B}_i(0)$, and the open ball of radius i centered at $x \in M$ is denoted by $B_i(x)$.
We fix an orthonormal basis $(e_1, \ldots, e_n)$ of the tangent space T_xM, and we consider the pull-back metric tensor on $\tilde{B}_i(0)$:

$$g_{ij} = g((\exp_x)_* e_i, (\exp_x)_* e_j).$$

Definition (Manifold of bounded geometry) One says that M has *bounded geometry at order* $s \in [0, \infty]$ if the following conditions hold:

(i) The injectivity radius $i = \mathrm{inj}(M)$ of M satisfies $i > 0$. In other words, for every $x \in M$, the exponential map $\exp_x : B_i(0) \subset T_xM \to B_i(x) \subset M$ is a diffeomorphism.

(ii) There exists a constant $C > 0$ such that for any $l \leq s$, the following property holds in normal coordinates in any ball of radius $i/2$:

$$|D^\alpha g_{\mu\nu}| \leq C \text{ and } |D^\alpha g^{\mu\nu}| \leq C.$$

Here D^α is any differential operator $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ with $\sum \alpha_i \leq l$.

Remark 1.7 Any manifold with bounded geometry is in particular complete, since the injectivity radius is bounded below.

The above definition of bounded geometry involves normal coordinates. One can give an intrinsic characterization, using curvature bounds. Let ∇ be the Levi-Civita connection on M. The following result can be found in [PRS08]:

Theorem 1.36 *Let (M,g) be a Riemannian manifold with* $\mathrm{inj}(M) > 0$*. Suppose that the covariant derivatives of the Riemannian tensor are uniformly bounded up to order s, i.e. there exists a constant $C' > 0$ such that*

$$\|\nabla^j R\| \leq C', \quad \forall j \leq s.$$

Then (M,g) has bounded geometry at order s. On the other hand, if (M,g) has bounded geometry at order s, then the covariant derivatives of the Riemannian tensor are uniformly bounded up to order $s-2$.

The following examples illustrate the notion of homogeneity that a manifold with bounded geometry satisfies:

Examples a) Any compact manifold has bounded geometry.

b) If $\tilde{M}$ is the universal covering of a compact manifold M, then the injectivity radii and curvature tensor of M and $\tilde{M}$ coincide, and thus $\tilde{M}$ has bounded geometry.

c) If M is a Lie group with a left-invariant metric, then M has bounded geometry.

d) More generally, if M is homogeneous, then M has bounded geometry.

Remark 1.8 The name of *uniform geometry* is more accurate (and Kanai uses a similar term). However, the name *manifold with bounded geometry* is widely accepted. Sometimes they are also called *thick spaces* (e.g. by P. Pansu).

There are several maps which make a category out of manifolds with bounded geometry. For example, the category of differentiable bilipschitz maps, or differentiable quasi-isometries.

Now let us introduce a similar notion for simplicial complexes.

Definition (Simplicial complex of bounded geometry) One says that the simplicial complex K has *bounded geometry* if it satisfies the following properties:

(i) There exists constants $D_1, D_2 > 0$ such that for any k-simplex $\Delta^k \in K$,

$$D_1 \leq \mathrm{Vol}^k(\Delta^k) \leq D_2$$

Where $\mathrm{Vol}^k(\Delta^k)$ is the volume of Δ^k, i.e. its k-dimensional Lebesgue measure.

(ii) For each vertex, the number of simplices containing it is uniformly bounded.

Definition (The category of bounded geometry simplicial complexes) We will denote by BGSC the category of euclidean bounded geometry simplicial complexes, together with uniformly continuous quasi-isometries $f : |K| \to |L|$ between their geometric realizations.

Definition Two uniformly continuous quasi-isometries $f, g : |K| \to |L|$ are *homotopic* if there exists a uniformly continuous quasi-isometric homotopy $F : [0,1] \times |K| \to |L|$ from f to g.

We now relate manifolds with bounded geometry and simplicial complexes with bounded geometry. Roughly speaking, a manifold has bounded geometry if and only if it admits a bilipschitz triangulation by a bounded geometry simplicial complex. First, let us recall the notion of triangulation of a differentiable manifold.

Definition Let M be a smooth manifold. A *smooth triangulation* of M is a pair (K, τ) where

(i) K is a locally finite simplicial complex of dimension $n = \dim(M)$ that we assume to be geometrically realized in $\mathbb{R}^N$ for some N, and,

(ii) $\tau : |K| \to M$ is a homeomorphism such that for any simplex $\Delta \in K$, there is an open subset U of the affine hull of Δ in $\mathbb{R}^N$ and a smooth extension τ_U of $\tau|_U$ such that the differential $d\tau_{Ux} : T_xU \to T_{\tau(x)}M$ is injective for any $x \in T$. In other words, $\tau|_U$ is a smooth embedding.

We will consider a special class of triangulations, for manifolds with bounded geometry. These triangulation will preserve most of the geometry of the manifold.

Definition A smooth triangulation (K, τ) of a Riemannian manifold M is *uniform* if

(i) K has bounded geometry;

(ii) $\tau : |K| \to M$ is bilipschitz in the following sense: there exists a constant $C > 0$ such that for any simplex Δ^k of dimension k of K and any $x \in \Delta^k$, one has

$$\frac{1}{C} \langle v, v \rangle_{\mathbb{R}^k} \leq g_{\tau(x)} \left(d\tau_{Ux} v, d\tau_{Ux} v \right) \leq C \langle v, v \rangle_{\mathbb{R}^k} .$$

Remark 1.9 If K is a bounded geometry simplicial complex, then any barycentric subdivision of K is uniform and has bounded geometry as well.

It is well known that any smooth manifold admits a triangulation. Furthermore, the following result belongs to the folklore:

Theorem 1.37 *A Riemannian manifold (M, g) admits a smooth uniform triangulation if and only if it has C^2-bounded geometry.*

□

A sketch of the proof may be found in [Att94]. See also the discussion in [PRS08]. Dodziuk attributes a similar result to Calabi.

Let us go back to simplicial morphisms for a while. Recall that a *morphism of bounded geometry simplicial complexes* is a uniformly continuous quasi-isometry $f : K \to L$. The following lemma allows us to approximate such a map by a simplicial uniformly continuous quasi-isometry:

Lemma 1.38 *Let $f : |K| \to |L|$ be a uniformly continuous quasi-isometry between bounded geometry simplicial complexes. There exists a barycentric subdivision K' of K and a simplicial uniformly continuous quasi-isometry $g : |K'| \to |L|$ such that for any $x \in |K|$, $f(x)$ and $g(x)$ belong to a same simplex of L.*

Proof : Let w be a vertex of L, and let us denote by $\mathrm{St}(w)$ its open star[1]. When w goes throughout the vertices of L, we obtain an open covering $\{U_w\}_w$ of $|K|$, with $U_w = f^{-1}(\mathrm{St}(w))$.
Let us show that $\{U_w\}$ admits a positive Lebesgue number. The diameter of each $\mathrm{St}(w)$ and of each intersection $\mathrm{St}(w) \cap \mathrm{St}(v)$ is uniformly bounded above and below, since L has bounded geometry. Moreover, f is a quasi-isometry, and thus the diameter of each U_w and of each intersection $U_v \cap U_w$ is uniformly bounded above and below. We choose $\delta > 0$ such that $\mathrm{diam}(U_v \cap U_w) > \delta$ for any choice of v, w. Then $\delta/2$ is a Lebesgue number of our covering $\{U_w\}$.
Now let K' be a barycentric subdivision of K such that for any vertex v of K', the diameter of $\mathrm{St}(v)$ is less than $\delta/2$. For any vertex v of K', there exists a vertex w of L such that $\mathrm{St}(v) \subset U_w$. Choosing such a w for each v, one obtains a map g that sends the vertices of K' to vertices of L. Moreover, g sends simplices to simplices. Indeed, if $v_0, \ldots, v_k$ are the vertices of a simplex Δ^k, then $\mathrm{St}(v_0) \cap \ldots \cap \mathrm{St}(v_k) \neq \emptyset$. Consequently $U_{v_0} \cap \ldots \cap U_{v_k} \neq \emptyset$, hence $\mathrm{St}(w_0) \cap \ldots \cap \mathrm{St}_(w_k) \neq \emptyset$. Thus $g(v_0), \ldots, g(v_k)$ belong to a same simplex of L. We thus have proved that g sends vertices to vertices and simplices to simplices. Hence, it can be extended to a simplicial map $g : K \to L$, by linear extension on each simplex. This assures that for any x, the points $f(x)$ and $g(x)$ always belong to a same simplex of L.

It remains to be shown that g is a uniformly continuous quasi-isometry. We know that f is a quasi-isometry. Hence there exists constants $\alpha > 1$ and $\beta \geq 0$ such that for any $x, x' \in |K|$,

$$\frac{1}{\alpha} d(x, x') - \beta \leq d(f(x), f(x')) \leq \alpha d(x, x') + \beta.$$

Let also $D > 0$ be such that $\mathrm{diam}\left(\Delta^k\right) < D$ for any simplex Δ^k of L. For any $x, x \in K$,

$$\begin{aligned} d(g(x), g(x')) &\leq d(g(x), f(x)) + d(f(x), f(x')) + d(f(x'), g(x')) \\ &\leq \alpha d(x, x') + 2D \end{aligned}$$

[1] i.e. the union of all open simplexes having w as vertex.

Moreover, one has

$$d(f(x), f(x')) \leq d(f(x), g(x)) + d(g(x), g(x')) + d(g(x'), f(x')).$$

Hence

$$d(g(x), g(x')) \geq d(f(x), f(x')) - d(f(x), g(x)) - d(f(x'), g(x')).$$

This tells us that

$$d(g(x), g(x')) \geq \frac{1}{\alpha} d(x, x') - \beta - 2D.$$

Hence g is a quasi-isometry, with constants $\alpha, (\beta + 2D)$. □

Definition (BGSC simplicial approximation) g is a *BGSC simplicial approximation* of f.

Remark 1.10 Let $f : |K| \to |L|$ be a uniformly continuous quasi-isometry, and g be a BGSC simplicial approximation of f. Let us define $F : [0,1] \times |K| \to |L|$ by the formula $F(t,x) = tg(x) + (1-t)f(x)$. Then F is a uniformly continuous quasi-isometric homotopy from f to g.

Definition Let K and L be bounded geometry simplicial complexes, and $f : |K| \to |L|$ be a uniformly continuous quasi-isometry. Let also $g : K \to L$ be a BSGC simplicial approximation of f. Let us moreover suppose that π is a sequence of real numbers $1 \leq p_k \leq \infty$ such that $p_{k+1} \leq p_k$. We define $f^* : H^k_\pi(L) \to H^k_\pi(K)$ the *linear map induced in L_π cohomology by f* by setting $f^* = g^*$.

In order to give sense to this definition, we must show that if g_1 and g_2 are two BGSC simplicial approximations of f, then $g_1^* = g_2^*$ at the cohomology level. We already know that g_1 and g_2 are homotopic in the BGSC sense, i.e. there exists a uniformly continuous quasi-isometric homotopy $F : [0,1] \times |K| \to |L|$ from g_1 to g_2. Hence the following lemma gives us our conclusion:

Lemma 1.39 *Let K, L be two bounded geometry simplicial complexes, and $f, g : K \to L$ be two simplicial uniformly continuous quasi-isometries. Suppose that there exists a BGSC homotopy $F : [0,1] \times |K| \to |L|$ from f to g. Let also π be a sequence of real numbers $1 \leq p_k \leq \infty$ such that $p_{k+1} \leq p_k$.*

Then there exists a linear map $A : C^k_\pi(L) \to C^{k-1}_\pi(K)$ such that

$$f^* - g^* = A\delta - \delta A.$$

Proof : Let $c \in C^k(L)$ be a k-cochain, and Δ^k a simplex. One has

$$\begin{aligned}(f^*c - g^*c)(\Delta^k) &= F^*c(1 \times \Delta^k) - F^*c(0 \times \Delta^k) \\ &= (F^*c)(1 \times \Delta^k - 0 \times \Delta^k) \\ &= F^*c(\partial I \times \Delta^k)\end{aligned}$$

Hence

$$(f^*c - g^*c)(\Delta^k) = (F^*c)(\partial(I \times \Delta^k) - I \times \partial\Delta^k).$$

Observe that in this formula, $I \times \Delta^k$ is not a simplex but a chain that triangulizes the polyhedron $I \times \Delta^k$.
Let $Ac : C_{k-1}(K) \to \mathbb{R}$ be the cochain defined by $(Ac)(\Delta^{k-1}) = (F^*c)(I \times \Delta^{k-1})$. One has

$$\begin{aligned}A(\delta c)(\Delta^k) &= (F^*\delta c)(I \times \Delta^k) \\ &= (\delta F^*c)(I \times \Delta^k) \\ &= (F^*c)(\partial(I \times \Delta^k))\end{aligned}$$

On the other hand,

$$(Ac)(\partial\Delta^k) = (F^*c)(I \times \partial\Delta^k).$$

Hence,

$$\begin{aligned}(f^*c - g^*c)(\Delta^k) &= A(\delta c)(\Delta^k) - (Ac)(\partial\Delta^k) \\ &= A(\delta c)(\Delta^k) - \delta(Ac)(\Delta^k) \\ &= (A(\delta c) - \delta(Ac))\,(\Delta^k)\end{aligned}$$

Thus $f^* - g^* = A\delta - \delta A$. If $c \in C^k_{p_k}(L)$, then $Ac \in C^{k-1}_{p_k}(K)$ which is included in $C^{k-1}_{p_{k-1}}(K)$ whenever $p_{k-1} \geq p_k$. □

Chapter 2

The De Rham isomorphism for L_π-cohomology

This chapter is dedicated to the proof of an isomorphism theorem between the de Rham L_π-cohomology and simplicial L_π-cohomology:

de Rham isomorphism theorem: *Let (M, g) be a non-compact, orientable, complete and connected Riemannian manifold, and assume that M admits a uniform triangulation $\tau : |K| \to M$. Let $\pi = (p_0, \dots, p_k, \dots)$ be a sequence of numbers satisfying one of the following hypothesis :*

(1) $1 < p_k < \infty$ *and* $0 \leq \dfrac{1}{p_k} - \dfrac{1}{p_{k-1}} \leq \dfrac{1}{n}$, *or*

(2) $1 \leq p_k < \infty$ *and* $0 \leq \dfrac{1}{p_k} - \dfrac{1}{p_{k-1}} < \dfrac{1}{n}$.

Then for any k there are vector space isomorphisms

$$H^k_\pi(M) = H^k_\pi(K) \qquad \textit{and} \qquad \overline{H}^k_\pi(M) = \overline{H}^k_\pi(K)$$

and the latter is continuous.

Here is how we prove this result. First, we introduce a complex of piecewise forms on K, called the *Sullivan complex*. We give a L_π version of it, and we establish two correspondances: one between the L_π-Sullivan complex and the simplicial L_π-complex of K, and one between the L_π-Sullivan complex and the de Rham L_π-complex of M. We then prove that these correspondances give rise to isomorphisms at the cohomology level.

The correspondances between the Sullivan complex and the cochain complexes are given by integration and Whitney transformation, and the correspondances between the Sullivan complex and the de Rham complex are given by inclusion and by regularization.

The Sullivan complex

Let (M, g) be a Riemannian manifold, and $\tau : |K| \to M$ be a uniform triangulation of M. We identify M and $|K|$ via this homeomorphism. Recall that K is realized in some euclidean space $\mathbb{R}^N$, thus each open face of K is a submanifold without boundary of $\mathbb{R}^N$.

Let us introduce two more vector spaces of differential forms on a manifold:

$$L^\infty(M, \Lambda^k) = \left\{ \omega \in L^1_{\mathrm{loc}}(M, \Lambda^k) \;\middle|\; \|\omega\|_\infty = \operatorname{esssup} |\omega(x)| < \infty \right\},$$

$$\Omega^k_\infty(M) = \left\{ \omega \in L^\infty(M, \Lambda^k) \;\middle|\; d\omega \in L^\infty(M, \Lambda^{k+1}) \right\}.$$

We denote respectively by $L^\infty_{\mathrm{loc}}(M, \Lambda^k)$ and $\Omega^k_{\infty,\mathrm{loc}}(M)$ the local versions of these spaces.

Terminology: the elements of the Banach space $\Omega^k_\infty(M)$ are the *flat forms.*

The theorem below allows us to take the pullback of a flat form by a Lipschitz map. Observe that this does not imply the existence of such a pullback for L^p forms.

Theorem 2.1 *Let $f : M \to N$ a Lipschitz map between manifolds. Then for any flat form $\omega \in \Omega^k_\infty(N)$, the form $f^*\omega$ is well defined and is a flat form. Furthermore, $df^\star\omega = f^\star d\omega$.*

Proof : See [Whi57]. See also the discussion in [Hei05]. □

Definition (Sullivan form) : A *Sullivan form of degree k* on K is a collection $\omega = \{\omega_\Delta\}_{\Delta \in K}$, where $\omega_\Delta \in \Omega^k_\infty(\Delta)$ for each $\Delta \in K$, satisfying the following condition: if Δ' is a simplex contained in Δ, we have $\omega_{\Delta'} = \omega_\Delta\,|_{\Delta'}$.

Here the restriction $\omega_\Delta\,|_{\Delta'}$ is the pullback $j^*_{\Delta',\Delta}\omega_\Delta$ where $j_{\Delta',\Delta}$ is the injection of Δ' into Δ. It is well defined by theorem 2.1.

Notation and terminology:

a) We denote by $S^k(K)$ the vector space of Sullivan forms of degree k on K.

b) ω_Δ is the Δ-*component* of the form ω.

Remark 2.1 One can define the exterior differential form of a Sullivan form, taking the exterior derivative component by component. Since $dj^*_{\Delta',\Delta} = j^*_{\Delta',\Delta}d$, one has $d(S^k(K)) \subset S^{k+1}(K)$. Hence $S^*(K)$ together with the exterior differential is a cochain complex of vector spaces.

Definition (Sullivan complex) The *Sullivan complex* of the simplicial complex K is the space $S^\bullet(K)$ together with the differential. We denote by $H^k(S^\bullet(K))$ its cohomology space of degree k.

The following result is due to Gol'dshteĭn, Kuz'minov and Shvedov (see [GKS88]):

Lemma 2.2 *There is a vector space isomorphism* $\phi : \Omega^k_{\infty,\mathrm{loc}}(M) \to S^k(K)$.

