THE LOOP GROUP AND THE COBAR CONSTRUCTION

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(Communicated by Brooke Shipley)

ABSTRACT. We prove that for any 1-reduced simplicial set X, Adams' cobar construction ΩCX on the normalised chain complex of X is naturally a strong deformation retract of the normalised chains CGX on the Kan loop group GX. In order to prove this result, we extend the definition of the cobar construction and actually obtain the existence of such a strong deformation retract for all 0-reduced simplicial sets.

INTRODUCTION

There are two classical differential graded algebra models for the loop space on a 1-reduced simplicial set X: Adams' cobar construction ΩCX on the normalised chain complex [1], and the normalised chains CGX on the Kan loop group GX [7]. Both of these models are (weakly) equivalent to $C\Omega|X|$, the chains on the loop space of the realisation $|X|$.

In this article we show that ΩCX is actually a strong deformation retract of CGX, opening up the possibility of applying the tools of homological algebra to transferring perturbations of algebraic structure from the latter to the former.

Theorem. *For any* 1*-reduced simplicial set* X *there is a natural strong deformation retract of chain complexes*

$$
\text{(1)} \qquad \Omega CX \underset{\psi}{\overset{\phi}{\rightleftarrows}} CGX . \;\, \Phi
$$

Here $\psi\phi$ *is the identity map on* ΩCX *and* Φ *is a chain homotopy from* $\phi\psi$ *to the identity map on* CGX. *Furthermore both* ϕ *and* ψ *are homomorphisms of differential graded algebras.*

In particular, ΩCX is isomorphic to a subdifferential graded algebra of CGX, and both ϕ and ψ induce isomorphisms of algebras in homology.

Remark 1. Let X and Y be 1-reduced simplicial sets, and let $f : GX \to GY$ be a simplicial map (not necessarily a homomorphism). The theorem above gives us a natural way to construct a chain-level model of f. Indeed, if we set

$$
\xi = \psi \circ Cf \circ \phi : \Omega CX \to \Omega CY,
$$

then

$$\begin{CD} \Omega CX @>{\xi}>> \Omega CX \\ @V{\phi}VV @VV{\phi}V \\ CGX @>{Cf}>> GCX \end{CD}$$

commutes up to natural chain homotopy.

Remark 2. It was proved in [6] that if X is a simplicial suspension, then ΩCX is naturally a primitively generated Hopf algebra, and the chain algebra map $\phi : \Omega CX \to CGX$ also respects comultiplicative structure. In this case the strong deformation retract of the theorem above is actually *Eilenberg-Zilber data* [3], which implies that the chain algebra map $\psi : CGX \to \Omega CX$ is also *strongly homotopy comultiplicative* [ibid.].

In order to prove the theorem above, we extend the definition of the cobar construction and actually obtain the existence of such a strong deformation retract for all 0-reduced simplicial sets.

The homomorphism ϕ, which we recall in the first section of this article, was first described by Szczarba [11] in the language of twisting cochains. Given a simplicial set X that is 0-reduced but not necessarily 1-reduced, he gives an explicit, though somewhat complicated, formula for a twisting cochain,

$$\alpha_\phi : CX \to CGX,$$

that is based on the universal twisting function $\tau : X \to GX$ and that gives rise in the usual way to an algebra homomorphism

$$\phi : \Omega CX \to CGX.$$

In degree zero the cobar construction is a free associative algebra on symbols given by the nondegenerate 1-simplices of X, while the right-hand side is the group ring on the free group on the same symbols. In the first section of the paper we observe that if X is not 1-reduced, then we may perform a change of rings along ϕ_0, obtaining an extended cobar construction $\hat{\Omega}CX$, together with an algebra homomorphism

$$\phi : \hat{\Omega}CX \to CGX$$

that is an isomorphism in degree zero.

In the second section we introduce the retraction map ψ from the chains on the loop group to the extended cobar construction, for which we provide an explicit recursive formula. We prove in fact that ψ is a natural homomorphism of chain algebras and a one-sided inverse of the Szczarba map ϕ. It is surprising that such a map has not been previously observed in the literature.

In the third section we complete the strong deformation retraction (1) by defining the natural homotopy Φ. For this, we use the acyclic models for the loop group on 0-reduced simplices studied by Morace and Prouté [10].

1. Preliminaries

1.1. **Simplicial notions and notation.** A simplicial set X is a contravariant functor from the category of finite nonempty ordinals Δ to the category of sets;

more prosaically it is a sequence of sets X_n, $n \geq 0$, and specified face and degeneracy operators

$$d_i : X_n \to X_{n-1}, \qquad s_i : X_n \to X_{n+1} \qquad (0 \leq i \leq n)$$

satisfying the simplicial identities; see for example [9]. A simplicial set is n-reduced if $X_k \cong \{*\}$ for $k \leq n$. The notions of simplicial group and simplicial objects in other categories are analogous.

Given an element $x \in X_n$ and any composite θ of simplicial face and degeneracy operators, represented by a monotonic function $f : \{0, \ldots, m\} \to \{0, \ldots, n\}$ in Δ, we also write

$$\theta(x) \quad = \quad x_{(f(0), \ldots, f(m))}$$

for the corresponding element of X_m. We may write the face and degeneracy maps themselves, for example, as

$$\begin{aligned} d_i(x) &= x_{(0, \ldots, \widehat{i}, \ldots, n)}, \\ s_i(x) &= x_{(0, \ldots, i, i, \ldots, n)}. \end{aligned}$$

The simplicial relations imply that any simplicial operator $X_n \to X_m$ has normal form

$$\theta \quad = \quad s_{i_1} \ldots s_{i_q} d_{j_1} \ldots d_{j_r}$$

with $i_k > i_{k+1}$ and $j_k < j_{k+1}$ for all k. In this form, the corresponding *derived operator* $X_{n+1} \to X_{m+1}$ is

$$\theta' \quad = \quad s_{i_1+1} \ldots s_{i_q+1} d_{j_1+1} \ldots d_{j_r+1}.$$

An operator θ is *frontal* if it contains no d_0; such operators satisfy $\theta' s_0 = s_0 \theta$.

1.2. **The cobar construction.** We introduce a slightly extended definition of the cobar construction, which will be better suited for applying to the normalised chain complex on a 0-reduced simplicial set. Our definition generalises the classical construction of Adams, with which it agrees for simply connected coalgebras.

