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Singularly perturbed piecewise deterministic games. (English summary)
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A hybrid game is examined that consists of a stochastic jump process together with a deterministic part governed by differential equations. Specific results are obtained when the time constants of the deterministic part are small compared with the time between two events of the jump process. Under such circumstances, the solution to the hybrid game is built from the solution of a purely stochastic game—the limit game—by selecting equilibrium strategies of a set of infinite horizon open-loop deterministic differential games defined by this limit game. These deterministic games, in turn, are solved each time the jump process switches mode. Such a decomposition principle is similar to what occurs in singularly perturbed dynamical systems.

The hybrid game G^ε, with $\varepsilon > 0$, is played by m players, each player $j \in M = \{1, \dots, m\}$ selecting a value of $u_j(t)$ in a compact set $U_j^i \subset \mathbb{R}^{\nu_j}$. The game evolves according to the dynamics $\varepsilon \dot{x}_j(t) = f_j^i(x_j(t), u_j(t))$ where the state $x_j(t)$ remains in a compact set $X_j \subset \mathbb{R}^{n_j}$ for all times. The functions $f_j^i(x_j, u_j)$ are activated depending on the current active mode i determined by a continuous-time jump process $\xi(\cdot)$ with state set $I = \{0, \dots, p-1\}$ and transition rates $q_{kl}(x(t)) = \lim_{dt \to 0} \frac{1}{dt} P[\xi(t+dt) = l \mid \xi(t) = k, x_1(t), \dots, x_m(t)]$. This means that while in mode i, each player's state x_j evolves independently from the other players.

However, coupling is introduced through the reward rates $L_j^i(x(t), u_j(t))$ that are given to each player j where $x(t) = (x_1(t), \dots, x_n(t))$. All players decide simultaneously upon their strategies after the jump process has occurred. At jump times $\tau^o = 0, \tau^1, \dots, \tau^\nu, \dots,$ of the ξ process, the players observe the state of the system $s^\nu = (\xi^\nu, x_1^\nu, \dots, x_n^\nu)$ and select absolute continuous functions $y_j \colon [0, \infty) \to X_j$ with initial conditions $y_j(0) = x_j^\nu$.

Unfortunately, when ε becomes small, the game becomes ill-conditioned. Nevertheless, it is possible to define a purely stochastic limit game G^0 as a controlled Markov chain on the discrete set I. The strategy of player j is defined by a vector $\tilde{x}_j = (x_j^i \colon i \in I)$ with $x_j^i \in X_j$, $j \in M$. The discounted payoff for player j (with discount rate ρ_j) is

$$V_j(\tilde{x}, i) = E_{\tilde{x}} \left[\int_0^\infty e^{-\rho_j t} \mathcal{L}_j^{\xi(t)}(x^{\xi(t)}, 0) dt \mid \xi(0) = i \right],$$

where $\mathcal{L}_j^i(x, z_j)$ is obtained by selecting the supremum value of the reward rates $L_j^i(x, u_j)$ for all possible input choices u_j constrained by the state velocity $z_j = f_j^i(x_j, u_j)$, that is

$$\mathcal{L}_j^i(x, z_j) = \begin{cases} -\infty & x_j \notin X_j \text{ or } F_j^i(x_j, z_j) = \varnothing; \\ \sup_{u_j \in F_j^i(x_j, z_j)} L_j^i(x, u_j) & \text{otherwise}, \end{cases}$$

with $F_j^i(x_j, z_j) = \{u_j \mid z_j = f_j^i(x_j, u_j)\}$. Because the game is the limit game, the state velocity z_j is set to zero.

It is assumed that the state evolution equation $\dot{x}_j = f^i_j(x_j, u_j)$, in the stretched out time scale $\varepsilon = 1$, is controllable to such an extent that there exists a feedback law determining u_j for which $\dot{x}_j = s^i_j(x_j, x^0, x^f) = f(x_j, u_j)$ reaches a ball around the final state x^f in bounded time (uniform reachability). Therefore, for any strategy $\tilde{x}$ of the limit game G^0, a strategy $\tilde{\gamma}^\varepsilon = \sigma^\varepsilon(\tilde{x})$ for G^ε can be associated as follows: When in mode $i \in I$, select $x^i(\cdot)\colon [0,\infty) \to X$, where $x_j(t)$ is a solution of $s^i_j(x_j(t), x^0, \tilde{x}) = \varepsilon \dot{x}_j(t)$ with $x^i_j(0) = x^0_j$.

