Comment on "An analytic solution of capillary rise restrained by gravity" by N. Fries and M. Dreyer

D.A. Barry [1,*], L. Wissmeier [1], J.-Y. Parlange [2], G.C. Sander [3], D.A. Lockington [4]

[1] *Laboratoire de technologie écologique, Institut d'ingénierie de l'environnement, Station 2, Ecole polytechnique fédérale de Lausanne, CH-1015 Lausanne, Switzerland. Ph. +41 (21) 693-5576; +41 (21) 693-5727; Fax. +41 (21) 693-5670; E-mail addresses: andrew.barry@epfl.ch, laurin.wissmeier@epfl.ch*

[2] *Department of Biological and Environmental Engineering, Cornell University, Ithaca, New York 14853-5701 USA. Ph. +1 (607) 255-2476; Fax. +1 (607) 255-4080; E-mail address: jp58@cornell.edu*

[3] *Department of Civil and Building Engineering, Loughborough University, Loughborough, LE11 3TU United Kingdom. Ph. +44 (1509) 223-777; Fax. +44 (1509) 223-981; Email address: g.sander@lboro.ac.uk*

[4] *School of Engineering, University of Queensland, Brisbane, Queensland 4072 Australia. Ph. +61 (7) 3365-4054; Fax. +61 (7) 3365-4599; Email address: d.lockington@mailbox.uq.edu.au*

Submitted to: *Journal of Colloid and Interface Science* 13 February 2009

Revised and resubmitted 29 March 2009

Revised and resubmitted 3 June 2009

[*] Author to whom all correspondence should be addressed

1. Introduction

Fries and Dreyer [4] state "We derive an analytic solution for the capillary rise of liquids in a cylindrical tube or a porous medium in terms of height h as a function of time t. The implicit $t(h)$ solution by Washburn is the basis for these calculations and the Lambert W function is used for its mathematical rearrangement."

In this comment we point out (i) that the porous medium model presented by Fries and Dreyer [4] does not cite the porous medium model of Green and Ampt [5] developed before the capillary tube model of Washburn [16], with both leading to the same solution; and (ii) that the Lambert W solution they provide is, in the case of porous media flow, identical to the solution earlier presented by Barry et al. [2] as an explicit form of the equation by Green and Ampt [5].

Below, we use FD and the appropriate number to refer to equations and figures presented in [4].

2. Brief Description of the Contribution of Green and Ampt [5]

In their famous paper, Green and Ampt [5] presented theory and experimental data addressing the question of liquid flow in a uniform porous medium. Their theory was developed by consideration of the Hagen-Poiseuille law, with explicit expressions derived for vertically upward and downward flow, as well as horizontal flow. Their work predates that of Lucas [8] and the equally famous paper of Washburn [17], which is even more cited than the Green and Ampt [5] paper. In addition, in another early paper Bell and Cameron [3] validated the square-root-of-time dependence of distance travelled by liquids in horizontal capillaries prior to [8] and [17].

Fries and Dreyer [4] consider capillary rise, for which Eq. (10) of Green and Ampt [5] applies equally well. The capillary rise equation of Green and Ampt [5] is (keeping as much as possible their original notation):

$$\frac{dl}{dt} = \frac{P}{S_{GA}} \frac{K_{GA} - l}{l}, \tag{1}$$

where l is the wetted distance in the soil, t is time, S_{GA} is porosity (notation changed from the original paper to avoid confusion with S, used below), K_{GA} (notation changed from original paper to avoid confusion with K in [4]) "is a constant of the soil depending on capillary forces acting on the moving boundary of the water" (pg. 6, [5]) and

$$P = \frac{\pi g s}{8 \eta A} \sum r^4, \tag{2}$$

which, corresponding to Darcy's law, is an expression for the saturated hydraulic conductivity, K_s, of the capillary tube "porous medium". Thus, in normal porous media notation, $P = K_s$.

