CONTROL THEORY, DIRICHLET SERIES AND IRRATIONAL NUMBERS

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ABSTRACT. In this paper we present a control theory approach applied to some irrationality problems involving certain Dirichlet series.

1. INTRODUCTION

The aim of this paper is to analyse from a control theory viewpoint some irrationality problems arising from certain convergent series related to the Riemann zeta function and more generally related to Dirichlet series.

The usual way to prove the irrationality of a certain real number is to prove that good rational approximations with sufficiently small denominators exist.

The motivation of this paper is to provide a control theory viewpoint in this sense: starting from a partial sum of a series (a *state* from a control theory point of view), we provide a way to control the denominators of certain rational approximations, by suitably acting on the subsequent coefficients of the series.

A classical problem in number theory considers sequences ω whose values ω_t for $t = 1, 2, \ldots$ are in $\mathbb{C}$, and the function

$$\alpha(\omega, s) = \sum_{t=1}^{\infty} \frac{\omega_t}{t^s}, \qquad s \in \mathbb{C}.$$

Given a certain sequence ω, a certain $s \in \mathbb{C}$, is $\alpha(\omega, s) = 0$? If $\alpha(\omega, s)$ is real, is it irrational? Series of this form are called Dirichlet series [2]. When ω is a Dirichlet character, the series is a Dirichlet L-series. When $\omega \equiv 1$ the corresponding L-series, denoted by $\zeta(s)$, is the Riemann zeta function. This series, defined for Re $(s) > 1$, can be completed to a meromorphic function defined on the complex plane and is the object

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of a wide research motivated mainly by the Riemann hypothesis on the distribution of its zeros.

The domain of investigation of this paper are the Dirichlet series for which $\omega_t = 1$ or -1, and $s \in \mathbb{N}$. In particular we will analyse some general questions of convergence and irrationality of this category of Dirichlet series. The Riemann zeta function for integers $s \geq 2$ is included in this category. It is well known that $\zeta(s)$ is irrational for every even positive integer, and Apéry proved that $\zeta(3)$ is irrational [1]. It is conjectured $\zeta(s)$ is irrational also for every odd integer $s \geq 5$.

For $\omega_t = (-1)^{n+1}$ and $s = 1$ the corresponding series converges to $\log 2$, which is an irrational number. Other examples of sequences ω correspond to other well known values of the corresponding Dirichlet series (see Table 1).

s	ω	convergence
1	$+1, -1, +1, -1, \ldots$	$\log 2$
2	$1, 1, 1, 1 \ldots$	$\zeta(2) = \pi^2/6$
	$1, -1, 1, -1, 1, -1, \ldots$	$\pi^2/12$
	$1, 1, -1, 1, 1, -1, 1, 1, -1, \ldots$	$7\pi^2/54$
	$1, -1, 1, -1, 1, 1, 1, -1, 1, 1, 1, 1, 1, 1, 1, -1, 1 \ldots$	$(\pi^2 - 4)/6$
	$1, -1, -1, 1, -1, -1, -1, -1, 1, -1, -1, \ldots$	$\pi^2(2\pi^2 - 15)/90$
3	$1, 1, 1, 1 \ldots$	$\zeta(3)$
	$1, -1, 1, -1, 1, -1, \ldots$	$3\zeta(3)/4$
	$1, 1, -1, 1, 1, -1, 1, 1, -1, \ldots$	$25\zeta(3)/27$

Tab. 1. Some examples of Dirichlet series with $\omega_t = 1$ or -1.

2. The control theory viewpoint

In this section we will use the classical notation from the control theory (see for exemple [3]). Let the system $\Sigma_s = (\mathcal{T}, \mathcal{X}, \mathcal{U}, \phi)$ be defined by

(i) the time set $\mathcal{T} = \mathbb{Z}$
(ii) the state space $\mathcal{X} = \mathbb{R}$
(iii) the input-value space $\mathcal{U} = (a, b)$, a bounded interval in $\mathbb{R}$.
(iv) the transition map ϕ defined on

$$\mathcal{D}_\phi = \{(\tau, \sigma, x, \omega) | \sigma, \tau \in \mathcal{T}, 1 \leq \sigma \leq \tau, x \in \mathcal{X}, \omega \in \mathcal{U}^{[\sigma,\tau)}\},$$

by

$$\phi(\tau, \sigma, x, \omega) = x + \sum_{t=\sigma}^{\tau-1} \frac{\omega_t}{t^s}, \tag{1}$$

where we denoted $\omega_t := \omega(t)$.

Note that for every x, each $\omega \in \mathcal{U}^{[\sigma,\infty)}$ is admissible for x [3, Definition 2.1.4.]. Such an ω can be regarded as a control of infinite lenght. The time varying discrete system Σ_s is complete [3, Definition 2.1.7.]. It applies to the state x a control ω. whose control at time t is boudend by c/t^s for a suitable constant c. For $s > 1$ the series

$$\sum_{t=\sigma}^{\infty} \frac{\omega_t}{t^s}$$

is absolutely convergent, so the control asymptotically forces the system to reach the value

$$\phi(\infty, \sigma, x, \omega) = \lim_{t \to \infty} \phi(t, \sigma, x, \omega|_{[\sigma,t)}) \in \mathbb{R}.$$

Note also that since Σ_s is a discrete-time system of class $\mathcal{C}^\infty$ (see [3, Definition 2.5.1.]), it is also a topological system in the sense of [3, Definition 3.7.1], with the topology of $\mathbb{R}$.