A key result is that

$$\lim_{\varepsilon \to 0} |V^\varepsilon_j(\sigma^\varepsilon(\tilde{x}), i, x^0) - V_j(\tilde{x}, i)| = 0,$$

with the G^ε-game payoff

$$V^\varepsilon_j(\gamma, i, x^0) = \\ E_\gamma \left[e^{-\rho_j t} \mathcal{L}^{\xi(t)}_j(x(t), \varepsilon \dot{x}_j(t)) dt \mid (\xi(t_0) = i, x(t_0) = x^0) \right],$$

which fully justifies the definition of G^0 as the limit game.

Moreover, let $\tilde{x}^*$ be an equilibrium of G^0. Then for all positive ζ, there exists ε_0 such that for all $0 < \varepsilon \leq \varepsilon_0$, the strategy m-tuple $\sigma^\varepsilon(\tilde{x}^*)$ defines a ζ-Nash equilibrium for the game G^ε, that is

$$V^\varepsilon_j([\sigma^\varepsilon_{M-j}(\tilde{x}^*), \gamma^\varepsilon_j], i, x^0) \leq V^\varepsilon_j(\sigma^\varepsilon(\tilde{x}^*), i, x^0) + \zeta.$$

(The strategy vector $[\sigma^\varepsilon_{M-j}, \gamma^\varepsilon_j]$ is obtained from the vector σ^ε by replacing the j-th strategy component with γ^ε_j.)

Another fundamental aspect about G^0 is that its resolution (providing the equilibrium payoffs $V^*_j(l)$, $l \in I$) can be used to define a family of infinite-horizon open-loop differential games (IHOLDGs) with payoffs over the time interval $[0, \Theta)$ given as

$$J^\Theta_j[x^0; x(\cdot)] = \int_0^\Theta \left\{ \mathcal{L}^i_j(x(\tau), \dot{x}_j(\tau)) + \sum_{l \in I} q_{il}(x(\tau)) V^*_j(l) \right\} d\tau, \\ j \in M.$$

When $\Theta \to \infty$, an M-trajectory $(x^*(\tau), \tau \geq 0)$ such that

$$\lim \inf_{\Theta \to \infty} (J^\Theta_j[x^0; x^*(\cdot)] - J^\Theta_j[x^0; [x^*_{M-j}(\cdot), x_j(\cdot)]]) \geq 0$$

is defined to be an overtaking equilibrium for the open-loop game.

It is then shown that whenever—(i) the jump rates $q_{il}(x)$ are affine in x, (ii) the reward functions $\mathcal{L}_j(x, z_j)$ are concave in a strong sense (strict diagonal concavity), (iii) in mode i, the overtaking trajectories of player j, emanating from different initial states x^0_j, can be synthesized—then the Nash equilibria of these IHOLDGs define a piecewise open-loop strategy for the G^ε game which is a ζ-equilibrium if ε is small enough.

This gives a decomposition principle to play the hybrid game: At a high level, the limit stochastic game G^0 is solved. For each player j, both an equilibrium steady state $\tilde{x}_j$ and an equilibrium potential function $V^*_j(\cdot)$ are obtained. These potential functions are transmitted to all players. The G^ε game is then played by observing the state $(\xi(t^{0+}), x(t^{0+})) = (i, x^i)$ at a jump time t^0 and making a time translation to get $t^0 = 0$. This allows the authors to solve the IHOLDG so as to find the unique Nash overtaking equilibrium and then to follow this trajectory, as long as the jump process remains in state i.

The paper ends with two interesting examples. The first one is a duopoly example for which the authors provide a complete, numerical solution. The second example shows how a climate-change model can be cast as a hybrid game of the kind studied in the paper.

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