Capacity of Deterministic Z-Chain Relay-Interference Network

Soheil Mohajer[†], Suhas N. Diggavi[†], Christina Fragouli[†], and David Tse[‡]

[†] School of Computer and Communication Sciences
École Polytechnique Fédéral de Lausanne, Lausanne, Switzerland.
Email: {soheil.mohajer, suhas.diggavi, christina.fragouli}@epfl.ch
[‡] Wireless Foundations, UC Berkeley, Berkeley, California, USA.
Email: dtse@eecs.berkeley.edu

Abstract

The wireless multiple-unicast problem is considered over a layered network, where the rates of transmission are limited by the relaying and interference effect. The deterministic model introduced in [3] is used to capture the broadcasting and multiple access effects. The capacity region of the Z-chain relay-interference network is fully characterized. In order to solve the problem, we introduce a new achievability scheme based on "interference neutralization" and a new analysis technique to bound the number of non-interfering (pure) signals.

I. INTRODUCTION

Unlike a wired network, a transmitted signal is broadcast in a wireless system, and hence causes interference between simultaneously transmitted signals. The interference can be used for cooperation, but also causes competition between distinct information flows. Hence, a fundamental question is how to manage interference in a wireless network.

In a general multiple unicast setup, many sources transmit messages, and each is of interest for one of the receivers. The interference channel problem [1] is the very basic example of such situation. The best known achievable region for this problem is due to Han and Kobayashi [1]. Over the past few decades, several techniques have been devised for transmission on the interference channels; among them, superposition of information, power allocation, and interference suppression are the most well-known ones. Recently, the capacity region of the interference channel has been characterized for some regimes by building on an approximate characterization (within 1 bit) given for the whole regime in [2].

The deterministic approach, studied by Avestimehr, Diggavi, and Tse [3], simplifies the wireless network interaction model by eliminating the noise. This approach was successfully applied to the relay network in [3], and resulted in insight in terms of transmission techniques. These insights also led to an approximate characterization of the noisy wireless relay network problem [4].

This model was also applied to a two-layers two-unicast system, called ZZ network in [5] and the capacity region was fully characterized under the deterministic model[1]. Moreover, a new transmission technique called *interference neutralization* was introduced to deal with the interference. In this technique, the interference caused by the two layers of the network are used to (partially) neutralize each other. It is shown that interference neutralization is crucial to achieve the performance of the network for some regime of parameters.

In this paper, we generalize the above mentioned network, and consider a chain with arbitrary number of Z channels, and characterize the region of the admissible rates under the deterministic model. We first develop a genie-aided outer bound for the rate-region.

We then show that this rate region is achievable using linear operations, where decoding a message of rate r is possible if and only if r non-interfered linearly independent equations describing the message are available at the receiver. In our achievability proof, we use a new technique, called *analysis of pure equations*, where we keep track of the number of the equations involving bits of each of the interfering messages at the relay nodes in the different layers of the network. We show that among all possible encoding schemes at the layers, there *exists* at least one which guarantee to provide the desired number of proper equations.

The rest of this paper is organized as follows. We describe the transmission model and the problem in Section II. The outer bound and the achievability analysis are given in Sections III and IV, respectively. Finally we conclude in Section V.

II. THE PROBLEM STATEMENT

Consider the network shown in Fig. 1, which is formed by cascading N consecutive Z channels. The transmitters S_1 and S_2 wish to communicate at rates r_1 and r_2 to the destination nodes D_1 and D_2, respectively.

---

[1]This idea also gives us insight to obtain an approximate (within constant number of bits) characterization for the Gaussian ZZ network.
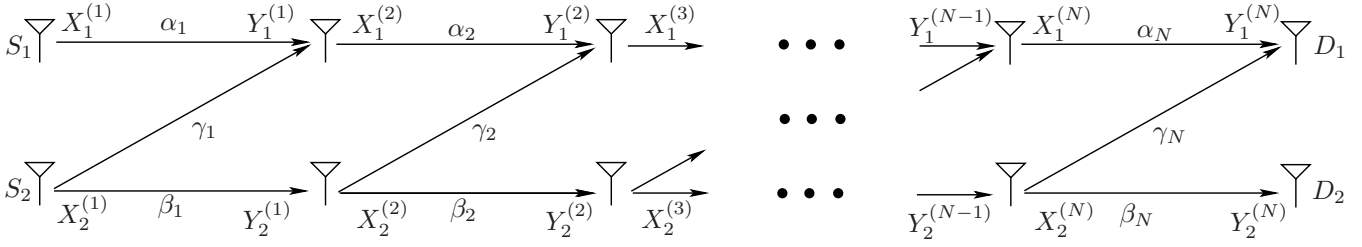
Fig. 1: Transmission model: the Z-chain network.