From this perspective, sequences in Table 1 can be regarded as controls applied to the state $x = 0$.

In the next section we will introduce certain controllers in order to provide, given an event (x, t) where x is a state and t is a time value, a feedback law defined by a suitable next-state mapping. So, with the designation of a controller, given an event (x, t), the control uniquely determines the evolution of the states at subsequent times: the system becomes a classical dynamical system.

3. Convergence as asymptotical controllability

Theorem 3.1. *Let $s = 2$. For every $\alpha \in (2 - \zeta(2), \zeta(2))$, there exist a sequence $\{\omega_t\}_{t\in\mathbb{N}}$ with $\omega_t = 1$ or -1 and such that*

$$\sum_{t=1}^{\infty} \frac{\omega_t}{t^2} = \alpha.$$

Proof. Let $\omega_1 = 1$. For $t \geq 2$, define by recurrence:

$$\omega_t = \begin{cases} 1 & \text{if } \displaystyle\sum_{k=1}^{t-1} \frac{\omega_k}{k^2} < \alpha \\ -1 & \text{else} \end{cases} \tag{2}$$

The decision rule defined by (2) plays the role of a feedback controller. To prove the theorem note that:

$$(2 - \zeta(2), \zeta(2)) = \left(1 - \sum_{k=2}^{\infty} \frac{1}{k^2}, 1 + \sum_{k=2}^{\infty} \frac{1}{k^2}\right) \simeq (0.355, 1.645).$$

The theorem is then a consequence of the fact that for every $t \geq 2$ one has:

$$\frac{1}{t^2} < \sum_{k=t+1}^{\infty} \frac{1}{k^2}. \tag{3}$$

To prove prove (3) let

$$f(t) = \frac{1}{t^2} - \sum_{k=t+1}^{\infty} \frac{1}{k^2}.$$

We will prove that $f(t) < 0$ for $t \geq 2$. We have

$$f(2) = \frac{1}{4} - \sum_{k=3}^{\infty} \frac{1}{k^2} = \frac{3}{2} - \zeta(2) \simeq -0.145$$

and

$$f(3) = \frac{1}{9} - \sum_{k=4}^{\infty} \frac{1}{k^2} = \frac{5}{4} + \frac{2}{9} - \zeta(2) \simeq -0.173.$$

For $t \geq 3$ we have:

$$f(t+1) - f(t) = \frac{2t^2 - (t+1)^2}{t^2(t+1)^2} > 0.$$

So for $t \geq 3$ the function $f(t)$ is increasing and its limit is 0. Since $f(3) < 0$, this proves that $f(t) < 0$ for every $t \geq 3$.

□

From the control theory viewpoint, let Σ'_s the system defined as Σ_s in which controls are allowed only in $\mathcal{U} = \{-1, +1\}$. In the transition map (1) coefficients ω_t can only be 1 or -1. The preceding theorem can be reformulated in terms of asymptotical controllability (see [3, Definition 5.5.1.]). In this framework, in the system Σ'_2 the event $(0,0) \in \mathcal{X} \times \mathcal{T}$ can be asymptotically controlled to α for every α in $(2 - \zeta(2), \zeta(2))$.

The Theorem 3.1 can be stated in analogous forms for other values of s. For exemple if $s = 3$ it becomes:

Theorem 3.2. *Let $s = 3$. For every $\alpha \in (2.25 - \zeta(3), \zeta(3)) \simeq (1.048, 1.202)$ and for every $\alpha \in (2 - \zeta(3), \zeta(3) - 0.25) \simeq (0.798, 0.952)$, there exists a sequence $\{\omega_t\}_{t \in \mathbb{N}}$ with $\omega_t = 1$ or -1 and such that*

$$\sum_{t=1}^{\infty} \frac{\omega_t}{t^3} = \alpha.$$

In other terms, in the system Σ_3' the event $(0,0)$ cannot be asymptotically controlled to a state in the interval $(\zeta(3)-0.25, 2.25-\zeta(3)) \simeq (0.952, 1.048)$, because

$$\frac{1}{8} - \sum_{k=3}^{\infty} \frac{1}{k^3} = \frac{5}{4} - \zeta(3) \simeq 0.048 > 0.$$

Apart from that, all other states in $(2-\zeta(3), \zeta(3)$ can be asymptotically reached because for $t \geq 3$, one has

$$\frac{1}{t^3} < \sum_{k=t+1}^{\infty} \frac{1}{k^3}.$$

For values of $s \geq 4$, the interval $(2-\zeta(s), \zeta(s))$ is a neighbourhood of $x = 1$ whose size decreases with s, and for α in a certain union of subintervals of $(2-\zeta(s), \zeta(s))$, the event $(0,0)$ can be asymptotically controlled to α.