Proof : For $\omega \in \Omega^k_{\infty,\mathrm{loc}}(M)$ and Δ^k a simplex, let $\omega_{\Delta^k} = (\tau|_{\Delta^k})^*\omega$. Let us denote $\phi_\tau\omega = \{\omega_{\Delta^k}\}_{\Delta^k\in K}$. We shall prove that ϕ_τ is an isomorphism. It is clearly injective and linear, therefore we only need to prove that it is onto. Let $\{\theta_\Delta\}$ be a closed Sullivan form of degree k on K.
There exists forms $\omega \in L^\infty_{\mathrm{loc}}(M,\Lambda^k), \omega' \in L^\infty_{\mathrm{loc}}(M,\Lambda^{k+1})$ such that $(\tau|_{\Delta^n})^*\omega = \theta_{\Delta^n}$ and $(\tau|_{\Delta^n})^*\omega' = d\theta_{\Delta^n}$, where $n = \dim(M)$. We need to prove that $d\omega = \omega'$, i.e. for any compactly supported smooth form u of degree $n-k-1$, one has

$$\int_M \omega \wedge du = (-1)^{k+1}\int_M \omega' \wedge u.$$

For a pair Δ', Δ of simplexes of K, let us write

$$[\Delta' : \Delta] = \begin{cases} 1 & \text{if } \Delta' \text{ is a face of } \Delta \text{ with induced orientation} \\ -1 & \text{if } \Delta' \text{ is a face of } \Delta \text{ with opposite orientation} \\ 0 & \text{else.} \end{cases}$$

One has

$$\begin{aligned} \int_M \omega\wedge du + (-1)^k\int_M \omega'\wedge u &= \sum_{\Delta^n}\left(\int_{\Delta^n}\theta_{\Delta^n}\wedge d(\tau^*u) + (-1)^k\int_{\Delta^n} d\theta_{\Delta^n}\wedge\tau^*u\right) \\ &\overset{(1)}{=} \sum_{\Delta^n}\int_{\Delta^n} d(\theta_{\Delta^n}\wedge\tau^*u) \\ &\overset{(2)}{=} \sum_{\Delta^n}\sum_{\Delta^{n-1}}\left[\Delta^{n-1}:\Delta^n\right]\int_{\Delta^{n-1}} j^*_{\Delta^{n-1},\Delta^n}\theta_{\Delta^n}\wedge\tau^*u \\ &\overset{(3)}{=} 0. \end{aligned}$$

Equality (1) is due to Leibniz's Formula. Equality (2) is Stokes theorem, and equality (3) is due to the following fact : each integral in the sum appears twice, each one corresponding to a different orientation of each $(n-1)$-simplex, and all terms vanish. □

Remark 2.2 The Δ-component of $\phi(\omega)$ is simply the pullback of ω by the triangulation mapping τ extended to a neighborhood of Δ.

We now introduce a L_π version of the Sullivan complex.

Definition (L_π-Sullivan complex) Let us denote by $S^k_\pi(K)$ the space of Sullivan forms of degree k for which the norm $\|\omega\|_{S^k_\pi(K)}$ is finite, where

$$\|\omega\|_{S^k_\pi(K)} = \left(\sum_{\Delta\in K} \operatorname{esssup} |\omega_\Delta|^{p_k}\right)^{\frac{1}{p_k}} + \left(\sum_{\Delta\in K} \operatorname{esssup} |d\omega_\Delta|^{p_{k+1}}\right)^{\frac{1}{p_{k+1}}}.$$

Proposition 2.3 *The space $S^k_\pi(K)$ is a Banach space.*

Proof : Let $(\omega^j) \subset S^k_\pi(K)$ be a Cauchy sequence, and for any simplex Δ let us consider the restriction ω^j_Δ. Each sequence (ω^j_Δ) is a Cauchy sequence in $\Omega^k_\infty(\Delta)$, which is a Banach space. In particular, for each Δ there exists a limit $\omega_\Delta \in \Omega^k_\infty(\Delta)$. We shall prove that the Sullivan form $\{\omega_\Delta\}_\Delta$ belongs to $S^k_\pi(K)$.

Let us enumerate the simplicies $\Delta_\mu, \mu \in \mathbb{N}$. There is a map $\phi : S^k(K) \to \mathbb{R}^{\mathbb{N}} \oplus \mathbb{R}^{\mathbb{N}}$ defined by

$$\phi(\omega) = \left(\left(\|\omega_{\Delta_\mu}\|_{L^\infty(\Delta_\mu,\Lambda^k)}\right)_\mu, \left(\|d\omega_{\Delta_\mu}\|_{L^\infty(\Delta_\mu,\Lambda^{k+1})}\right)_\mu\right).$$

This map sends continuously $S^k_\pi(K)$ into the subspace $\ell^{p_k}(\mathbb{N}) \bigoplus \ell^{p_{k+1}}(\mathbb{N})$. In particular,

$$\phi(\omega) = \phi\left(\lim_{j\to\infty} \omega^j\right) = \lim_{j\to\infty} \phi\left(\omega^j\right).$$

Since $\phi\left(\omega^j\right) \in \ell^{p_k}(\mathbb{N}) \bigoplus \ell^{p_{k+1}}(\mathbb{N})$, one has $\phi(\omega) \in \ell^{p_k}(\mathbb{N}) \bigoplus \ell^{p_{k+1}}(\mathbb{N})$. But this exactly means that $\{\omega_\Delta\}_\Delta \in S^k_\pi(K)$. □

Notations :

1. $H^k(S^\bullet_\pi(K))$ denotes the cohomology of the Banach complex $S^\bullet_\pi(K)$.

2. If K is a simplicial complex, and L is a subcomplex of K, we denote by $C^k(K,L)$ the Banach subspace of elements of $C^k(K)$ which are 0 on L. Similarly, $C^k_\pi(K,L)$ stands for the elements of $C^k_\pi(K)$ which vanish on L. We denote by $H^k_\pi(K,L)$ the resulting cohomology.

3. If K is a simplicial complex triangulating a manifold and L a subcomplex of K we denote by $S^k(K,L)$ the subspace of elements in $S^k(K)$ which are 0 on L. Similarly, $S^k_\pi(K,L)$ stands for the elements of $S^k_\pi(K)$ which vanish on L. We denote by $H^k(S^\bullet(K,L))$ and $H^k(S^k_\pi(K,L))$ the respective resulting cohomologies.

The integration morphism

Definition (Integration morphism) : Let K be a simplicial complex realized in some euclidean space $\mathbb{R}^N$. The integration morphism is the linear map $I : S^k(K) \to C^k(K)$

defined by the relation

$$(I\omega)(\Delta^k) = \int_{\Delta^k} \omega$$

where $\omega \in S^k(K),\ \Delta^k \in K$.

Lemma 2.4 *Let K be a simplicial complex. We have the relation $\delta \circ I = I \circ d$. Moreoever, if K has bounded geometry, then I sends $S^k_\pi(K)$ to $C^k_\pi(K)$ continuously.*

Proof : The fact that $\delta \circ I = I \circ d$ is a direct corollary of Stokes theorem and of the definition of δ, which is dual to the boundary operator. Let $\omega \in S^k_\pi(K)$, so that

$$\|\omega\|_{S^k_\pi(K)} = \left(\sum_{\Delta \in K} \operatorname{esssup} |\omega_\Delta|^{p_k}\right)^{\frac{1}{p_k}} + \left(\sum_{\Delta \in K} \operatorname{esssup} |d\omega_\Delta|^{p_{k+1}}\right)^{\frac{1}{p_{k+1}}} < \infty.$$

Let $D > 0$ be such that $\mathrm{Vol}^k(\Delta^k) < D$ for any simplex Δ^k of dimension k. We have

$$\begin{aligned} \|I(\omega)\|_{p_k}^{p_k} &= \sum_{\Delta^k \in K} \left| I(\omega)(\Delta^k) \right|^{p_k} \\ &= \sum_{\Delta^k \in K} \left| \int_{\Delta^k} \omega_{\Delta^k} \right|^{p_k} \\ &\leq \sum_{\Delta^k \in K} \left(\mathrm{Vol}^k(\Delta^k) \sup_{\Delta^k} |\omega_{\Delta^k}| \right)^{p_k} \\ &\leq D^{p_k} \sum_{\Delta \in K} \sup_{\Delta} |\omega_\Delta|^{p_k}. \end{aligned}$$

Hence

$$\|I(\omega)\|_{p_k} \leq D \left(\sum_{\Delta \in K} \sup_\Delta |\omega_\Delta|^{p_k} \right)^{\frac{1}{p_k}}.$$

Similarly, since $\delta \circ I = I \circ d$, we have also

$$\|\delta I(\omega)\|_{p_{k+1}} \leq D \left(\sum_{\Delta \in K} \sup_\Delta |d\omega_\Delta|^{p_{k+1}} \right)^{\frac{1}{p_{k+1}}}.$$

Hence $\|I(\omega)\|_{C^k_\pi(K)} \leq D \cdot \|\omega\|_{S^k_\pi(K)}$. Therefore, $I\left(S^k_\pi(K)\right) \subset C^k_\pi(K)$, and $I : S^k_\pi(K) \to C^k_\pi(K)$ is bounded with norm at most D. □

The Whitney transformation

We now introduce Whitney forms on a simplicial complex. For their construction we follow [ST76]. Let $n = \dim(K)$, and let us denote by $(e_i)_{i \in \mathbb{N}}$ the vertices of K, by $\mathrm{St}(e_i)$ the star of e_i, and by b_i the barycentric coordinate function of e_i, that is the map defined as follow : for any $x \in |K|$, there exists a unique open simplex Δ such that $x \in \Delta$. Let $e_{i_0}, \dots, e_{i_k}$ be the vertices of Δ. For any $j \notin \{i_0, \dots, i_k\}$, we set $b_j(x) = 0$. For any $j \in \{i_0, \dots, i_k\}$, the real numbers $b_j(x) \in [0,1]$ are then uniquely determined by

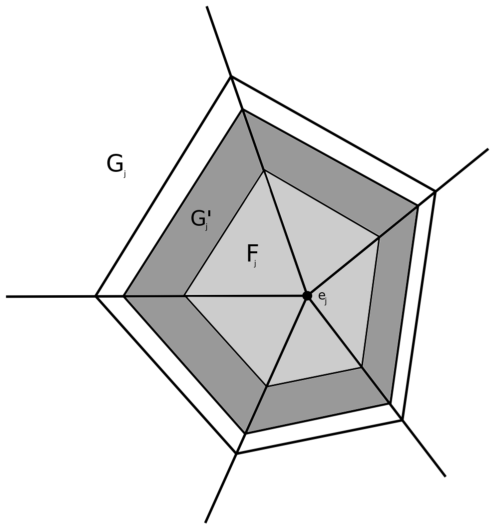
$$x = \sum_{\mu=0}^{k} b_{i_\mu}(x) e_{i_\mu}.$$

Let

$$F_i = \left\{ x \in |K| \;\middle|\; b_i(x) \geq \frac{1}{n+1} \right\}$$
$$G_i = \left\{ x \in |K| \;\middle|\; b_i(x) \leq \frac{1}{n+2} \right\}$$

Let also G_i' be the complementary set of G_i in $|K|$. Note that F_i is compact, since it is closed and contained in the bounded set $\mathrm{St}(e_i)$. Let f_i be a smooth function such that $f_i > 0$ on F_i and $f_i = 0$ on G_i. Observe that all simplices are bilipschitz-equivalent one to each other, with uniform Lipschitz functions. Hence, the functions f_i can be chosen independently of i: i.e. we can define f_i for one i and define it for all other i by composition with a bilipschitz diffeomorphism.

It is clear that (G_i') is a locally finite open covering of $|K|$, and moreover the function f_i has its support contained in G_i'.



Hence for any $x \in |K|$, the sum $\sum_{i=1}^{\infty} f_i(x)$ has only a finite number of non-zero terms. In particular, the following expression defines a smooth function $\beta_i : |K| \to \mathbb{R}$:

$$\beta_i = \frac{f_i}{\sum_{j=1}^{\infty} f_j}.$$

Now let Δ be a s-simplex of K, and Δ^k a k-face of Δ, with $k \leq s$. Let $e_{i_0}, \cdots, e_{i_s}$ be the vertices of Δ with $i_0 < \cdots < i_s$, and $e_{j_0}, \cdots, e_{j_k}$ with $i_0 \leq j_0 < \cdots < j_k \leq i_s$. Let us consider the following form:

$$\gamma_{\Delta^k \, \Delta}(x) = k! \sum_{r=0}^{k} (-1)^r \beta_{j_r} d\beta_{j_0} \wedge \cdots \wedge \widehat{d\beta_{j_r}} \wedge \cdots \wedge d\beta_{j_k}$$

where the symbol $\widehat{\ }$ means that the corresponding term is omitted. It is immediate to see that this form has degree k, is defined on all M and is zero outside of Δ.

Properties:

a) If Δ' is a face of Δ'', $\gamma_{\Delta,\Delta'} = \gamma_{\Delta,\Delta''}$,

b) For any simplex Δ_1^k and any face Δ_2^ℓ,

$$\int_\Delta \gamma_{\Delta_1^k, \Delta_2^\ell} = \mathrm{const}_{k,\ell}$$

where the constant $\mathrm{const}_{k,\ell}$ depends only on k and ℓ.

Definition (Whitney forms and Whitney transformation) (i) Let $c \in C^k(K)$ be a cochain on a triangulation K of a manifold M. The *Whitney form associated to c* is the Sullivan form of degree k given by

$$w(c)_\Delta = \sum_{\Delta^k < \Delta} c(\Delta^k) \gamma_{\Delta^k, \Delta}$$

where the notation $\Delta^k < \Delta$ means that Δ^k is a face of Δ.

(ii) The *Whitney transformation* is the linear map $w : C^k(K) \to S^k(K)$ defined by

$$w(c) = (w(c)_\Delta)_{\Delta \in K} \, .$$

The following result is classical in the proof of the usual de Rham isomorphism theorem:

Lemma 2.5 *(1)* $w \circ \delta = d \circ w$

(2) $I \circ w = \mathrm{Id}_{C^k(K)}$.

Proof : These are the points (1) and (2) of Lemma 1, chapter 6.2 of [ST76]. □

Lemma 2.6 *In the case where the triangulation K is uniform, the Whitney transformation w sends $C^k_\pi(K)$ to $S^k_\pi(K)$ continuously.*

Proof : Since the simplices are uniformly bilipschitz equivalent, there exists a constant $\kappa > 0$ such that $\left|\gamma_{\Delta',\Delta}\right|, \left|d\gamma_{\Delta',\Delta}\right| \leq \kappa$. Let $N = \dim(M)$. We have

$$\begin{aligned}
\sum_{\Delta\in K} \sup_{\Delta} |w(c)|^{p_k} &= \sum_{\Delta\in K} \sup_{\Delta} \left| \sum_{\Delta^k<\Delta} c(\Delta^k)\gamma_{\Delta^k,\Delta} \right|^{p_k} \\
&= \sum_{\Delta\in K} \sup_{\Delta} \left| \sum_{\Delta^k} [\Delta^k : \Delta] c(\Delta^k)\gamma_{\Delta^k,\Delta} \right|^{p_k} \\
&\overset{(1)}{\leq} \sum_{\Delta\in K} \sup_{\Delta} \left\{ \left(\frac{(\dim(\Delta)+1)!}{(\dim(\Delta)-k)!(k+1)!} \right)^{p_k} \sum_{\Delta^k} \left|[\Delta^k : \Delta]\right| \left|c(\Delta^k)\right|^{p_k} \left|\gamma_{\Delta^k,\Delta}\right|^{p_k} \right\} \\
&\overset{(2)}{\leq} \kappa^{p_k}(N+1)!^{p_k} \sum_{\Delta^k} \sum_{\Delta\in K} \left|[\Delta^k : \Delta]\right| \left|c(\Delta^k)\right|^{p_k} \\
&\leq \kappa^{p_k}(N+1)!^{p_k} \left\|c\right\|_{p_k}^{p_k}.
\end{aligned}$$

In inequality (1), the constant

$$\frac{(\dim(\Delta)+1)!}{(\dim(\Delta)-k)!(k+1)!}$$

is simply the number of k-faces of Δ, i.e. the number of terms in the sum

$$\sum_{\Delta^k} [\Delta^k : \Delta] c(\Delta^k)\gamma_{\Delta^k,\Delta}.$$

This constant is bounded by $(N+1)!$, which explains inequality (2).

Similarly,

$$\sum_{\Delta\in K} \sup_{\Delta} |dw(c)|^{p_{k+1}} \leq \kappa^{p_{k+1}}(N+1)!^{p_{k+1}} \left\|\delta c\right\|_{p_{k+1}}^{p_{k+1}}.$$

We thus obtain

$$\left(\sum_{\Delta\in K} \sup_{\Delta} |w(c)|^{p_k} \right)^{\frac{1}{p_k}} + \left(\sum_{\Delta\in K} \sup_{\Delta} |dw(c)|^{p_{k+1}} \right)^{\frac{1}{p_{k+1}}} \leq \kappa(N+1)! \left(\|c\|_{C^*_{p_k}(K)} + \|\delta c\|_{C^*_{p_{k+1}}(K)} \right).$$

Hence

$$\|w(c)\|_{S^*_\pi(K)} \leq \kappa(N+1)!\|c\|_{C^k_\pi(K)}.$$

$\square$

Lemma 2.7 (Inclusion) *The isomorphism $\phi^{-1} : S^k(K) \to \Omega^k_{\infty,\mathrm{loc}}(M)$ of lemma 2.2 sends $S^k_\pi(K)$ onto $\Omega^k_\pi(M)$. Moreover, the operator $\phi^{-1} : S^k_\pi(K) \to \Omega^k_\pi(M)$ is bounded.*

Proof : Let $\omega \in S^k_\pi(K)$ be a L_π-Sullivan form of degree k, and let θ be defined by $\theta = \phi^{-1}(\omega) \in \Omega^k_{\infty,\mathrm{loc}}(M)$. Then

$$\begin{aligned}
\|\omega\|_{\Omega^k_\pi(M)} &= \left(\int_M |\omega|^{p_k}\right)^{\frac{1}{p_k}} + \left(\int_M |d\omega|^{p_{k+1}}\right)^{\frac{1}{p_{k+1}}} \\
&= \left(\sum_\Delta \int_\Delta |\theta|^{p_k}\right)^{\frac{1}{p_k}} + \left(\sum_\Delta \int_\Delta |d\theta|^{p_k}\right)^{\frac{1}{p_k}} \\
&\leq \mathrm{cte}\, \|\theta\|_{S^*_\pi(K)}
\end{aligned}$$

This shows the result. □

We can now prove our isomorphism theorem.

