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Distortion of mappings and $L_{q,p}$-cohomology

Vladimir Gol'dshtein · Marc Troyanov

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Abstract We study some relation between some geometrically defined classes of diffeomorphisms between manifolds and the $L_{q,p}$-cohomology of these manifolds. We apply these results to the $L_{q,p}$-cohomology of a manifold with a cusp.

Keywords $L_{q,p}$-cohomology · Differential forms · Distortion of mappings

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1 Introduction

The $L_{q,p}$-cohomology is an invariant of Riemannian manifolds defined to be the quotient of the space of p-integrable closed differential k-forms on the manifold modulo the exact forms having a q-integrable primitive:

$$H^k_{q,p}(M) = \{\omega \mid \omega \text{ is a } k\text{-form}, \quad |\omega| \in L^p(M) \text{ and } d\omega = 0\}/\{d\theta \mid |\theta| \in L^q(M)\}.$$

This invariant has been first defined for the special case $p = q = 2$ in the 1970s and has been intensively studied since then, we refer to the book [16] for an overview of L_2-cohomology. The $L_{q,p}$-cohomology has been introduced in the early 1980's as an invariant of the Lipschitz structure of manifolds, see [1]. During the next two decades, the main interest was focused on the case $p = q$, i.e., on L_p-cohomology, and the last chapter of the book by Gromov [9] is devoted to this subject; see also [2,9,20–23] for more geometrical applications of L_p-cohomology.

V. Gol'dshtein
Department of Mathematics, Ben Gurion University, P.O. Box 653, Beer Sheva 84105, Israel
e-mail: vladimir@bgu.ac.il

M. Troyanov (✉)
Section de Mathématiques, École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland
e-mail: marc.troyanov@epfl.ch

Although the $L_{q,p}$-cohomology with $q \neq p$ has attracted less attention, it possesses a richer structure. The subject is also motivated by its connections with Sobolev type inequalities [6] and quasiconformal geometry [7]. See also [2,5,13,14] for other results on $L_{q,p}$-cohomology.

When an invariant of a geometric object has been defined, it is important to investigate its functorial properties, i.e., its behavior under various classes of mappings. It is one of our goal in the present paper to describe a natural class of maps which induces morphisms at the level of $L_{q,p}$-cohomology. Our answer is restricted to the case of diffeomorphisms and is given in Theorem 6.1(C) below.

A diffeomorphisms will behave functorially for $L_{q,p}$-cohomology, if its *distortion* is controlled in some specific way. To explain what is meant by the distortion, consider a diffeomorphism $f : M \to \tilde{M}$ between two Riemannian manifolds. One then define for any k the *principal invariant* of f as

$$\sigma_k(f,x) = \sum_{i_1<i_2<\cdots<i_k} \lambda_{i_1}(x)\lambda_{i_2}(x)\cdots\lambda_{i_k}(x),$$

where the λ_i's are the singular values of df_x, i.e., the eigenvalues of $\sqrt{(df_x)^*(df_x)}$. One then say that f has *bounded* (s,t)*-distortion in degree* k, and we write $f \in \mathrm{BD}^k_{(s,t)}(M,\tilde{M})$, if

$$(\sigma_k(f,x))^s \cdot J_f^{-1}(x) \in L^t(M)$$

where J_f is the Jacobian of f.

The class $BD^1_{n,\infty}$ (where n is the dimension of M) is exactly the class of quasiconformal diffeomorphisms (also called mappings with bounded distortion), which has been introduced by Y. Reshetnyak in the early 1960s and has been intensively studied since then. The classes $BD^1_{s,\infty}$ has been studied by different authors and under various names, see [3,15,17–19,24,26,28,30]. The class $BD^{n-1}_{s,\infty}$ also appears in [26], where some obstructions to their existence are given.

As a preliminary step to the study of functoriality in $L_{q,p}$-cohomology, we study diffeomorphisms $f : M \to \tilde{M}$ that induce bounded operator between the Banach spaces of $\tilde{p}$-integrable differential k-forms. The result is formulated in Proposition 4.1 : it states that *a diffeomorphism* $f \in \mathrm{BD}^k_{(\tilde{p},t)}(M,\tilde{M})$ *induces a bounded operator* $f^* : L^{\tilde{p}}(\tilde{M},\Lambda^k) \to L^p(M,\Lambda^k)$ *if* $p \le \tilde{p} < \infty$ *and* $t = \frac{p}{\tilde{p}-p}$. Let us note that finer information is available in the case $k=1$, see [3,4,29,30].

To obtain a functoriality in $L_{q,p}$-cohomology, we need to control the distortion of the map f both on k-forms and on $(k-1)$ forms. This is formulated in Theorem 6.1(C), which states in particular that *a diffeomorphism* $f \in \mathrm{BD}^{n-k+1}_{(\tilde{q}',r)}(M,\tilde{M}) \cap \mathrm{BD}^k_{(\tilde{p},t)}(M,\tilde{M})$ *induces a well defined linear map* $f^* : H^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to H^k_{q,p}(M)$ *if* $p \le \tilde{p}$, $q \le \tilde{q}$, $t = \frac{p}{\tilde{p}-p}$, $u = \frac{q}{\tilde{q}-q}$ *and* $\tilde{q}' = \frac{\tilde{q}}{\tilde{q}-1}$.

This is a quite technical result, and it would be nice to be able to give conditions under which the map f^* is injective at the level of $L_{q,p}$-cohomology. But unfortunately, the results we give in Sect. 5 strongly suggest that it will be hard or impossible to find conditions for injectivity, except for the special cases of quasiconformal or bilipschitz maps. However we have the following result [Theorem 6.1(B)], which allows us to prove some vanishing results in $L_{q,p}$-cohomology without requiring the functoriality : *If there exists a diffeomorphism* $f \in \mathrm{BD}^{n-k+1}_{(\tilde{q}',r)}(M,\tilde{M}) \cap \mathrm{BD}^k_{(\tilde{p},t)}(M,\tilde{M})$ *with* $p \le \tilde{p}$, $q \le \tilde{q}$, $t = \frac{p}{\tilde{p}-p}$, $r = \frac{q(\tilde{q}-1)}{q-\tilde{q}}$, *and* $\tilde{q}' = \frac{\tilde{q}}{\tilde{q}-1}$, *then* $H^k_{q,p}(M) = 0$ *implies* $H^k_{\tilde{q},\tilde{p}}(\tilde{M}) = 0$. We show in section 7 how this result can be used to prove a vanishing results in $L_{q,p}$-cohomology.