Let (C, ∂, Δ) be a connected differential graded coalgebra over a commutative ring R, so that $C_0 = R$. We suppose furthermore that C is R-free in each degree. Consider the ring B given by the free associative R-algebra on the desuspension of C_1,

$$B \quad = \quad T(s^{-1} C_1) \quad = \quad \bigoplus_{k \geq 0} (s^{-1} C_1)^{\otimes k}.$$

Fix an R-basis $\{x_j \mid j \in J\}$ of C_1, and let A be the ring obtained from B by freely adjoining inverses ξ_j of all elements of the form $1 + s^{-1} x_j$, for all $j \in J$. Explicitly,

$$A \quad = \quad T_B\big(\xi_j \mid j \in J\big) \ / \ \big(\xi_j \otimes (1 + s^{-1} x_j) = 1 = (1 + s^{-1} x_j) \otimes \xi_j\big).$$

Observe that the relations may also be expressed in the form

$$\xi_j \otimes s^{-1} x_j \quad = \quad 1 - \xi_j \quad = \quad s^{-1} x_j \otimes \xi_j.$$

The ring A may be regarded as a differential graded algebra concentrated in degree zero. The graded algebra underlying the extended cobar construction $(\hat{\Omega} C, \partial^\Omega)$ is then

$$\begin{aligned} \hat{\Omega} C \quad &= \quad T_A(s^{-1} C_{\geq 2}) \\ &= \quad \bigoplus_{k \geq 0} A \otimes (s^{-1} C_{\geq 2} \otimes A)^{\otimes k}. \end{aligned}$$

Each R-module $(\hat{\Omega}C)_n$ is generated by words

$$a = a_1 \otimes \cdots \otimes a_r, \qquad |a_i| = n_i, \quad |a| = n = \sum n_i,$$

where either $a_i = s^{-1}c$ for some R-basis element $c \in C_{n_i+1}$ or $n_i = 0$ and $a_i = \xi_j$ for some $j \in J$. This R-module is free on those words in which ξ_j does not appear adjacent to $s^{-1}x_j$. The algebra multiplication is induced by a concatenation of words, extended bilinearly to $\hat{\Omega}C$, modulo the relation that ξ_j is inverse to $1+s^{-1}x_j$.

The differential ∂^{Ω} on $\hat{\Omega}C$ is the derivation of A-algebras that is specified by

$$\partial_n^{\Omega}s^{-1}c \quad = \quad -s^{-1}dc \quad + \quad (-1)^{|c_i|}s^{-1}c_i \otimes s^{-1}c^i$$

for all basis elements $c \in C_{n+1}$ and all $n \geq 1$, where $\Delta(c) = 1 \otimes c + c \otimes 1 + c_i \otimes c^i$ (using the Einstein summation convention). Note that ∂^{Ω} is necessarily zero on elements of A.

The unit $1 \in (\hat{\Omega}C)_0$ is identified with the empty word, via the isomorphism $R \cong (s^{-1}C_1)^{\otimes 0}$.

Remark 3. If C is simply connected so that $C_1 = \{0\}$ and therefore $A = B = R$, then $\hat{\Omega}C$ coincides with the usual cobar construction ΩC defined by Adams.

1.3. **The Kan loop group.** Let X be a 0-reduced simplicial set and G a simplicial group. A *twisting function* $\tau : X \to G$ is a collection of functions of degree -1,

$$\{\tau : X_{n+1} \to G_n \mid n \geq 0\},$$

satisfying

$$\tau s_0 x = 1\,,$$

$$(2) \qquad\qquad s_i\tau x = \tau s_{i+1}x\,,$$

$$(3) \qquad\qquad d_0\tau x = \tau d_0 x^{-1} \cdot \tau d_1 x\,,$$

$$(4) \qquad\qquad d_i\tau x = \tau d_{i+1}x \quad \text{if } i \geq 1\,.$$

Let GX denote the Kan loop group on X, which is a simplicial group that models the space of based loops on the geometric realization of X (see [7, 9]). There is a *universal* twisting function

$$\tau : X_{n+1} \twoheadrightarrow (GX)_n = \mathbf{F}(X_{n+1})/\mathbf{F}(s_0X_n),$$

$$\tau x \ = \ [x],$$

sending $x \in X_{n+1}$ to the class of the corresponding generator in the quotient of free groups, and the simplicial structure on GX is defined by (2)–(4).

Recall that the normalised chain complex CG on a simplicial group G also has a differential graded algebra structure, with multiplication given by the shuffle map and the multiplication in G,

$$m : CG \otimes CG \longrightarrow C(G \times G) \longrightarrow CG;$$

that is,

$$(5) \quad m(g \otimes h) = \sum(-1)^{\text{sgn}\,(\mu,\nu)}s_{\mu(q)}\ldots s_{\mu(1)}g \cdot s_{\nu(p)}\ldots s_{\nu(1)}h, \quad g \in G_p, h \in G_q,$$

where the summation is over all (p,q)-shuffles $(\mu,\nu) \in \text{Shuff}(p,q)$.

The following proposition is the motivation for our extension of the cobar construction. Recall that for any simplicial set X, the degree n part of its normalised

chain complex, $C_n X$, is the free abelian group on the set of all nondegenerate n-simplices of X and that CX has a comultiplication $\Delta : CX \to CX \otimes CX$ given by the Alexander-Whitney diagonal approximation

$$\Delta(x) \; = \; x_{(0)} \otimes x \; + \; \sum_{i=1}^{n-1} x_{(0,\ldots,i)} \otimes x_{(i,\ldots,n)} \; + \; x \otimes x_{(n)}$$

for $x \in X_n$, $n \geq 1$, and with $\Delta(x) = x \otimes x$ for $x \in X_0$. In particular, if X is 0-reduced, then CX is a connected, differential graded coalgebra over $\mathbb{Z}$.

Proposition 4. *Let X be a 0-reduced simplicial set and GX its Kan loop group. Then there is an isomorphism of rings*

$$(\hat{\Omega} CX)_0 \underset{\psi_0}{\overset{\phi_0}{\underset{\cong}{\rightleftarrows}}} (CGX)_0$$

determined by

$$\psi_0(\tau x) = \xi_x, \qquad\qquad \psi_0(\tau x^{-1}) = 1 + s^{-1}x,$$
$$\phi_0(s^{-1}x) = \tau x^{-1} - 1, \qquad\qquad \phi_0(\xi_x) = \tau x.$$