4. Controllability and irrational numbers

In the previous section we showed that for an integer $s \geq 2$, and every α (rational or irrational) in a certain union of intervals, there exists a Dirichlet series with coefficients 1 or -1 that for that value of s converges to α. In particular this is true for every irrational number in that intervals.

This was done by introducing a certain feedback control depending on α. In this section we will provide sequences of controls ω_t with $\omega_t = 1$ or -1 whose definition is independent of an auxiliary real number α and whose corresponding Dirichlet series for a fixed value of the integer s converges to an irrational number. We will prove that given a finite sequence of $\omega_1, \ldots, \omega_{t_0}$ arbitrarily defined in $\{-1, 1\}$ it is always possible to complete that sequence to an infinite sequence with $\omega_t = 1$ or -1 whose Dirichlet series for a fixed value of s converges to an irrational number.

Theorem 4.1. *Let s be an integer with $s \geq 2$. Let $\omega_1, \ldots, \omega_{t_0}$ be arbitrarily defined in $\{-1, 1\}$. Let $\varepsilon > 0$ and define recursively t_h by*

$$t_{h+1} = 1 + [\ \mathrm{lcm}(1, 2, \ldots, t_h)]^{1+\varepsilon}.$$

For every $h \geq 0$ let $\omega_{t_h+1} = 1$. For $t = t_h + 2, t_h + 3, \ldots, t_{h+1}$ define:

$$\omega_t = \begin{cases} +1 & \text{if} \quad -\dfrac{1}{t^s} + \displaystyle\sum_{k=t_h+1}^{t-1} \frac{\omega_k}{k^s} \leq 0 \\ \\ -1 & \text{else} \end{cases}$$

The number α defined by

$$\alpha = \sum_{t=1}^{\infty} \frac{\omega_t}{t^s} \tag{4}$$

is irrational.

Proof. The series (4) defining α is an absolute convergent series, so in particular it is convergent. For every $h = 0, 1, \ldots$ consider the rational approximations p_h/q_h of α defined by

$$\frac{p_h}{q_h} = \sum_{k=1}^{t_h} \frac{\omega_t}{t^s} \in \mathbb{Q}.$$

We assume that $(p_h, q_h) = 1$ and that q_h positive. According to that, it is $q_h \leq \text{lcm }(1, 2, \ldots, t_h)^s$.

Note that $p_{h+1}/q_{h+1} > p_h/q_h$. This is a consequence of the fact that for every $t = t_h + 1, t_h + 2, \ldots, t_{h+1}$, is by definition

$$\sum_{k=t_h+1}^{t} \frac{\omega_k}{k^s} > 0.$$

This means that for every h

$$\frac{p_h}{q_h} < \alpha$$

and

$$\alpha = \lim_{h \to \infty} \frac{p_h}{q_h} = \sup_{h \in \mathbb{N}} \frac{p_h}{q_h}.$$

Moreover we have:

$$0 \neq \left| \alpha - \frac{p_h}{q_h} \right| = \alpha - \frac{p_h}{q_h} = \sum_{k=t_h+1}^{\infty} \frac{\omega_k}{k^s} = \sum_{k=t_h+1}^{t_{h+1}} \frac{\omega_k}{k^s} + \sum_{k=t_{h+1}+1}^{\infty} \frac{\omega_k}{k^s}.$$

Note that for sufficiently large h,

$$\sum_{k=t_h+1}^{t_{h+1}} \frac{\omega_k}{k^s} \leq \frac{2}{t_{h+1}^s}$$

and that

$$\sum_{k=t_{h+1}+1}^{\infty} \frac{\omega_k}{k^s} \leq \frac{1}{t_{h+1}^s},$$

so for sufficiently large h,

$$0 < \left| \alpha - \frac{p_h}{q_h} \right| \leq \frac{3}{t_{h+1}^s} \leq \frac{3}{\text{lcm }(1, 2, \ldots, t_h)^{s(1+\varepsilon)}} \leq \frac{3}{q_h^{1+\varepsilon}}. \tag{5}$$

On the other hand if α was rational, then $\alpha = M/N$ for certain fixed $M, N \in \mathbb{N}$, and since $p_h/q_h \neq \alpha$, we should have

$$\left|\alpha - \frac{p_h}{q_h}\right| \geq \frac{1}{Nq_h},$$

and this is in contraddiction with (5) for sufficiently large h. So α is irrational. □

Remark 4.2. From the control theory point of view, the preceding theorem can be easily seen in more generality as follows. In the system Σ_s' an arbitrary event $(x, t) \in \mathcal{X} \times \mathcal{T}$ with $x \in \mathbb{Q}$ can be asymptotically controlled to a state $\alpha \in \mathcal{X}$ corresponding to a suitable irrational number.

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References

[1] Apry, R. (1979), *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque 61: 11–13 .

[2] Hardy, G.H. and Wright, E.M. *An Introduction to the Theory of Numbers*, Oxford Clarendon Press, 1979.

[3] Sontag, E.D. *Mathematical Control Theory, Deterministic Finite Dimensional Systems*, Second Edition, Springer, New York, 1998.

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