Proof of the de Rham theorem

First, we prove that there is an isomorphism in cohomology between $H^k(S^*_\pi(M))$ and $H^k_\pi(M)$.

Lemma 2.8 *Let (M, g) be a Riemannian manifold, and assume that M admits a uniform triangulation $\tau : |K| \to M$. Let π be a sequence of real numbers such that one of the following conditions hold :*

(1) $1 < p_k < \infty$ and $\frac{1}{p_k} - \frac{1}{p_{k-1}} \leq \frac{1}{n}$, or

(2) $1 \leq p_k < \infty$ and $\frac{1}{p_k} - \frac{1}{p_{k-1}} < \frac{1}{n}$.

Then for any k there is a vector space isomorphism

$$H^k\left((S^\bullet_\pi(K)\right) = H^k_\pi\left(M\right).$$

Proof : By point 2 of the regularization theorem 1.6, we know that $R^M_\varepsilon : \Omega^\bullet_\pi(M) \to \Omega^\bullet_\pi(M)$ is a morphism of Banach complexes. By point 1, it has its image contained in the subcomplex $S^\bullet_\pi(K)$, and by point 6, it is homotopic to the identity $\mathrm{Id}_{S^\bullet_\pi(K)}$. By proposition A.2 of chapter 4, we can conclude. □

The results still holds in reduced cohomology:

Lemma 2.9 *Let (M, g) be a Riemannian manifold, and assume that M admits a uniform triangulation $\tau : |K| \to M$. Let π be a sequence of real numbers such that one of the following conditions hold :*

(1) $1 < p_k < \infty$ *and* $\frac{1}{p_k} - \frac{1}{p_{k-1}} \le \frac{1}{n}$*, or*

(2) $1 \le p_k < \infty$ *and* $\frac{1}{p_k} - \frac{1}{p_{k-1}} < \frac{1}{n}$.

$$\overline{H}^k \left(S^\bullet_\pi(K)\right) = \overline{H}^k_\pi \left(M\right).$$

Proof : A homotopy is a weak homotopy as well, and the operators R^M_ε of the regularization theorem are continous. Hence by point 2 of proposition A.3, we have the conclusion. □

Lemma 2.10 *Suppose that* Δ^ℓ *is a standard simplex of dimension* ℓ*. Then*

$$H^k(S^\bullet(\Delta^\ell, \partial\Delta^\ell)) = \begin{cases} 0 \text{ if } k \neq \ell \\ \mathbb{R} \text{ if } k = \ell \end{cases}$$

where the isomorphism $H^k(S^\bullet(\Delta^k, \partial\Delta^k)) \to \mathbb{R}$ *is given by*

$$\theta \mapsto \int_{\mathbb{B}^k} \theta.$$

Proof : The proof is similar to the computation of the classical de Rham cohomology of smooth forms. In the classical case, it relies essentially on:

- Two short exact sequences in cohomology (the Mayer-Vietoris sequence and the relative cohomology sequence);
- Algebraic work (diagram chasing) to derive long exact sequences from the two above.

The Mayer Vietoris sequence still exists for L^p forms and flat forms: indeed, it only involves the existence of a restriction operator (to an open subset). Moreover, it is still exact. Hence, the long exact Mayer Vietoris sequence still exists.

Since flat forms may be restricted, the short exact sequence in relative cohomology still exists for flat forms as well, as it relies only on the existence of a restriction operator.

□

Now we prove that there is an isomorphism in cohomology between $H^k(S^\bullet_\pi(K))$ and $H^k(C^\bullet_\pi(K))$.

Lemma 2.11 *Let* $1 \le p_k < \infty$ *a non-increasing sequence of real numbers. For a Riemannian manifold* (M, g) *with bounded geometry and a bounded triangulation* K *of* M*, we have*

$$H^k(S^\bullet_\pi(K)) = H^k_\pi(K).$$

Proof : By lemmas 2.5 and 2.6, the map

$$I : S^k_\pi(K) \to C^k_\pi(K)$$

is continuous and surjective. Hence, the map induced in cohomology

$$I : H^k(S^\bullet_\pi(K)) \to H^k_\pi(K)$$

is continuous and surjective as well. We have to prove that it is also injective. In fact we will prove more generally that $I : H^k\left(S^\bullet_\pi(K, L)\right) \to H^k_\pi(K, L)$ is injective, where L is a subcomplex.

For this purpose, we must show the following : if $\theta \in S^k_\pi(K, L)$ is a Sullivan form of degree k, such that $d\theta = 0$, and that $I\theta = \delta c$ for some $c \in C^{k-1}_\pi(K, L)$, there exists $\omega \in S^{k-1}_\pi(K, L)$ such that $d\omega = \theta$.

In the case where $k = 0$, the form θ is simply a function, and the condition $d\theta = 0$ means that θ is constant. Moreover, the condition $I\theta = \delta c$ tells us that $I\theta = 0$, for 0 is the only exact simplicial cochain of degree -1. But for any 0-simplex (i.e. vertex) v, one has

$$I\theta(v) = \int_v \theta = \theta(v).$$

Hence $\theta = 0$, and this form is thus exact.

We can suppose now that $k > 0$. Let us denote by $K^{(j)}$ the j-th skeleton of K, and by K_j the subcomplex $K^{(j)} \cup L$. We construct for each $j \geq k$ a $(k-1)$-form $\omega_j \in S^{k-1}_\pi(K_j, L)$ such that $d\omega_j = j^*_{K_j,K}\theta$. Since $\dim(K)$ is finite, this procedure will end up after a finite number of steps.

We distinguish three cases: $j \leq k-1$, $j = k$ and $j > k$.

(a) First, for any $j \leq k-1$, we can simply set ω to be 0 on the j-skeleton. Hence, let us set

$$\omega_j = 0 \text{ for any } j \leq k-1.$$

(b) Suppose now that $j = k$. We must define ω_{k,Δ^k} for any k-simplex Δ^k.

We know that $d\theta = 0$. In particular, θ_{Δ^k} is an element of $H^k(S^\bullet(\Delta^k, \partial\Delta^k))$. Moreover, $I\theta_{\Delta^k} = 0$ in cohomology. But by lemma 2.10, I establishes an isomorphism between $H^k(S^\bullet(\Delta^k, \partial\Delta^k))$ and $\mathbb{R}$. Hence at the cohomology level, $\theta_{\Delta^k} = 0$, i.e. θ_{Δ^k} is exact. It means that $\theta_{\Delta^k} \in B^k(\Delta^k)$, where $B^k(\Delta^k)$ is the space of exact forms of $S^k(\Delta^k, \partial\Delta^k)$. Finding a primitive yet doesn't suffices, as we have to control its norm.

By lemma 2.10, $H^k\left(S^\bullet(\Delta^k, \partial\Delta^k)\right)$ is a finite-dimensional vector space. In particular, the space $B^k(\Delta^k)$ is closed and the map $d : S^{k-1}(\Delta^k, \partial\Delta^k) \to B^k(\Delta^k)$ is thus a continuous and surjective map between Banach spaces. By the open map theorem, there exists a constant C_{Δ^k} such that for any $\alpha \in B^k(\Delta^k)$, there exists $\beta_{\Delta^k} \in S^{k-1}(\Delta^k, \partial\Delta^k)$ verifying $d\beta_{\Delta^k} = \alpha$ and

$$\|\beta_{\Delta^k}\|_\infty \leq C_{\Delta^k} \|\alpha\|_\infty .$$

Moreover, the constant C_{Δ^k} can be chosen uniformly : indeed, all the simplices of a given dimension are the same up to a Bilipschitz change of coordinates, with uniform Lipschitz constants. We can thus chose $C \geq C_{\Delta^k}$ for any Δ^k.

Let us apply this result to $\alpha = \theta_{\Delta^k}$. There exists a form $\omega_{k,\Delta^k} \in S^{k-1}(\Delta^k, \partial\Delta^k)$ such that $d\omega_{k,\Delta^k} = \theta_{\Delta^k}$ and

$$\left\|\omega_{k,\Delta^k}\right\|_\infty \leq C\left\|\theta_{\Delta^k}\right\|_\infty.$$

Let $\omega_k = \left(\omega_{k,\Delta^k}\right)_{\Delta^k}$. We claim that $\omega_k \in S_\pi^{k-1}(K_k, L)$. Let us recall that for $p_{k-1} \geq p_k$, there exists a continuous inclusion $\ell^{p_k}(\mathbb{N}) \subset \ell^{p_{k-1}}(\mathbb{N})$ with norm at most 1. The set of simplices Δ^k of $K^{(k)}$ is countable and one has $p_{k-1} \geq p_k$, hence

$$\begin{aligned}
\left(\sum_{\Delta^k\in K_k} \|\omega_{k,\Delta}\|_\infty^{p_{k-1}}\right)^{\frac{1}{p_{k-1}}} &\leq \left(\sum_{\Delta^k\in K_k} \|\omega_{k,\Delta}\|_\infty^{p_k}\right)^{\frac{1}{p_k}} \\
&\leq \left(\sum_{\Delta\in K_k} \left(C\left\|\theta_\Delta\right\|_\infty\right)^{p_k}\right)^{\frac{1}{p_k}} \\
&\leq C\left(\sum_{\Delta\in K_k} \|\theta_\Delta\|_\infty^{p_k}\right)^{\frac{1}{p_k}} < \infty.
\end{aligned}$$

Moreover, since $d\omega_{k,\Delta} = \theta_\Delta$, one has

$$\left(\sum_{\Delta\in K_k} \|d\omega_{k,\Delta}\|_\infty^{p_k}\right)^{\frac{1}{p_k}} = \left(\sum_{\Delta\in K_k} \|\theta_\Delta\|_\infty^{p_k}\right)^{\frac{1}{p_k}} < \infty.$$

Those two inequalities yield the fact that $\omega_k \in S_\pi^{k-1}(K_k, L)$. More precisely, one has

$$\|\omega_k\|_{S_\pi^{k-1}(K_k,L)} \leq (C+1)\|\theta\|_{S_\pi^k(K_k,L)}.$$

The form ω_k thus satisfies our conditions.

(c) We still have to construct ω_j for $j > k$. Let us suppose that we have so far constructed a form $\omega_{j-1} \in S_\pi^{k-1}(K_{j-1}, L)$ such that $d\omega_{j-1} = j^*_{K_{j-1},K}\theta$. Let $\omega' \in S_\pi^{k-1}(K_j, L)$ be an extension of ω_{j-1} to K_j. We are going to add to ω' a "primitive" of $\theta - d\omega'$.

Suppose that Δ^j is a simplex of dimension j. Its boundary is a sum of simplices of dimension $j-1$, hence we have $j^*_{\partial\Delta,K_j}(\theta - d\omega') = 0$. We have $d(\theta - d\omega') = 0$ on Δ^j, thus $\theta - d\omega'$ is closed. Moreover, it is an element of $S^k(\Delta^j, \partial\Delta^j)$, and by lemma 2.10, one has

$$H^k\left(S^\bullet(\Delta^j, \partial\Delta^j)\right) = 0.$$

Hence $\theta - d\omega' \in B^k(\Delta^j)$ which is a Banach space, and by the open map theorem, there exists $\omega'' \in S^k_\pi(K_j, L)$ such that

$$d\omega'' = \theta - d\omega'.$$

Let $\omega_j = \omega' + \omega'' \in S^k_\pi(K_j, L)$. We have

$$\begin{aligned} d\omega_j &= d\omega' + d\omega'' \\ &= d\omega' + \theta - d\omega' \\ &= \theta. \end{aligned}$$

By induction, for j large enough, $\omega = \omega_j$ is the one we are looking for.

□

Observe that the inequality

$$\|\omega_k\|_{S^{k-1}_\pi(K,L)} \leq (C+1)\|\theta\|_{S^k_\pi(K,L)}$$

establishes the continuity of our construction.
In particular, our isomorphism restricts to the reduced cohomology setting:

Lemma 2.12 *Let $1 \leq p_k < \infty$ a non-increasing sequence of real numbers. For a Riemannian manifold (M, g) with bounded geometry and a bounded triangulation K of M, we have*

$$\overline{H}^k(S^\bullet_\pi(K)) = \overline{H}^k_\pi(K)$$

□

Let us summarize the situation. We have three Banach complexes together with morphisms

$$\Omega^\bullet_\pi(M) \underset{\iota}{\overset{R^M}{\rightleftarrows}} S^\bullet_\pi(K) \underset{w}{\overset{I}{\rightleftarrows}} C^\bullet_\pi(K)$$

where ι is the inclusion. These morphisms induce linear maps at the cohomology and reduced cohomology level :

$$H^k_\pi(M) \underset{\iota}{\overset{R^M}{\rightleftarrows}} H^k\left(S^\bullet_\pi(K)\right) \underset{w}{\overset{I}{\rightleftarrows}} H^k_\pi(K) \qquad \overline{H}^k_\pi(M) \underset{\iota}{\overset{R^M}{\rightleftarrows}} \overline{H}^k\left(S^\bullet_\pi(K)\right) \underset{w}{\overset{I}{\rightleftarrows}} \overline{H}^k_\pi(K)$$

Using the isomorphisms given by 2.8, 2.9, 2.11, 2.12, we now have the following theorem:

Theorem 2.13 *[de Rham isomorphism for L_π-cohomology] Let (M, g) be a non-compact, orientable, complete and connected Riemannian manifold, and assume that M admits a uniform triangulation $\tau : |K| \to M$. Let π be a sequence of numbers satisfying one of the following hypothesis :*

(1) $1 < p_k < \infty$ *and* $0 \leq \frac{1}{p_k} - \frac{1}{p_{k-1}} \leq \frac{1}{n}$, *or*

(2) $1 \leq p_k < \infty$ *and* $0 \leq \frac{1}{p_k} - \frac{1}{p_{k-1}} < \frac{1}{n}$.

Then for any k there are vector space isomorphisms

$$H^k_\pi(M) = H^k_\pi(K) \qquad \textit{and} \qquad \overline{H}^k_\pi(M) = \overline{H}^k_\pi(K)$$

and the latter is continuous.

□

Monotonicity for non-compact manifolds

As corollary, we can adapt the monotonicity results 1.34 and 1.35 to the Riemannian setting:

Lemma 2.14 *Let M be a Riemannian manifold with bounded geometry, p, q_1, q_2 three real numbers satisfying one of the following hypothesis :*

(1) $1 \leq p, q_1, q_2 < \infty$, *as well as* $0 \leq \frac{1}{p} - \frac{1}{q_2} < \frac{1}{n}$, $\quad 0 \leq \frac{1}{p} - \frac{1}{q_1} < \frac{1}{n}$ *and* $q_2 \geq q_1$, *or*

(2) $1 < p, q_1, q_2 < \infty$, *as well as* $0 \leq \frac{1}{p} - \frac{1}{q_2} \leq \frac{1}{n}$, $\quad 0 \leq \frac{1}{p} - \frac{1}{q_1} \leq \frac{1}{n}$ *and* $q_2 \geq q_1$.

Then the following inclusions hold:

$$H^k_{q_2p}(M) \subset H^k_{q_1p}(M) \qquad \textit{and} \qquad \overline{H}^k_{q_2p}(M) \subset \overline{H}^k_{q_1p}(M).$$

Proof : For a uniform triangulation K of M, there exist vectors space isomorphisms

$$H^k_{q_2p}(K) = H^k_{q_2p}(M) \text{ and } H^k_{q_1p}(K) = H^k_{q_1p}(M),$$
$$\overline{H}^k_{q_2p}(K) = \overline{H}^k_{q_2p}(M) \text{ and } \overline{H}^k_{q_1p}(K) = \overline{H}^k_{q_1p}(M).$$

By lemma 1.34, one has $H^k_{q_2p}(K) \subset H^k_{q_1p}(K)$ and $\overline{H}^k_{q_2p}(K) \subset \overline{H}^k_{q_1p}(K)$, hence

$$H^k_{q_2p}(M) \subset H^k_{q_1p}(M) \text{ and } \overline{H}^k_{q_2p}(M) \subset \overline{H}^k_{q_1p}(M).$$

□

Lemma 2.15 *Let M be a Riemannian manifold with bounded geometry, p, q_1, q_2 three real numbers satisfying one of the following hypothesis :*

(1) $1 \leq p, q_1, q_2 < \infty$, *as well as* $0 \leq \frac{1}{p_2} - \frac{1}{q} < \frac{1}{n}$, $\quad 0 \leq \frac{1}{p_1} - \frac{1}{q} < \frac{1}{n}$ *and* $p_2 \leq p_1$, *or*

(2) $1 < p, q_1, q_2 < \infty$, *as well as* $0 \leq \frac{1}{p_2} - \frac{1}{q} \leq \frac{1}{n}$, $\quad 0 \leq \frac{1}{p_1} - \frac{1}{q} \leq \frac{1}{n}$ *and* $p_2 \leq p_1$.

Then

$$H^k_{q,p_1}(M) = 0 \Rightarrow H^k_{q,p_2}(M) = 0.$$

Proof : For any uniform triangulation K of M, one has

$$H^k_{q,p_1}(M) = H^k_{q,p_1}(K) \text{ and } H^k_{q,p_2}(M) = H^k_{q,p_2}(K).$$

Moreover, by lemma 1.35, we know that $H^k_{q,p_1}(K) = 0 \Rightarrow H^k_{q,p_2}(K) = 0$. Hence

$$H^k_{q,p_1}(M) = 0 \Rightarrow H^k_{q,p_2}(M) = 0.$$

□

Chapter 3

Quasi-isometry invariance

In this chapter, we define a L_π-cohomology notion for graphs, and we prove, following a strategy of Gábor Elek, that under suitable assumptions on a sequence π of real numbers, the L_π-cohomologies of two uniformly contractible quasi-isometric Riemannian manifolds with bounded geometry coincide.

The strategy is the following: first, we prove that the so-called coarse L_π-cohomology of two graphs is a quasi-isometry invariant. Then, to each simplicial complex K with bounded geometry, we attach a graph G (namely its 0-skeleton together with the distance induced by its 1-skeleton) and prove that the L_π simplicial cohomology of K coincides with the coarse L_π-cohomology of G. If K and K' are quasi-isometric simplicial complexes, their 0-skeleta G and G' are also quasi-isometric, and thus $H^k_\pi(K) = H^k_\pi(K')$. This result implies that for quasi-isometric Riemannian manifolds M, M' admitting a good triangulation, $H^k_\pi(M) = H^k_\pi(M')$, since de Rham theorem allows to induce the quasi-isometry on the simplicial setting.