The network is formed by N layers, where the k-th one connects the relay nodes A_{k-1} and B_{k-1} to A_k and B_k, for $k = 0, 2, \ldots, N$, where the relays in layers zero and N are the transmitters and receivers, respectively. We denote the parameters of the network by a family of triples, $\{(\alpha_k, \beta_k, \gamma_k)\}_{k=1}^N$, where α_k and β_k are the gains of the first and second direct links of the k-th layer of the network and γ_k is the gain of the cross link of the same layer.

We denote the inputs of the layer k by $X_1^{(k)}$ and $X_2^{(k)}$ and received vectors of this layer by $Y_1^{(k)}$ and $Y_2^{(k)}$. Transmission model in the k-th layer of the Z-chain network can be written as

$$Y_1^{(k)} = N_{11}^{(k)} X_1^{(k)} + N_{12}^{(k)} X_2^{(k)},$$
$$Y_2^{(k)} = N_{22}^{(k)} X_2^{(k)},$$

where $\alpha_k = \text{rank } N_{11}^{(k)}$, $\beta_k = \text{rank } N_{22}^{(k)}$, and $\gamma_k = \text{rank } N_{12}^{(k)}$. Using shift deterministic model, the channel matrices would be of the form $N_{ij}^{(k)} = \mathbf{D}^{q-n_{ij}^{(k)}}$ where

$$D = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{q \times q},$$

q is a constant, and $n_{ij}^{(k)}$ is the rank of $N_{ij}^{(k)}$ (and equals α_k, β_k, or γ_k).

The relay node A_k forms its encoded message for the next layer, $X_1^{(k+1)}$ as a function of its received signal $Y_1^{(k)}$, and similarly for B_k.

The rate pair (r_1, r_2) is called admissible if and only if, for some large enough n, there exist codes of length n to be used at the relays such that $W_1 \in \{1, \ldots, 2^{nr_1}\}$ and $W_2 \in \{1, \ldots, 2^{nr_2}\}$ can be transmitted to the destination nodes, respectively, with vanishing error probability. Characterization of $\mathcal{R}$, the set of all such rate pairs, is the main question solved in this work.

III. THE OUTER BOUND

The following theorem provides an outer bound for the capacity region by upper bounding the individual rates as well as the sum-rate. The main idea behind the result is to give the relay at the k-th layer the interfering signal from all the other layers. This genie-aided side-information is shown to allow the relay node A_k to decode message W_1 if D_1 is able to do so.

Theorem 1: Let

$$\mathcal{R}^U \triangleq \{(r_1, r_2) : r_1 \leq \alpha_k \qquad \forall k, \tag{1}$$
$$r_2 \leq \beta_k \qquad \forall k, \tag{2}$$
$$r_1 + r_2 \leq \Psi_k + \Gamma_N - 2\gamma_k \qquad \forall k, \} \tag{3}$$

where $\Psi_k \triangleq \max(\alpha_k, \gamma_k) + \max(\beta_k, \gamma_k)$ and $\Gamma_k \triangleq \sum_{i=1}^k \gamma_i$ for $k = 1, 2, \ldots, N$. Then $\mathcal{R} \subseteq \mathcal{R}^U$.

Proof: Let the rate pair (r_1, r_2) is achievable. Therefore, there exists a code with block length n which can be used to transmit messages W_1 and W_2 of rates r_1 and r_2, respectively. The transmitters encode the messages into sequences $X_1^{(1)}[1, \ldots, n]$ and $X_2^{(1)}[1, \ldots, n]$, where we have used the variables $1, \ldots, n$ to denote the corresponding time instance. The relays in the first layer receive sequences $Y_1^{(1)}[1, \ldots, n]$ and $Y_2^{(1)}[1, \ldots, n]$. They apply their encoding functions on the received sequences to form the transmitting sequences $X_1^{(2)}[2, \ldots, n+1]$ and $X_2^{(2)}[2, \ldots, n+1]$. Note that the system is casual and $X_i^{(2)}[j]$ is a only a function of $Y_i^{(1)}[1, \ldots, j-1]$. Similarly, the transmitted sequences corresponding to W_1 and W_2 in the k-th layer if the network are $X_1^{(k)}[k, \ldots, n+k-1]$ $X_2^{(k)}[k, \ldots, n+k-1]$. We may drop the time indices, and use bold face variable to denote the n-tuples for brevity.