Proof. The proof is straightforward. Note that if x is the degenerate element $s_0(*)$, then the four equations yield $\psi_0(1) = 1$, $\phi_0(0) = 0$, $\phi_0(1) = 1$. In degree 0 the multiplication (5) is just $m(g \otimes h) = g \cdot h$, and we have

$$\psi_0(\tau x_1^{\alpha_1} \ldots \tau x_r^{\alpha_r}) = \psi_0(\tau x_1^{\alpha_1}) \otimes \cdots \otimes \psi_0(\tau x_r^{\alpha_r}),$$
$$\phi_0(a_1 \otimes \cdots \otimes a_r) = \phi_0(a_1) \cdots \phi_0(a_r),$$

where $x_i \in X_1$, $\alpha_i = \pm 1$ and $a_i = s^{-1}x_i$ or ξ_{x_i}. This is well defined: $\psi_0(g)$ is inverse to $\psi_0(g^{-1})$ for all $g \in (GX)_0$ and $\phi_0(\xi_x)$ is inverse to $\phi_0(1 + s^{-1}x)$ for all $x \in X_1 \smallsetminus \{s_0(*)\}$. It is also clear that the composites $\phi_0\psi_0$ and $\psi_0\phi_0$ are the respective identity maps:

$$\phi_0\psi_0(\tau x) = \phi_0(\xi_x) = \tau x, \qquad\qquad \phi_0\psi_0(\tau x^{-1}) = \phi_0(1 + s^{-1}x) = \tau x^{-1},$$
$$\psi_0\phi_0(s^{-1}x) = \psi_0(\tau x^{-1} - 1) = s^{-1}x, \qquad \psi_0\phi_0(\xi_x) = \psi_0(\tau x) = \xi_x. \qquad \square$$

1.4. **The Szczarba map.** A map of differential graded algebras

$$\phi : \Omega CX \to CGX$$

was given explicitly by Szczarba in the language of twisting cochains. The following definition, lemma and theorem are from sections 2 and 3 of Szczarba's paper [11], adapted slightly to define a map on the extended cobar construction

$$\phi : \hat{\Omega} CX \to CGX$$

that extends the isomorphism ϕ_0 of Proposition 4.

Let S_n be the set of $n!$ sequences of integers

$$i = (i_1, \ldots, i_n) \quad \text{such that} \ \ 0 \leq i_k \leq n - k \text{ for each } k.$$

In particular, $i_n = 0$. The *sign* of such a sequence $i \in S_n$ is

$$(-1)^{\sum i}, \quad \text{where} \quad \sum i \; = \; i_1 + i_2 + \cdots + i_n.$$

Definition 5. Given a twisting function $\tau : X \to G$, the *Szczarba operators* are the functions

$$\mathrm{Sz}_i : X_{n+1} \longrightarrow G_n , \qquad i = (i_1, \ldots, i_n) \in S_n ,$$

given by the following product in G_n:

$$\mathrm{Sz}_i\, x \;=\; D_{0;\,i}^{n+1}\, \tau x^{-1} \cdot D_{1;\,i}^{n+1}\, \tau d_0 x^{-1} \cdots D_{n;\,i}^{n+1}\, \tau d_0^n x^{-1} \,.$$

Here the operators $D_{j;\,i}^{n+1} : G_{n-j} \to G_n$ for $i \in S_n$, $j = 0, \ldots, n$, are defined as

$$D_{0;\,()}^{1} \;=\; Id_{G_0},$$

$$(6) \qquad D_{j;\,i_1,\ldots,i_n}^{n+1} \;=\; \begin{cases} D_{j;\,i_2,\ldots,i_n}^{n}{}' \, s_0 \, d_{i_1-j} & \text{if } j < i_1 , \\[4pt] D_{j;\,i_2,\ldots,i_n}^{n}{}' & \text{if } j = i_1 , \\[4pt] D_{j-1;\,i_2,\ldots,i_n}^{n}{}' \, s_0 & \text{if } j > i_1 . \end{cases}$$

As simplicial operators these are all frontal: defining $D_{j;\,i}^{n+1} : X_{n-j} \to X_n$ in the same way, one has $D_{j;\,i}^{n+1} \tau = \tau\, D_{j;\,i}^{n+1}{}' : X_{n-j+1} \to G_n$.

Lemma 6. *The Szczarba operators satisfy*

$$
\begin{aligned}
d_0\, \mathrm{Sz}_{i_1,\ldots,i_n} &= \mathrm{Sz}_{i_2,\ldots,i_n}\, d_{i_1+1} , \\
d_k\, \mathrm{Sz}_{i_1,\ldots,i_k,i_{k+1},\ldots,i_n} &= d_k\, \mathrm{Sz}_{i_1,\ldots,i_{k+1},i_k-1,\ldots,i_n} \quad \text{if } i_k > i_{k+1} , \\
d_n\, \mathrm{Sz}_{i_1,\ldots,i_n}\, x &= s_\mu\, \mathrm{Sz}_{i'}\, x_{(0,\ldots,r)} \cdot s_\nu\, \mathrm{Sz}_{i''}\, x_{(r,\ldots,n+1)} \,.
\end{aligned}
$$

In the last equation the sequences i', i'', the integer r and the $(r-1, n-r)$-shuffle (μ, ν) are defined by a certain bijection

$$
\begin{aligned}
S_n &\cong \bigcup_{r=1}^{n} \mathrm{Shuff}(r-1, n-r) \times S_{r-1} \times S_{n-r}, \\
i &\mapsto \quad\quad ((\mu, \nu), i', i'')
\end{aligned}
$$

(see [11, Lemma 3.3]*), which respects parity as follows:*

$$n + \sum i \;=\; r + \mathrm{sgn}(\mu, \nu) + \sum i' + \sum i'' \pmod 2 .$$

Note that Szczarba's sign conventions differ slightly from ours and that his inductively-defined parity $\epsilon(i, n+1)$ is in fact just $n + \sum i$.

Theorem 7. *For any twisting function $\tau : X \to G$ on a 0-reduced simplicial set X, there is a canonical homomorphism of differential graded algebras defined by*

$$
\begin{aligned}
\phi : \hat{\Omega}CX &\longrightarrow CG, \\
\phi_0(\xi_{x_1}) &= \tau x_1, \\
\phi_0(s^{-1}x_1) &= \tau(x_1)^{-1} - 1, \\
\phi_n(s^{-1}x_{n+1}) &= \sum_{i \in S_n} (-1)^{\sum i}\, \mathrm{Sz}_i\, x \quad (n \geq 1),
\end{aligned}
$$

for $x_{n+1} \in X_{n+1}$.