Coarse L_π-cohomology

Let G be a metric graph, and let us recall that V_G denotes the set of vertices of G, together with the metric induced by G. In the sequel, we consider graphs that have *bounded geometry*:

Definition (Graph with bounded geometry) Let G be a graph. One says that G has *bounded geometry* if there is a uniform bound on the number of neighbors of a vertex.

For example, the 1-skeleton of a simplicial complex with bounded geometry, with the length metric, is such a graph.

Definition Let G be a graph with bounded geometry. For $k \in \mathbb{N}$ and $R > 0$, the *penumbra* of radius R and order k of G is the set

$$\mathrm{Pen}(G,R) = \left\{ (x_0, \ldots, x_k) \in V_G^{k+1} \;\middle|\; d(x_i, x_j) \leq R, 0 \leq i,j \leq k \right\}.$$

Among other characterizations, it is the R-neighborhood of the diagonal in V_G^{k+1}.

Let $1 \leq p < \infty$. We define

$$CX_p^k(G) = \left\{ \alpha : V_G^{k+1} \to \mathbb{R} \;\middle|\; \sum_{(x_o,\dots,x_k)\in \mathrm{Pen}(G,R)} |\alpha(x_0,\dots,x_k)|^p < \infty \text{ for any } R > 0 \right\}.$$

We endow $CX_p^k(G)$ with the Frechet topology given by the family of semi-norms ρ_R given by

$$\rho_R(\alpha) = \left(\sum_{(x_o,\dots,x_k)\in \mathrm{Pen}(G,R)} |\alpha(x_0,\dots,x_k)|^p \right)^{\frac{1}{p}}.$$

The space $CX_p^k(G)$ is thus a topological vector space, and is metrizable: indeed, a metric inducing its topology is for example:

$$d(\alpha,\beta) = \sum_{R\in\mathbb{N}} \frac{\rho_R(\alpha-\beta)}{1+\rho_R(\alpha-\beta)}.$$

The differential map defined by

$$d\alpha(x_0,\dots,x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i \alpha(x_0,\dots,\widehat{x_i},\dots,x_{k+1})$$

sends obviously $CX_p^k(G)$ onto $CX_p^{k+1}(G)$ in a continuous way. For $1 \leq q,p < \infty$, let

$$\Omega X_{q,p}^k(G) = \left\{ \alpha \in CX_q^k(G) \;\middle|\; d\alpha \in CX_p^{k+1}(G) \right\}.$$

If π is as usual a sequence of real numbers $1 \leq p_k < \infty$, one denotes by $\Omega X_\pi^k(G)$ the vector space $\Omega X_{p_k p_{k+1}}^k(G)$.

With these notations, one has a cochain complex of vector spaces

$$\cdots \longrightarrow \Omega X_\pi^{k-1}(G) \xrightarrow{d} \Omega X_\pi^k(G) \xrightarrow{d} \Omega X_\pi^{k+1}(G) \longrightarrow \cdots$$

Let us denote by $HX_\pi^k(G)$ the cohomology in degree k of this complex, that is

$$HX_\pi^k(G) = ZX_{p_k}^k(G)/BX_{p_{k-1}p_k}^k(G)$$

Where

$$ZX_{p_k}^k(G) = \ker(d) \cap CX_{p_k}^k(G)$$

and

$$BX_{p_{k-1}p_k}^k(G) = dCX_{p_{k-1}}^{k-1}(G) \cap CX_{p_k}^k(G) = d\left(\Omega X_\pi^{k-1}(G)\right).$$

We will also use the notations

$$ZX_\pi^k(G) = ZX_{p_k}^k(G) \text{ and } BX_\pi^k(G) = BX_{p_{k-1}p_k}^k(G).$$

Definition The cohomology $HX_\pi^*(G)$ is called the L_π-*coarse-cohomology* of G.

Let $\overline{BX}_\pi^k(G)$ the closure of $BX_\pi^k(G)$, i.e. the space of coarse cochains $\alpha \in \Omega X_\pi^k(G)$ such that there exists a sequence $(u_n) \subset \Omega X_\pi^{k-1}(G)$ with

$$\sum_{(x_0,\dots,x_k)\in \mathrm{Pen}(G,n)} |(du_n - \alpha)(x_0,\dots,x_k)|^{p_k} \le \frac{1}{2^n}.$$

This leads to the following definition:

Definition (Coarse reduced L_π-cohomology of a graph) Let G be a graph. The *(coarse) reduced L_π-cohomology of* G is the quotient

$$\overline{HX}_\pi^k(G) = Z_\pi^k(G)/\overline{B}_\pi^k(G).$$

Quasi-isometry invariance of the coarse cohomology

Let $\phi : G \to H$ be a map between graphs and $\alpha : V_H^{k+1} \to \mathbb{R}$ be a real-valued map, $\phi^*\alpha$ denotes the real-valued map $\phi^*\alpha : V_G^{k+1} \to \mathbb{R}$ given by $\phi^*\alpha(x_0,\dots,x_k) = \alpha(\phi(x_0),\dots,\phi(x_k))$. Let us also observe that $d\phi^*\alpha = \phi^* d\alpha$ for any $\phi : G \to H$ and $\alpha : V_H^{k+1} \to \mathbb{R}$. In particular, any map $\phi : G \to H$ acts at the cohomology level.

Definition A map $\phi : V \to W$ is (L, C)-*quasi-Lipschitz* if there exist constants $L, C > 0$ such that:

$$d(\phi(x), \phi(y)) \le L \cdot d(x, y) + C.$$

The map has *bounded multiplicity* if

$$M = \max_{y\in W} \mathrm{Card}\left(\phi^{-1}(y)\right) < \infty.$$

M is the *multiplicity* of the map.

Lemma 3.1 *Let G, H be two graphs, and $\phi : G \to H$ a (L, C)-quasi-Lipschitz map with bounded multiplicity. Then $\phi^*\left(CX_p^k(H)\right) \subset CX_p^k(G)$. Moreover, ϕ^* sends $CX_p^k(H)$ into $CX_p^k(G)$ continuously.*

Proof : Observe first that for any $R > 0$, one has

$$\phi\left(\mathrm{Pen}(G, R)\right) \subset \mathrm{Pen}\left(H, CR + L\right).$$

For any $R > C$, one thus has

$$\begin{aligned}
\rho_{\frac{R-C}{L}}(\phi^*\beta)^p &= \sum_{(x_0,\dots,x_k)\in \mathrm{Pen}(G,\frac{R-C}{L})} |(\phi^*\beta)(x_0,\dots,x_k)|^p \\
&\le M^{k+1} \sum_{(y_0,\dots,y_k)\in \mathrm{Pen}(H,R)} |\beta(y_0,\dots,y_k)|^p \\
&\le M^{k+1} \cdot \rho_R(\beta)^p
\end{aligned}$$

where M is the multiplicity of ϕ. This inequality yields the continuity.

□

Remark 3.1 As noticed above, $\phi^* : CX_p^k(H) \to CX_p^k(G)$ is a chain map: $\phi^* d = d\phi^*$. Hence if $\phi : G \to H$ is a (C, L)-quasi-Lipschitz map, it sends $CX_p^k(H)$ into $CX_p^k(G)$ for any choice of k, p and therefore induces a map at the cohomology level

$$\phi^* : HX_\pi^k(H) \to HX_\pi^k(G).$$

Definition Two maps $f, g : V \to W$ between metric spaces are *parallel* if

$$\sup_{x \in V} d(f(x), g(x)) < \infty.$$

Lemma 3.2 *Let G, H be two graphs with bounded geometry, and $\phi, \psi : G \to H$ two parallel (C, L)-quasi-Lipschitz maps. Let also $1 \le p < \infty$ and $k \in \mathbb{N}$. Then the map $T : CX_p^{k+1}(H) \to CX_p^k(G)$ defined by*

$$T\beta(x_0, \dots, x_{k-1}) = \sum_{\mu=0}^{k-1} (-1)^\mu \beta(f(x_0), \dots, f(x_\mu), g(x_\mu), \dots, g(x_{k-1}))$$

has the following properties:

(1) T is continuous;

(2) T is a homotopy from f^* to g^* in the following sense:

$$f^* - g^* = dT + Td.$$

Proof : Let us first prove that T is continuous. We can assume that f and g are (L, C) quasi-lipshitz and C-parallel, i.e. $d(f(x), g(x)) \le C$ for any $x \in V$. If $E \subset V$ is a set of diameter r, then $f(E) \subset W$ and $g(E) \subset W$ have at most a diameter $Lr + C$ and $f(E) \cup g(E) \subset W$ has diameter at most $Lr + 2C$ because f and g are C-parallel. This implies that $(x_0, \dots, x_{k-1}) \in P_{k,r}(V) \Rightarrow ((f(x_0), \dots, f(x_\mu), g(x_\mu), g(x_{\mu+1})) \in P_{k+1, Lr+2C}(W)$, thus

$$\rho_{\frac{r-2C}{L}}(T\beta) \le k\rho_r(\beta)$$

and T is thus continuous.
We now prove the identity (2). To simplify the calculation, we shall write $y_i = f(x_i)$ and $z_j = g(x_j)$, thus

$$T\beta(x_0, \dots, x_{k-1}) = \sum_{\mu=0}^{k-1} (-1)^\mu \beta(y_0, \dots, y_\mu, z_\mu, \dots z_{k-1}).$$

Thus $T(d\beta)$ is the following sum containing $(k+2)(k+1)$ terms:

$$T(d\beta)(x_0,\ldots,x_k) = \sum_{\mu=0}^{k}(-1)^{\mu}d\beta(y_0,\ldots,y_\mu,z_\mu,\ldots z_k)$$

$$= \sum_{\mu=0}^{k}\left(\sum_{j=0}^{\mu}(-1)^{j+\mu}\beta(y_0,\ldots,\widehat{y_j},\ldots,y_\mu,z_\mu,\ldots z_k)\right.$$
$$\left. + \sum_{j=\mu}^{k}(-1)^{j+\mu+1}\beta(y_0,\ldots,y_\mu,z_\mu,\ldots,\widehat{z_j},\ldots z_k)\right),$$

which can be rewritten as

$$T(d\beta)(x_0,\ldots,x_k) =$$
$$\sum_{0\le j\le\mu\le k}(-1)^{j+\mu}\beta(y_0,\ldots,\widehat{y_j},\ldots,y_\mu,z_\mu,\ldots z_k)$$
$$- \sum_{0\le\mu\le j\le k}(-1)^{j+\mu}\beta(y_0,\ldots,y_\mu,z_\mu,\ldots,\widehat{z_j},\ldots z_k).$$

Likewise $d(T\beta)$ is the following sum containing $k(k+1)$ terms:

$$d(T\beta)(x_0,\ldots,x_k) = \sum_{j=0}^{k}(-1)^{j}(T\beta)(x_0,\ldots,\widehat{x_j},\ldots x_k)$$

$$= \sum_{j=0}^{k}\left(\sum_{\mu=j+1}^{k}(-1)^{j+\mu-1}\beta(y_0,\ldots,\widehat{y_j},\ldots,y_\mu,z_\mu,\ldots z_k)\right.$$
$$\left. + \sum_{\mu=0}^{j-1}(-1)^{j+\mu}\beta(y_0,\ldots,y_\mu,z_\mu,\ldots,\widehat{z_j},\ldots z_k)\right).$$

And this can be rewritten as

$$d(T\beta)(x_0,\ldots,x_k) =$$
$$- \sum_{0\le j<\mu\le k}(-1)^{j+\mu}\beta(y_0,\ldots,\widehat{y_j},\ldots,y_\mu,z_\mu,\ldots z_k)$$
$$+ \sum_{0\le\mu<j\le k}(-1)^{j+\mu}\beta(y_0,\ldots,y_\mu,z_\mu,\ldots,\widehat{z_j},\ldots z_k).$$

Adding now $T(d\beta)+d(T\beta)$ kills all terms with $\mu\neq j$ leaving us with the sum of $2(k+1)$ terms corresponding to $\mu=j$

$$(T(d\beta)+d(T\beta))(x_0,\ldots,x_k) =$$
$$\sum_{\mu=0}^{k}\left(\beta(y_0,\ldots,\widehat{y_\mu},z_\mu,\ldots z_k) - \beta(y_0,\ldots,y_\mu,\widehat{z_\mu},\ldots z_k)\right).$$

Observe that for any $\mu = 0, , k-1$, we have

$$\beta(y_0, \ldots, y_\mu, \widehat{z_\mu}, \ldots z_k) = \beta(y_0, \ldots, \widehat{y_{\mu+1}}, z_{\mu+1}, \ldots z_k).$$

The previous sum enjoys a telescoping cancelation and we finally obtain

$$\begin{aligned}(T(d\beta) + d(T\beta))(x_0, \ldots, x_k) &= \beta(z_0, \ldots z_k) - \beta(y_0, \ldots y_k) \\ &= (g^*(\beta) - f^*(\beta))(x_0, \ldots, x_k).\end{aligned}$$

□

Corollary 3.3 *Let $f, g : G \to H$ be two parallel (C, L)-quasi-Lipschitz maps between graphs with bounded geometry, and π a decreasing sequence of real numbers $1 \leq p_{k+1} \leq p_k < \infty$. Then f and g induce the same linear maps at the cohomology level:*

$$f^* = g^* : HX^k_\pi(H) \to HX^k_\pi(G).$$

Proof : Let $\beta \in \Omega X^k_\pi(H)$. Then $\beta \in CX^k_{p_k}(H)$, hence $T\beta \in CX^{k-1}_{p_k}(G) \subset CX^{k-1}_{p_{k-1}}(G)$ and $dT\beta \in CX^k_{p_k}(G)$. Consequently, if $d\beta = 0$, one has

$$f^*\beta - g^*\beta = d\gamma$$

with $\gamma = T\beta \in \Omega X^{k-1}_\pi(G)$. Hence $f^*\beta - g^*\beta \in d\Omega X^{k-1}_\pi(G) = BX^k_\pi(G)$. This shows that $f^* - g^* = 0$ at the cohomology level. □

We may now prove that the coarse L_π-cohomology of a graph is a quasi-isometry invariant.

Theorem 3.4 *Let G, H be two graphs, π a decreasing sequence of real numbers $1 < p_k < \infty$, and let $\phi : G \to H$ be a quasi-isometry. Then*

$$\phi^* : HX^k_\pi(H) \to HX^k_\pi(G)$$

is an isomorphism of vector spaces.

Proof : Since $\phi : G \to H$ is a quasi-isometry, there exists a quasi-isometry $\psi : H \to G$ such that

$$\sup_{x \in G} d(x, \psi \circ \phi(x)) < \infty$$

and

$$\sup_{y \in H} d(y, \psi \circ \phi(y)) < \infty.$$

That is, $\phi \circ \psi$ and $\psi \circ \phi$ are parallel to $\mathrm{Id}_H, \mathrm{Id}_G$ respectively. Moreover, since $\phi \circ \psi$ and $\psi \circ \phi$ are quasi-isometries (as composition of quasi-isometries), they are in particular quasi-Lipschitz. Hence by lemma 3.3, the maps $\phi \circ \psi$ and $\psi \circ \phi$ coincide with identities at the cohomology level:

$$(\phi \circ \psi)^* = \mathrm{Id} : HX_G^k \to HX_H^k$$

and

$$(\psi \circ \phi)^* = \mathrm{Id} : HX_H^k \to HX_G^k.$$

By functoriality, this means that $\phi^* \circ \psi^* = \mathrm{Id}$ and $\psi^* \circ \phi^* = \mathrm{Id}$, i.e. ϕ^* and ψ^* are inverse one to each other.

□

The case of reduced cohomology

From lemma 3.1, we know that if ϕ is quasi-Lipschitz, then $\phi^* : CX_p^k(H) \to CX_p^k(G)$ is continuous with respect to the Frechet topology of $CX_p^k(H)$ and $CX_p^k(G)$. In particular, ϕ^* sends $\overline{BX}_\pi^k(H)$ onto $\overline{BX}_\pi^k(G)$, and thus induces a map at the reduced cohomology level:

$$\phi^* : \overline{HX}_\pi^k(H) \to \overline{HX}_\pi^k(G).$$

Since the map T is continuous, the result 3.4 still stands in non-reduced cohomology:

Lemma 3.5 *Let G, H be two graphs, π a decreasing sequence of real numbers $1 < p_k < \infty$, and let $\phi : G \to H$ be a quasi-isometry. Then*

$$\phi^* : \overline{HX}_\pi^k(H) \to \overline{HX}_\pi^k(G)$$

is an isomorphism of vector spaces. Moreover, it is continuous with respect to the Frechet topology.

□

Uniformly contractible metric spaces

We will need a restriction on the topology and geometry of our objects:

Definition A metric space (X, d) is *uniformly contractible* if the following condition holds: there exists a function $R : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any ball $B(x_0, r)$, there exists a homotopy $F : [0,1] \times B(x_0, r) \to B(x_0, R(r))$ from the identity to the constant map x_0. In other words, any ball of radius r retracts to a point within a ball of radius $R(r)$.

Any uniformly contractible metric space is clearly contractible. However, the converse is not true. The following examples go back to Gromov (see [Gro93]): for any integer $n \geq 1$, let $\mathbb{S}_r^2$ be the 2-sphere of radius r and let S_n be the space obtained by removing a disk with euclidean perimeter 2π and containing the north pole (see picture 3.1).

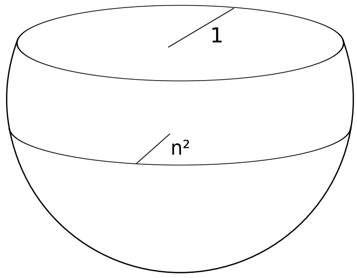


Figure 3.1: A cutsphere

At each integer point n of the real line, let us attach the space S_n from its south pole (see figure 3.2 below). Then we obtain a contractible space, which is however not uniformly contractible. Indeed, a circle of radius 1 located near the boundary of a sphere will eventually need to go through the equator in order to be contracted onto a point. Since the equators can be as large as desired, this forbids this space to be uniformly contractible.

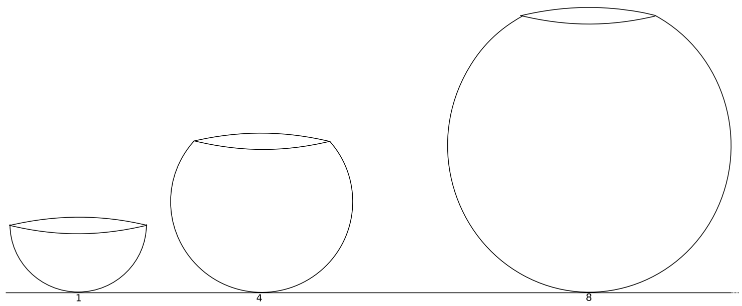


Figure 3.2: A contractible yet non-uniformly contractible space

A second example can be obtained in the following way: one takes $\mathbb{R}^2$, and gives it a distance which makes it isometric to the standard cylinder $\mathbb{S}^1 \times [0, \infty[$ outside of a ball of finite radius.
This copy of $\mathbb{R}^2$ is of course contractible, but non-uniformly contractible.