The proof of the individual rate inequalities goes through the cut-set bounds. More precisely, (1) is just the evaluation of the amount of information can be transmitted through the cut formed by the links $\{\gamma_1, \ldots, \gamma_{k-1}, \alpha_k\}$ (the dashed-line cut in Fig. 2). More precisely,
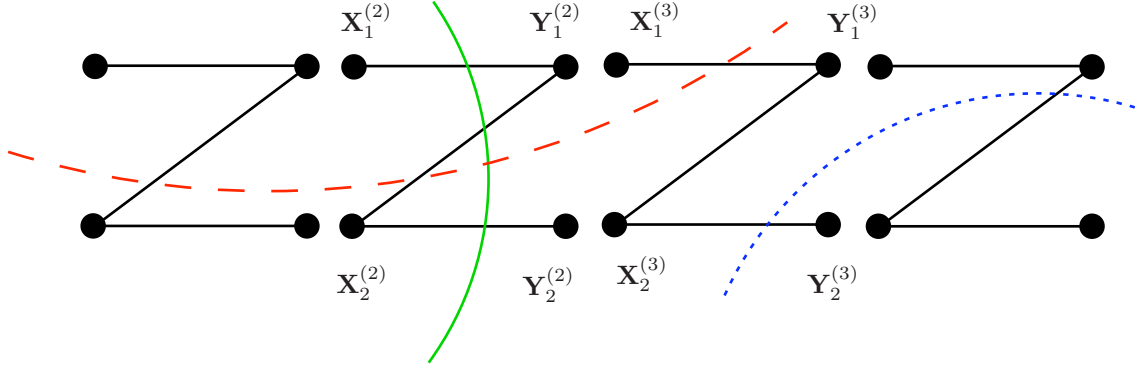
Fig. 2: Three different kinds of cuts

$$
\begin{aligned}
nr_1 &\leq I(\mathbf{Y}_1^{(k)}, \ldots, \mathbf{Y}_1^{(N)}, \mathbf{Y}_2^{(1)}, \ldots, \mathbf{Y}_2^{(N)}; \mathbf{X}_1^{(1)}, \ldots, \mathbf{X}_1^{(k)} | \mathbf{X}_1^{(k+1)}, \ldots, \mathbf{X}_1^{(N)}, \mathbf{X}_2^{(1)}, \ldots, \mathbf{X}_2^{(N)}) + n\varepsilon_n \\
&\overset{(a)}{=} H(\mathbf{Y}_1^{(k)}, \ldots, \mathbf{Y}_1^{(N)}, \mathbf{Y}_2^{(1)}, \ldots, \mathbf{Y}_2^{(N)} | \mathbf{X}_1^{(k+1)}, \ldots, \mathbf{X}_1^{(N)}, \mathbf{X}_2^{(1)}, \ldots, \mathbf{X}_2^{(N)}) + n\varepsilon_n \\
&\overset{(b)}{=} H(\mathbf{Y}_1^{(k)} | \mathbf{X}_1^{(k+1)}, \ldots, \mathbf{X}_1^{(N)}, \mathbf{X}_2^{(1)}, \ldots, \mathbf{X}_2^{(N)}) + n\varepsilon_n \\
&\leq H(\mathbf{Y}_1^{(k)} | \mathbf{X}_2^{(k)}) + n\varepsilon_n \\
&\leq \sum_{i=1}^{n} H(Y_1^{(k)}[i] | X_2^{(k)}[i]) + n\varepsilon_n \\
&\leq n\alpha_k + n\varepsilon_n
\end{aligned}
\tag{4}
$$

where $\varepsilon_n \to 0$ as n grows, (a) holds since the network is deterministic, and (b) follows from the fact that

$$
H(\mathbf{Y}_1^{(k+1)}, \ldots, \mathbf{Y}_1^{(N)}, \mathbf{Y}_2^{(1)}, \ldots, \mathbf{Y}_2^{(N)} | \mathbf{X}_1^{(k+1)}, \ldots, \mathbf{X}_1^{(N)}, \mathbf{X}_2^{(1)}, \ldots, \mathbf{X}_2^{(N)}) = 0.
$$