Proof. The map ϕ extends linearly and multiplicatively via

$$\phi_{p+q}(a \otimes b) = m(\phi_p(a) \otimes \phi_q(b)), \qquad |a| = p, |b| = q,$$

where m is the multiplication (5), to all of $\hat{\Omega}CX$. We show ϕ is a chain map, i.e., that $\partial_n \phi_n = \phi_{n-1} \partial_n^\Omega$. For $x \in X_2$ we can write

$$
\begin{aligned}
\partial_1^\Omega s^{-1} x &= -s^{-1} d_0 x + s^{-1} d_1 x - s^{-1} d_2 x - s^{-1} x_{(0,1)} \otimes s^{-1} x_{(1,2)} \\
&= (1 + s^{-1} d_1 x) - (1 + s^{-1} x_{(0,1)}) \otimes (1 + s^{-1} x_{(1,2)})
\end{aligned}
$$

and so we have, by Lemma 6,

$$
\begin{aligned}
\partial_1 \phi_1 s^{-1} x &= \partial_1 \mathrm{Sz}_0 x = d_0 \mathrm{Sz}_0 x - d_1 \mathrm{Sz}_0 x = \mathrm{Sz}_{()} d_1 x - \mathrm{Sz}_{()} x_{(0,1)} \cdot \mathrm{Sz}_{()} x_{(1,2)} \\
&= \tau(d_1 x)^{-1} - \tau(x_{(0,1)})^{-1} \cdot \tau(x_{(1,2)})^{-1} = \phi_0 \partial_1^\Omega s^{-1} x.
\end{aligned}
$$

For $x \in X_{n+1}$ the argument is essentially the same. We have

$$
\begin{aligned}
\partial_n \phi_n s^{-1} x &= \sum_{i \in S_n} (-1)^{\sum i} \partial_n \mathrm{Sz}_i x \\
&= \sum_{i \in S_n} (-1)^{\sum i} \left(\sum_{k=0}^n (-1)^k d_k \mathrm{Sz}_i x \right),
\end{aligned}
$$

where, by Lemma 6, all the terms for $0 < k < n$ cancel, and the terms for $k = 0, n$ may be rewritten as

$$
\begin{aligned}
&\sum_{\substack{0 \le i_1 \le n-1 \\ i \in S_{n-1}}} (-1)^{i_1 + \sum i} \mathrm{Sz}_i d_{i_1+1} x + \sum_{\substack{1 \le r \le n \\ (\mu,\nu),\, i',\, i''}} (-1)^{n + \sum i} s_\mu \mathrm{Sz}_{i'} x_{(0,\ldots,r)} \cdot s_\nu \mathrm{Sz}_{i''} x_{(r,\ldots,n+1)} \\
&= \sum_{1 \le r \le n} (-1)^{r-1} \phi_{n-1} s^{-1} d_r x + \sum_{1 \le r \le n} (-1)^r m(\phi_{r-1} s^{-1} x_{(0,\ldots,r)} \otimes \phi_{n-r} s^{-1} x_{(r,\ldots,n+1)}) \\
&\qquad\qquad\qquad\qquad - 1 \otimes \phi_{n-1} s^{-1} x_{(1,\ldots,n+1)} + (-1)^n \phi_{n-1} s^{-1} x_{(0,\ldots,n)} \otimes 1 \\
&= \phi_{n-1} \left(\sum_{r=0}^{n+1} (-1)^{r-1} s^{-1} d_r x + \sum_{r=1}^n (-1)^r s^{-1} x_{(0,\ldots,r)} \otimes s^{-1} x_{(r,\ldots,n+1)} \right) \\
&= \phi_{n-1} \partial_n^\Omega x. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

We will need one further property of the Szczarba operators.

Lemma 8. *For all $x \in X_{n+1}$ and $i \in S_n$, the following product in G_n is degenerate:*

$$
D_{0;\,i}^{n+1} \tau x \cdot \mathrm{Sz}_i x = D_{1;\,i}^{n+1} \tau d_0 x^{-1} \cdots D_{n;\,i}^{n+1} \tau d_0^n x^{-1}.
$$

Proof. For any sequence $i \in S_n$ we will show by induction that $D_{j;\,i}^{n+1}$ is $s_{\kappa(i)-1}$-degenerate for all $j > 0$, where $\kappa(i)$ is the least integer such that $i_{\kappa(i)} = 0$. If $i_1 = 0$, so that $\kappa(i) = 1$, then by (6),

$$
D_{j;\,i}^{n+1} = D_{j-1;\,i_2,\ldots,i_n}^{n\,\prime} s_0 = s_0 D_{j-1;\,i_2,\ldots,i_n}^n
$$

for all $j > i_1 = 0$. If $i_1 > 0$, so that $\kappa(i) > 1$, then $\kappa(i_2,\ldots,i_n) = \kappa(i) - 1$ and we know that $D_{j;\,i_2,\ldots,i_n}^n$ (if $j > 0$) and $D_{j-1;\,i_2,\ldots,i_n}^n$ (if $j > 1$) are $s_{\kappa(i)-2}$-degenerate by the inductive hypothesis. The corresponding derived operators are therefore $s_{\kappa(i)-1}$-degenerate and, by (6), so is $D_{j;\,i}^{n+1}$ for all $j > 0$. $\qquad \square$

2. The retraction map

2.1. **Definition of the map.** Let X be a 0-reduced simplicial set. We introduce in this section a map of differential graded algebras

$$\psi : CGX \longrightarrow \hat{\Omega} CX$$

between the chains on the loop group and the extended cobar construction, which is a retraction of the Szczarba map ϕ. The map ψ is uniquely determined by the relation

$$(7) \qquad \psi_n(\tau x \cdot g) \;=\; \psi_n(g) \;-\; \sum_{i=0}^{n} s^{-1} x_{(0,\dots,i+1)} \otimes \psi_{n-i}(\tau d_1^i x \cdot d_0^i g)$$

for $x \in X_{n+1}$ and $g \in (GX)_n$. Note that the $i = 0$ term on the right-hand side is $s^{-1} x_{(0,1)} \otimes \psi_n(\tau x \cdot g)$. In fact ψ may be expressed inductively, on the degree n and the word length in $(GX)_n$.

Lemma 9. *The definition of ψ in* (7) *may be rewritten as*

$$\psi_n(\tau x \cdot g) = \xi_{x_{(0,1)}} \otimes \left(\psi_n(g) - \sum_{i=1}^{n} s^{-1} x_{(0,\dots,i+1)} \otimes \psi_{n-i} \omega_i(x,g) \right),$$

$$\psi_n(\tau x^{-1} \cdot h) = (1 + s^{-1} x_{(0,1)}) \otimes \psi_n(h) + \sum_{i=1}^{n} s^{-1} x_{(0,\dots,i+1)} \otimes \psi_{n-i} \overline{\omega}_i(x,h),$$

where

$$\omega_i(x,g) = \tau d_1^i x \cdot d_0^i g \;\in\; (GX)_{n-i},$$
$$\overline{\omega}_i(x,h) = \omega_i(x, \tau x^{-1} \cdot h) = \tau d_0 d_2^{i-1} x \cdot d_0^i h.$$

Proof. Collecting the terms in (7) involving $\psi_n(\tau x \cdot g)$ and dividing by $1 + s^{-1} x_{(0,1)}$ gives the first equation. The second is obtained by taking $g = \tau x^{-1} \cdot h$ in the first. $\square$

From these formulae it is straightforward to give the map ψ explicitly in low degrees.

