Asynchronous variational integrators as a mechanical tool for dimensioning thin-sell structures

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Objectives

Thin-shell and rod theory using discrete mechanics applied to structures in civil engineering. The aim is to apply structure preserving algorithms to concrete problems in construction. The major objectives of this interdisciplinary work is the search and the development of a practical tool to study irregular surfaces. Goals of this work are to:

- Provide a mathematical model for thin-shells
- Provide a 2-dimensional simulation

Notation and Definitions

The reference configuration $\mathcal{B} \subset \mathbb{R}^3$ is a closed 2-submanifold. A ***configuration*** of $\mathcal{B}$ is an embedding

$$\Phi : \mathcal{B} \to \mathcal{S} = \mathbb{R}^3;$$

it represents a deformed state of the shell whose material configuration is $\mathcal{B}$. Let $\mathcal{C}$ be the set of all configurations of $\mathcal{B}$. A ***motion*** of $\mathcal{B}$ is a curve $t \in \mathbb{R} \mapsto \Phi_t = \Phi(t, \cdot) \in \mathcal{C}$.

If $\{\theta^\alpha\}$, $\alpha = 1, 2$, is a coordinate system on $\mathcal{B}$, the configuration Φ is written as

$$(\theta^1, \theta^2) \in \mathcal{B} \overset{\Phi}{\longmapsto} (z^1(\theta^1, \theta^2), z^2(\theta^1, \theta^2), z^3(\theta^1, \theta^2)) \in \mathbb{R}^3,$$

where $\{z^j(\theta^1, \theta^2)\}$ are the Euclidean coordinates of the point $\mathbf{x} = \Phi(\mathbf{X})$, $\mathbf{X} \in \mathcal{B}$.

With $\{\hat{\mathbf{i}}_j\}$ the standard basis vectors in $\mathbb{R}^3$, the basis vectors of $T_{\mathbf{x}}(\Phi(\mathcal{B}))$ associated to the embedding Φ are

$$\mathbf{e}_\alpha := \frac{\partial z^j}{\partial \theta^\alpha} \hat{\mathbf{i}}_j, \qquad \alpha = 1, 2, \qquad j = 1, 2, 3.$$

Let $\{Z^j(\theta^1, \theta^2) \mid j = 1, 2, 3\}$ be the Euclidean coordinates of $\mathbf{X} \in \mathcal{B} \subset \mathbb{R}^3$. Let $\{\mathbf{E}_\alpha \mid \alpha = 1, 2\}$ be the basis of $T_{\mathbf{X}}\mathcal{B}$ associated to the coordinate system $\{\theta^1, \theta^2\}$. We define $\mathbf{E}_3 \perp T_{\mathbf{X}}\mathcal{B}$ by

$$\mathbf{E}_3 = \frac{\mathbf{E}_1 \times \mathbf{E}_2}{|\mathbf{E}_1 \times \mathbf{E}_2|}$$

to be the reference shell director[a] $\mathbf{T}$.

In this work we consider the simplest properly invariant isotropic[b] constitutive relations for the effective membrane and shear stress resultants. So we define the unit normal to the deformed surface $\Phi(\mathcal{B})$

$$\mathbf{e}_3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|}$$

to be the deformed director $\mathbf{t}$.

Denote by $\langle \cdot, \cdot \rangle_{\mathbf{x}}$ the standard inner product in $\mathbb{R}^3$ for vectors based at $\mathbf{x} \in \mathcal{S} = \mathbb{R}^3$ and by $\langle \cdot, \cdot \rangle_{\mathbf{X}}$ the standard inner product in $\mathbb{R}^3$ for vectors based at $\mathbf{X} \in \mathcal{B}$. The components $g_{\alpha\beta}$ of the metric tensor on $\Phi(\mathcal{B})$ (obtained by pulling back to $\Phi(\mathcal{B})$ by the inclusion map the inner product $\langle \cdot, \cdot \rangle_{\mathbf{x}}$ on $\mathbb{R}^3$) are defined by $g_{\alpha\beta}(\mathbf{x}) := \langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle_{\mathbf{x}}$. Similarly define the components $G_{\alpha\beta}$ of the metric on $\mathcal{B}$ by $G_{\alpha\beta}(\mathbf{X}) := \langle \mathbf{E}_\alpha, \mathbf{E}_\beta \rangle_{\mathbf{X}}$. Let $[G^{\alpha\beta}] := [G_{\alpha\beta}]^{-1}$ and $[g^{\alpha\beta}] := [g_{\alpha\beta}]^{-1}$.