2 Preliminary notions

2.1 $L_{q,p}$-cohomology

We recall the definition of $L_{q,p}$-cohomology, see [6] for more details. Given an oriented Riemannian manifold (M, g), we denote by $L^p(M, \Lambda^k)$ the Banach space of differential forms such that

$$\|\theta\|_p = \left(\int_M |\theta|^p dx\right)^{\frac{1}{p}} < \infty.$$

Any element in $L^p(M, \Lambda^k)$ defines a current, in particular we can define the subspace of (weakly) closed forms $Z_p^k(M) = L^p(M, \Lambda^k) \cap \ker d$, it is a closed subspace of $L^p(M, \Lambda^k)$. We then introduce the space

$$\Omega_{q,p}^{k-1}(M) = \{\omega \in L^q(M, \Lambda^{k-1}) \mid d\omega \in L^p(M, \Lambda^k)\},$$

which is a Banach space for the norm $\|\omega\|_{q,p} = \|\omega\|_{L^q} + \|d\omega\|_{L^p}$. We also define the space $B_{q,p}^k(M) = d\Omega_{q,p}^{k-1}(M) \subset Z_p^k(M)$ of exact forms in L^p having a primitive in L^q:

$$B_{q,p}^k(M) = d\left(L^q(M, \Lambda^{k-1})\right) \cap L^p(M, \Lambda^k).$$

Definition 2.1 The reduced and unreduced $L_{q,p}$-*cohomology* of (M, g) (where $1 \leq p, q \leq \infty$) are defined as

$$H_{q,p}^k(M) = Z_p^k(M)/B_{q,p}^k(M) \quad \text{and} \quad \overline{H}_{q,p}^k(M) = Z_p^k(M)/\overline{B}_{q,p}^k(M),$$

where $\overline{B}_{q,p}^k(M)$ is the closure of $B_{q,p}^k(M)$.

The reduced cohomology $\overline{H}_{q,p}^k(M)$ is naturally a Banach space. We also define the *torsion* to be quotient $T_{q,p}^k(M) = \overline{B}_{q,p}^k(M)/B_{q,p}^k(M)$. The torsion vanishes if and only if the cohomology space $H_{q,p}^k(M)$ is a Banach space (in this case $\overline{H}_{q,p}^k(M) = H_{q,p}^k(M)$).

2.2 Linear map between Euclidean spaces

Recall that a Euclidean vector space (E, g) is a finite dimensional real vector space equipped with a scalar product. Two linear mappings $A, B \in L(E_1; E_2)$ between two Euclidean vector spaces (E_1, g_1) of dimension n and m are said to be *orthogonally equivalent* if there exist orthogonal transformations $Q_1 \in O(E_1)$ and $Q_2 \in O(E_2)$ such that $B = Q_2^{-1} A Q_1$, i.e., the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{A} & E_2 \\ Q_1 \uparrow & & \uparrow Q_2 \\ E_2 & \xrightarrow{B} & E_2 \end{array}$$

commutes. Given a linear mapping $A : (E_1, g_1) \to (E_2, g_2)$, its (right) *Cauchy-Green tensor* $\mathbf{c}$ is the symmetric bilinear form on E_1 defined by $\mathbf{c}(x, y) = g_2(Ax, Ay)$. The *adjoint* of A is the linear map $A^\# : E_2 \to E_1$ satisfying

$$g_2(x, Ay) = g_1(A^\# x, y)$$

for all $x \in E_1$ and $y \in E_2$. The Cauchy-Green tensor and the adjoint are related by

$$\mathbf{c}(x, y) = g_2(Ax, Ay) = g_1(A^{\#}Ax, y).$$

Let us denote the eigenvalues of $A^{\#}A$ by $\mu_1, \mu_2, \dots, \mu_n$. Then $\mu_i \in [0, \infty)$, for all i, and there exists orthonormal basis $e_1, e_2, \dots, e_n$ of E_1 and $e'_1, e'_2, \dots, e'_m$ of E_2 such that $Ae_i = \sqrt{\mu_i}e'_i$ for all i. The matrix of $A^{\#}A$ with respect to an orthonormal basis $e_1, e_2, \dots, e_n$ of E_1 coincides with the matrix $\mathbf{C}$ of the Cauchy-Green tensor $\mathbf{c}$ in the same basis.

Definition 2.2 The numbers $\lambda_i = \sqrt{\mu_i}$ are called the *principal distortion coefficients* of A or the *singular values* of A.

The principal distortion coefficients can be computed from the distortion polynomial which is defined as follows:

Definition 2.3 Given an arbitrary basis $e_1, e_2, \dots, e_n$ of E_1, we associate to g_1 and $\mathbf{c}$, the $n \times n$ matrices $\mathbf{G} = (g_1(e_i, e_j))$ and $\mathbf{C} = (\mathbf{c}(e_i, e_j))$. The *distortion polynomial* of A is the polynomial

$$P_A(t) = \frac{\det(\mathbf{C} - t\mathbf{G})}{\det \mathbf{G}}.$$

The distortion polynomial $P_A(t)$ is independent of the choice of the basis $\{e_i\}$, it coincides with the characteristic polynomial of $AA^{\#}$ and has nonnegative roots. In particular, the roots of P_A are the eigenvalues μ_i of $AA^{\#}$ and the $\lambda_i = \sqrt{\mu_i}$ are the principal distortion coefficients of A. The distortion polynomial can thus be written in terms of the principal distortion coefficients as

$$P_A(t) = \prod_i (t - \lambda_i^2).$$

The following notion is also useful:

Definition 2.4 The *principal invariants of A* are the elementary symmetric polynomials in the λ_i's, They are thus defined by $\sigma_0(A) = 1$ and

$$\sigma_k(A) = \sum_{i_1 < i_2 < \dots < i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$$

for $k = 1, \dots, 2 \dots, n$.

The following result is well known, see e.g., [25, p. 57].

Proposition 2.1 *Two linear mappings $A, B \in L(E_1; E_2)$ are orthogonally equivalent if and only if they have the same principal invariants*: $\sigma_k(A) = \sigma_k(B)$ *for* $k = 1, 2, \dots, n$.

The principal invariants of A are related to the action of $A \in L(E_1; E_2)$ on the exterior (Grassmann) algebras: recall that if E is an Euclidean vector space, then the exterior algebra ΛE is equipped with a canonical scalar product. If $e_1, e_2, \dots, e_n$ is an orthonormal basis of E_1, then the $\binom{n}{k}$ multi-vectors $\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}\}(i_1 < i_2 < \dots < i_k)$ form an orthonormal basis of $\Lambda^k E$. To any linear map $A \in L(E_1; E_2)$ we associate a linear map $\Lambda^k A \in L(\Lambda^k E_1; \Lambda^k E_2)$, and we have

$$\frac{1}{\binom{n}{k}}\sigma_k \leq \left\| \Lambda^k A \right\| \leq \sigma_k \tag{2.1}$$

Indeed, suppose that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the principal distortion coefficients of A, then we have $\|\Lambda^k A\| = \lambda_{n-k+1}\lambda_{n-k+2}\cdots\lambda_n$ and $\sigma_k = \sum_{i_1<i_2<\cdots<i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$.

If $E_1 = E_2 = \mathbb{R}^n$ and A is a diagonal matrix with nonnegative entries, then we have $\sigma_k = \mathrm{Trace}(\Lambda^k A)$.