Similarly, the bound in (2) can be obtained by bounding the maximum information can be passed through the cut specified by the links of gains $\{\beta_k, \gamma_{k+1}, \ldots, \gamma_N\}$ (the dotted-line cut in Fig. 2). We have

$$
\begin{aligned}
nr_2 &\leq I(\mathbf{Y}_2^{(k)}, \ldots, \mathbf{Y}_2^{(N)}; \mathbf{X}_1^{(1)}, \ldots, \mathbf{X}_1^{(N)}, \mathbf{X}_2^{(1)}, \ldots, \mathbf{X}_2^{(k)} | \mathbf{X}_2^{(k+1)}, \ldots, \mathbf{X}_2^{(N)}) + n\varepsilon_n \\
&\overset{(c)}{=} H(\mathbf{Y}_2^{(k)}, \ldots, \mathbf{Y}_2^{(N)} | \mathbf{X}_2^{(k+1)}, \ldots, \mathbf{X}_2^{(N)}) + n\varepsilon_n \\
&\overset{(d)}{\leq} H(\mathbf{Y}_2^{(k)}) + n\varepsilon_n \\
&\leq \sum_{i=1}^{n} H(Y_2^{(k)}[i]) + n\varepsilon_n \\
&\leq n\beta_k + n\varepsilon_n
\end{aligned}
\tag{5}
$$

where (c) is due to the deterministic structure of the network, and (d) holds since $\mathbf{Y}_2^{(\ell)} = N_{22}^{(\ell)} \mathbf{X}_2^{(\ell)}$ is a deterministic function of $\mathbf{X}_2^{(\ell)}$ for $\ell = k+1, \ldots, N$.

The sum-rate bound is obtained by bounding the amount of information can be passed through the k-th layer of the network, namely, the edges $\{\alpha_k, \beta_k, \gamma_k\}$ (the solid-line cut in Fig.2).

In the following chain of inequalities we will use a new random variable which is the interference observed [2] by the relay node A_k, and defined as $t^{(k)} = N_{12}^{(k)} X_2^{(k)}$ for $k = 1, 2, \ldots, N$. We also use bold-face symbol $\mathbf{t}$ to denote a block of length n of t.

This bound is essentially a genie-aided bound, where a genie provides the output of all the cross links except the k-th one $(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(k-1)}, \mathbf{t}^{(k+1)}, \ldots, \mathbf{t}^{(N)})$ at A_k. Intuitively, we capture the maximum possible interference neutralization, and argue that having such aid from the genie, A_k can decode the message W_1 if the destination node D_1 can do.

---

[2]One can think of these random variables as the output of some auxiliary channels with the same behavior as the cross links.

$$n(r_1 + r_2) \leq I(\mathbf{Y}_1^{(k)}, \ldots, \mathbf{Y}_1^{(N)}, \mathbf{Y}_2^{(k)}, \ldots, \mathbf{Y}_2^{(N)}; \mathbf{X}_1^{(1)}, \ldots, \mathbf{X}_1^{(k)}, \mathbf{X}_2^{(1)}, \ldots, \mathbf{X}_2^{(k)} | \mathbf{X}_1^{(k+1)}, \ldots, \mathbf{X}_1^{(N)}, \mathbf{X}_2^{(k+1)}, \ldots, \mathbf{X}_2^{(N)}) + 2n\varepsilon_n$$

$$= H(\mathbf{Y}_1^{(k)}, \ldots, \mathbf{Y}_1^{(N)}, \mathbf{Y}_2^{(k)}, \ldots, \mathbf{Y}_2^{(N)} | \mathbf{X}_1^{(k+1)}, \ldots, \mathbf{X}_1^{(N)}, \mathbf{X}_2^{(k+1)}, \ldots, \mathbf{X}_2^{(N)}) + 2n\varepsilon_n$$

$$= H(\mathbf{Y}_1^{(k)}, \mathbf{Y}_2^{(k)} | \mathbf{X}_1^{(k+1)}, \ldots, \mathbf{X}_1^{(N)}, \mathbf{X}_2^{(k+1)}, \ldots, \mathbf{X}_2^{(N)}) + 2n\varepsilon_n$$