Discrete Variational Mechanics

Let $\Phi(\mathcal{B}) \times \Phi(\mathcal{B})$ be the ***discrete state space*** associated to the deformed surface $\Phi(\mathcal{B})$ and define the ***discrete path space*** by $\mathcal{C}_d(\Phi(\mathcal{B})) := \{\mathbf{x}_d : \{t_k\}_{k=0}^N \mapsto \{\mathbf{x}_k := \mathbf{x}_d(t_k)\} \in \Phi(\mathcal{B})^N\}$.

A discrete path $\mathbf{x}_d \in \mathcal{C}_d$ is said to be a solution of the discrete Euler-Lagrange equations if

$$(D_2 L_d(\mathbf{x}_{k-1}, \mathbf{x}_k) + D_1 L_d(\mathbf{x}_k, \mathbf{x}_{k+1})) \cdot \delta \mathbf{x}_k = 0$$

for all variations $\delta \mathbf{x}_d \in T_{\mathbf{x}_d}\mathcal{C}_d(\Phi(\mathcal{B})) = \{v_{\mathbf{x}} : \{t_k\}_{k=0}^N \mapsto \{v_{\mathbf{x}}(t_k)\} \in T\Phi(\mathcal{B})^N\}$, where L_d is the ***discrete Lagrangian***.

We use uniform B-splines associated with the set of discrete paths to be able to define $T_{\mathbf{x}_d}\mathcal{C}_d(\Phi(\mathcal{B}))$ and the discrete Lagrangian (of order r) of a thin-shell

$$L_d(\mathbf{x}_k, \mathbf{x}_{k+1}, \Delta t) = \int_{t_k}^{t_{k+1}} L(\mathbf{x}, \dot{\mathbf{x}}) dt + \mathcal{O}(\Delta t)^{r+1}$$

where L is the Lagrangian of the continuous systems and $\mathbf{x}(t)$ is the solution of the Euler-Lagrange equations satisfying $\mathbf{x}(t_k) = \mathbf{x}_k$ and $\mathbf{x}(t_{k+1}) = \mathbf{x}_{k+1}$.

[a] We assume the existence of a traction vector $\mathbf{t}$ for the motion of $\mathcal{B}$ in $\mathcal{S}$.

[b] Isotropic at x_0 if $SO(3)$ is a subset of material symmetries.

AVI as a mechanical tool for rod' study

The ***strain mesures*** relative to the dual spatial surface basis [Simo and Fox, 1987, p 287]:

$$\epsilon_{ij} := \frac{1}{2}(\langle \mathbf{e}_i, \mathbf{e}_j \rangle - \langle \mathbf{E}_i, \mathbf{E}_j \rangle)$$

$$\rho_{\alpha\beta} := \left\langle \frac{\partial \mathbf{E}_\alpha}{\partial \theta^\beta}, \mathbf{E}_3 \right\rangle - \left\langle \frac{\partial \mathbf{e}_\alpha}{\partial \theta^\beta}, \mathbf{e}_3 \right\rangle$$

For the simplest properly invariant isotropic constitutive relations we postulate the existence of a stored energy function of the displacement field $\mathbf{u}$ of the form

$$W(\mathbf{u}) = \frac{1}{2}\left(\frac{Eh}{1-\nu^2}\right) H^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \frac{1}{2}\left(\frac{Eh^3}{12(1-\nu^2)}\right) H^{\alpha\beta\gamma\delta} \rho_{\alpha\beta}\rho_{\gamma\delta}$$

where E is Young's modulus, ν is Poisson's ratio, h is the thickness of the shell, and

$$H^{\alpha\beta\gamma\delta} = \nu\, G^{\alpha\beta} G^{\gamma\delta} + \frac{1}{2}(1-\nu)\,(G^{\alpha\gamma}G^{\beta\delta} + G^{\alpha\delta}G^{\beta\gamma}).$$