Using the fact that the principal distortion coefficients of A^{-1} are the inverse of the principal distortion coefficients of A, we obtain the following

Lemma 2.2 *If* $\dim(E_1) = \dim(E_2) = n$ *and* A *is invertible, then for any* $0 \leq m \leq n$, *we have*

$$\sigma_m(A^{-1}) = \frac{\sigma_{n-m}(A)}{J_A}.$$

3 Diffeomorphism and $L_{q,p}$-cohomology

Let (M, g) and $(\tilde{M}, \tilde{g})$ be two smooth oriented n-dimensional Riemannian manifolds and $f : M \to \tilde{M}$ be a diffeomorphism such that the induced operator

$$f^* : L^{\tilde{p}}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)$$

is bounded for some specified $p, \tilde{p} \in [0, \infty)$. Then the condition $f^*d = df^*$ implies that

$$f^* : Z^k_{\tilde{p}}(\tilde{M}) \to Z^k_p(M)$$

is a well defined bounded operator. In the framework of $L_{q,p}$-cohomology there are two natural questions which then arise:

(i) Suppose that $\omega \in B^k_{\tilde{q},\tilde{p}}(\tilde{M})$. Under what conditions does this imply that $f^*\omega \in B^k_{q,p}(M)$, i.e., that

$$f^*(B^k_{\tilde{q},\tilde{p}}(\tilde{M})) \subset B^k_{q,p}(M)\ ?$$

(ii) Suppose that $f^*\omega \in B^k_{q,p}(M)$. Under what conditions can we conclude that $\omega \in B^k_{\tilde{q},\tilde{p}}(\tilde{M})$, i.e., that

$$(f^{-1})^*(B^k_{q,p}(M)) \subset B^k_{\tilde{q},\tilde{p}}(\tilde{M})\ ?$$

A positive answer to the first question gives us a well defined linear map

$$f^* : H^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to H^k_{q,p}(M),$$

and a positive answer to both questions implies the injectivity of this linear map.

In this section we give an answer to these questions in terms of boundedness of the operators f^*, and $f_* = (f^{-1})^*$. We begin with the second question.

Theorem 3.1 *Let* $f : M \to \tilde{M}$ *be a diffeomorphism,* $1 \leq p \leq \tilde{p} < \infty$ *and* $1 \leq \tilde{q} \leq q < \infty$. *Assume that both operators*

$$f^* : L^{\tilde{p}}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k), \quad \textit{and} \quad f_* : L^q(M, \Lambda^{k-1}) \to L^{\tilde{q}}(\tilde{M}, \Lambda^{k-1})$$

are bounded. Then for any $\omega \in Z^k_{\tilde{p}}(\tilde{M})$, *we have* $f^*\omega \in Z^k_p(M)$. *Furthermore, if* $[f^*\omega] = 0$ *in* $H^k_{q,p}(M)$ *then* $[\omega] = 0$ *in* $H^k_{\tilde{q},\tilde{p}}(\tilde{M})$ *(thus* $H^k_{q,p}(M) = 0 \Rightarrow H^k_{\tilde{q},\tilde{p}}(\tilde{M}) = 0$).

Remark We should not conclude that $f^*: H^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to H^k_{q,p}(M)$ is an injective map, because this map is a priory not even well defined.

Proof Choose $\omega \in Z^k_{\tilde{p}}(\tilde{M})$. Because $f^*: L^{\tilde{p}}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)$ is a bounded operator, $f^*\omega \in L^p(M, \Lambda^k)$, and since $d(f^*\omega) = f^* d\omega = 0$ we have $f^*\omega \in Z^k_p(M)$. Suppose now that $\left[f^*\omega\right] = 0$ in $H^k_{q,p}(M)$, then $f^*\omega \in B^k_{q,p}(M)$, that is there exists $\theta \in L^q(M, \Lambda^{k-1})$ such that $d\theta = f^*\omega$. But by the second hypothesis the operator $f_*: L^q(M, \Lambda^k) \to L^{\tilde{q}}(\tilde{M}, \Lambda^k)$ is bounded and therefore $f_*\theta \in L^{\tilde{q}}(\tilde{M}, \Lambda^k)$. We then have

$$\omega = f_*\left(f^*\omega\right) = f_* d\theta = d\left(f_*\theta\right) \in B^k_{\tilde{q},\tilde{p}}(\tilde{M})$$

Therefore $[\omega] = 0$ in $H^k_{\tilde{q},\tilde{p}}(\tilde{M})$. □

The argument of the previous proof is illustrated in the following commutative diagrams:

$$\begin{array}{ccc} Z^k_{\tilde{p}}(\tilde{M}) & \xrightarrow{f^*} & Z^k_p(M) \\ d\uparrow & & \uparrow d \\ L^{\tilde{q}}(\tilde{M}, \Lambda^{k-1}) & \xleftarrow{f_*} & L^q(M, \Lambda^{k-1}) \end{array} \qquad \begin{array}{ccc} \omega & \xrightarrow{f^*} & f^*\omega \\ d\uparrow & & \uparrow d \\ f_*\theta & \xleftarrow{f_*} & \theta \end{array}$$

The next result gives us sufficient conditions for a diffeomorphism to behave functorially at the $L_{q,p}$-cohomology level.

Theorem 3.2 *Let $f: M \to \tilde{M}$ be a diffeomorphism and $1 \le p \le \tilde{p} < \infty$ and $1 \le q \le \tilde{q} < \infty$. Assume that*

$$f^*: L^{\tilde{p}}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k), \quad \textit{and} \quad f^*: L^{\tilde{q}}(\tilde{M}, \Lambda^{k-1}) \to L^q(M, \Lambda^{k-1})$$

are bounded operators. Then

(a) $f^*: \Omega^{k-1}_{\tilde{q},\tilde{p}}(\tilde{M}) \to \Omega^{k-1}_{q,p}(M)$ *is a bounded operator,*

(b) $f^*: H^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to H^k_{q,p}(M)$ *is a well defined linear map,*

(c) $f^*: \overline{H}^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to \overline{H}^k_{q,p}(M)$ *is a well defined bounded operator,*

Proof (a) By definition $\omega \in \Omega^{k-1}_{\tilde{q},\tilde{p}}(\tilde{M})$ if $\omega \in L^{\tilde{q}}(\tilde{M}, \Lambda^{k-1})$ and $d\omega \in L^{\tilde{p}}(\tilde{M}, \Lambda^k)$. Because both operators $f^*: L^{\tilde{p}}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)$, $f^*: L^{\tilde{q}}(\tilde{M}, \Lambda^{k-1}) \to L^q(M, \Lambda^{k-1})$ are bounded and $f^*d\omega = df^*\omega$ we obtain that $f^*\omega \in \Omega^{k-1}_{q,p}(M)$. The operator $f^*: \Omega^{k-1}_{\tilde{q},\tilde{p}}(\tilde{M}) \to \Omega^{k-1}_{q,p}(M)$ is clearly bounded.

(b) The condition $f^*d = df^*$ and the boundedness of the operators $f^*: L^{\tilde{p}}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)$ implies that $f^*\left(Z^k_{\tilde{p}}(\tilde{M})\right) \subset Z^k_p(M)$. Using the boundedness of the operator $f^*: \Omega^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to \Omega^k_{q,p}(M)$ and the condition $f^*d = df^*$ we see that

$$f^*\left(B^k_{\tilde{q},\tilde{p}}(\tilde{M})\right) = f^*\left(d\Omega^{k-1}_{\tilde{q},\tilde{p}}(\tilde{M})\right) = df^*\left(\Omega^{k-1}_{\tilde{q},\tilde{p}}(\tilde{M})\right) \subset d\left(\Omega^{k-1}_{q,p}(M)\right) = B^k_{q,p}(M).$$

The inclusions

$$f^*\left(Z^k_{\tilde{p}}(\tilde{M})\right) \subset Z^k_p(M), \qquad f^*\left(B^k_{\tilde{q},\tilde{p}}(\tilde{M})\right) \subset B^k_{q,p}(M) \tag{3.1}$$

imply that the linear map

$$f^*: H^k_{\tilde{q},\tilde{p}}(\tilde{M}) = Z^k_{\tilde{p}}(\tilde{M})/B^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to Z^k_p(M)/B^k_{q,p}(M) = H^k_{q,p}(M)$$

is well defined.