Fries and Dreyer [4] use an averaged single-capillary version of Eq. (2) to connect the Hagen-Poiseuille and Darcy flow laws. In Eq. (2), g is the magnitude of gravitational acceleration, $\sum r^4$ is the sum of radii of a bundle of capillaries, A is the cross-sectional area, s is the liquid's density and η is its viscosity. The model in Eq. (1), along with its counterparts for vertically downward and horizontal flow, was used by Green and Ampt [5] to analyze corresponding laboratory experiments wherein K_{GA} and P were determined. Green and Ampt [5], in their Eq. (11), also provide the solution to Eq. (1) for the condition $l(0) = 0$. Their solution is (correcting an obvious typographical error):

$$\frac{Pt}{S_{GA}} = K_{GA}\ln\left(\frac{K_{GA}}{K_{GA} - l}\right) - l. \tag{3}$$

To account for a non-vertical capillary rise, one merely changes g in Eq. (2) to $g\sin\psi$, where ψ is the angle of inclination as given in Fig. FD2. In Eq. (1), we make the substitutions $l = h$, $PK_{GA}/S_{GA} = a$ and $P/S_{GA} = b$ to obtain Eq. (FD8):

$$\frac{dh}{dt} = \frac{a}{h} - b. \tag{4}$$

This basically concludes our first point. However, because of differences in notation that can easily lead to confusion we add the following.

If notation differences are accounted for, when the definition of P in Eq. (2) is simplified to the "single capillary tube" porous medium found in [4], the expression for b in Eq. (4) that is deduced from $P/S_{GA} = b$ is identical to that given in Eq. (FD7).

Turning to the parameter a in Eq. (4), in [4] it is given by Eq. (FD6):

$$a = \frac{2\sigma\cos\theta}{\phi\mu}\frac{K}{R}, \tag{5}$$

where the notation is identical to that in [4]. Since Green and Ampt [5] did not provide a precise mathematical definition of K_{GA}, it is clear that their model and Eq. (FD8) are identical. Indeed, a "capillary tube" definition for K_{GA} is provided by Eqs. (FD6), (FD7) and (FD24):

$$K_{GA} = \left(\frac{2\sigma\cos\theta}{R}\right)\left(\frac{1}{\rho g\sin\psi}\right) = \frac{a}{b} = h_{max}, \tag{6}$$

where the first term in parentheses on the right side is the capillary pressure for a liquid a tube as given by the Young-Laplace law while the second is the inverse of the body force per unit volume experienced by the liquid in the tube. The latter equalities in Eq. (6) are simply Eq. (FD24).

The expression for K_{GA} in Eq. (6) is based on the analogy, used in [4], that a porous medium can be modeled as an average single capillary. Clearly, this correspondence, while useful, is a limited representation of a porous medium. For a porous medium the constant K_{GA} is more appropriately defined in terms of macroscopic properties that can be determined experimentally. Horizontal movement of a liquid in a porous medium is controlled by capillarity. For that case, Eq. (13) of Green and Ampt [5] is, for an initially dry medium:

$$\frac{PK_{GA}}{S_{GA}}t = \frac{1}{2}l^2. \tag{7}$$

The cumulative infiltration per unit cross-sectional area of porous medium, $I(t)$, i.e., the total amount of water entering the porous medium since $t = 0$, is the product of the porosity and $l(t)$, assuming the medium is initially dry. Then, Eq. (7) can be written as:

$$I(t) = \sqrt{2K_s S_{GA} K_{GA}}\sqrt{t}, \tag{8}$$

where P in Eq. (7) has been replaced by K_s to conform to commonly used notation, as already mentioned above. Eq. (8) shows that imbibition into a horizontal porous medium is proportional to $\sqrt{t}$, as is of course the corresponding law of Bell and Cameron [3], Lucas [8] and Washburn [17] for horizontal capillary tubes. Washburn [17] also considered briefly horizontal flow in porous media, again showing the $\sqrt{t}$ dependence of I.

As noted, for purely horizontal imbibition the only mechanism driving the flow is capillarity. Thus, the coefficient of $\sqrt{t}$ in Eq. (8) is simply a measure of the capacity of the porous medium's capillarity to drive flow. This quantity, which depends on the medium's initial moisture content and pressure head applied at the boundary, is called the sorptivity, S [15], a quantity that has a voluminous literature devoted to it (e.g., [1],[7],[10],[11],[12],[13],[14],[16]). Thus, in standard notation as commonly used, K_{GA} in Eq. (8) is given by:

$$K_{GA} = \frac{S^2}{2\phi K_s}, \tag{9}$$

where, using the notation of [4], ϕ is the porous medium porosity. The aforementioned expression $PK_{GA}/S_{GA} = a$ then can be used to express a is terms of standard porous medium properties, as shown in Eq. (13) below.