As mentioned earlier and to ensure that the bending energy is finite we use B-splines[a]. We used quadratic uniform B-splines whose general form is:

$$\mathbf{x}_a(u) = \frac{1}{2}(u^2, u, 1)\begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}(\mathbf{x}_{a-1}, \mathbf{x}_a, \mathbf{x}_{a+1})^T, \quad \text{with} \quad 0 \leqslant u \leqslant 1,$$

where a is a node of the triangulation $\mathcal{T}$ of the rod and $a-1, a+1$ are the neighboring nodes of a. For a 1-simplex $K \in \mathcal{T}$ associated to the rod, the local Lagrangian has the form

$$L_K(\mathbf{x}_K, \dot{\mathbf{x}}_K, t) = T_K(\dot{\mathbf{x}}_K) - V_K(\mathbf{x}_K, t),$$

where $\mathbf{x}_K$ is the vector of positions of all nodes in the element K, $V_K(\mathbf{x}_K, t)$ is the elemental potential energy, and

$$T_K(\dot{\mathbf{x}}_K) = \frac{1}{2}\dot{\mathbf{x}}_K^t \mathbf{m}_K \dot{\mathbf{x}}_K,$$

where $\mathbf{m}_K$ is the mass matrix element (symmetric positive definite).

A particular choice of discrete Lagrangian, resulting in explicit integrators of the central-difference type, is given by

$$L_K^j = \int_{t_K^j}^{t_K^{j+1}} T_K(\dot{\mathbf{x}}_K(t))\, dt - \left(t_K^{j+1} - t_K^j\right) V_K\left(\mathbf{x}_K^{j+1}, t_K^{j+1}\right)$$

where L_K^j is defined on the interval $[t_K^j, t_K^{j+1}]$ for $t_K^j = j\Delta t_K$ (the jth time step for element K). Indeed, theorically it is necessary to guarantee that a valid time step for each element will allways be determined for each t_j, to guarantee energy conservation. But, as it was noted in [Lew, Marsden, Ortiz, and West, 2004, p. 199], if we fix the time step Δt_K for each simplex K, the total energy of the system oscillates around a constant value without growth or decay, so, as a first try, we have made this choice in our experiment, for simplicity.

The corresponding discrete action sum is

$$S_d = \sum_{K \in \mathcal{T}} \sum_{1 \leqslant j < N_K} L_K^j \approx \sum_a \sum_{i=0}^{N_a - 1} \frac{1}{2} m_a (t_a^{i+1} - t_a^i) \left\| \frac{\mathbf{x}_a^{i+1} - \mathbf{x}_{a+1}^i}{t_a^{i+1} - t_a^i} \right\|^2 - \sum_K \sum_{j=0}^{N_K - 1} (t_K^{j+1} - t_K^j) V_K(\mathbf{x}_K^{j+1}),$$

where $\mathbf{x}_a^i$ is the position of the node a at time t_a^i. The position and the time are dependent of two 1-simplices, one on each side.

The discrete version of Hamilton's principle states that the discrete trajectory having prescribed initial and final endpoints renders the discrete action sum stationary with respect to admissible variations of the nodal coordinates $\mathbf{x}_a$. It leads to the discrete Euler-Lagrange equations

$$D_a^i S_d = 0$$

for all $a \in \mathcal{T}$ such that $t_{initial} < t_a^i < t_{final}$ and $a \in \mathcal{T} \setminus \partial_d \mathcal{B}$, where $\partial_d \mathcal{B}$ is the boundary of the reference seen as a complex simplicial boundary.

Thus, the element K accumulates and memorizes the impulses I_K^j over the time interval (t_K^{j-1}, t_K^j). At the end of the interval, the element releases its memorized impulses by imparting percussions on its nodes.

In all the experiments, the value of the time step for each element is computed as $\Delta t = f\frac{h}{c}$ where $f = \frac{1}{100}$ (or close to it), and h is the radius of the largest ball contained in the element.

[a] In $\mathbb{R}^3$, if we want smalest element in the mesh the subdivision of the surface obtained by Loop or Catmull-Clark subdivisions are guaranted to be H^2.

Simulations

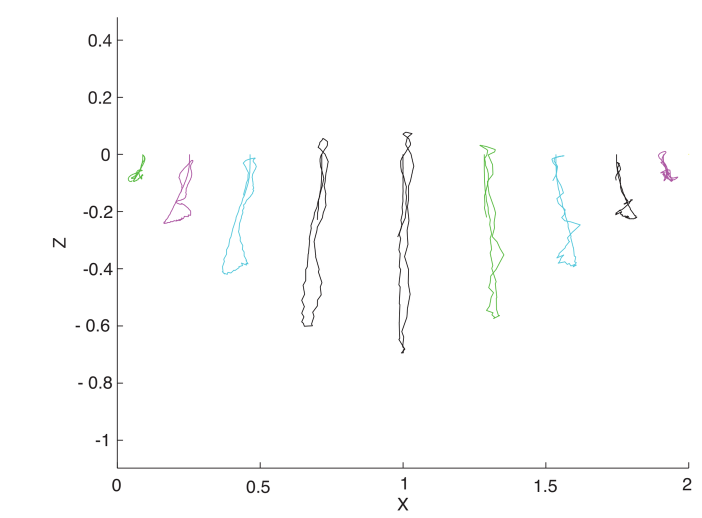
Fig. 1 to 6 display the behavior of several beams with increasing stiffness. These results were obtained using a module of elasticity E ranging from $E = 11$ to $E = 11000$, $\nu = 0.3$, with a beam of length $L = 2m$, width $h = 1mm$, and density $\rho = 400kg/m^3$. The time of the experiment is $1s$ in each case.