(c) Using the inclusions (3.1) and the continuity of the operator $f^*: \Omega^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to \Omega^k_{q,p}(M)$, we have

$$f^*\left(\overline{B^k_{\tilde{q},\tilde{p}}(\tilde{M})}\right) \subset \overline{f^*\left(B^k_{\tilde{q},\tilde{p}}(\tilde{M})\right)} \subset \overline{B^k_{q,p}(M)}. \tag{3.2}$$

Therefore, the operator

$$f^*: \overline{H}^k_{\tilde{q},\tilde{p}}(\tilde{M}) = Z^k_{\tilde{p}}(\tilde{M})/\overline{B^k_{\tilde{q},\tilde{p}}(\tilde{M})} \to Z^k_p(M)/\overline{B^k_{q,p}(M)} = \overline{H}^k_{q,p}(M)$$

is well defined and bounded. □

Using the two previous theorems, we have the following result:

Theorem 3.3 *Let $f: M \to \tilde{M}$ be a diffeomorphism and $1 \le p \le \tilde{p} \le \infty$ and $1 \le \tilde{q} = q \le \infty$. Assume that the operator $f^*: L^{\tilde{p}}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)$ is bounded and that $f^*: L^q(\tilde{M}, \Lambda^{k-1}) \to L^q(M, \Lambda^{k-1})$ is an isomorphism of Banach spaces. Then the linear map*

$$f^*: H^k_{q,\tilde{p}}(\tilde{M}) \to H^k_{q,p}(M)$$

is well defined and injective.

The proof is immediate. □

Corollary 3.4 *Let $f: M \to \tilde{M}$ satisfying the hypothesis of the previous theorem. If $T^k_{q,p}(M) = 0$ then $T^k_{\tilde{q},\tilde{p}}(\tilde{M}) = 0$.*

Proof Since $T^k_{q,p}(M) = 0$, we have $\overline{B^k_{q,p}(M)} = B^k_{q,p}(M)$. The hypothesis of Theorem 3.2 are satisfied, thus the inclusions (3.2) holds and we thus have

$$f^*\left(\overline{B^k_{\tilde{q},\tilde{p}}(\tilde{M})}\right) \subset \overline{B^k_{q,p}(M)} = B^k_{q,p}(M).$$

Choose now an arbitrary element $\omega \in \overline{B^k_{\tilde{q},\tilde{p}}(\tilde{M})}$. We have $f^*\omega \in B^k_{q,p}(M)$ by the previous inclusion, this means that $[f^*\omega] = 0 \in H^k_{q,p}(M)$, but $f^*: H^k_{\tilde{q},\tilde{p}}(\tilde{M}) \to H^k_{q,p}(M)$ is injective by the previous theorem and therefore $[\omega] = 0$ in $H^k_{q,\tilde{p}}(\tilde{M})$, that is $\omega \in B^k_{q,\tilde{p}}(\tilde{M})$. Since ω was arbitrary, we have shown that $\overline{B^k_{q,\tilde{p}}(\tilde{M})} = B^k_{q,\tilde{p}}(\tilde{M})$, i.e., $T^k_{q,\tilde{p}}(\tilde{M}) = 0$. □

Remark The hypothesis in Theorem 3.3 seems to be very restrictive, the results of Sect. 5 suggest that it will be difficult to find diffeomorphisms satisfying these hypothesis and which aren't bilipshitz or quasiconformal. See the discussion at the end of Sect. 5.

4 Diffeomorphisms with controlled distortion

Let (M, g) and $(\tilde{M}, \tilde{g})$ be two smooth oriented Riemannian manifolds. In this section we study classes of diffeomorphisms $f: M \to \tilde{M}$ with bounded distortion of an integral type that induce bounded operators $f^*: L^{\tilde{p}}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)$ for $1 \le p \le \tilde{p} \le \infty$. To define these classes we use the notation

$$\sigma_k(f, x) = \sigma_k(df_x)$$

for the kth principal invariant of the differential df_x. We also write $\sigma_k(f)$ when there is no risk of confusion, observe that $\sigma_n(f) = J_f$, where J_f is the Jacobian of f.

Definition 4.1 A diffeomorphism $f : M \to \tilde{M}$ is said to be of *bounded* (s,t)*-distortion in degree* k, and we write $f \in \mathrm{BD}^k_{(s,t)}(M, \tilde{M})$, if

$$(\sigma_k(f))^s \, J_f^{-1} \in L^t(M).$$

It is assumed that $1 \le s < \infty$ and $0 < t \le \infty$.

It is convenient to introduce the quantity

$$K_{s,t,k}(f) = \left\| \frac{(\sigma_k(f))^s}{J_f(x)} \right\|_{L^t(M)},$$

the mapping f belongs then to $\mathrm{BD}^k_{(s,t)}(M, \tilde{M})$, if and only if $K_{s,t,k}(f) < \infty$.

Proposition 4.1 *Let* $f : M \to \tilde{M}$ *be a diffeomorphism. Suppose* $p \le \tilde{p} < \infty$ *and for any* $\omega \in L^{\tilde{p}}(\tilde{M}, \Lambda^k)$ *we have*

$$\|f^*\omega\|_{L^p(M,\Lambda^k)} \le \left(K_{\tilde{p},t,k}(f)\right)^{1/\tilde{p}} \|\omega\|_{L^{\tilde{p}}(\tilde{M},\Lambda^k)}$$

where $t = \frac{p}{\tilde{p}-p}$. *In particular if* $f \in \mathrm{BD}^k_{(\tilde{p},t)}(M, \tilde{M})$*, then the operator*

$$f^* : L^{\tilde{p}}(\tilde{M}, \Lambda^k) \to L^p(M, \Lambda^k)$$

is bounded.

Proof Without loss of generality we can suppose that $J_f(x) > 0$. Using the fact that $|(f^*\omega)_x| \le \sigma_k(f,x) \cdot \left|\omega_{f(x)}\right|$, we have

$$\begin{aligned} \|f^*\omega\|^p_{L^p(M,\Lambda^k)} &= \int_M \left|(f^*\omega)_x\right|^p dx \le \int_M (\sigma_k(f,x))^p \left|\omega_{f(x)}\right|^p dx \\ &\le \int_M \left\{ \left(\sigma_k(f,x)\, J_f^{-1/\tilde{p}}(x)\right)^p \cdot \left(\left|\omega_{f(x)}\right| J_f^{1/\tilde{p}}(x)\right)^p \right\} dx. \end{aligned}$$