$$\leq H(\mathbf{Y}_1^{(k)}, \mathbf{Y}_2^{(k)})$$

$$\leq H(\mathbf{Y}_1^{(k)}, \mathbf{Y}_2^{(k)}, \mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(k-1)}, \mathbf{t}^{(k+1)}, \ldots, \mathbf{t}^{(N)})$$

$$\leq H(\mathbf{Y}_1^{(k)}) + H(\mathbf{t}^{(1)}) + \cdots + H(\mathbf{t}^{(k-1)}) + H(\mathbf{t}^{(k+1)}) + \cdots + H(\mathbf{t}^{(N)}) + H(\mathbf{Y}_2^{(k)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k}), \tag{6}$$

where we have used $\mathcal{T}_{<k} = (\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(k-1)})$ and $\mathcal{T}_{>k} = (\mathbf{t}^{(k+1)}, \ldots, \mathbf{t}^{(N)})$ for shortening the notations. Here, the main idea is that if the relay node A_k was able to decode the message sent by A_{k-1}, it could also decode the part of the second message W_2 transmitted by B_{k-1} over the cross link γ_k. However, there is no decodability requirement for this relay, and such argument does not work. However, since the sequence received at D_1 (which can be used to decode W_1) is a deterministic function of $\mathbf{Y}_1^{(k)}$, and $\mathbf{t}^{(k+1)}, \ldots, \mathbf{t}^{(N)}$, the relay node A_k would be able to decode W_1 by providing it with $\mathbf{t}^{(k+1)}, \ldots, \mathbf{t}^{(N)}$ as side information.

On the other hand, the message transmitted by A_{k-1} is not only a function of W_1, but also a function of all interferences received at the previous layers. Hence, having W_1 and the messages $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(k-1)}$, the relay A_k can decode $\mathbf{X}_1^{(k-1)}$, and therefore $\mathbf{t}^{(k)}$, the interfering message (partially) describing W_2. At the end, decoding $\mathbf{t}^{(k)}$ allows us to bound the innovative information about W_2 encoded in $\mathbf{Y}_2^{(k)}$.

This argument can be formally written as the following. We may start with

$$H(\mathbf{Y}_1^{(N)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k}) \leq H(\mathbf{Y}_1^{(N)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{>k})$$

$$= H(\mathbf{Y}_1^{(N)} - \mathbf{t}^{(N)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{>k})$$

$$= H(\mathbf{X}_1^{(N-1)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{>k})$$

$$\leq H(\mathbf{Y}_1^{(N-1)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{>k}).$$

Similarly by using induction we obtain

$$H(\mathbf{Y}_1^{(N)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{>k}) \leq H(\mathbf{Y}_1^{(k)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{>k}) = 0. \tag{7}$$

Combining (7) with Fano's inequality, $H(W_1 | \mathbf{Y}_1^{(N)}) \leq n\varepsilon_n$, we have

$$H(W_1 | \mathbf{Y}_1^{(k)}, \mathcal{T}_{>k}) \leq H(W_1, \mathbf{Y}_1^{(N)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{>k})$$

$$= H(\mathbf{Y}_1^{(N)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{>k}) + H(W_1 | \mathbf{Y}_1^{(N)}, \mathbf{Y}_1^{(k)}, \mathcal{T}_{>k})$$

$$\leq n\varepsilon_n, \tag{8}$$

which means W_1 can be decoded using $\mathbf{Y}_1^{(k)}$ and $\mathcal{T}_{>k}$. We also know that $\mathbf{X}_1^{(k)}$ the transmitted sequence by A_k is a function of $\mathbf{Y}_1^{(k-1)}$. Hence,

$$H(\mathbf{X}_1^{(k)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k}) \leq H(\mathbf{Y}_1^{(k-1)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k})$$

$$= H(\mathbf{Y}_1^{(k-1)} - \mathbf{t}^{(k-1)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k})$$

$$\leq H(\mathbf{X}_1^{(k-1)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k}).$$