Rips thickenings The main difference between examples of figure 3 and 3 is the following: the first one admits no uniformly contractible *Rips thickening*. We define this notion for a graph, but it is the same for a metric space.

Definition (Rips thickening of a Graph) Let G be a graph, and $r > 0$. The *Rips thickening of radius r of G* is the simplicial complex $P_r(G)$ defined as follows:

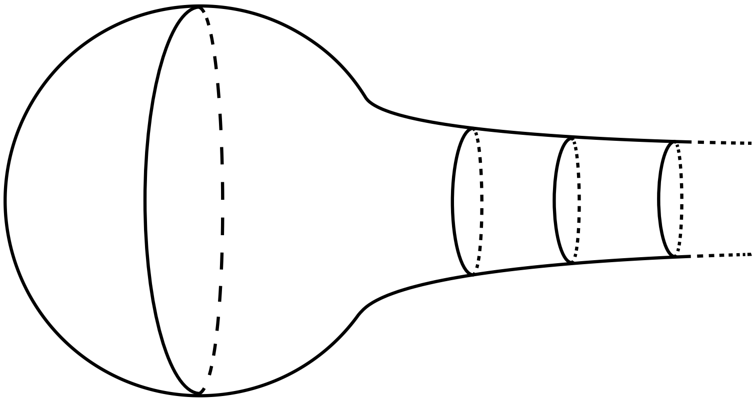


Figure 3.3: The space $\mathbb{R}^2$... somehow.

(i) The vertices of $P_r(G)$ are the vertices of G;

(ii) The $(k+1)$-tuple $(x_0, \ldots, x_k)$ defines a k-simplex T of $P_r(G)$ if and only if the following condition is satisfied:

$$0 < d(x_i, x_j) \leq r \quad \forall i \neq j.$$

If K is a simplicial complex, its *Rips thickening* of radius r is the Rips thickening $P_r(G_K)$ of its 1-skeleton.
By notation abuse, we also denote $P_0(K) = K$ for a simplicial complex K.

Remark 3.2 The Rips thickening $P_r(G)$ differs from $\mathrm{Pen}(G, r)$ since it does not contain the diagonal.

Lemma 3.6 *If G has bounded geometry, then for any $r > 0$, the simplicial complex $P_r(G)$ has bounded geometry.*

Proof : Since G has bounded geometry, there exists a constant N such that any vertex has at most N neighbors in G. The ball $B(v, r)$ centered in v and of radius r contains at most $\sum_{i=0}^{r} N^i$ vertices of G. Hence in $P_r(G)$ a vertex v has at most $\sum_{i=0}^{r} N^i$ neighbors. □

Sequences of Rips complexes: We suppose in the sequel that K is a simplicial complex with bounded geometry. We assume as well that K has its edges of length at most 1, which can always be obtained up to a bilipschitz homeomorphism.

For each integer $r \geq 1$, let $\mu_r : P_r(K) \to P_{r+1}(K)$ be the natural inclusion. Moreover, let us define define a map $\mu_0 : K \to P_1(K)$ by setting $\mu_0(v) = v$ for each vertex v, and by extending μ_0 by linearity on each simplex. Each μ_r is a uniformly continuous quasi-isometry, hence we have a sequence in the category of bounded geometry simplicial complexes (BGSC):

$$K \xrightarrow{\mu_0} P_1(K) \xrightarrow{\mu_1} P_2(K) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{r-1}} P_r(K) \xrightarrow{\mu_r} \cdots$$

By functoriality of H^k_π (see 1.39), such a sequence induces two sequences in simplicial L_π-cohomology:

$$H^k_\pi(K) \xleftarrow{\mu_0^*} H^k_\pi(P_1(K)) \xleftarrow{\mu_1^*} H^k_\pi(P_2(K)) \xleftarrow{\mu_2^*} \cdots \xleftarrow{\mu_{i-1}^*} H^k_\pi(P_i(K)) \xleftarrow{\mu_i^*} \cdots$$

$$\overline{H}^k_\pi(K) \xleftarrow{\mu_0^*} \overline{H}^k_\pi(P_1(K)) \xleftarrow{\mu_1^*} \overline{H}^k_\pi(P_2(K)) \xleftarrow{\mu_2^*} \cdots \xleftarrow{\mu_{i-1}^*} \overline{H}^k_\pi(P_i(K)) \xleftarrow{\mu_i^*} \cdots$$

If one defines for any $j > i$ the map $\lambda_{ji} = \mu_{j-1} \circ \ldots \circ \mu_i : P_i(K) \to P_j(K)$, of course for $i < j < l$ one has $\lambda_{lj} \circ \lambda_{ji} = \lambda_{li}$ and thus $\lambda^*_{ji} \circ \lambda^*_{lj} = \lambda^*_{li}$. The two sequences described above thus form projective systems.

Definition A map $f : X \to Y$ between two metric spaces is said to be *bornologous* if there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ for any $x, x' \in X$, one has

$$d(f(x), f(x')) < \rho\left(d(x, x')\right).$$

Definition A bornologous map $f : X \to Y$ together with a bornologous map $g : Y \to X$ form a *bornotopy equivalence* if $g \circ f$ and $f \circ g$ are parallel to Id_X and Id_Y respectively.

The following proposition will be helpful :

Proposition 3.7 *For any $r > 0$, there exists a map $g_r : P_{r+1}(K) \to K$ such that:*

(1) g_r is a uniformly continous quasi-isometry,

(2) There is a uniformly continuous homotopy between $g_r \circ (\mu_r \circ \ldots \circ \mu_0)$ and Id_K.

Proof : The proof rests on two lemmas Higson and Roe (see [HR95]):

Lemma 3.8 *Let $f : X \to Y$ be a bornologous map, where X is a finite dimensional metric simplicial complex and Y is uniformly contractible. Then there exists a uniformly continuous map $g : X \to Y$ that is parallel to f. Moreover, if f is already uniformly continuous on a subcomplex X', then we may take $g = f$ on X'.*

Let us prove this lemma. We construct g by induction on the skeleton. Let $X^{(k)}$ denote the k-skeleton of X, and let us set $g = f$ on $X^{(0)} \cup X'$. This initializes our induction. Let us now suppose that g has been defined on $X^{(k)} \cup X'$, and matches the conditions. For any $(k+1)$-simplex Δ^{k+1}, the map g is defined on $\partial\Delta^{k+1}$. Since Y is uniformly contractible, one can extend $g|_{\partial\Delta^{k+1}}$ to a map $\Delta^{k+1} \to Y$ whose image lies within a bounded distance of the image of the vertex set of Δ. This construction defines the map g on $X^{(k+1)}$. This finishes our induction step, and proves lemma 3.8. $\square$

The map $\widehat{\mu_r} = \mu_r \circ \ldots \circ \mu_0 : K \to P_{r+1}(K)$ is a uniformly continuous quasi-isometry. In particular, there exists a quasi-isometry $\widehat{\mu_r}^- : P_{r+1}(K) \to K$ such that $\widehat{\mu_r} \circ \widehat{\mu_r}^-$ and $\widehat{\mu_r}^- \circ \widehat{\mu_r}$ are parallel to $\mathrm{Id}_{P_{r+1}(K)}$ and Id_K respectively.

The map $\widehat{\mu_r}^- : P_{r+1}(K) \to K$ is bornologous, the simplicial complex K is uniformly contractible and $P_{r+1}(K)$ is finite dimensional. Hence, there exists a uniformly continuous map $g_r : P_{r+1}(K) \to K$ parallel to $\widehat{\mu_r}^-$.

Since $\widehat{\mu_r}^-$ is a quasi-isometry and g_r is parallel to $\widehat{\mu_r}^-$, the map g_r is itself a quasi-isometry. Moreover, the maps $\widehat{\mu_r} \circ \widehat{\mu_r}^-$ and $\widehat{\mu_r}^- \circ \widehat{\mu_r}$ are parallel to $\mathrm{Id}_{P_{r+1}(K)}$ and Id_K respectively, hence the maps $\widehat{\mu_r} \circ g_r$ and $g_r \circ \widehat{\mu_r}$ are themselves parallel to $\mathrm{Id}_{P_{r+1}(K)}$ and Id_K respectively. We thus have two quasi-isometric uniformly continuous maps

$$\widehat{\mu_r} : K \to P_{r+1}(K) \text{ and } g_r : P_{r+1}(K) \to K.$$

that form a bornotopy equivalence.
We now use the following lemma from Higson and Roe [HR95]:

Lemma 3.9 *Let X be a finite dimensional metric simplicial complex and Y be a uniformly contractible metric space. Then two uniformly continuous parallel bornologous maps f, g from X to Y are uniformly continously homotopic.*

Let us prove this lemma.

Observe that if Y is uniformly contractible, and if $f, g : X \to Y$ are bornologous parallel maps, then there exists a bornologous map $F : X \times [0,1] \to Y$ such that $F(\cdot, 0) = f, F(\cdot, 1) = g$. Indeed, one can set e.g.

$$F(x,t) = \begin{cases} f(x) & \text{if } t \leq \frac{1}{2} \\ g(x) & \text{else.} \end{cases}$$

Such a map is called a *bornotopy* from f to g. Let $F : X \times [0,1] \to Y$ be a bornotopy from f to g. We can suppose that the map F is uniformly continous on the subcomplex $X \times \{0,1\}$. By lemma 3.8, we thus can assume that F is uniformly continuous on $X \times [0,1]$. $\square$

Apply now lemma 3.9 to the maps $g_i \circ \widehat{\mu_r} : K \to K$ and Id_K to obtain that they are uniformly continously homotopic. $\square$

We will also need the following lemma :

Lemma 3.10 *Let X, Y be two simplicial complexes of bounded geometry, and $f, g : X \to Y$ two simplicial, uniformly continuous and parallel quasi-isometries. Then there exists $r > 0$ and a simplicial uniformly continuous quasi-isometric homotopy $F : X \times [0,1] \to P_r(Y)$ from $\mu_{r-1} \circ \ldots \mu_0 \circ f$ to $\mu_{r-1} \circ \ldots \mu_0 \circ g$.*

Proof : : Since $f, g : X \to Y$ are parallel, there exists $K > 0$ such that for any $x \in X$, one has

$$|f(x) - g(x)| < K.$$

In particular, if x is in a fixed simplex Δ, there exists $r > 0$ large enough such that $\mu_{r-1} \circ \ldots \mu_0 \circ f(x)$ and $\mu_{r-1} \circ \ldots \mu_0 \circ g(x)$ always belong a same simplex of $P_r(X)$. Moreover, r depends on the diameter of Δ. Since X has bounded geometry, there exists a uniform r such that for any $x \in X$, the points $\mu_{r-1} \circ \ldots \mu_0 \circ f(x)$ and $\mu_{r-1} \circ \ldots \mu_0 \circ g(x)$ belong to the same simplex of $P_r(Y)$.
We can then simply define a linear homotopy from $\mu_{r-1} \circ \ldots \mu_0 \circ f$ to $\mu_{r-1} \circ \ldots \mu_0 \circ g$. Such a homotopy is uniformly continuous, and it is quasi-isometric. It suffices to take a simplicial approximation of it. □

Coarse cohomology and simplicial cohomology

We now relate the simplicial cohomology of a simplicial complex and the coarse cohomology of its 1-skeleton. We begin by showing that the simplicial cohomology and reduced simplicial cohomology of the complex can be expressed as the inverse limit of the cohomology groups of its Rips thickenings.

Let K be a simplicial complex, with bounded geometry. We denote by G_K its 1-skeleton together with the length metric. It is a graph, whose vertices and edges coincide with those of K. Moreover, from the fact that K has bounded geometry we deduce that it also is the case for G_K.

We are going to study the inverse limits

$$\lim_{\leftarrow} H^k_\pi\left(P_i(K)\right) \qquad \text{and} \qquad \lim_{\leftarrow} \overline{H}^k_\pi\left(P_i(K)\right).$$

Proposition 3.11 *If K is a uniformly contractible simplicial complex with bounded geometry, There exist vector spaces isomorphisms*

$$\pi_0 : \lim_{\leftarrow} H^k_\pi\left(P_i(K)\right) \cong H^k_\pi(K) \qquad \textit{and} \qquad \pi_0 : \lim_{\leftarrow} \overline{H}^k_\pi\left(P_i(K)\right) \cong \overline{H}^k_\pi(K).$$

Proof : : Let $(r_i)_{i \geq 1}$ be an increasing sequence of positive integers, and $r_0 = 0$. Let

$$f_i : P_{r_i}(K) \to P_{r_{i+1}}(K) \text{ and } h_i : K \to P_{r_{i+1}}(K)$$

be defined by

$$f_i = \mu_{r_{i+1}-1} \circ \ldots \circ \mu_{r_i+1} \circ \mu_{r_i} \text{ and } h_i = f_i \circ f_{i-1} \circ \ldots \circ f_1 \circ f_0.$$

In the lemma below we will prove that for a suitable choice of (r_i), there exists linear maps $q_i : H^*_\pi(K) \to H^*_\pi(P_{r_{i+1}}(K))$ such that

- $h_i^* \circ q_i = \mathrm{Id}_{H^*_\pi(K)}$;
- $q_{i-1} \circ h_i^* = f_i^*$.

Then the natural map $\pi_0 : \lim\limits_{\leftarrow} H^*_\pi(P_{r_i}(K)) \to H^*_\pi(K)$ is an isomorphism. Indeed, let $\beta \in H^*_\pi(K)$ and let us denote by $\beta_{i+1} = q_i(\beta) \in H^*_\pi(P_{r_{i+1}}(K))$. One has

$$\begin{aligned} f_i^*(\beta_{i+1}) &= f_i^* \circ q_i(\beta) \\ &= q_{i-1} \circ h_i^* \circ q_i(\beta) \\ &= q_{i-1}(\beta) \\ &= \beta_i \end{aligned}$$

This tells us that the sequence (β_i) defines a unique element $\gamma \in \lim\limits_{\leftarrow} H^*_\pi(P_{r_i}(K))$ such that $\pi_i(\gamma) = \beta_i$, where π_i is the natural map. One has $\pi_0(\gamma) = \beta_0 = \beta$, hence π_0 is surjective. Let us now show the injectivity of π_0. Fix $\gamma \in \lim\limits_{\leftarrow} H^*_\pi(P_{r_i}(K))$, $\gamma \neq 0$, and let us show that $\pi_0(\gamma) \neq 0$. Let (β_i) represent γ. Since $\gamma \neq 0$, there exists i such that $\beta_{i-1} := \pi_{i-1}(\beta) \neq 0$. One necessarily has $\pi_i(\beta) \neq 0$. Moreover,

$$\begin{aligned} 0 &\neq f_{i-1}^*(\beta_i) \\ &= q_{i-2} \circ h_{i-1}^*(\beta_i) \end{aligned}$$

Since $h_{i-1}^*(\beta_i) = \beta$, one has

$$q_{i-2}(\beta) \neq 0.$$

Hence $\beta \neq 0$ by linearity of q_{i-2}, i.e. $\ker(\pi_0) = \{0\}$.

□

We still have to show that the linear maps q_i exist. This is the goal of the following lemma:

Lemma 3.12 *Let K be a uniformly contractible simplicial complex with bounded geometry. Then there exists an increasing sequence of integers $r_i > 0$ and linear maps $q_i : H^*_\pi(K) \to H^*_\pi(P_{r_{i+1}}(K))$ such that*

- $h_i^* \circ q_i = Id_{H^*_\pi(K)}$;
- $q_{i-1} \circ h_i^* = f_i^*$.

Proof : Let $r > 0$ and let g_r be the uniformly continuous quasi-isometry constructed in proposition 3.7. In particular, since there is a uniformly continuous homotopy between $g_r \circ (\mu_r \circ \ldots \circ \mu_0)$ and Id_K, we have the following equality in cohomology:

$$(g_r \circ (\mu_r \circ \ldots \circ \mu_0))^* = \mathrm{Id}_{H^*_\pi(K))}\,.$$

By functoriality (see 1.39), it says that $(\mu_r \circ \ldots \circ \mu_0)^* \circ g_r^* = \mathrm{Id}_{H_\pi^*(K)}$. Hence the map $q_i = g_{r_i}^*$ satisfies the first condition for any choice of r_i.

Let us chose $r_0 = 0, r_1 = 1, q_0 = g_0^*$. The second condition is naturally satisfied.

We proved in the proof of lemma 3.8 that $\widehat{\mu_r} : K \to P_{r+1}(K)$ and $g_r : P_{r+1}(K) \to K$ form a bornotopy equivalence. In particular, there exists $K > 0$ such that $d(\mu_0 \circ g_0(x), x) < K$. Hence $\mu_0 \circ g_0 : P_1(K) \to P_1(K)$ and $\mathrm{Id} : P_1(K) \to P_1(K)$ are parallel maps. Thus by lemma 3.10, for r_2 large enough $\mu_{r_2-1} \circ \ldots \circ \mu_1$ is homotopic to $\mu_{r_2-1} \circ \ldots \circ \mu_0 \circ g_0$. We can fix $g_2 = g'_{r_2}$ and $q_2 = g_2^*$: this allows the second condition to be satisfied. Now we proceed inductively. □

Proposition 3.13 *Let G be a graph. There is a vector space isomorphism*

$$\phi^* : \overline{HX}_\pi^k(G) \cong \lim_{\leftarrow} \overline{H}_\pi^k(P_i(G)).$$

Proof : Let $\phi_i : CX_\pi^k(G) \to C_\pi^k(P_i(G))$ be defined as follows: for any $\alpha \in CX_\pi^k(G)$ and any simplex $(x_0, \ldots, x_k)$ of $P_i(G)$, we set

$$\phi_i(\alpha)(x_0, \ldots, x_k) = \alpha(x_0, \ldots, x_k).$$

Then, we extend $\phi_i(\alpha)$ to all simplicial chains and ϕ_i to all simplicial cochains by linearity. Observe that ϕ_i is continuous.

Claim 1: One has $\delta\phi_i = \phi_i d$.