Lemma 10. *The map $\psi_0 : (GCX)_0 \to (\hat{\Omega} CX)_0$ agrees with that defined in Proposition 4, and the map $\psi_1 : (GCX)_1 \to (\hat{\Omega} CX)_1$ is given for $x, x_i \in X_2$ and $\alpha, \alpha_i = \pm 1$ by*

$$\psi_1(\tau x_1^{\alpha_1} \dots \tau x_r^{\alpha_r}) = \sum_{i=1}^{r} \psi_0 d_1(\tau x_1^{\alpha_1} \dots \tau x_{i-1}^{\alpha_{i-1}}) \otimes \psi_1(\tau x_i^{\alpha_i}) \otimes \psi_0 d_0(\tau x_{i+1}^{\alpha_{i+1}} \dots \tau x_r^{\alpha_r})$$

$$\text{with } \psi_1(\tau x^{\alpha}) = \begin{cases} -\psi_0(\tau x_{(0,1)}) \otimes s^{-1} x \otimes \psi_0(\tau x_{(0,2)}) & (\alpha = +1) \\ s^{-1} x \otimes \psi_0(\tau x_{(1,2)}) & (\alpha = -1). \end{cases}$$

Lemma 11. *ψ is well-defined.*

Proof. We show that $\psi_n(w) = 0$ if w is degenerate, by induction on n and on the word length in GX. Suppose $0 \le j \le n-1$ and

$$w = s_j(\tau x^{\alpha} \cdot g),$$

where $\alpha = \pm 1$ and $\tau x^{\alpha} \cdot g$ is a reduced word in $(GX)_{n-1}$. For $\alpha = +1$ we have

$$(1 + s^{-1}x_{(0,1)}) \otimes \psi_n(w) = (1 + s^{-1}x_{(0,1)}) \otimes \psi_n(\tau s_{j+1}x \cdot s_j g)$$

$$= \psi_n s_j g - \sum_{i=1}^{n}(s_{j+1}x)_{(0,\dots,i+1)} \otimes \psi_{n-i}\omega_i(s_{j+1}x, s_j g)$$

in which the first term is zero inductively. Each term in the summation is also zero since $(s_{j+1}x)_{(0,\dots,i+1)}$ is degenerate for $j < i$ and $\omega_i(s_{j+1}x, s_j g) = s_{j-i}\omega_i(x, g)$ for $j \geq i$. Since $1 + s^{-1}x_{(0,1)}$ is invertible, we have $\psi_n(w) = 0$.

The argument for $\alpha = -1$ is similar. $\qquad\square$

2.2. Properties of the retraction map.

We now prove that ψ is a morphism of differential graded algebras and a retraction of the Szczarba map ϕ.

Proposition 12. *ψ is a chain map.*

Proof. We will show that for all $x \in X_{n+1}$ and $g \in (GX)_n$,

$$\partial_n^{\Omega}\psi_n(\tau x \cdot g) \;=\; \psi_{n-1}\partial_n(\tau x \cdot g),$$

by induction on n and on the word length in GX. We first observe that

$$\psi_{n-1}(d_0(\tau x \cdot g) - \tau d_1 x \cdot d_0 g) \;=\; \psi_{n-1}(\tau d_0 x^{-1} \cdot \tau d_1 x \cdot d_0 g) - \psi_{n-1}(\tau d_1 x \cdot d_0 g)$$

$$= s^{-1}x_{(1,2)} \otimes \psi_{n-1}(\tau d_1 x \cdot d_0 g) + \sum_{i=1}^{n-1} s^{-1}x_{(1,\dots,i+2)} \otimes \psi_{n-1-i}\overline{\omega}_i(d_0 x, \tau d_1 x \cdot d_0 g),$$

using the second formula in Lemma 9. Now since $\tau d_1 x \cdot d_0 g = \omega_1(x, g)$, and also $\overline{\omega}_i(d_0 x, \tau d_1 x \cdot d_0 g) = \overline{\omega}_i(d_1 x, \tau d_1 x \cdot d_0 g) = \omega_i(d_1 x, d_0 g) = \omega_{i+1}(x, g)$, we get

$$(8) \qquad \psi_{n-1}(d_0(\tau x \cdot g) - \tau d_1 x \cdot d_0 g) \;=\; \sum_{k=1}^{n} s^{-1}x_{(1,\dots,k+1)} \otimes \psi_{n-k}\omega_k(x, g)$$

and substituting $d_1^{r-1}x$ and $d_0^{r-1}g$ for x and g respectively, we obtain

$$(9) \qquad \psi_{n-r}(d_0\omega_{r-1}(x, g) - \omega_r(x, g)) \;=\; \sum_{k=r}^{n} s^{-1}x_{(r,\dots,k+1)} \otimes \psi_{n-k}\omega_k(x, g).$$

Now, using Lemma 9, we know that

$$(1 + s^{-1}x_{(0,1)}) \otimes \partial_n^{\Omega}\psi_n(\tau x \cdot g) \;=\; \partial_n^{\Omega}\big((1 + s^{-1}x_{(0,1)}) \otimes \psi_n(\tau x \cdot g)\big)$$

$$= \partial_n^{\Omega}\left(-\sum_{i=1}^{n} s^{-1}x_{(0,\dots,i+1)} \otimes \psi_{n-i}\omega_i(x, g) \; + \psi_n g\right)$$

$$= -\sum_{i=1}^{n} \partial_i^{\Omega}s^{-1}x_{(0,\dots,i+1)} \otimes \psi_{n-i}\omega_i(x, g)$$

$$- \sum_{i=1}^{n-1}(-1)^i s^{-1}x_{(0,\dots,i+1)} \otimes \psi_{n-1-i}\partial_{n-i}\omega_i(x, g) \; + \; \psi_{n-1}\partial_n g,$$

since inductively $\partial^{\Omega}_{n-i}\psi_{n-i} = \psi_{n-i-1}\partial_{n-i}$ and $\partial^{\Omega}_n\psi_n g = \psi_{n-1}\partial_n g$. Now expanding the operators ∂^{Ω} and ∂ we get

$$(1 + s^{-1}x_{(0,1)}) \otimes \partial^{\Omega}_n \psi_n(\tau x \cdot g)$$

$$= -\sum_{i=1}^{n}\sum_{r=0}^{i+1}(-1)^{r+1}s^{-1}x_{(0,\ldots,\hat{r},\ldots i+1)} \otimes \psi_{n-i}\omega_i(x,g)$$