Using Hölder's inequality for $s = \frac{\tilde{p}}{\tilde{p}-p}$ and $s' = \frac{\tilde{p}}{p}$ (so that $\frac{1}{s} + \frac{1}{s'} = 1$), and the change of variable formula, we obtain

$$\begin{aligned} \|f^*\omega\|^p_{L^p(M,\Lambda^k)} &\le \left(\int_M \left(\sigma_k^{\tilde{p}}(f,x) J_f^{-1}(x)\right)^{\frac{p}{\tilde{p}-p}} dx\right)^{\frac{\tilde{p}-p}{\tilde{p}}} \cdot \left(\int_M \left(\left|\omega_{f(x)}\right|^{\tilde{p}} J_f(x)\right) dx\right)^{\frac{p}{\tilde{p}}} \\ &\le \left(K_{\tilde{p},t,k}(f)\right)^{\frac{p}{\tilde{p}}} \left(\int_{\tilde{M}} \left|\omega_y\right|^{\tilde{p}} dy\right)^{\frac{p}{\tilde{p}}}, \end{aligned}$$

that is

$$\|f^*\omega\|_{L^p(M,\Lambda^k)} \le \left(K_{\tilde{p},t,k}(f)\right)^{1/\tilde{p}} \|\omega\|_{L^{\tilde{p}}(\tilde{M},\Lambda^k)}.$$

□

Remark Every diffeomorphism belongs to the class $BD^n_{1,\infty}$, i.e., $BD^n_{1,\infty}(M, \tilde{M}) = \mathrm{Diff}(M, \tilde{M})$.The previous proposition states in particular the well known fact that the condition for an n-form to be integrable is invariant under diffeomorphism and therefore independent of the choice of a Riemannian metric.

The next proposition describes the inverse of diffeomorphisms in $BD^k_{s,t}$.

Proposition 4.2 *Let* $f : M \to \tilde{M}$ *be a diffeomorphism,* $0 \le m \le n$. *Let* $1 \le \alpha < \infty$ *and* $0 < \beta \le \infty$ *with* $\beta(\alpha - 1) > 1$. *Then the equivalence*

$$f^{-1} \in \mathrm{BD}^m_{(\alpha,\beta)}(\tilde{M}, M) \quad \Leftrightarrow \quad f \in \mathrm{BD}^{n-m}_{(s,t)}(M, \tilde{M})$$

holds if and only if

$$s = \frac{\alpha}{\alpha - 1 - \frac{1}{\beta}} \quad \textit{and} \quad t = \beta(\alpha - 1) - 1. \tag{4.1}$$

Proof Without loss of generality we can suppose that $J_f > 0$.

Assume first that $\beta < \infty$, then the condition $f^{-1} \in \mathrm{BD}^m_{(\alpha,\beta)}(\tilde{M}, M)$ means that

$$\int\limits_{\tilde{M}} \left\{ \sigma_m^\alpha(f^{-1}, y)\, J_{f^{-1}}^{-1}(y) \right\}^\beta dy < \infty.$$

By Lemma 2.2, we have

$$\sigma_m(f^{-1}, f(x)) = \frac{\sigma_{n-m}(f, x)}{J_f(x)} \tag{4.2}$$

at $y = f(x)$ and for any $0 \le m \le n$. Using the relations (4.1), which can be rewritten as

$$\alpha\beta = st = t + \beta + 1,$$

together with the change of variable formula with the standard relations $dy = J_f(x)dx$, $J_{f^{-1}}(f(x)) = J_f^{-1}(x)$, we can rewrite the latter integral as

$$\begin{aligned} \int\limits_M \left\{ \left(\frac{\sigma_{n-m}(f, x)}{J_f(x)} \right)^\alpha J_f(x) \right\}^\beta J_f(x)\, dx &= \int\limits_M (\sigma_{n-m}(f, x))^{\alpha\beta} \left(J_f(x) \right)^{1+\beta-\alpha\beta} dx. \\ &= \int\limits_M \left\{ (\sigma_{n-m}(f, x))^s \left(J_f(x) \right)^{-1} \right\}^t dx. \end{aligned}$$

This integral is finite if and only if $f \in \mathrm{BD}^{n-m}_{(s,t)}(M, \tilde{M})$.

Assume now that $\beta = \infty$, then we also have $t = \infty$. The condition $f^{-1} \in \mathrm{BD}^m_{(\alpha,\infty)}(\tilde{M}, M)$ means in that case that

$$\sigma_m^\alpha(f^{-1})\, J_{f^{-1}}^{-1} \quad \text{is uniformly bounded.} \tag{4.3}$$

Using the relation $s = \frac{\alpha}{\alpha-1}$, Eq. (4.2) and $J_{f^{-1}} = J_f^{-1}$, we have

$$\left(\sigma_m(f^{-1})\right)^\alpha J_{f^{-1}}^{-1} = \left(\sigma_m(f^{-1})\right)^\alpha J_f = \left(\sigma_m(f^{-1}) J_f\right)^\alpha J_f^{1-\alpha} = \left\{ \sigma_{n-m}^s(f)\, J_f^{-1} \right\}^{\alpha-1}$$

Thus (4.3) holds if and only if $\sigma_{n-m}^s(f)\, J_f^{-1}$ is bounded, i.e., $f \in \mathrm{BD}^{n-m}_{(s,t)}(M, \tilde{M})$. □

Corollary 4.3 *If* $\tilde{q} \le q$ *and the diffeomorphism* f *belongs to* $\mathrm{BD}^{n-m}_{(s,t)}(M, \tilde{M})$ *with*

$$s = \frac{\tilde{q}}{\tilde{q} - 1}, \qquad t = \frac{q(\tilde{q} - 1)}{q - \tilde{q}},$$

then the operator

$$f_*: L^q(M, \Lambda^m) \to L^{\tilde q}(\tilde M, \Lambda^m)$$

is bounded.

Proof This follows immediately from Proposition 4.1 and the previous proposition with $\alpha = q$ and $\beta = \frac{\tilde q}{q - \tilde q}$. □

Corollary 4.4 *If the diffeomorphism $f: M \to \tilde M$ satisfies $f \in BD^k_{(q,\infty)}(M, \tilde M) \cap \mathrm{BD}^{n-k}_{(q',\infty)}(M, \tilde M)$ with $q' = \frac{q}{q-1}$ then $f^*: L^q(\tilde M, \Lambda^k) \to L^q(M, \Lambda^k)$ is an isomorphism*

Proof It follows at once from the Propositions 4.2 and 4.1. □

5 Relation with quasiconformal and bilipschitz diffeomorphisms

Recall that an orientation preserving diffeomorphism[1] $f: (M, g) \to (\tilde M, \tilde g)$, between two oriented n-dimensional Riemannian manifolds is said to be *quasiconformal* if

$$\frac{|df|^n}{J_f} \in L^\infty(M).$$

Lemma 5.1 *For the diffeomorphism $f: (M, g) \to (\tilde M, \tilde g)$, the following properties are equivalent*

(i) *f is quasiconformal;*
(ii) *f^{-1} is quasiconformal;*
(iii) *If $\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)$ are the principal distortion coefficients of df_x, then*

$$\sup_{x \in M} \frac{\max\{\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)\}}{\min\{\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)\}} < \infty.$$

The proof of this lemma is standard and easy.

Let us denote by $\mathrm{QC}(M, \tilde M)$ the class of all quasiconformal diffeomorphisms, it is clear that $\mathrm{QC}(M, \tilde M) = \mathrm{BD}^1_{n,\infty}(M, \tilde M)$, but, more generally:

Proposition 5.2 *We have*

$$\mathrm{QC}(M, \tilde M) = \mathrm{BD}^k_{\frac{n}{k},\infty}(M, \tilde M)$$

for any $1 \le k \le n-1$.