The time steps in seconds for each 1-simplex, from left to the right are:

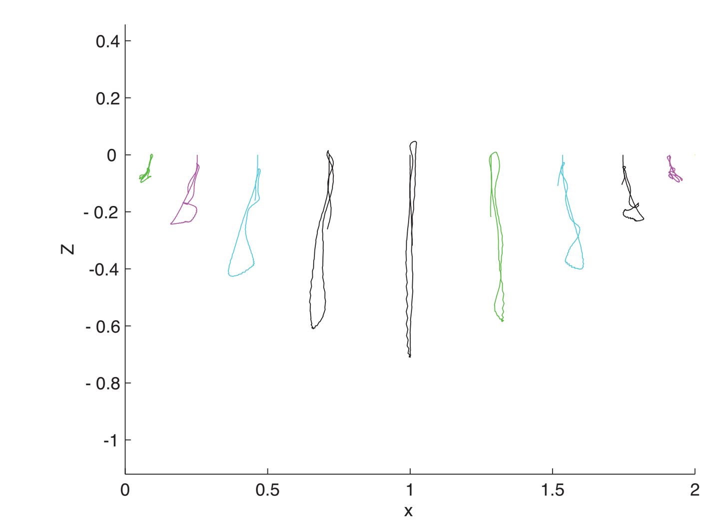
$$0.0046, 0.0066, 0.0098, 0.0120, 0.0139, 0.0148, 0.0139, 0.0120, 0.0098, 0.0066.$$

If N is the number of nodes on the rod, define the position n_a of the node a at time $t = 0$ on the rod by using the following rules, the first for the first $\frac{N}{2}$ nodes and the second for the last $\frac{N}{2}$ nodes:

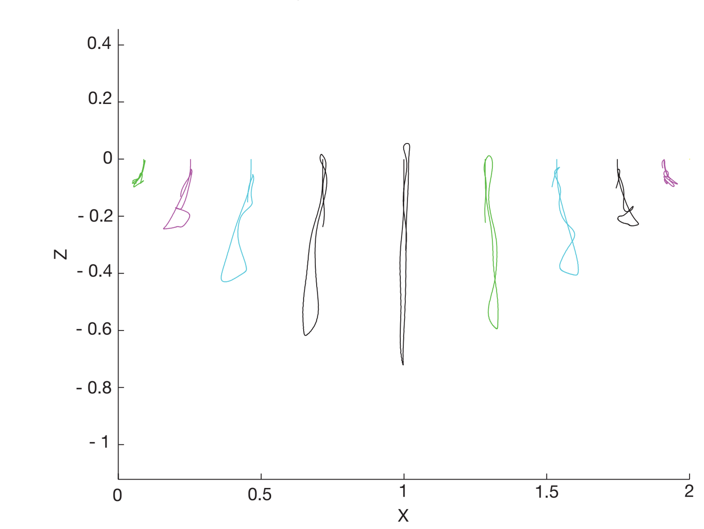
$$\mathbf{X}_a = 2^{S-1}\left(\frac{n_a - 1}{N - 1}\right)^S L, \qquad \mathbf{X}_a = \left(1 - 2^{S-1}\left(\frac{N - n_a}{N - 1}\right)^S\right) L$$