3. Lambert *W* solution of Barry et al. [2]

Fries and Dreyer [4] state that Barry et al. [2] take a "simplified form" of Eq. (FD8) – i.e., Eq. (4) above – in which (in the variables of [4]) the final b is replaced by a, see Eq. (FD21). This, of course, is impossible since a and b have different dimensions, so Fries and Dreyer [4] must mean some dimensionless form such as they themselves take in Eq. (FD35), which corresponds to non-dimensionalizing Eq. (FD8) such that a and b become equal. Indeed, Eq. (18) of Barry et al. [2] is:

$$I(t) = -\alpha^* K_s A(t). \tag{10}$$

where $I = -\phi h$ (negative sign because flow is upward) is the cumulative infiltration due to capillary rise in the porous medium,

$$A(t) = 1 + W\left[-\exp\left(-\frac{t}{\alpha^*} - 1\right)\right], \tag{11}$$

and

$$\alpha^* = \frac{S^2}{2K_s^2}. \tag{12}$$

With the changes in notation discussed earlier, in Eq. (FD8) this corresponds to

$$a = \frac{S^2}{2\phi^2} \tag{13}$$

and

$$b = \frac{K_s}{\phi}, \tag{14}$$

which are certainly not equal.

4. Conclusion

The literature on flow in porous media tends to cite either the porous medium approach of Green and Ampt [5], or the capillary tube model of Washburn [17]. Here, we have attempted to clarify the porous medium aspects of the work of Green and Ampt [5] since it was missing in the contribution of Fries and Dreyer [4] as well as many other similar papers, e.g., see [9]. There is also some confusion due to the change in notation between that used by Green and Ampt [5], by Washburn [17] and by the more recent concepts of hydraulic conductivity and, especially, sorptivity. We conclude that this confusion may have led Fries and Dreyer [4] to overlook that their Lambert W porous medium solution was presented earlier by Barry et al. [2].

References

[1] D.A. Barry, J.-Y. Parlange, Transp. Porous Media 14 (1994) 89–92.

[2] D.A. Barry, J.-Y. Parlange, G.C. Sander, M. Sivaplan, J. Hydrol. 142 (1993) 29–46.

[3] J.M. Bell, F.K. Cameron, J. Phys. Chem. 10 (1906) 658–674.

[4] N. Fries, M. Dreyer, J. Colloid Interface Sci. 320 (2008) 259–263.

[5] W.H. Green, G.A. Ampt, J. Agri. Sci. 4 (1911) 1–24.

[6] D.A. Lockington, J.-Y. Parlange, J. Colloid Interface Sci. 278 (2004) 404–409.

[7] D. Lockington, J. Parslow, J.-Y. Parlange, Soil Sci. Soc. Am. J. 52 (1988) 903–908.

[8] R. Lucas, Kolloid-Z. 23 (1918) 15–22.

[9] R. Masoody, K.M. Pillai, P.P. Varanasi, AIChE J. 53 (2007) 2769-2782.

[10] J.-Y. Parlange, Proc. Soil Sci. Soc. Am. 39 (1975) 415–418.

[11] J.-Y. Parlange, D.A. Barry, M.B. Parlange, R. Haverkamp, Water Resour. Res. 28 (1992) 2529–2531.

[12] J.-Y. Parlange, D.A. Barry, M.B. Parlange, D.A. Lockington, R. Haverkamp, Transp. Porous Media 15 (1994) 197–208.

[13] J.-Y. Parlange, R. Haverkamp, C.W. Rose, I. Phillips, Soil Sci. 145 (1988) 385–388.

[14] J.-Y. Parlange, M. Vauclin, R. Haverkamp, I. Lisle, Soil Sci. 139 (1985), 458–461.

[15] J.R. Philip, Soil Sci. 84 (1957) 257–264.

[16] J.-P. Vandervaere, M. Vauclin, D.E. Elrick, Soil Sci. Soc. Am. J. 64 (2000) 1272–1284.

[17] E.W. Washburn, Phys. Rev. 17 (1921) 273–283.