Proof Suppose that $f: (M, g) \to (\tilde M, \tilde g)$ is quasiconformal. Let us assume that $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$, then by condition (iii) of the previous lemma, there exists a constant C such that

$$\sigma_k(f, x) \le C \cdot (\lambda_1(x))^k.$$

Since $J_f = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$, we have

$$\frac{(\sigma_k(f))^{n/k}}{J_f} \le C \cdot \frac{(\lambda_1^k)^{n/k}}{J_f} \le C \cdot \frac{(\lambda_1)^n}{(\lambda_1 \cdot \lambda_2 \cdots \lambda_n)} \le C,$$

[1] It is usual, and important, to consider not only diffeomorphisms, but more generally homeomorphisms in $W^{1,n}_{loc}$ when defining quasiconformal maps. In our present context, diffeomorphisms are sufficient, see, however, the discussion in Sect. 8.

i.e., $f \in \mathrm{BD}^k_{\frac{n}{k},\infty}(M, \tilde{M})$. We have thus shown that $\mathrm{QC}(M, \tilde{M}) \subset \mathrm{BD}^k_{\frac{n}{k},\infty}(M, \tilde{M})$.

To prove the converse inclusion, we distinguish three cases: $k = \frac{n}{2}$, $1 \le k < \frac{n}{2}$ and $\frac{n}{2} < k < n$.

Let us first assume that $k = \frac{n}{2}$, then we have

$$\frac{\lambda_n}{\lambda_1} \le \frac{(\lambda_{n-k+1}\cdots\lambda_n)}{(\lambda_1\cdots\lambda_k)} \le \frac{(\lambda_{n-k+1}\cdots\lambda_n)^2}{(\lambda_1\cdots\lambda_k)(\lambda_{n-k+1}\cdots\lambda_n)} \le \frac{(\sigma_k(f))^2}{J_f},$$

which implies that $\mathrm{BD}^{n/2}_{2,\infty}(M, \tilde{M}) \subset \mathrm{QC}(M, \tilde{M})$.

Assume now that $1 \le k < \frac{n}{2}$, i.e., $k+1 \le n-k$. Observe that

$$(\lambda_{k+1}\cdots\lambda_{n-k}) \le (\lambda_{n-k})^{n-2k} \le (\lambda_{n-k+1}\cdots\lambda_n)^{(n-2k)/k},$$

therefore

$$\begin{aligned} J_f &= (\lambda_1 \cdot \lambda_2 \cdots \lambda_n) \\ &= (\lambda_1\cdots\lambda_k)(\lambda_{k+1}\cdots\lambda_{n-k})(\lambda_{n-k+1}\cdots\lambda_n) \\ &\le (\lambda_1\cdots\lambda_k)(\lambda_{n-k+1}\cdots\lambda_n)^{\frac{n-2k}{k}+1} \\ &= (\lambda_1\cdots\lambda_k)(\lambda_{n-k+1}\cdots\lambda_n)^{\frac{n}{k}-1}. \end{aligned}$$

Because $\sigma_k \ge \lambda_{n-k+1}\cdots\lambda_n$, we have from the previous inequality

$$\frac{(\sigma_k(f))^{n/k}}{J_f} \ge \frac{(\lambda_{n-k+1}\cdots\lambda_n)^{\frac{n}{k}}}{J_f} \ge \frac{(\lambda_{n-k+1}\cdots\lambda_n)}{(\lambda_1\cdots\lambda_k)}.$$

Since

$$\frac{\lambda_{n-k+1}}{\lambda_k}, \frac{\lambda_{n-k+2}}{\lambda_{k-1}}, \ldots, \frac{\lambda_{n-1}}{\lambda_2} \ge 1,$$

we finally have

$$\frac{\lambda_n}{\lambda_1} \le \frac{(\lambda_n\cdots\lambda_{n-k+1})}{(\lambda_1\cdots\lambda_k)} \le \frac{(\sigma_k(f))^{n/k}}{J_f},$$

from which it follows that $\mathrm{BD}^k_{\frac{n}{k},\infty}(M, \tilde{M}) \subset \mathrm{QC}(M, \tilde{M})$.

If $k > \frac{n}{2}$, then $n-k < \frac{n}{2}$ and we have from the previous argument and Proposition 4.2

$$f^{-1} \in \mathrm{BD}^{n-k}_{\frac{n}{n-k},\infty}(\tilde{M}, M) \subset \mathrm{QC}(\tilde{M}, M),$$

and we deduce from Lemma 5.1 that $f \in \mathrm{QC}(M, \tilde{M})$. □

The next result relates our class of maps to bilipschitz ones.

Proposition 5.3 *If $f \in BD^k_{(q,\infty)}(M, \tilde{M}) \cap \mathrm{BD}^{n-k}_{(q',\infty)}(M, \tilde{M})$ with $q' = \frac{q}{q-1}$, then f is quasiconformal. Furthermore if $q \neq \frac{n}{k}$, then f is bilipschitz.*

Proof Using the same notations and convention as in the previous proof, we have

$$\begin{aligned} \frac{\lambda_n}{\lambda_1} &\le \frac{(\lambda_n\cdots\lambda_{n-k+1})}{(\lambda_1\cdots\lambda_k)} = \frac{(\lambda_{n-k+1}\cdots\lambda_n)(\lambda_{k+1}\cdots\lambda_n)}{(\lambda_1\cdots\lambda_k)(\lambda_{k+1}\cdots\lambda_n)} \\ &\le \frac{\sigma_k(f)\cdot\sigma_{n-k}(f)}{J_f} = \left(\frac{(\sigma_k(f))^q}{J_f}\right)^{\frac{1}{q}}\left(\frac{(\sigma_{n-k}(f))^{q'}}{J_f}\right)^{\frac{1}{q'}}, \end{aligned}$$

because $\frac{1}{q}+\frac{1}{q'}=1$. It follows from this computation that any map f in $BD^k_{(q,\infty)}(M,\tilde{M})\cap \mathrm{BD}^{n-k}_{(q',\infty)}(M,\tilde{M})$ is quasiconformal.

We now prove that f is bilipschitz if $q\neq\frac{n}{k}$: Because f is quasiconformal, there exists a constant c such that $\lambda_n\le c\cdot\lambda_1$. Since $\lambda_1^k\le\sigma_k(f)$ and $J_f\le\lambda_n^k$, we have

$$|df|^{qk-n}=\lambda_n^{qk-n}\le\frac{(c\lambda_1)^{kq}}{\lambda_n^n}\le c^{kq}\cdot\frac{(\sigma_k(f))^q}{J_f},$$

this implies that any quasiconformal map in $BD^k_{(q,\infty)}(M,\tilde{M})$ is Lipschitz if $qk>n$. If $qk<n$, then $q'(n-k)<n$ and the same argument shows that any quasiconformal map in $BD^{n-k}_{(q',\infty)}(M,\tilde{M})$ is Lipschitz. Thus any $f\in BD^k_{(q,\infty)}(M,\tilde{M})\cap\mathrm{BD}^{n-k}_{(q',\infty)}(M,\tilde{M})$ with $q\neq\frac{n}{k}$ is Lipschitz. But Proposition 4.2 implies that $f^{-1}\in BD^k_{(q,\infty)}(\tilde{M},M)\cap\mathrm{BD}^{n-k}_{(q',\infty)}(\tilde{M},M)$, hence f^{-1} is also a lipshitz map if $q\neq\frac{n}{k}$. □

An open question. The previous result and the Corollary 4.4 suggest the following question: *Suppose a diffeomorphism $f:M\to\tilde{M}$ induces an isomorphism $f^*:L^q(\tilde{M},\Lambda^k)\to L^q(M,\Lambda^k)$. Can we conclude that f is quasiconformal for $q=\frac{n}{k}$ and bilipshitz otherwise?*

If $k=1$, the answer to the above question is positive, see [3,4,29,30].