Indeed, let $\alpha \in CX_\pi^k(G)$ and $(x_0, \ldots, x_{k+1})$ a simplex of $P_i(G)$. Then

$$\begin{aligned}
\delta\phi_i(\alpha)(x_0, \ldots, x_{k+1}) &= \phi_i(\alpha)(\partial(x_0, \ldots, x_{k+1})) \\
&= \phi_i(\alpha)\left(\sum_{j=0}^{k+1}(-1)^j(x_0, \ldots, \widehat{x_j}, \ldots, x_{k+1}\right) \\
&= \sum_{j=0}^{k+1}(-1)^j\phi_i(\alpha)\,(x_0, \ldots, \widehat{x_j}, \ldots, x_{k+1}) \\
&= \sum_{j=0}^{k+1}(-1)^j\alpha\,(x_0, \ldots, \widehat{x_j}, \ldots, x_{k+1}) \\
&= d\alpha(x_0, \ldots, x_{k+1}) \\
&= \phi_i d\alpha(x_0, \ldots, x_{k+1}).
\end{aligned}$$

In particular, ϕ_i induces a continuous linear map

$$\phi_i^* : \overline{HX}_\pi^k(G) \to \overline{H}_\pi^k(P_i(G))$$

Claim 2: There is a linear map induced on the inverse limit

$$\phi^* = \lim_{\leftarrow} \phi_i^* : \overline{HX}_\pi^k(G) \to \lim_{\leftarrow} \overline{H}_\pi^k(P_i(G)).$$

Indeed, one has $\mu_i^* \circ \phi_{i+1}^* = \phi_i^*$.

Claim 3: $\phi^* : \overline{HX}_\pi^k(G) \to \lim_{\leftarrow} \overline{HX}_\pi^k(P_i(G))$ is a vector space isomorphism, continuous with respect to the Frechet structure.

We need to exhibit find an inverse of ϕ^*. Let $\beta \in \lim_{\leftarrow} \overline{HX}_\pi^k(P_i(G))$. One can represent β by a sequence $(\beta_i)_{i\geq 0}$ with $\beta_i \in \overline{HX}_\pi^k(P_i(G))$ such that $\mu_i^*(\beta_{i+1}) = \beta_i$. Let us chose $z_i \in Z_\pi^k(G)$ representing β_i. We claim that there exists $z_2 \in Z_\pi^k(P_2(G))$ representing β_2 such that

$$\|\mu_1^* z_2 - z_1\|_\pi < \frac{1}{2}.$$

Indeed, let w_2 be any cocycle representing β_2. Then $\mu_1^* w_2$ represents z_1, hence we can write $\mu_1^* w_2 - z_1 = \delta u_1 + \alpha_1$ where:

- $u_1 \in C_\pi^{k-1}(P_1(G))$;
- $\alpha_1 \in C_\pi^k(P_1(G))$ and $\|\alpha_1\|_\pi < \frac{1}{2}$.

Let $u_2 \in C^{k-1}(P_2(G))$ be defined by

$$u_2(x_0, \dots, x_{k-1}) = \begin{cases} u_1(x_0, \dots, x_{k-1}) & \text{if } (x_0, \dots, x_{k-1}) \in P_1(G) \\ 0 & \text{else} \end{cases}.$$

One has $u_2 \in C_\pi^{k-1}(P_2(G))$, and $\mu_1^* u_2 = u_1$. Hence,

$$\begin{aligned} \mu_1^*(w_2 - \delta u_2) &= \mu_1^* w_2 - \partial u_1 \\ &= \alpha_1 + z_1. \end{aligned}$$

Let $z_2 = w_2 - \delta u_2$. Then z_2 represents β_2, and moreover

$$\mu_1^* z_2 = \alpha_1 + z_1.$$

Hence as we claimed above,

$$\|\mu_1^* z_2 - z_1\|_\pi < \frac{1}{2}.$$

We now can inductively construct a sequence z_i of cocycles $z_i \in Z_\pi^k(P_i(G))$ such that z_i represents β_i and satisfying $\|\mu_i^* z_{i+1} - z_i\|_\pi \leq \frac{1}{2^i}$. Let $\psi^*(\beta) : V_G^{k+1} \to \mathbb{R}$ be defined by

$$\psi^*(\beta)(x_0, \dots, x_k) = \lim_{i\to\infty} z_i(x_0, \dots, x_k).$$

One has $\psi^*(\beta) \in ZX^k_\pi(G)$. We claim that for any β, the class $[\psi^*\beta]$ does not depend on the particular choice of the sequence z_i. Indeed, let (z'_i) be another sequence of cocycles z'_i representing β_i such that $\|\mu_i^* z_{i+1} - z_i\|_\pi < \frac{1}{2^i}$. Let

$$\alpha(x_0, \dots, x_k) = \lim_{i\to\infty} (z_i - z'_i)(x_0, \dots, x_k).$$

We need to prove that $\alpha \in \overline{BX}^k(G)$. Since $z_i - z'_i$ represents the zero cohomology, there exists $w_i \in C^k_\pi(P_i(G))$ such that $\left\|\partial w_i - (z_i - z'_i)\right\|_\pi < \frac{1}{2^i}$. Let $t_i \in \Omega X^{k-1}_\pi(G)$ be defined by

$$t_i(x_0, \dots, x_{k-1}) = \begin{cases} w_i(x_0, \dots, x_{k-1}) & \text{if } (x_0, \dots, x_{k-1}) \in P_i(G) \\ 0 & \text{else.} \end{cases}$$

Hence for any $i > 0$,

$$\begin{aligned} \sum_{(x_0,\dots,x_k)\in \mathrm{Pen}(G,i)} \left|(\partial t_i - \alpha)(x_0, \dots, x_k)\right|^{p_k} &= \sum_{(x_0,\dots,x_k)\in \mathrm{Pen}(G,i)} \left|(\partial w_i - \alpha)(x_0, \dots, x_k)\right|^{p_k} \\ &\leq \|\partial w_i - \alpha\|_\pi^{p_k} \\ &\leq \left(\frac{1}{2^i}\right)^{p_k}. \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{(x_0,\dots,x_{k+1})\in \mathrm{Pen}(G,i)} \left|(\partial \circ \partial t_i - \partial\alpha)(x_0, \dots, x_{k+1})\right|^{p_{k+1}} &= \sum_{(x_0,\dots,x_{k+1})\in \mathrm{Pen}(G,i)} \left|\partial\alpha(x_0, \dots, x_k)\right|^{p_k} \\ &\leq \|\partial\alpha\|_{p_{k+1}} \\ &= 0. \end{aligned}$$

Thus α is $1/2^i$ close in $\|\cdot\|_\pi$ norm to ∂t_i, with $t_i \in C^{k-1}_\pi(G)$. Hence, $\alpha \in \overline{BX}^k_\pi(G)$. This tells us that ψ^* is well defined at the cohomology level.

Now, from the definition of ψ^*, it is clear that is is the inverse of ϕ^* at the cohomology level.

□

Corollary 3.14 *If K is a uniformly contractible euclidean simplicial complex with bounded geometry, there is a vector space isomorphism*

$$\overline{HX}^k_\pi(G_K) \cong \overline{H}^k_\pi(K).$$

Proof : One has

$$\begin{aligned}\overline{HX}^k_\pi(G_K) &= \lim_{\leftarrow} \overline{H}^k_\pi(P_i(G_K)) \\ &= \lim_{\leftarrow} \overline{H}^k_\pi(P_i(K)) \\ &= \overline{H}^k_\pi(K).\end{aligned}$$

The first equality is due to proposition 3.13. The second one is evident since $P_i(G_K) = P_i(K)$, and the third one is due to 3.11. □

This result also stands for non-reduced cohomology:

Proposition 3.15 *Let K be a uniformly contractible simplicial complex with bounded geometry. One has*

$$HX^k_\pi(G_K) = H^k_\pi(K)$$

Proof : Once again we consider the Rips thickening sequence:

$$K \xrightarrow{\mu_0} P_1(K) \xrightarrow{\mu_1} P_2(K) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{r-1}} P_r(K) \xrightarrow{\mu_r} \cdots$$

Let $C^k_\pi(K) = Z^k_\pi(K) \oplus U^k_\pi(K)$ be a linear decomposition of $C^k_\pi(K)$ and suppose that there exists linear maps $s^k_i : C^k_\pi(K) \to C^k_\pi(P_{i+1}(K))$ such that

(i) $\mu_i^* \circ s^k_i = s^k_{i-1}$;

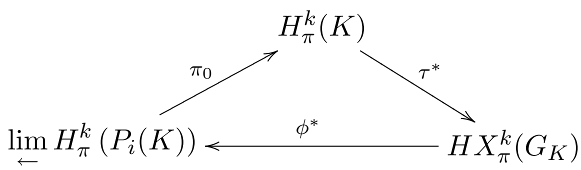
(ii) For any $v \in U^{k-1}_\pi$, one has $s^k_i(\partial v) = \partial s^{k-1}_i(v)$.

By (i), there is a limit map $\tau : Z^k_\pi(K) \to ZX^k_\pi(G_K)$ defined by

$$\tau(z)(x_0, \ldots, x_k) = \lim_{i\to\infty} s^k_i(z)(x_0, \ldots, x_k).$$

By (ii), τ induces induces a map $\tau^* : H^k_\pi(K) \to HX^k_\pi(K)$ at the cohomology level.

Let us recall from proposition 3.11 that the projection $\pi_0 : \lim\limits_{\leftarrow} H^k_\pi\,(P_i(K)) \to H^k_\pi(K)$ is a vector space isomorphism. In the proof of proposition 3.13, we also have constructed a vector space isomorphism $\phi^* : HX^k_\pi(G_K) \to \lim\limits_{\leftarrow} H^k_\pi\,(P_i(K))$. Hence we have the following diagram:



Examinating the definitions of ϕ^* and τ, we see that this diagram commutes. Hence $\tau^* : H^k_\pi(K) \to HX^k_\pi(K)$ is an isomorphism.

We still have to construct the maps s^k_i. We begin by the construction of auxiliary maps. Let $\nu^k_i : C^k_\pi(K) \to C^k_\pi(P_{i+1}(K))$ be defined by the following formula for $c \in C^k_\pi(K)$ and $(x_0, \ldots, x_k) \in P_{i+1}(K)$:

$$\nu^k_i(c)(x_0, \ldots, x_k) = \begin{cases} c(x_0, \ldots, x_k) & \text{if } (x_0, \ldots, x_k) \in K \\ 0 & \text{else} \end{cases}.$$

Claim 1: $\mu^*_i \circ \nu^k_i = \nu^k_{i-1}$.

Indeed, let $(x_0, \ldots, x_k)$ and $c \in C^k_\pi(K)$. If $(x_0, \ldots, x_k) \notin K$, then $\nu^k_i(c)(x_0, \ldots, x_k) = 0$ and $\nu^k_{i-1}(x_0, \ldots, x_k) = 0$ as well. The map μ^*_i being linear, one thus has

$$\mu^*_i \circ \nu^k_i(c)(x_0, \ldots, x_k) = 0 = \nu^k_{i-1}(c)(x_0, \ldots, x_k).$$

Suppose then that $(x_0, \ldots, x_k) \in K$. Then for each vertex x_j of K, we have $\mu_i(x_j) = x_j$, hence

$$\begin{aligned} \mu^*_i \circ \nu^k_i(c)(x_0, \ldots, x_k) &= \nu^k_i(c)(\mu_0(x_0), \ldots, \mu_0(x_k)) \\ &= \nu^k_i(c)(x_0, \ldots, x_k) \\ &= c(x_0, \ldots, x_k) \\ &= \nu^k_{i-1}(c)(x_0, \ldots, x_k) \end{aligned}$$

This proves our claim 1.

Claim 2: For any $c \in C^{k-1}_\pi(K)$, one has

$$\mu^*_i \left(\delta(\nu^{k-1}_i(c)) \right) = \delta\nu^{k-1}_{i-1}(c).$$

Indeed,

$$\begin{aligned} \delta\nu^k_{i-1}(c)(x_0, \ldots, x_k) &= \nu^k_{i-1}(c)(\partial(x_0, \ldots, x_k)) \\ &= \mu^*_i \circ \nu^k_i(c)(\partial(x_0, \ldots, x_k)) \\ &= \mu^*_i \circ \delta\nu^{k-1}_{i-1}(c)(x_0, \ldots, x_k) \end{aligned}$$

This proves our claim 2.

We know modify the maps ν^k_i. Let $C^k_\pi(K) = Z^k_\pi(K) \oplus U^k_\pi(K)$ be the linear decomposition we chose earlier. There is a surjective linear map $d : C^{k-1}_\pi(K) \to B^k_\pi(K)$, with kernel $Z^{k-1}_\pi(K)$. Hence, there is a vector space isomorphism

$$d : U^{k-1}_\pi(K) \to B^k_\pi(K).$$

One can write $Z^k_\pi(K) = H^k_\pi(K) \oplus B^k_\pi(K)$, and thus $Z^k_\pi(K) = H^k_\pi(K) \oplus dU^{k-1}_\pi(K)$. One thus has the following decomposition:

$$C^k_\pi(K) = H^k_\pi(K) \oplus dU^{k-1}_\pi(K) \oplus U^k_\pi(K)$$

Hence any $c \in C^k_\pi(K)$ can be written

$$c = \sum_{l\in A} \lambda_l z^l_0 + dv + u$$

Where $v \in U^{k-1}_\pi(K), u \in U^k_\pi(K)$, $\lambda_l \in \mathbb{R}$ and where $(z^l_0)_{l\in A}$ is a collection of cocycles z^l_0, each one representing a cohomology class β^l_0, the collection $(\beta^l_0)_{l\in A}$ being Hamel basis for the vector space $H^k_\pi(K)$. Here A is some finite index set.
Now for each $l \in A$ and each $i \geq 1$, let $z^l_i \in Z^k_\pi(P_{i+1}(K))$ be a cocycle chosen in such a way that $\mu^*_i z^l_i = z^l_{i-1}$.
Now let us set

$$s^k_i\left(\sum_{l\in A} \lambda_l z^l_0 + dv + u\right) = \sum_{l\in A} \lambda_l z^l_i + d\mu^{k-1}_i(v) + \mu^k_i(u).$$

The maps s^k_i constructed this way satisfy (i) and (ii).

□

As a corollary, reduced and non-reduced simplicial cohomologies are quasi-isometry invariants:

Theorem 3.16 *Let K, L be uniformly contractible simplicial complexes with bounded geometry, and suppose that they are quasi-isometric. Then for any non-increasing sequence π of real numbers $1 < p_k < \infty$, there exist vector space isomorphisms*

$$H^k_\pi(K) = H^k_\pi(L) \qquad \textit{and} \qquad \overline{H}^k_\pi(K) = \overline{H}^k_\pi(L).$$

Proof : Since K and L are quasi-isometric, their 1-skeleta G_K and G_L are quasi-isometric as well. Hence

$$\begin{aligned} H^k_\pi(K) &= HX^k_\pi(G_K) \\ &= HX^k_\pi(G_L) \\ &= H^k_\pi(L). \end{aligned}$$

The same list of equalities holds for reduced cohomology. □

Quasi-isometry invariance for Riemannian manifolds

Using de Rham's theorem, this extends to quasi-isometric riemannian manifolds:

Theorem 3.17 *Let M, N be m, n-Riemannian manifolds $(n \geq m)$ with bounded geometry, admitting uniformly contractible simplicial complexes. Suppose that they are quasi-isometric. Then for any sequence π of real numbers satisfying one of the following hypothesis :*

(1) $1 < p_k < \infty$ *and* $0 \leq \frac{1}{p_k} - \frac{1}{p_{k-1}} \leq \frac{1}{n}$, *or*

(2) $1 \leq p_k < \infty$ *and* $0 \leq \frac{1}{p_k} - \frac{1}{p_{k-1}} < \frac{1}{n}$.

there exist vector space isomorphisms

$$\overline{H}^k_\pi(N) = \overline{H}^k_\pi(M).$$

$$H^k_\pi(N) = H^k_\pi(M).$$

Proof : Let K, L be two uniform triangulations of M, N respectively. Since M and N are quasi-isometric, with bounded geometry and uniformly contractible, it is the case for K and L as well. Hence, one has

$$H^k_\pi(K) = H^k_\pi(L)$$

and

$$\overline{H}^k_\pi(K) = \overline{H}^k_\pi(L).$$

It then suffices to apply the L_π de Rham isomorphism theorem. □

Application to the quasi-isometry invariance of Sobolev inequalities and Isoperimetric inequality

The classical isoperimetric inequality states that for a bounded domain Ω of the euclidean space $\mathbb{R}^n$, there exists a constant $c_n > 0$ such that

$$(\mathrm{Vol}\,\Omega)^{\frac{1}{n}} \leq c_n \cdot (\mathrm{area}\partial\Omega)^{\frac{1}{n-1}} .$$

The *isoperimetric constant* $I_m(M)$ of a Riemannian manifold M is defined by the formula

$$I_m(M) = \inf_{\Omega} \frac{\mathrm{area}\partial\Omega}{(\mathrm{Vol}\,\Omega)^{\frac{m-1}{m}}}$$

where Ω runs through all bounded domains in M. The classical isoperimetric inequality may be rewritten in the form $I_n(\mathbb{R}^n) > 0$. More generally, a n-manifold M *satisfies an isoperimetric inequality of order* m if $I_(M) > 0$.
In [Kan86], M. Kanai shows that for bounded geometry Riemannian manifolds, satisfying an isoperimetric inequality is a quasi-isometry invariant. More precisely, we have the following theorem:

Theorem 3.18 *Let M, N be two Riemannian manifolds with bounded geometry, and suppose that M and N are quasi-isometric. Then for any integer $m \geq \max\{\dim M, \dim N\}$, one has*

$$I_m(M) > 0 \text{ if and only if } I_m(N) > 0.$$

We shall see that this theorem can be obtained by the use of the quasi-isometry invariance of L_π-cohomology. First, let us recall the link between Sobolev inequalities and isoperimetric inequalities. The *analytic constants* $\mathrm{Sob}_{m,l}(M)$ of a Riemmannian manifold M are defined by

$$\mathrm{Sob}_{m,l}(M) = \inf_{\alpha \in C_0^\infty(M)} \frac{\|du\|_l}{\|u\|_{\frac{m-1}{m}}}, \qquad m > 1$$

The manifold M satisfies the Sobolev inequalities if $S_{m,l}(M) > 0$. A result due to Federer-Fleming and Maz'ya (see [Kan86]) says that

$$\left(\mathrm{Sob}^0_{q,1}\right)^{-1} = I_{\frac{q}{1-q}}(M)$$

In [GT06], Gol'dshteĭn and Troyanov establish the following link between $L_{q,p}$-cohomology and Sobolev inequalities:

Theorem 3.19 *Let $1 \leq p < \infty, 1 < q < \infty$. Let $1 \leq p < \infty, 1 < q < \infty$. Then $T^k_{q,p}(M) = 0$ if, and only if $\mathrm{Sob}^k_{p,q}(M) > 0$.*

Let us consider the case where $k = 1$, and θ has compact support. In this case, θ is a function, as well as ζ. Since ζ belongs to $Z^{k-1}_q(M)$, one has $d\zeta = 0$, hence ζ is constant. If M is non-compact, then ζ must be zero, for it is integrable. Hence our estimate comes out to be $\|\theta\|_q \leq C\|d\theta\|_p$. As a consequence, the theorem above can be rewritten:

Theorem 3.20 *Let $1 \leq p < \infty, 1 < q < \infty$. Then $T^0_{q,p}(M) = 0$ if, and only if $\mathrm{Sob}^0_{q,p}(M) > 0$.*

Moreoever, since both cohomology and reduced cohomology are quasi-isometry invariants, the torsion is a quasi-isometry invariant as well. In particular, let M and N be two uniformly contractible quasi-isometric manifolds, with dimension $n > 1$. Then

$$
\begin{aligned}
I_n(M) > 0 &\iff \mathrm{Sob}_{n,1}(M) > 0 \\
&\iff T^0_{n,1}(M) = 0 \\
&\iff T^0_{n,1}(N) = 0 \\
&\iff \mathrm{Sob}_{n,1}(N) > 0 \\
&\iff I_n(N) > 0
\end{aligned}
$$

Hence the existence of an isoperimetric inequality is a quasi-isometry invariant.