(10) $$\quad -\sum_{i=1}^{n}\sum_{q=1}^{i}(-1)^q s^{-1}x_{(0,\ldots,q)} \otimes s^{-1}x_{(q,\ldots,i+1)} \otimes \psi_{n-i}\omega_i(x,g)$$

$$\quad -\sum_{j=1}^{n-1}\sum_{t=0}^{n-j}(-1)^{j+t}s^{-1}x_{(0,\ldots,j+1)} \otimes \psi_{n-1-j}d_t\omega_j(x,g) \;+\; \sum_{k=0}^{n}(-1)^k\psi_{n-1}d_k g.$$

Collecting together the terms for which either $i = 1$, $k = 0$, $r = 0$, or $q = 1$ gives

$$\left((1 + s^{-1}x_{(0,1)}) - (1 + s^{-1}x_{(0,2)})\right) \otimes \psi_{n-1}\omega_1(x,g) \quad + \quad \psi_{n-1}d_0 g$$

$$+\sum_{i=1}^{n}(1 + s^{-1}x_{(0,1)}) \otimes s^{-1}x_{(1,\ldots i+1)} \otimes \psi_{n-i}\omega_i(x,g)$$

$$= (1 + s^{-1}x_{(0,1)}) \otimes \psi_{n-1}d_0(\tau x \cdot g) - (1 + s^{-1}x_{(0,2)}) \otimes \psi_{n-1}\omega_1(x,g) + \psi_{n-1}d_0 g$$

by (8), and by Lemma 9 the last two terms here cancel exactly with the terms for $r = 1$ and $i > 1$ in (10).

Now collecting the terms for $r = i + 1$ and $i > 1$ in (10), together with all the (i,q)-indexed terms not already considered, gives

$$-\sum_{i=2}^{n}\left((-1)^i s^{-1}x_{(0,\ldots,i)} + \sum_{q=2}^{i}(-1)^q s^{-1}x_{(0,\ldots,q)} \otimes s^{-1}x_{(q,\ldots,i+1)}\right) \otimes \psi_{n-i}\omega_i(x,g)$$

$$= -\sum_{r=2}^{n}(-1)^r s^{-1}x_{(0,\ldots,r)} \otimes \left(\psi_{n-r}\omega_r(x,g) + \sum_{k=r}^{n} s^{-1}x_{(r,\ldots,i+1)} \otimes \psi_{n-i}\omega_i(x,g)\right)$$

$$= -\sum_{r=2}^{n}(-1)^r s^{-1}x_{(0,\ldots,r)} \otimes \psi_{n-r}d_0\omega_{r-1}(x,g)$$

by (9), and this cancels exactly with the terms for $t = 0$ in (10).

Thus expression (10) is equal to

$$(1 + s^{-1}x_{(0,1)}) \otimes \psi_{n-1}d_0(\tau x \cdot g) \;+\; \sum_{k=1}^{n}(-1)^k\psi_{n-1}d_k g$$

(11) $$\quad -\sum_{2 \le r \le i \le n}(-1)^{r+1}s^{-1}x_{(0,\ldots,\hat{r},\ldots i+1)} \otimes \psi_{n-i}\omega_i(x,g)$$

$$\quad -\sum_{1 \le j < j+t \le n}(-1)^{j+t}s^{-1}x_{(0,\ldots,j+1)} \otimes \psi_{n-1-j}d_t\omega_j(x,g).$$

Now to complete the proof it remains to show that expression (11) is equal to

$$\begin{aligned}(1 + s^{-1}x_{(0,1)}) \otimes \psi_{n-1}\partial_n(\tau x \cdot g) \;=\;& (1 + s^{-1}x_{(0,1)}) \otimes \psi_{n-1}d_0(\tau x \cdot g)\\ &+ \sum_{i=1}^{n}(-1)^i(1 + s^{-1}x_{(0,1)}) \otimes \psi_{n-1}(\tau d_{i+1}x \cdot d_i g).\end{aligned}$$

The first term is as required, and by Lemma 9 the summation is

$$\sum_{i=1}^{n}(-1)^i\left(\psi_{n-1}d_i g - \sum_{k=1}^{n-1}s^{-1}(d_{i+1}x)_{(0,\dots,k+1)} \otimes \psi_{n-1-k}\omega_k(d_{i+1}x, d_i g)\right).$$

The result therefore follows from the observations that

$$(d_{i+1}x)_{(0,\dots k+1)} = \begin{cases} x_{(0,\dots,\widehat{i+1},\dots,k+2)}, \\ x_{(0,\dots,k+1)}, \end{cases} \qquad \omega_k(d_{i+1}x, d_i g) = \begin{cases} \omega_{k+1}(x,g) & (i \le k), \\ d_{i-k}\omega_k(x,g) & (i > k). \end{cases}$$

$\square$

Proposition 13. *The map ψ is an algebra homomorphism.*

Proof. Let $v \in (GX)_p$ and $w \in (GX)_q$ and consider $v \otimes w \in (CGX \otimes CGX)_n$, $n = p + q$. To show that ψ is multiplicative we must prove that

$$\psi_n m(v \otimes w) \;=\; \sum_{(\mu,\nu)}(-1)^{\mathrm{sgn}(\mu,\nu)}\psi_n(s_\mu v \cdot s_\nu w) \;=\; \psi_p v \otimes \psi_q w$$

in $\hat{\Omega}CX$ by induction on p and the word length of v. For $v = *$, or p or $q = 0$, there is nothing to prove; suppose inductively that $p, q \ge 1$ and $v = \tau x \cdot g$ for $x \in X_{p+1}$ (the argument for $v = \tau x^{-1} \cdot g$ is similar). Then by Lemma 9,

$$(1 + s^{-1}x_{(0,1)}) \otimes \psi_n(s_\mu(\tau x \cdot g) \cdot s_\nu w) \;=\; (1 + s^{-1}x_{(0,1)}) \otimes \psi_n(\tau s'_\mu x \cdot (s_\mu g \cdot s_\nu w))$$

$$= \psi_n(s_\mu g \cdot s_\nu w) - \sum_{i=1}^{n}d_{i+2}^{n-i}s'_\mu x \otimes \psi_{n-i}\omega_i(s'_\mu x, s_\mu g \cdot s_\nu w).$$