For a more complete discussion of quasiconformal maps in the context of differential forms, we refer to [7] .

6 $L_{q,p}$-cohomology and BD-diffeomorphisms

Combining the results of the two previous sections, we obtain the following theorem.

Theorem 6.1 *Suppose $p\le\tilde{p}<\infty$, and let $f:M\to\tilde{M}$ be a diffeomorphism of the class $\mathrm{BD}^k_{(\tilde{p},t)}(M,\tilde{M})$ where $t=\frac{p}{\tilde{p}-p}$. Then the following holds:*

(A) *$f^*:L^{\tilde{p}}(\tilde{M},\Lambda^k)\to L^p(M,\Lambda^k)$ is a bounded operator and $f^*(Z^k_{\tilde{p}}(\tilde{M}))\subset Z^k_p(M)$.*

(B) *If $q\ge\tilde{q}>1$ and $f\in\mathrm{BD}^{n-k+1}_{(\tilde{q}',r)}(M,\tilde{M})\cap\mathrm{BD}^k_{(\tilde{p},t)}(M,\tilde{M})$ with $\tilde{q}'=\frac{\tilde{q}}{\tilde{q}-1}$, $r=\frac{q(\tilde{q}-1)}{q-\tilde{q}}$, then $[f^*\omega]=0$ in $H^k_{q,p}(M)$ implies $[\omega]=0$ in $H^k_{\tilde{q},\tilde{p}}(\tilde{M})$ (thus $H^k_{q,p}(M)=0\Rightarrow H^k_{\tilde{q},\tilde{p}}(\tilde{M})=0$).*

(C) *If $q\le\tilde{q}$ and $f\in\mathrm{BD}^{k-1}_{(\tilde{q},u)}(M,\tilde{M})\cap\mathrm{BD}^k_{(\tilde{p},t)}(M,\tilde{M})$ where $u=\frac{q}{\tilde{q}-q}$ and $t=\frac{p}{\tilde{p}-p}$, then*

   (a) *$f^*:\Omega^{k-1}_{\tilde{q},\tilde{p}}(\tilde{M})\to\Omega^{k-1}_{q,p}(M)$ is a bounded operator,*

   (b) *$f^*:H^k_{\tilde{q},\tilde{p}}(\tilde{M})\to H^k_{q,p}(M)$ is a well defined linear map,*

   (c) *$f^*:\overline{H}^k_{\tilde{q},\tilde{p}}(\tilde{M})\to\overline{H}^k_{q,p}(M)$ is a bounded operator.*

Proof The statement (A) follows immediately from Proposition 4.1 and the fact that $df^*\omega=f^*d\omega$, whereas the assertion (B) follows from Propositions 4.1, 4.2 and Theorem 3.1. Finally, the property (C) follows from Proposition 4.1 and Theorem 3.2. □

Part (C) of the Theorem gives us sufficient conditions on a map f to have a functorial behavior in $L_{q,p}$-cohomology.

7 An example: manifold with a cusp

In this section, we show how Theorem 6.1 can be used to produce a vanishing result in $L_{q,p}$-cohomology. We consider the Riemannian manifold $(\tilde{M}, g)$ such that M is diffeomorphic to $\mathbb{R}^n$ and $\tilde{g}$ is a Riemannian metric such that in polar coordinates, we have

$$\tilde{g} = dr^2 + e^{-2r} \cdot h$$

for large enough r, where h denotes the standard metric on the sphere $\mathbb{S}^{n-1}$. Let us also consider the identity map $f : \mathbb{R}^n \to \tilde{M}$, where $\mathbb{R}^n$ is given its standard Euclidean metric, which writes in polar coordinates as

$$ds^2 = dr^2 + r^2 \cdot h.$$

Proposition 7.1 *If $s > \frac{n-1}{m-1}$, then the above map $f : \mathbb{R}^n \to \tilde{M}$ belongs to the class $\mathrm{BD}^m_{s,t}(\mathbb{R}^n, \tilde{M})$ for any $0 < t \leq \infty$.*

Proof For r large enough, we have the following principal distortion coefficients for f:

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = \cdots = \lambda_n = \frac{e^{-r}}{r}.$$

In particular $J_f = (\frac{e^{-r}}{r})^{n-1}$ and

$$\sigma_m(f) = \left(\frac{e^{-r}}{r}\right)^m + \binom{n-1}{m-1}\left(\frac{e^{-r}}{r}\right)^{m-1} \leq C_1 \left(\frac{e^{-r}}{r}\right)^{m-1}.$$

and thus

$$\frac{(\sigma_m(f))^s}{J_f} \leq C_2 \left(\frac{e^{-r}}{r}\right)^{s(m-1)-(n-1)}$$

outside a compact set in $\mathbb{R}^n$. Therefore, $\int_{\mathbb{R}^n} (\frac{(\sigma_m(f))^s}{J_f})^t dx < \infty$ if and only if

$$\int_1^\infty \left(\frac{e^{-r}}{r}\right)^{t(s(m-1)-(n-1))} \cdot r^{n-1} dr < \infty$$

which is the case when $s \geq \frac{n-1}{m-1}$. This implies that $f \in \mathrm{BD}^m_{s,t}(\mathbb{R}^n, \tilde{M})$ for any $0 < t < \infty$.

It is also clear that $f \in \mathrm{BD}^m_{s,\infty}(\mathbb{R}^n, \tilde{M})$, since $\frac{(\sigma_m(f))^s}{J_f}$ is bounded when $s \geq \frac{n-1}{m-1}$. □

Corollary 7.2 *If $\tilde{q} < \dfrac{n-1}{k-1} < \tilde{p}$, then $H^k_{\tilde{q},\tilde{p}}(\tilde{M}) = 0$.*

Proof We will use Theorem 6.1(B) with the previous Proposition. We have $f \in \mathrm{BD}^k_{(\tilde{p},t)}$ $(\mathbb{R}^n, \tilde{M})$ for any $t > 0$, since we have $\tilde{p} > \frac{n-1}{k-1}$ by hypothesis. We also have $f \in \mathrm{BD}^{n-k+1}_{(\tilde{q}',r)}(\mathbb{R}^n, \tilde{M})$ if $\tilde{q}' > \frac{n-1}{n-k}$. But this inequality is equivalent to

$$\tilde{q} = \frac{\tilde{q}'}{\tilde{q}'-1} < \frac{n-1}{k-1},$$

and this also holds by hypothesis. We thus have $f \in \mathrm{BD}^{n-k+1}_{(\tilde{q}',r)}(\mathbb{R}^n, \tilde{M}) \cap \mathrm{BD}^k_{(\tilde{p},t)}(\mathbb{R}^n, \tilde{M})$ for any $\tilde{q} < \frac{n-1}{k-1} < \tilde{p}$.