Appendix A

Appendix : background

This chapter is an addendum : it explains the notions of *Banach complexes*, of *quasi-isometries*, and gives some classical technical results cited throughout the text. We begin by Banach complexes.

Banach Complexes

Complexes, morphisms and homotopy Recall that a *Banach space* is a real or complex vector space F together with a norm $\|\cdot\|$ which makes it complete as a metric space, i.e. all Cauchy sequences converge.

Definition (Banach Complexes) A *Banach complex* (one should say *cocomplex*) (F^*, d) is a countable collection $(F_i, \|\cdot\|_i)_{i\in\mathbb{N}}$ of Banach spaces together with continuous linear maps $d_k : F_k \to F_{k+1}$ such that $d_{k+1} \circ d_k = 0$.

We write $F^* = \bigoplus_{i\in\mathbb{N}} F_i$. In this case, $d : F^* \to F^*$ is the evident linear map defined on each element of the sum by d_i, and this allows us to write d for any d_i. As in the case of cochain complexes, we can represent a complex by a simple diagram:

$$\cdots \longrightarrow F_{k-1} \xrightarrow{d} F_k \xrightarrow{d} F_{k+1} \longrightarrow \cdots$$

The Banach complexes form a category:

Definition (Morphims of Banach complexes) Let (F^*, d) and (G^*, d) be Banach complexes. A *morphisms of Banach complexes* $f : F^* \to G^*$ is a collection of morphisms $f_k : F^k \to G^k$ such that one has $d \circ f = f \circ d$, where this equality is to be understood as $d_k \circ f_k = f_k \circ d_k$ for any k.

In other terms, each square of the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & F^{k-1} & \xrightarrow{d} & F^k & \xrightarrow{d} & F^{k+1} & \xrightarrow{d} \cdots \\
& \downarrow{\scriptstyle f_{k-1}} & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k+1}} & \\
\cdots \longrightarrow & G^{k-1} & \xrightarrow{d} & G^k & \xrightarrow{d} & G^{k+1} & \xrightarrow{d} \cdots
\end{array}$$

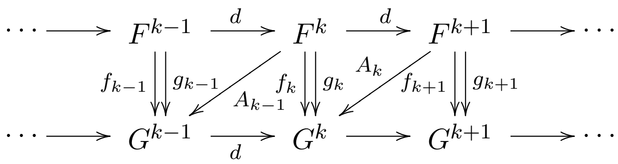
The notation $f : F^* \to G^*$ is non ambiguous, as the map f defined on the direct sum $F^* = \bigoplus_{i \in \mathbb{N}} F_i$ has meaning, and is a bounded operator between Banach spaces. However, it is obviously not true that *any* bounded operator $f : F^* \to G^*$ defines a morphism of Banach complexes.

We will generaly simply write $f : F^k \to G^k$ for the map $f_k : F^k \to G^k$.

There are two notions of (chain)-homotopy in this category:

Definition (Homotopy in Banach complexes) Let F, G be Banach complexes and $f, g : F \to G$ be morphisms of Banach complexes. A *homotopy* is a collection of bounded operators $\left\{A_k : F^k \to G^{k-1}\right\}$ such that

$$f_k - g_k = d \circ A_k + A_{k+1} \circ d.$$



And this diagram "commutes" if we replace the vertical arrows by their differences.

Definition (Weak homotopy in Banach complexes) Let F, G be Banach complexes and $f, g : F \to G$ be morphisms of Banach complexes. A *weak homotopy* is a collection of families of bounded operators $\left\{A_{i,k} : F^k \to G^{k-1}\right\}_{i \in \mathbb{N}}$ such that for any $x \in F$, one has

$$\lim_{i \to \infty} \left\| \left(d \circ A_{i,k} + A_{i,k+1} \circ d\right)(x) - \left(f_k - g_k\right)(x) \right\| = 0.$$

Definition (Subcomplex and Banach subcomplex) Let (F^*, d) be a Banach complex. A *subcomplex* G of F is a collection of (non-necessarily closed) vector spaces $G^k \subset F^k$ such that $dG^k \subset G^{k+1}$. If each G^k is closed, then G is itself a Banach complex, which we call a *Banach subcomplex* of F^*.

Remark A.1 Let $f : F^* \to F^*$ be a morphism from a Banach complex to itself. Then

(a) The image $f(F^*) = \left\{\bigoplus_k f_k(F^k)\right\}$ of f is a subcomplex of F^*;

(b) If f is closed, $f(F^*)$ is a Banach subcomplex of F^*.

(c) The kernel $\ker(f) = f^{-1}(0)$ is always a Banach subcomplex of F^*.

To each complex, we can attach a sequence of vector spaces and a sequence of Banach spaces in a functorial way:

Cohomology and induced morphisms

Definition (Cohomology of a Banach complex) Let us write $Z^k(F^*, d) = \ker d \cap F_k$ and $B^k(F^*, d) = dF^{k-1}$. Let us also introduce the notations $Z^*(F^*, d) = \bigoplus_k Z^k(F^*, d)$ and $B^*(F^*, d) = \bigoplus_k B^k(F^*, d)$ and Since $d \circ d = 0$, we have $B^k \subset Z^k$, and the quotient $Z^k(F^*, d)/B^k(F^*, d)$ is a vector space, called the *space of cohomology of degree k of* (F^*, d).

Observe that Z^k is closed in F_k but $B^k \subset Z^k$ is not generaly a Banach space. The closure $\overline{B^k}(F^*, d)$ is however closed by definition, and is still a subspace of $Z^k(F^*, d)$. This leads to the following definition:

Definition (Reduced cohomology of a Banach complex) The quotient space

$$Z^k(F^*, d)/\overline{B^k}(F^*, d)$$

which is always a Banach space, is called the *reduced cohomology space of degree k of* (F^*, d).

The *torsion* measures the difference between cohomology and reduced cohomology:

Definition The *torsion* of degree k of (F^*, d) is the space

$$H^k(F^*, d)/\overline{H}^k(F^*, d) = \overline{B^k}(F^*, d)/B^k(F^*, d).$$

Induced morphisms in cohomology: Let $f : F \to G$ be a morphism of Banach complexes. Since $df = fd$, one has the following facts:

- If $z \in Z^k(F, d)$, then $0 = f(dz) = df(z)$ and thus $f(z) \in Z^k(G, d)$. This says that $f\left(Z^k(Z, d)\right) \subset Z^k(G, d)$.
- The same argument leads to $f(B^k) \subset B^k$ and $f(\overline{B}^k) \subset B^k$.

Now pick up $[\xi] \in H^k(F, d)$. If $\xi \in Z^k(F, d)$ represents $[\xi]$, one has $f(\xi) \in Z^k(G, d)$ and therefore $[f(\xi)]$ has a meaning. Moreover, if ξ' is another representant of $[\xi]$, one has $\xi - \xi' = d\eta$ for some $\eta \in F^{k-1}$, and thus $\xi - \xi' \in B^k(F, d)$, which implies that $f(\xi) - f(\xi') = f(\xi - \xi') \in B^k(G, d)$. This means that $f(\xi)$ and $f(\xi')$ both represent the same cohomology class, that is $[f(\xi)] = [f(\xi')]$. The map $[\xi] \mapsto f([\xi])$ is thus well defined, and it is straightforward to check that it is linear. We call it the *map induced by f in cohomology*, and denote it by $H^k f : H^k(F, d) \to H^k(G, d)$. In a similar way, one can introduce a linear bounded map $\overline{H}^k f : \overline{H}^k(F, d) \to \overline{H}^k(G, d)$.

The verification of the following proposition is straightforward.

Proposition A.1 (Functoriality) *H^k is a contravariant functor from the category of Banach complexes to the category of vector spaces, and $\overline{H}^k$ is a contravariant functor from the category of Banach complexes to the category of Banach spaces.*

□

Proposition A.2 (Homotopical morphisms) *Let $f : F^* \to G^*$ be a morphism of Banach complexes, such that $f(F^*) \subset G^*$ where G^* is a subcomplex of F^*. Suppose that there exists a homotopy $\left\{A_k : F^k \to F^{k-1}\right\}$ between f and the identity operator Id_{F^*}. Then one has an isomorphism of vector spaces*

$$H^k(F^*, d) = H^k(G^*, d).$$

Proof : Let $x \in Z^k$. Since $\left\{A_k : F^k \to F^{k-1}\right\}$ between f and the identical operator Id_{F^*}, one has

$$f(x) - x = d \circ A_k(x) + A_{k+1} \circ d(x).$$

Since we assumed $x \in Z^k$, one has $dx = 0$ and thus

$$f(x) - x = d \circ A_k(x).$$

This means that $f(x) - x \in B^k$, which assures that at the cohomology level one has $[f(x)] = [\mathrm{Id}(x)]$, that is in cohomology, $H^k f = H^k \,\mathrm{Id}$. But $H^k \,\mathrm{Id} = \mathrm{Id}$, and thus $H^k f = \mathrm{Id}$. In particular, $H^k f$ is a vector space isomorphism. □

This proposition can be generalized

Proposition A.3 (Homotopical morphisms, revisited) *Let $f, g : F^* \to G^*$ be morphisms of Banach complexes.*

(1) If there exists a homotopy $\left\{A_k : F^k \to G^{k-1}\right\}$ between f and g, then at the cohomology level the maps coincide:

$$H^k f = H^k g : H^k(F^*, d) \to H^k(G^*, d).$$

(2) If there exists a weak homotopy $\left\{A_{i,k} : F^k \to G^{k-1}\right\}$ between f and g, then at the reduced cohomology level the maps coincide:

$$\overline{H}^k f = H^k g : \overline{H}^k(F^*, d) \to \overline{H}^k(G^*, d).$$

Proof :

(1) As in the proof of proposition A.2, since $f - g = d \circ A_k(x) + A_k \circ d(x)$, one has $H^k f = H^k g$ at the cohomology level.

(2) One has

$$\lim_{i \to 0} \left\| \left(d \circ A_{i,k} + A_{i,k+1} \circ d\right)(x) - (f - g)(x) \right\| = 0.$$

As a consequence, for any $x \in Z^k$,

$$\lim_{i \to 0} \|d \circ A_{i,k}(x) - (f-g)(x)\| = 0.$$

This means that $(f-g)(x)$ belongs to $\overline{B}^k(G^*, d)$, and thus in cohomology the classes coincide: $[f(x)] = [g(x)]$. One thus has $\overline{H}^k f = \overline{H}^k g$.

□

Sobolev inequalities for Banach complexes The three propositions that we prove here can be found in [GT06]. We can call them Sobolev inequalities for Banach complexes.

Proposition A.4 *Let (F^*, d) be a Banach complex. The following assertions are equivalent:*

(i.) $\dim T^k < \infty$*;*

(ii.) $T^k = 0$*;*

(iii.) $H^k(F^*, d)$ *is a Banach space;*

(iv.) $d : F^{k-1} \longrightarrow F^k$ *is a closed operator.*

Proof :

- (i) $\Rightarrow$ (ii): T^k is the quotient of a Banach space by the image of a dense subspace. Hence it is finite-dimensional if and only if it is trivial.
- (ii) $\Rightarrow$ (iii): If $T^k = 0$, then H^k and $\overline{H}^k$ coincide. Since $\overline{H}^k$ is a Banach space, it is the case for H^k as well.
- (iii) $\Rightarrow$ (iv): Conversely, if H^k is a Banach space, then it coincides with $\overline{H}^k$. In particular, this means that $B^k = \overline{B}^k$. Hence the image of d is closed.
- (iv) $\Rightarrow$ (i): if d is closed, then B^k is closed. Hence $H^k = \overline{H}^k$ and the torsion must be zero.

□

Proposition A.5 *Let $(F^\star, d)$ be a Banach complex. The following assertions are equivalent:*

(i.) $H^k = 0$*;*

(ii.) $d_{k-1} : F^{k-1}/Z^{k-1} \longrightarrow Z^k$ *admits a bounded inverse;*

(iii.) *There exists a constant $C > 0$ such that for any k-cocycle $\phi \in Z^k$, there exists a cochain $\psi \in F^{k-1}$ such that $d\psi = \phi$ and $\|\psi\| \leq C_k \|\phi\|$.*

Proof :

- (i.) $\Rightarrow$ (ii.) : Since $H^k = 0$, then $B^k = Z^k$. Hence the bounded and surjective operator $d_{k-1} : F^{k-1} \to B^k$ is in fact $d_{k-1} : F^{k-1} \to Z^k$. Modding out the kernel, we obtain a bounded and bijective operator between Banach spaces.

$$d_{k-1} : F^{k-1}/Z^{k-1} \longrightarrow Z^k.$$

  By the open map theorem, it is a homeomorphism.

- (ii.) $\Rightarrow$ (iii.) : Let C' be the norm of $d_{k-1} : F^{k-1}/Z^{k-1} \longrightarrow Z^k$. Let $\phi \in Z^k$, and $[\eta] = d_{k-1}^{-1}(\phi) \in F^{k-1}/Z^{k-1}$. By hypothesis, we have

$$\|[\eta]\| \leq C'_k \|\phi\|.$$

  Yet,

$$\|[\eta]\| = \inf\{\|\eta - \varepsilon\| \mid \varepsilon \in Z^{k-1}\}.$$

  Hence,

$$\|[\eta]\| = \inf\{\|\eta - \varepsilon\| \mid \varepsilon \in Z^{k-1}\} \leq C' \|\phi\|.$$

  Let ε such that $\|\eta - \varepsilon\| \leq 2C'\|\phi\|$. We see that one can choose $\psi = \eta - \varepsilon$ et $C = 2C'$.

- (iii.) $\Rightarrow$ (i.) This is trivial.

□

Proposition A.6 *Let $(F^\star, d)$ be a Banach complex. The following assertions are equivalent:*

(i.) *$T^k = 0$;*

(ii.) *$d_{k-1} : F^{k-1}/Z^{k-1} \longrightarrow B^k$ admits a bounded inverse;*

(iii.) *There exists a constant $C' > 0$ such that for any $\xi \in F^{k-1}$, there exists a cochain $\zeta \in Z^{k-1}$ such that $\|\xi - \zeta\|_{F^{k-1}} \leq C' \|d\xi\|_{F^k}$.*

A proof can be found in [GT06].

Quasi-isometry and some invariants

Now let us take a look at the notion of quasi-isometries between metric spaces. We start with Hausdorff and Gromov-Hausdorff distances.

Let (X, d) be a metric space, and $A \subset X$. For any $\varepsilon > 0$, let A_ε designate the ε-neighborhood of A in X, that is

$$A_\varepsilon = \{x \in X \,| d(x, A) \leq \varepsilon\} .$$

Definition (Hausdorff distance) The *Hausdorff distance* $d_H(A, B)$ between two subsets $A, B \subset X$ by the formula

$$d_H(A, B) = \inf \{\varepsilon > 0 \,| A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\} .$$

It is a distance on the collection of compact, nonempty subsets of X.

Definition (Gromov-Hausdorff distance) The *Gromov-Hausdorff distance* $d_{GH}(X, Y)$ between two metric spaces X and Y is defined by the following property:

$d_{GH}(X, Y) \leq \varepsilon$ if, and only if, there exists a metric spaces Z and two subspaces X', Y' of Z isometric to X and Y respectively such that $d_H(X', Y') \leq \varepsilon$.

In other terms, the Gromov-Hausdorff distance between X and Y is the infimum of the Hausdorff distances of their images, taken over all isometric embeddings in a common space.

Definition (Net) Let X be a metric space. A *net* in X is a subset $N \subset X$ satisfying the following condition: there exists $\varepsilon > 0$ such that $N_\varepsilon = X$. In other terms, N is ε-dense in X for some $\varepsilon > 0$. We also use the terminology ε-net. For $\rho > 0$, a net $N \in X$ is *ρ-separated* if $d(x, y) > \rho$ for any choice of x, y in X. A net is *separated* if it is ρ-separated for some $\rho > 0$.

Definition (Relation) A *relation* between two sets X and Y is a subset $\mathcal{R} \subset X \times Y$ satisfying the following conditions:

(i) For any $x \in X$, there exists $y \in Y$ such that $(x, y) \in \mathcal{R}$;

(ii) For any $y \in Y$, there exists $x \in X$ such that $(x, y) \in \mathcal{R}$;

The graph of a surjective map is a relation, but the converse is generaly not true. However, given $x \in X$, one can chose $f(x) := y \in Y$ such that (x, y), thus obtaining a non-unique, non-surjective map $f : X \to Y$, whose graph is a subset of $\mathcal{R}$.

Definition (Distorsion) Let X, Y be two metric spaces, with metrics d_X and d_Y respectively.

- The *distorsion* of a relation $\mathcal{R} \subset X \times Y$ is

$$\operatorname{dis}(\mathcal{R}) = \sup_{(x_1,y_1),(x_2,y_2)\in\mathcal{R}} \{|d_X(x_1,x_2) - d_Y(y_1,y_2)|\}$$

- The *distorsion* of a map $f : X \to Y$ is

$$\operatorname{dis}(f) = \sup_{x_1,x_2\in X} \{|d_X(x_1,x_2) - d_Y(f(x_1),f(x_2))|\}$$

Definition Let X, Y be two metric spaces. A map $f : X \to Y$ is said to be

(i) An *ε-isometry*, $\varepsilon > 0$, if $\operatorname{dis}(f) \leq \varepsilon$ and if its image $f(X)$ is a ε-net in Y.