The term $d_{i+2}^{n-i}s'_\mu x$ will be degenerate unless $i \le p$ and (s_μ, s_ν) is of the form $(s_{i+\xi_q}s_{i+\xi_{q-1}} \dots s_{i+\xi_1}, \, s_0^i s_\zeta)$ for some $(p - i, q)$-shuffle (ξ, ζ), and we have

$$(1 + s^{-1}x_{(0,1)}) \otimes \psi_n m(v \otimes w)$$

$$= \sum_{(\mu,\nu)}(-1)^{\mathrm{sgn}(\mu,\nu)}\psi_n(s_\mu g \cdot s_\nu w) - \sum_{\substack{1 \le i \le p \\ (\xi,\zeta)}}(-1)^{\mathrm{sgn}(\xi,\zeta)}d_{i+2}^{p-i}x \otimes \psi_{n-i}(s_\xi \omega_i(x,g) \cdot s_\zeta w)$$

$$= \psi_n m(g \otimes w) \;-\; \sum_{i=1}^{p}d_{i+2}^{p-i}x \;\otimes\; \psi_{n-i}m(\omega_i(x,g) \otimes w)$$

$$= \left(\psi_p g \;-\; \sum_{i=1}^{p}d_{i+2}^{p-i}x \;\otimes\; \psi_{p-i}\omega_i(x,g)\right) \otimes \psi_q w, \quad \text{by the inductive hypothesis,}$$

$$= (1 + s^{-1}x_{(0,1)}) \;\otimes\; \psi_p v \;\otimes\; \psi_q w,$$

by Lemma 9. The result follows. $\square$

Proposition 14. *The map ψ is a retraction of ϕ; that is, the composite $\psi\phi$ is the identity.*

Proof. It is enough to prove this on algebra generators of $\hat{\Omega}CX$. For $x \in X_{n+1}$ and $i = (i_1, \ldots, i_n) \in S_n$ we will show that

$$\psi_n \mathrm{Sz}_i x = \begin{cases} x & \text{if } i_1 = \cdots = i_n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $x_{0;\,i}$ the element of X satisfying

$$D^{n+1}_{0;\,i} \tau x = \tau x_{0;\,i}.$$

Lemma 8 tells us that $\psi_n(\tau x_{0;\,i} \cdot \mathrm{Sz}_i x) = 0$, and so by Lemma 9 we have

$$\psi_n \mathrm{Sz}_i x = \sum_{k=1}^{n} d^{n-k}_{k+2} x_{0;\,i} \otimes \psi_{n-k} \omega_k(x_{0;\,i}, \mathrm{Sz}_i x).$$

From (6) we see that $D^{n+1}_{0,\,i} \tau x$ has an s_{k-1} degeneracy if $i_k \neq 0$. Thus $d^{n-k}_{k+2} x_{0;\,i}$ is degenerate except in the case $i_1 = \cdots = i_k = 0$. In this case we see from (6) and Lemma 6 that

$$\omega_k(x_{0;\,i}, \mathrm{Sz}_i x) = \tau d^k_1 x_{0;\,i} \cdot d^i_0 \mathrm{Sz}_i x = D^{n-k+1}_{0;\,i_{k+1},\ldots,i_n} \tau d^k_1 x \cdot \mathrm{Sz}_{i_{k+1},\ldots,i_n} d^k_1 x,$$

which is degenerate again by Lemma 8. The only nonzero term is therefore

$$\psi_n \mathrm{Sz}_{0,\ldots,0} x = x_{0;0,\ldots,0} \otimes \psi_0(*) = x,$$

and hence $\psi\phi x = x$, as required. $\square$

3. Deformation retraction of the loop group

Both the Kan functor G and the cobar construction model loop spaces. In the 1-reduced case it is easy to show that the Szczarba map $\phi : \Omega CX \to CGX$ is a weak equivalence, by applying Zeeman's comparison theorem to the map of spectral sequences associated with

(12)
$$\begin{array}{ccccc} \Omega CX & \longrightarrow & \Omega CX \otimes_{\alpha_\phi} CX & \longrightarrow & CX \\ \phi \downarrow & & \downarrow & & \| \\ CGX & \longrightarrow & C(GX \times_\tau X) & \longrightarrow & CX \end{array}$$

in which the total spaces are acyclic.

We prove here the following stronger result.

Theorem 15. *Let X be a 0-reduced simplicial set. Let ϕ be the Szczarba map and ψ the retraction map defined above.*

There is a natural strong deformation retraction of chain complexes

$$\hat{\Omega}CX \underset{\psi}{\overset{\phi}{\rightleftarrows}} CGX. \,\,\overset{\curvearrowright}{\Phi}$$

Recall that if A and B are chain complexes, $\nabla : A \to B$ and $f : B \to A$ are chain maps, and $h : B \to B$ is linear map of degree $+1$, then

$$A \underset{f}{\overset{\nabla}{\rightleftarrows}} B \,\,\overset{\curvearrowright}{h}$$

is a strong deformation retract if $f\nabla = Id_A$ and $\partial h + h\partial = \nabla f - Id_B$. Given a strong deformation retract, one can apply the machinery of homological perturbation theory to transfer perturbations of the structure B across to A, obtaining a new strong deformation retract; see [4], [5], [8].

Proof. According to Proposition 14, we need only to prove that there is a natural chain homotopy from the composite $\phi\psi$ to the identity map on CGX. The proof, an acyclic models argument, proceeds by induction on the degree.

The base step of the induction is trivial, by Proposition 4. We can simply set $\Phi_0 = 0 : C_0GX \to C_1GX$ for all 0-reduced simplicial sets X.

Suppose now that $\Phi_k : C_kGX \to C_{k+1}GX$ has been defined for all $0 \le k < n$ and for all 0-reduced simplicial sets X so that

(1.k) Φ_k is natural in X for all k, and

(2.k) $\partial\Phi_k + \Phi_{k-1}\partial = \phi\psi - Id_{CGX}$ for all k and all X,

where $n \ge 1$.

Let $\overline{\Delta}[n]$ denote the quotient of the standard simplicial n-simplex by its 0-skeleton. If $x = (k_0 \cdots k_j)$ is a j-simplex of $\Delta[n]$, let $x \cdot n$ denote the $(j+1)$-simplex $(k_0 \cdots k_j\, n)$. Let

$$h_i^n : \left(G\overline{\Delta}[n]\right)_i \to \left(G\overline{\Delta}[n]\right)_{i+1}$$

denote the group homomorphism specified by $h_i^n(\tau x) = \tau(x \cdot n)$ for all $x \in \overline{\Delta}[n]_{i+1}$. Let

$$\bar{h}^n : C_*G\overline{\Delta}[n] \to C_{*+1}G\overline{\Delta}[n]$$

denote the degree $+1$ linear map specified by $\bar{h}_i^n(w) = -h_i^n(w)$ for all $w \in (G\overline{\Delta}[n])_i$.

Morace and Prouté proved in [10] that for all $i \ge 1$,

$$\partial_{i+1}\bar{h}_i^n + \bar{h}_{i-1}^n\partial_i = Id,$$

i.e., that $\bar{h}^n$ is a contraction in positive degrees. It follows that $H_iG\overline{\Delta}[n] = 0$ for all $i \ge 1$.

Consider the infinite wedge

$$W(n+1) = \bigvee_{m\in\mathbb{N}} \overline{\Delta}[n+1].$$

There is a chain homotopy

$$\tilde{h}^{n+1} : C_*GW(n+1) \to C_{*+1}GW(n+1)$$

that is a contraction in positive degrees and that generalises Morace and Prouté's construction.