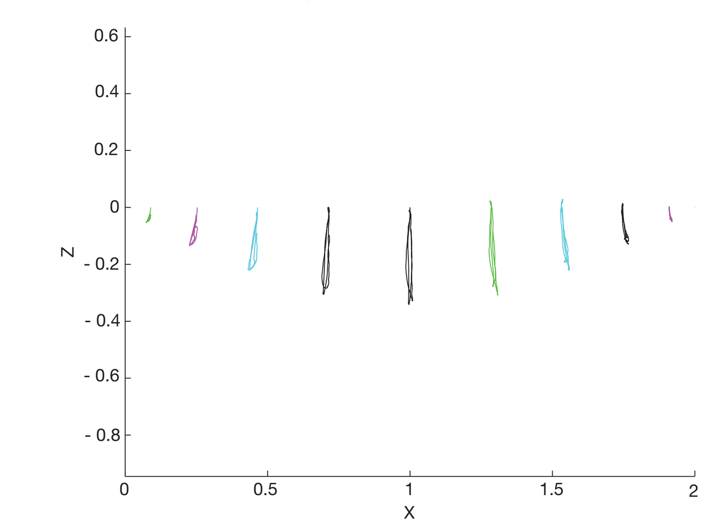
Module of elasticity E = 11, value of f = 0.015, skew S = 1.5



Module of elasticity E = 11, value of f = 0.01, skew S = 1.5



Module of elasticity E = 11, value of f = 0.005, skew S = 1.5



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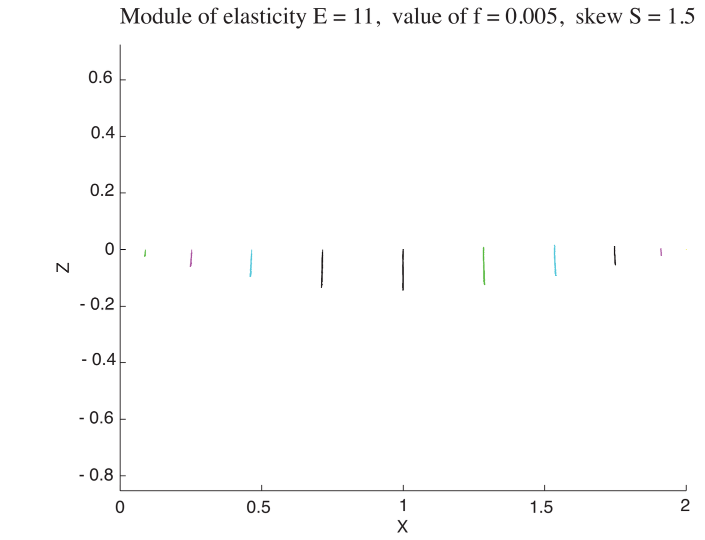
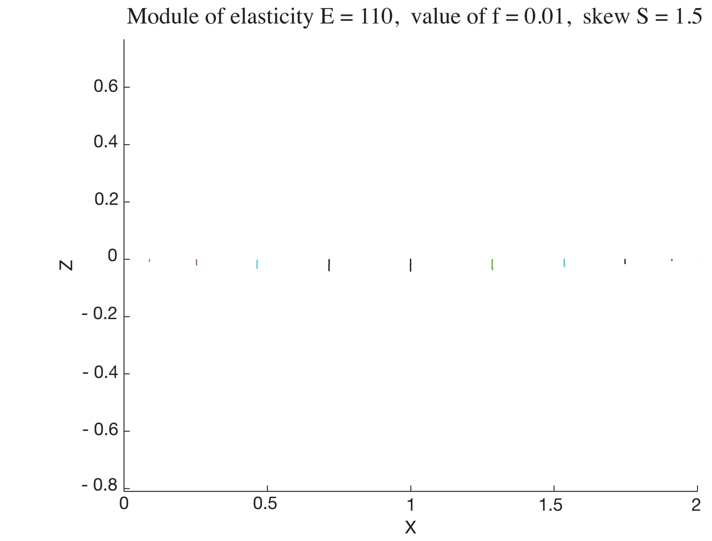


Module of elasticity E = 1100, value of f = 0.01, skew S = 1.5



Module of elasticity E = 11000, value of f = 0.01, skew S = 1.5

References

Lew, A., J. E. Marsden, M. Ortiz, and M. West [2003], Asynchronous variational integrators. *Arch. Rational Mech. Anal.* **167**, 85-146.

Lew, A., J. E. Marsden, M. Ortiz, and M. West [2004], Variational time integrators. *Int. J. Numer. Methods Engrg.* **60**, 153-212.

Marsden, J.E. and M. West [2001], Discrete mechanics and variational integrators. *Acta Numer.* **10**, 357–514.

Simo, J.C. and D.D. Fox [1989], On a stress resultant geometrically exact shell model. I. Formulation and optimal parametrization. *Comput. Methods Appl. Mech. Engrg.* **72**(3), 267–304.