Let us now set $p = \frac{n}{k}$ and $q = \frac{n}{k-1}$, and observe that $p \leq \frac{n-1}{k-1}$, hence $p \leq \tilde{p}$ and $q \geq \frac{n-1}{k-1}$, hence $q \geq \tilde{q}$.

In [27], it is proved that $H^k_{q,p}(\mathbb{R}^n) = 0$ if $p = \frac{n}{k}$ and $q = \frac{n}{k-1}$. Therefore by Theorem 6.1 we have $H^k_{\tilde{q},\tilde{p}}(\tilde{M}) = 0$ for any $\tilde{q} < \dfrac{n-1}{k-1} < \tilde{p}$. □

8 Non-smooth mappings

We have formulated our results for diffeomorphisms, but it is clear that Definition 4.1 makes sense for wider classes of maps such as Sobolev maps in $W^{1,1}_{loc}$ or maps which are approximately differentiable almost everywhere, we can thus consider the class of $W^{1,1}_{loc}$ homeomorphisms with bounded mean distortion. It is then natural and important to wonder whether our results still hold in this wider context.

Unfortunately, there is no elementary answer to this question. A careful look at our arguments show that we have used the following properties of diffeomorphisms:

(i) The *change of variables formula in integrals*: $\int_M u(f(x))\, J_f(x) dx = \int_{\tilde{M}} u(y)\, dy$ in Proposition 4.1).

(ii) The *change of variables formula for the inverse map* which is implicitly used in Corollary 4.3.

(iii) The naturality of the exterior differential $df^*\omega = f^*d\omega$ is used everywhere.

The change of variables formula in integrals holds for a homeomorphism f in $W^{1,1}_{loc}$ provided we assume the *Luzin* (N) *condition* to hold. This condition states that a subset of zero measure in M is mapped by f onto a set of zero measure in $\tilde{M}$. The map change of variables formula for the inverse map f^{-1} holds if the Luzin (N^{-1}) condition holds, that is the inverse image of subset of zero measure also has zero measure. The Luzin condition is widely studied in the literature (see, for example, [10–12, 29]). Concerning the naturality of the exterior differential, we refer to [8].

Let us finally mention that for the special case of locally quasiconformal maps, all these properties hold. The relation between the theory of quasiconformal mappings and L_{qp}-cohomology is studied in [7].

References

1. Gol'dshtein, V.M., Kuz'minov, V.I., Shvedov, I.A.: Differential forms on Lipschitz Manifolds. Siberian Math. J. **23**(2), 16–30 (1982). English translation in Siberian Math. J. **23**(2), 151–161 (1982)
2. Gol'dshtein, V.M., Kuz'minov, V.I., Shvedov, I.A.: L_p-cohomology of warped cylinder. Siberian Math. J. **31**(6), 55–63 (1990). English translation in Siberian Math. J. **31**(6), 716–727 (1990)
3. Gol'dshtein, V., Gurov, L., Romanov, A.: Homeomorphisms that induce monomorphisms of Sobolev spaces. Israel J. Math. **91**(1), 31–60 (1995)
4. Gol'dshtein, V.M., Romanov, A.S.: Transformations that preserve Sobolev spaces **25**(3), 382–388 (1984)
5. Gol'dshtein, V., Troyanov, M.: The L_{pq}-cohomology of SOL. Ann. Fac. Sci. Toulouse **Vii**(4) (1998)
6. Gol'dshtein, V., Troyanov, M.: Sobolev inequality for differential forms and $L_{q,p}$-cohomology. J. Geom. Anal. **16**(4), 597–631 (2006)
7. Gol'dshtein, V., Troyanov, M.: A conformal de Rham complex. arXiv:0711.1286
8. Gol'dshtein, V., Troyanov, M.: On the naturality of exterior differential arXiv:0801.4295. Math. Rep. Can. Acad. Sci. (to appear)
9. Gromov, M.: Asymptotic Invariants of Infinite Groups in Geometric Group Theory, vol. 2. London Math. Soc. Lecture Notes 182, Cambridge University Press (1992)

10. Heinonen, J., Koskela, P.: Sobolev mappings with integrable distortion. Arch. Rat. Mech. Anal. **125**, 81–97 (1993)
11. Hencl, S., Koskela, P.: Mapping of finite distortion: openess and discretness for quasilight mappings. Ann. Inst. H. Poincaré **22**, 331–342 (2005)
12. Kauhanen, J., Koskela, P., Maly, J.: Mappings of finite distortion: discretness and openness. Arch. Rat. Mech. Anal., **160**(2), 135–151 (2001)
13. Kopylov, Y.A.: $L_{q,p}$-Cohomology and normal solvability. Arch. Math. **89**(1), 87–96 (2007)
14. Kopylov, Y.A.: $L_{p,q}$-Cohomology of warped cylinders. arXiv:0803.3298v1
15. Lelong-Ferrand, L.: Etude d'une classe d'applications liées à des homomorphismes d'algèbres de fonctions et généralisant les quasi-conformes. Duke Math. J. **40**, 163–186 (1973)
16. Lück, W.: L^2-Invariants: Theory and Applications to Geometry and K-Theory. Springer, Berlin (2002)
17. Maz'ya, V.G.: Spaces. Springer, Heidelberg (1985)
18. Maz'ya, V.G., Shaposhnikova, T.: Theory of Multipliers in Spaces of Differentiable Functions. Pitman, London (1985)
19. Pansu, P.: Difféomorphismes de p-dilatation bornées. Ann. Acad. Sc. Fenn. **223**, 475–506 (1997)
20. Pansu, P.: Cohomologie L^p, espaces homogènes et pincement. Preprint, Orsay (1999)
21. Pansu, P.: L^p-cohomology and pinching. In: Rigidity in Dynamics and Geometry (Cambridge, 2000), pp. 379–389, Springer, Berlin (2002)
22. Pansu, P.: Cohomologie L^p en degré 1 des espaces homogénes. J. Potential Anal. **27**, 151–165 (2007)
23. Pansu, P.: Cohomologie L^p et pincement. Comment. Math. Helvetici. (to appear)
24. Reiman, M.: Uber harmonishe Kapazität und quasikonforme Abbildungen in Raum. Comm. Math. Helv. **44**, 284–307 (1969)
25. Reshetnyak, Yu.G.: Space Mappings with Bounded distortion. Translations of Mathematical Monographs, vol. 73, American Mathematical Society (1985)
26. Troyanov, M., Vodop'yanov, S.K.: Liouville type theorem for mappings with bounded co-distortion. Ann. Inst. Fourier **52**(6), 1754–1783 (2002)
27. Troyanov, M.: On the Hodge decomposition in $\mathbb{R}^n$. arXiv:0710.5414.
28. Vodop'yanov, S.K.: Topological and geometrical properties of mappings with an integrable Jacobian in Sobolev classes. Siberian Math. J. **41**(4), 19–39 (2000)
29. Vodop'yanov, S.K., Gol'dshtein, V.M.: Quasiconformal mappings and spaces of mfubctions with generalized first derivatives. Siberian Math. J. **17**(3), 515–531 (1976)
30. Vodop'yanov, S.K., Ukhlov, A.D.: Sobolev spaces and (p, q)-quasiconformal mappings of Carnot groups. Siberian Math. J. **39**(4), 776–795 (1998)