Let

$$w = \tau\delta_{m_1}^{\alpha_1} \cdots \tau\delta_{m_k}^{\alpha_k} \in \left(GW(n+1)\right)_n,$$

where δ_{m_i} denotes the unique nondegenerate $(n+1)$-simplex in the m_i^{th} copy of $\overline{\Delta}[n+1]$ in $W(n)$, and $\alpha_i = \pm 1$. Set

$$\Phi_n(w) = \tilde{h}^n\big(\phi\psi(w) - w - \Phi_{n-1}(\partial w)\big).$$

The induction hypothesis implies that $\phi\psi(w) - w - \Phi_{n-1}(\partial w)$ is a cycle and that

$$\begin{aligned}
\partial\Phi_n(w) &= -\tilde{h}^n\partial\big(\phi\psi(w) - w - \Phi_{n-1}(\partial w)\big) + \phi\psi(w) - w - \Phi_{n-1}(\partial w) \\
&= \phi\psi(w) - w - \Phi_{n-1}(\partial w).
\end{aligned}$$

Adding $\Phi_{n-1}(\partial w)$ to both sides of this equation, we obtain

$$\partial \Phi_n(w) + \Phi_{n-1}(\partial w) = \phi\psi(w) - w;$$

i.e., $(2.n)$ holds for all such w.

Let X be any 0-reduced simplicial set, and let

$$w = \tau x_1^{\alpha_1} \cdots \tau x_k^{\alpha_k}$$

be any nondegenerate n-simplex in GX, where $\alpha_i = \pm 1$ for all i. Let $\zeta_i : \overline{\Delta}[n+1] \to X$ be the simplicial map representing x_i.

Let $\gamma : \overline{\Delta}[n+1] \to X$ denote the simplicial map collapsing everything to the basepoint. Consider the morphism of simplicial groups

$$\Psi_w = G(\zeta_1 \vee \cdots \vee \zeta_k \vee \bigvee_{m>k} \gamma) : GW(n+1) \to GX.$$

Observe that

$$\Psi_w(\tau\delta_1^{\alpha_1} \cdots \tau\delta_k^{\alpha_k}) = w,$$

where δ_i denotes the unique nondegenerate n-simplex in the ith copy of $\overline{\Delta}[n+1]$ in $W(n+1)$.

Using the map Ψ_w constructed above for any generator w of C_nGX, we define $\Phi_n : C_nGX \to C_{n+1}GX$ for any 0-reduced simplicial set X by

$$\Phi_n(w) = C_{n+1}\Psi_w \circ \Phi_n(\tau\delta_1^{\alpha_1} \cdots \tau\delta_k^{\alpha_k}).$$

Note that if $X = W(n+1)$ and $w = \tau\delta_{j_1}^{\alpha_1} \cdots \tau\delta_{j_k}^{\alpha_k}$, then

$$\Psi_w : GW(n+1) \to GW(n+1)$$

is a homomorphism of simplicial groups given simply by permuting generators. It follows from the construction of the chain homotopy $\bar{h}^{n+1}$ and therefore of the chain homotopy $\tilde{h}^{n+1}$ that $\tilde{h}^{n+1}$ is natural with respect to homomorphisms that simply permute generators, so that

$$C_{n+1}\Psi_w \circ \tilde{h}_n^n = \tilde{h}_n^n \circ C_n\Psi_w.$$

Consequently, $\Phi_n(w)$ is indeed well-defined, since ϕ, ψ and, by the induction hypothesis, Φ_{n-1} are all natural with respect to simplicial maps.

Moreover,

$$\begin{aligned}
\partial\Phi_n(w) &= C_{n+1}\Psi_w \circ \partial\Phi_n(\tau\delta_1^{\alpha_1} \cdots \tau\delta_k^{\alpha_k}) \\
&= C_{n+1}\Psi_w \circ \left((\phi\psi - Id_{CGW(n)} - \Phi_{n-1}\partial)(\tau\delta_1^{\alpha_1} \cdots \tau\delta_k^{\alpha_k})\right) \\
&\overset{(\star)}{=} (\phi\psi - Id_{CGX} - \Phi_{n-1}\partial) \circ C_n\Psi_w(\tau\delta_1^{\alpha_1} \cdots \tau\delta_k^{\alpha_k}) \\
&= (\phi\psi - Id_{CGX} - \Phi_{n-1}\partial)(w),
\end{aligned}$$

where the equality $(\star)$ follows from naturality of ϕ, ψ and Φ_{n-1}. In other words,

$$\partial\Phi_n + \Phi_{n-1}\partial = \phi\psi - Id_{CGX},$$

for all X; i.e., condition $(2.n)$ holds.

To conclude, observe that condition $(1.n)$ holds as well, since for all simplicial maps $g : X \to Y$ between 0-reduced spaces and all $w \in GX$,

$$Gg \circ \Psi_w = \Psi_{Gg(w)} : GW(n+1) \to GY. \qquad\qquad \square$$

Remark 16. It is in order to be able to apply the chain homotopy of Morace and Prouté that we work with 0-reduced simplicial sets. There is no such chain homotopy in the 1-reduced case, so it seems we are obliged to prove the existence of the strong deformation retract in the 0-reduced case in order to conclude that it exists in the 1-reduced case as well.

Remark 17. As defined in the proof above, Φ_n is almost certainly not a derivation homotopy, since, as easy computations show, $\tilde{h}$ is not a derivation homotopy.

Remark 18. Morace and Prouté showed that $\bar{h}_{i+1}^n \circ \bar{h}_i^n = 0$ for all i and n, from which it follows that $\tilde{h}_{i+1}^n \circ \tilde{h}_i^n = 0$ as well and therefore that

$$\tilde{h}_{n+1}^{n+1} \circ \Phi_n(\tau\delta_{j_1}^{\alpha_1} \cdots \tau\delta_{j_k}^{\alpha_k}) = 0$$

for all $j_1, ..., j_k$ and $\alpha_1, ..., \alpha_k$.

Remark 19. The results in this paper generalise from chain complexes to crossed complexes. There is a crossed cobar construction $\Omega\pi X$ on the fundamental crossed complex πX, see [2], and we may define a 'crossed' Szczarba map of crossed chain algebras $\phi : \Omega\pi X \to \pi GX$ that forms part of a deformation retraction

$$\Omega\pi X \underset{\psi}{\overset{\phi}{\rightleftarrows}} \pi GX. \circlearrowright \Phi$$

The classical argument that ϕ is a weak equivalence, using (12), does not go through in this slightly nonabelian situation, since it seems there is no good notion of twisted tensor product of crossed complexes.

Acknowledgement

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