(ii) A *quasi-isometric embedding* if there exists real numbers $L > 1$ and $C > 0$ such that for any $x_1, x_2 \in X$,

$$\frac{1}{L} \cdot d(x_1,x_2) - C \leq d\left(f(x_1), f(x_2)\right) \leq L \cdot d(x_1,x_2) + C.$$

(iii) A quasi-isometry if it is both a quasi-isometric embedding whose image $f(X)$ is ε-dense in Y for some $\varepsilon > 0$.

A remark on the terminology: A quasi-isometry is sometimes called a *rough isometry*, whereas an old terminology designs bilipschitz maps by the expression "quasi-isometry“. The terminology we use here is the most common one.

Definition Two metric spaces X and Y are said to be *quasi-isometric* of there exist metrics spaces X' and Y' such that X' and Y' are bilipschiz-equivalent and

$$d_{GH}(X',X), d_{GH}(Y',Y) < \infty.$$

This is an equivalence relation between metric spaces, and equivalent metric spaces are simply said to be *quasi-isometric*. We will see further that being quasi-isometric is equivalent to the existence of a quasi-isometry between X and Y.

A quasi-isometry is in some way a map which is bilipschitz at large scales, and thus captures the "large-scale geometry“ of a metric space. A quasi-isometry needs not be continuous, and therefore carries no topological information. For instance, $\mathbb{Z}^n$ and $\mathbb{R}^n$ are quasi-isometric: the usual injection is a quasi-isometry. Moreover, this notion only allows to distinguish between non-compact spaces: two spaces with finite diameter are obviously quasi-isometric one to each other.

Proposition A.7 *For any two metric spaces X and Y, one has*

$$d_{GH}(X,Y) = \frac{1}{2} \inf_{\mathcal{R}} \operatorname{dis}(\mathcal{R}).$$

The infimum is taken over all relations between X and Y.

Proof : We separate the proof in parts.

(1) For any $r > d_{GH}(X,Y)$, there exists a relation $\mathcal{R}$ between X and Y with $\mathrm{dis}(\mathcal{R}) < 2r$. Indeed, let Z be a metric space containing isometric copies X' and Y of X and Y respectively, with $d_H(X',Y') < r$ in Z. Thus for any $x' \in X'$ there exists $y' \in Y'$ with $d(x',y') < r$ and similarly for any $y' \in Y'$, there exists $x' \in X'$ such that $d(x',y') < r$. Hence we can define the following relation between X' and Y':

$$\mathcal{R} = \left\{ (x,y) \in X' \times Y' \middle| \, d(x,y) < r \right\}.$$

Since $X = X'$ and $Y = Y'$, this gives us a relation $\mathcal{R}$ between X and Y. We just need to check that its distorsion is controlled. One has for $(x_1,y_1),(x_2,y_2) \in \mathcal{R}$:

$$\begin{aligned} d(x_1,x_2) - d(y_1,y_2) &\leq d(x_1,y_1) + d(y_1,x_2) - d(y_1,y_2) \\ &\leq d(x_1,y_1) + d(y_1,y_2) + d(y_2,x_2) - d(y_1,y_2) \\ &= d(x_1,y_1) + d(x_2,y_2) \\ &< 2r \end{aligned}$$

By a similar argument,

$$\begin{aligned} -d(x_1,x_2) + d(y_1,y_2) &\leq d(y_1,x_1) + d(y_2,x_2) \\ &< 2r \end{aligned}$$

Hence $|d(x_1,x_2) - d(y_1,y_2)| < 2r$, which assures that $\mathrm{dis}(\mathcal{R}) < 2r$. Hence we have

$$d_{GH}(X,Y) \geq \frac{1}{2} \inf_{\mathcal{R}} \mathrm{dis}(\mathcal{R}).$$

(2) It remains to be shown that $d_{GH}(X,Y) \leq \frac{1}{2} \inf_{\mathcal{R}} \mathrm{dis}(\mathcal{R})$. Let $\mathcal{R}$ be a relation between X and Y, and $r = \frac{1}{2} \cdot \mathrm{dis}(\mathcal{R})$. We want to show that $d_{GH}(X,Y) \leq r$, which can be done by finding a metric space Z containing isometric copies of X, Y, with Hausdorff distance between them lower that r. Let us set $Z = X \coprod Y$. We define the following distance d_Z on $X \coprod Y$: if z_1, z_2 both lie in X, then we set $d_Z(z_1,z_2) = d_X(z_1,z_2)$. Similarly, if z_1, z_2 both lie in Y, then we set $d_Z(z_1,z_2) = d_Y(z_1,z_2)$. If $z_1 \in X$ and $z_2 \in Y$, let

$$d_Z(z_1,z_2) = \inf_{(x,y)\in\mathcal{R}} \left\{ d_X(z_1,x) + d_Y(z_2,y) + r \right\}.$$

Then d_Z is a metric on Z. Indeed, it is of course symmetric and non-negative. Moreover, if $d(z_1,z_2)$, then both z_1 and z_2 must belong either to X or Y simultaneously, and thus $z_1 = z_2$ as their distance is given by d_X or d_Y respectively. One has moreover $z_1 = z_2 \Rightarrow d_Z(z_1,z_2) = 0$. Only the triangle inequality remains to be shown:

$$d_Z(z_1,z_2) + d_Z(z_2,z_3) \leq d_Z(z_1,z_3).$$

If z_1, z_2, z_3 all belong to X or Y, then there is nothing to show. Let us suppose that $z_1, z_2 \in X$ and $z_3 \in Y$. One has

$$\begin{aligned} d_Z(z_1, z_2) &= d_X(z_1, z_2) \\ d_Z(z_2, z_3) &= \inf_{(x,y)\in\mathcal{R}} \{d_X(z_2, x) + d_Y(z_3, y) + r\} \\ d_Z(z_1, z_3) &= \inf_{(x,y)\in\mathcal{R}} \{d_X(z_1, x) + d_Y(z_3, y) + r\} \end{aligned}$$

Hence

$$\begin{aligned} d_Z(z_1, z_2) + d_Z(z_2, z_3) &= d_X(z_1, z_2) + \inf_{(x,y)\in\mathcal{R}} \{d_X(z_2, x) + d_Y(z_3, y) + r\} \\ &= \inf_{(x,y)\in\mathcal{R}} \{d_X(z_1, z_2) + d_X(z_2, x) + d_Y(z_3, y) + r\} \\ &\geq \inf_{(x,y)\in\mathcal{R}} \{d_X(z_1, x) + d_Y(z_3, y) + r\} \\ &= d_Z(z_1, z_3). \end{aligned}$$

The other cases are symmetric. Hence d_Z is indeed a metric on Z. Let us finally show that for this metric, $d_H(X, Y) \leq r$ in Z. Let $z_1 \in X$. We know that $\mathrm{dis}(\mathcal{R}) = 2r$, and thus

$$\sup_{(x,y),(z_1,z_2)\in\mathcal{R}} \{|d(z_1, x) - d(z_2, y)|\} = 2r.$$

Let $z_2 \in Y$ such that $(z_1, z_2) \in \mathcal{R}$. One has

$$\begin{aligned} d(z_1, z_2) &= \inf_{(x,y)\in\mathcal{R}} (d(z_1, x) + d(z_2, y) + r) \\ &\leq d(z_1, z_1) + d(z_2, z_2) + r \\ &= r \end{aligned}$$

Thus $d_{GH}(X, Y) \leq r$.

□

Proposition A.8 *Let X, Y be two metric spaces. Then*

(i) If $d_{GH}(X, Y) < \varepsilon$, there exists a 2ε-isometry from X to Y.

(ii) If there exists a ε-isometry from X to Y, then $d_{GH}(X, Y) < 2\varepsilon$.

Proof :

(i) Let $d_{GH}(X,Y) < \varepsilon$. By Proposition A.7, there exists a relation $\mathcal{R} \subset X \times Y$ with distorsion $\mathrm{dis}(\mathcal{R}) \leq 2\varepsilon$. Let $x \in X$, and let us choose $y \in Y$ such that $(x,y) \in \mathcal{R}$. Let us denote $f(x) = y$. The distorsion of the map $f : X \to Y$ satisfies $\mathrm{dis}(f) \leq \mathrm{dis}(\mathcal{R})$, for the sup is taken on a smaller set. The only thing we have to prove is that $f(X)$ is a ε-net in Y. For any $y \in Y$, let $x \in X$ such that $(x,y) \in \mathcal{R}$. Since $(x,y),(x,f(x)) \in \mathcal{R}$, one has

$$|d(x,x) - d(y,f(x))| \leq \mathrm{dis}(\mathcal{R}) \leq 2\varepsilon.$$

Hence $d(y,f(x)) \leq 2\varepsilon$ and thus $y \in f(X)_{2\varepsilon}$.

(ii) Let $f : X \to Y$ be a ε-isometry, and $\mathcal{R} \subset X \times Y$ be defined by

$$\mathcal{R} = \{ (x,y) \in X \times Y | \, d(y,f(x)) \leq \varepsilon \} .$$

Since $f(X)_\varepsilon = Y$, this is a relation between X and Y. Moreover, $\mathrm{dis}(\mathcal{R}) \leq 2\varepsilon$, and thus

$$d_{GH}(X,Y) \leq \frac{3}{2}\varepsilon \leq 2\varepsilon.$$

□

This proposition can be used to establish a link between being quasi-isometric and the existence of a quasi-isometry:

Proposition A.9 *Let X and Y be two metric spaces. The following are equivalent:*

(i) X and Y are quasi-isometric;

(ii) There exists a quasi-isometry $f : X \to Y$;

(iii) X and Y contain bilipschitz homeomorphic separated nets.

Proof :

1. By definition, if X and Y are quasi-isometric, there exist X' and Y' bilipshitz-homeomorphic metric spaces with $d_{GH}(X',X), d_{GH}(Y',Y) \leq \varepsilon$, where ε is some positive real number. Let $\phi : X' \to Y'$ a bilipschitz homeomorphism, and let us also take 2ε-isometries $f_1 : X \to X'$ and $f_2 : Y' \to Y$, whose existence is guaranteed by Proposition A.8. We know that $\mathrm{dis}(f_1), \mathrm{dis}(f_2) \leq 2\varepsilon$.

   Let $\lambda \geq 1$ such that

$$\frac{1}{\lambda} d(x_1,x_2) \leq d(\phi(x_1),\phi(x_2)) \leq \lambda d(x_1,x_2) \text{ for any } x_1,x_2 \in X'$$

   Let $d_1 = \mathrm{dis}(f_1), d_2 = \mathrm{dis}(f_2)$, and let us denote $\psi = f_2 \circ \phi \circ f_1$. One has

$$\begin{aligned}
-d_2 + d\left(\phi \circ f_1(x_1), \phi \circ f_1(x_2)\right) &\leq d\left(\psi(x_1), \psi(x_2)\right) \leq && d_2 + d\left(\phi \circ f_1(x), \phi \circ f_1(x_2)\right) \\
-d_2 + \frac{1}{\lambda} d\left(f_1(x_1), f_1(x_2)\right) &\leq d\left(\psi(x_1), \psi(x_2)\right) \leq && d_2 + \lambda d\left(\phi \circ f_1(x), \phi \circ f_1(x_2)\right) \\
-d_2 \frac{1}{\lambda}\left(d(x_1, x_2) - d_1\right) &\leq d\left(\psi(x_1), \psi(x_2)\right) \leq && d_2 + \lambda d\left(d(x_1, x_2) + d_1\right) \\
\frac{1}{\lambda} d(x_1, x_2) - \left(d_2 + \frac{1}{\lambda} d_1\right) &\leq d\left(\psi(x_1), \psi(x_2)\right) \leq && \lambda d(x_1, x_2) + (d_2 + \lambda d_1)
\end{aligned}$$

With $L = \lambda \geq 1$ and $C = \max\left\{d_2 + \frac{1}{\lambda} d_1, d_2 + \lambda d_1\right\} \geq 0$, one has

$$\frac{1}{L} d(x_1, x_2) - C \leq d(\phi(x_1), \psi(x_2) \leq L d(x_1, x_2) + C.$$

That is, ψ is a quasi-isometric embedding. Since the images of f_1, f_2 are separated nets and since ϕ is a bilipschitz-homeomorphisms (thus sending separated nets to separated nets), ϕ is a quasi-isometry.

2. Let $f : X \to Y$ be a quasi-isometry with

$$\frac{1}{L} d(x_1, x_2) - C \leq d(f(x_1), f(x_2) \leq L d(x_1, x_2) + C$$

and let $\Delta > (2\lambda + 1)C$, and let S be a Δ-separated net in X. Then from the inequality above, one has for any $x_1, x_2 \in X$:

$$\frac{1}{2\lambda} d(x_1, x_2) \leq d(f(x_1), f(x_2)) \leq (\lambda + 1) d(x_1, x_2).$$

This means that f is a bilispschitz homeomorphism from S to $f(S)$, and the latter is still a net in Y.

3. Finally, S and $f(S)$ can be used as the bilipschitz-homeomorphic subspaces in the definition of quasi-isometric spaces.

□

This allows us to establish a nice criterion for metric spaces to be quasi-isometric:

Proposition A.10 *Let X and Y be quasi-isometric spaces. Then there exists a quasi-isometry $\Phi : X \to Y$, a quasi-isometry $\Psi : Y \to X$ and a constant $N > 0$ such that for any $x \in X$,*

$$d(\Phi \circ \Psi(x), x) \leq N$$

Proof : Let S and T be bilipschitz N-separated nets in X and Y respectively, and let $\phi : S \to T$ denote a bilipschitz homeomorphism.

One can define a (non necessarily continuous) map $\pi_S : X \to S$ such that $\pi_S|_S = \mathrm{Id}_S$ and such that for any $x \in X$, $d(x, \pi_S(x)) \leq N$. A similar map $\pi_T : Y \to T$ may be defined as well. Then it is clear that $\Phi = \phi \circ \pi_S$ and $\Psi = \phi^{-1} \circ \pi_T$ satisfy our hypothesis. □

The notion of quasi-isometry is initially due to Kanai and Gromov. They were the first ones to exhibit properties of metric spaces invariant under this particular class of maps. Here are some quasi-isometry invariants:

Growth of the volume Let (M, g) be a Riemannian manifold, $x \in M$ and $r > 0$. Let $B(x, r)$ denote the ball centered at x of radius r. The manifold M has *polynomial growth* if its volume is polynomial in r.
In [Kan85], Kanai shows the following result: if M and N are quasi-isometric Riemannian manifolds of bounded geometry, with Ricci curvature bounded in absolute value and a positive injectivity radius, M is of polynomial growth if and only if N has polynomial growth.

Gromov-hyperbolicity: Let X be a metric space, and $x, y, p \in X$. The *Gromov product* $(x|y)_p$ is

$$(x|y)_p = \frac{1}{2} \left(d(x, p) + d(y, p) - d(x, y) \right).$$

Given $d \geq 0$, the space X is d-hyperbolic if $(x|z)_p \geq \min \{(x|y)_p, (y|z)_p\} - d$, and one says that X is *(Gromov)-hyperbolic* if it is d-hyperbolic for some $d \geq 0$. In [Gro87], Gromov shows that being hyperbolic is a quasi-isometry invariant.

p-**hyperbolicity**: *p-capacity p-parabolic manifolds* Let (M, g) be a Riemannian manifold, $\Omega \subset M$ a connected domain, and $D \subset \Omega$ a compact set. For $1 \leq p \leq \infty$, the *p-capacity* of D in Ω is defined as follows:

$$\mathrm{Cap}_p(D, \Omega) = \inf \left\{ \int_\Omega |du|^p \,\middle|\, u \in W_0^{1,p}(\Omega) \cap C_0^0(\Omega), u \geq 1 \text{ on } D \right\}$$

where $W_0^{1,p}$ is the closure of the set C_0^1 of compactly supported smooth functions with respect to the norm

$$\|u\|_{1,p} = \|u\|_{L^p} + \|du\|_{L^p}.$$

A Riemannian manifold is *p-hyperbolic* if it contains a compact set of positive p-capacity, and *p-parabolic* otherwise. In [Kan86], Kanai shows that being 2-parabolic is preserved under quasi-isometries for manifolds with bounded geometry. In [Hol94], Holopainen extends this result to p-capacity.

Integral inequalities

The two following estimates for convolution are useful in the proof of the regularization theorem. Let $f \star g$ denote the convolution product of two real-valued measurable functions f and g.

Proposition A.11 (Young's inequality for convolution) *Let $1 \leq p, q, r < \infty$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. Let also U be an open subset of $\mathbb{R}^n$, and $f \in L^p(U), g \in L^q(U)$. Then $f \star g \in L^r(U)$, and moreover we have the estimate*

$$\|f \star g\|_r \leq \|f\|_p \cdot \|g\|_q.$$

Proof : See [Fol84], proposition 8.9. □

Proposition A.12 (Hardy-Littlewood-Sobolev inequality) *Let $1 < p, q < \infty$ be real numbers such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. Let also U be an open subset of $\mathbb{R}^n$, and $f \in L^p(U)$. Let us denote by g the function defined by $g(x) = \displaystyle\int_U f(y)(x-y)^{1-n} dy$. Then $g \in L^q(U)$ and moreover*

$$\|g\|_q \leq A_{q,p} \cdot \|f\|_p$$

where $A_{q,p}$ is a constant depending only on p, q and U.

Proof : See [Ste70], Theorem 1 of page 119. □

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Curriculum Vitae

I was born on July 21$^{\text{th}}$, 1979 in Thonon-les-Bains, France. I attended primary and secondary school in Ferney-Voltaire, and finally obtained the french secondary school degree 'Baccalauréat' in 1998. In 1999 I attended the "Cours de mathématiques spéciales" at the Ecole Polytechnique Fédérale de Lausanne (EPFL) and finally I undertook to study mathematics at EPFL in october 2000. For my master's thesis, I studied a problem of geometry in metric spaces under the direction of Prof. Marc Troyanov and eventually obtained a Master degree in Mathematical Sciences ('Diplôme d'ingénieur mathématicien') in april 2005. I was then hired as a teaching and research assistant and I began working on the present thesis in november 2005. I had the opportunity to attend several courses and international conferences, and to teach in different areas of geometry, analysis and algebra.