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The inversion formulae for automorphisms of polynomial algebras and rings of differential operators in prime characteristic. (English summary)


Inversion formulas for a class of algebras—over a field of prime characteristic—are derived. Among such algebras, the paper considers (i) the polynomial algebra $P_n := K[x_1, x_2, \ldots, x_n]$; (ii) the ring of differential operators $\mathcal{D}(P_n)$ on $P_n$, i.e., $\mathcal{D}(P_n) \otimes P_n$; (iii) the $n$-th Weyl algebra $A_n$; (iv) the algebra $P_n \otimes A_m$; (v) the power series algebra $K[[x_1, \ldots, x_n]]$; (vi) $T_{k_1, \ldots, k_n} \otimes P_m$, where $T_{k_1, \ldots, k_n}$ is the subalgebra of $\mathcal{D}(P_m)$ generated by $P_n$ and the higher derivations $\partial_i^{[j]}$, $0 \leq j < p^{k_i}$, $i = 1, \ldots, n$, where $k_1, \ldots, k_n \in \mathbb{N}$.

The general idea for finding the inversion formula is the following. Let $A$ be an algebra over the field $K$, $\sigma$ an automorphism over $A$, and $\{x^\alpha\}$ a $K$-basis of $A$. The identity of the algebra is first decomposed as $\text{id}_A(\cdot) = \sum \lambda_{\alpha,y_\alpha}(\cdot)x^{\alpha}$, where $\lambda_{\alpha,y_\alpha}$ are algebraic maps. Both this decomposition and the existence of the inverse are assumed to exist for $\sigma$. Applying $\sigma(\cdot)$ to the identity map also has the presentation $\text{id}_A(\cdot) = \sum \lambda_{\alpha,\sigma(y_\alpha)}(\cdot)\sigma(x^{\alpha})$. Then because $\lambda_{\alpha,\sigma(y_\alpha)}(A) \subseteq K$, one simply applies $\sigma^{-1}(\cdot)$ to obtain the inverse expressed as $\sigma^{-1}(\cdot) = \sum \lambda_{\alpha,\sigma(y_\alpha)}(\cdot)x^{\alpha}$. The whole difficulty is to find suitable maps $\lambda_{\alpha,\sigma(y_\alpha)}(\cdot)$.

So as to be more specific, some definitions and notation are required. Let $\delta$ be a $K$-derivation of an algebra $A$ over an arbitrary field $K$. A finite or infinite sequence $x = \{x^i, 0 \leq i \leq l - 1\}$ of elements in $A$ where $x^{[0]} = 1$ is called an iterative sequence of length $l$ if $x^{[i]}x^{[j]} = (i+j)^{-1}x^{[i+j]}$, $0 \leq i, j \leq i-1, i+j \leq l - 1$. An iterative $\delta$ descent is such a sequence for which $\delta(x^{[i]}) = x^{[i-1]}$, $0 \leq i \leq l - 1, x^{[-1]} = 0$. Whenever $\delta$ is nilpotent, i.e., $\delta^l = 0$ for some $l \geq 2$, two $K$-linear maps from $A$ to $A$ can be constructed starting from an iterative $\delta$ sequence $\{x^i, 0 \leq i < l\}$ in the following way: $\varphi := \sum_{i=0}^{l-1}(-1)^i x^{[i]} \delta^i(\cdot)$ and $\psi := \sum_{i=0}^{l-1}(-1)^i \delta^i(\cdot)x^{[i]}$. These maps are projection maps onto the kernel $A^{\delta}$ of $\delta$; that is, if $c \in A$ is written as $c = a + b$ with $a \in A^{\delta}$ and $b \in A_{+}$, which is always possible since $A = A^{\delta} \oplus A_{+}$ with $A_{+} := \bigoplus_{i=1}^{l-1} x^{[i]} A^{\delta}$, then $\psi(c) = \varphi(c) = a$.

In the case of an automorphism $\sigma$ that preserves the ring of invariants in the sense that $\sigma(A^{\delta}) = A^{\delta}$, the following concepts are required. Both a non-empty well-ordered set $I$ and a set of commuting locally nilpotent $K$-derivations $\delta := \delta_i, i \in I$, are given. Suppose that for each $i \in I$ there exists an iterative $\delta_i$-descent $\{x^i_\lambda, j \leq k \in I\} \subseteq \bigcap_{i \neq k \in I} A^{\delta_k}$. Define (i) $\delta_i := \sigma|_{A_i}$; (ii) the twisted derivations $\delta'$ as $\{\delta'_i := \sigma \delta_i \sigma^{-1}, \ i \in I\}$; and (iii) the images of the iterative descents $x^i_\lambda := \{x^i_\lambda := \sigma(x^i_\lambda), i \in I\}$. The inversion formula is shown to be

$$\sigma^{-1}(a) = \sum_{\alpha \in E} x^{[\alpha]} \sigma^{-1}_\sigma(\delta'^{\alpha}(a)) = \sum_{\alpha \in E} \sigma^{-1}_\sigma \psi_{\sigma}(\delta'^{\alpha}(a))x^{[\alpha]}.$$

The result for $\sigma \in \text{Aut}_K(\mathcal{D}(K[x_1, \ldots, x_n]))$ is more involved, but nevertheless still rests on suitable locally nilpotent derivations and their nil algebras. $\mathcal{D}(K[x_1, \ldots, x_n])$ is the ring of differential
operators on the polynomial algebra $K[x_1, \ldots, x_n]$. This algebra is a $K$-algebra generated by the elements $x_1, \ldots, x_n$ and commuting higher derivations $\partial_i^{[k]} := \frac{\partial^k}{\partial x_i^k}$, $i = 1, \ldots, n$ and $k \geq 1$, that satisfy the defining relations $[x_i, x_j] = [\partial_i^{[k]}, \partial_j^{[l]}] = 0$, $\partial_i^{[k]} \partial_i^{[l]} = (k+l)\partial_i^{[k+l]}$, $[\partial_i^{[k]}, x_j] = \delta_{ij} \partial_i^{[k-1]}$ for all $i, j = 1, \ldots, n$ and $k, l \geq 1$ where $\delta_{ij}$ is the Kronecker delta and $\partial_i^{[0]} := 1$, $\partial_i^{[1]} = \frac{\partial}{\partial x_i}$. Given two elements $x_i, x_j$, define the inner derivation as $[x_i, x_j] = (\text{ad} x_i)(x_j) := x_i x_j - x_j x_i$. The key projection maps are $\varphi_i = \sum_{k \geq 0} \partial_i^{[k]} (\text{ad} x_i)^k$, $\psi_i = \sum_{k \geq 0} (\text{ad} x_i)^k \partial_i^{[k]}$, $i = 1, \ldots n$, which are used to infer the validity of the inversion formula

$$
\sigma^{-1}(a) = \sum_{\alpha, \beta \in \mathbb{N}^n} (-1)^{|\alpha|} \varphi_\alpha (\delta^{[\beta]}(a) \partial^{[\beta]} x^\alpha =
\sum_{\alpha, \beta \in \mathbb{N}^n} \psi_\alpha (\delta^{[\alpha]} \delta^{[\beta]}(a)) x^\alpha \partial^{[\beta]},
$$

where $\delta^{[\beta]} := \prod_{i=1}^n (-\text{ad} x'_i)^{\beta_i}$, the primed quantities being $x'_i := \sigma(x_i)$ and $\partial_i^{[k]} := \sigma(\partial_i^{[k]})$.

Reviewed by Philippe A. Müllerhaupt

References


Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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