Note that in the last equality we have used the fact that $\mathbf{Y}_1^{(k-1)} = \mathbf{t}^{(k-1)} + \mathbf{N}_{11}^{(k-1)} \mathbf{X}_1^{(k-1)}$ where $\mathbf{N}_{ij}^k = I_n \otimes N_{ij}^{(k)}$ is the expansion of the channel matrix over a block of length n. Using similar argument recursively, we get

$$H(\mathbf{X}_1^{(k)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k}) \leq H(\mathbf{X}_1^{(1)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k})$$

$$\leq H(W_1 | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k}) \leq n\varepsilon. \tag{9}$$

where the last inequality is due to (8). Hence,

$$H(\mathbf{t}^{(k)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k}) \leq H(\mathbf{t}^{(k)}, \mathbf{X}_1^{(k)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k})$$

$$= H(\mathbf{X}_1^{(k)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k}) + H(\mathbf{t}^{(k)} | \mathbf{Y}_1^{(k)}, \mathbf{X}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k})$$

$$= H(\mathbf{X}_1^{(k)} | \mathbf{Y}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k}) + H(\mathbf{Y}_1^{(k)} - \mathbf{N}_{11}^{(k)} \mathbf{X}_1^{(k)} | \mathbf{Y}_1^{(k)}, \mathbf{X}_1^{(k)}, \mathcal{T}_{<k}, \mathcal{T}_{>k})$$
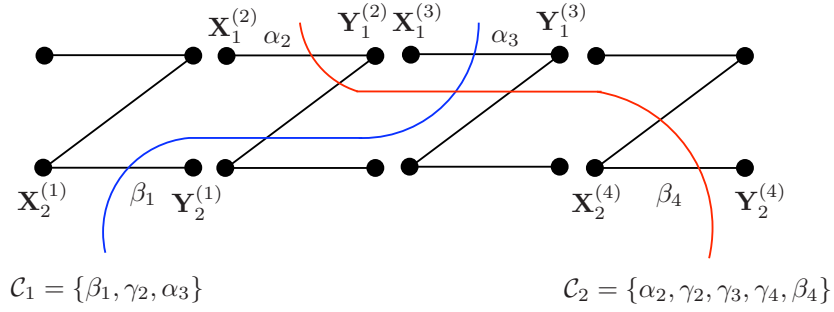
$$\leq n\varepsilon_n. \tag{10}$$

Fig. 3: The other cuts

Finally,

$$H(\mathbf{Y}_2^{(k)}|\mathbf{Y}_1^{(k)},\mathcal{T}_{<k},\mathcal{T}_{>k}) \leq H(\mathbf{Y}_2^{(k)},\mathbf{t}^{(k)}|\mathbf{Y}_1^{(k)},\mathcal{T}_{<k},\mathcal{T}_{>k})$$
$$= H(\mathbf{Y}_2^{(k)}|\mathbf{Y}_1^{(k)},\mathcal{T}_{<k},\mathbf{t}^{(k)},\mathcal{T}_{>k}) + H(\mathbf{t}^{(k)}|\mathbf{Y}_1^{(k)},\mathcal{T}_{<k},\mathcal{T}_{>k})$$
$$\leq H(\mathbf{Y}_2^{(k)}|\mathbf{t}^{(k)}) + n\varepsilon_n$$
$$\leq n(\beta_k - \gamma_k)^+ + n\varepsilon_n. \tag{11}$$

Replacing (11) in (6) we get

$$n(r_1 + r_2) \leq n\max(\alpha_k, \gamma_k) + \sum_{i \neq k} n\gamma_i + n(\beta_k - \gamma_k)^+ + n\varepsilon'$$

which yields in (3) after some simplifications. ∎

Remark: Note the bounds corresponding to the other cuts (e.g., bounds shown in Fig. 3) are implied by (1)-(3), and do not affect the capacity region.

IV. ACHIEVABILITY

The goal of this section is to show that any rate pair $(r_1, r_2) \in \mathcal{R}^U$ is achievable. We will show that such rate pair is achievable using only linear operations, hence, the signal at any relay or destination node would be a linear combination of the input bits of $W_1 = [b_1(1), b_1(2), \ldots, b_1(r_1)]^T$ and $W_2 = [b_2(1), b_2(2), \ldots, b_2(r_2)]^T$, the binary representations of the input messages. It is clear that destinations can decode if and only if the nodes D_1 (D_2) can obtain exactly r_1 (r_2) linearly independent equations which only involve the unknown bits of W_1 (W_2) from the set of received equations. In order to show achievability, we introduce a new interference management scheme we term *interference neutralization*. Here the interfering signal is eliminated when mixed over the air, without necessarily decoding it. This was also used in the ZZ network in [5], and is crucial in this work as well.

We focus on a special class of encoding schemes, where the relay nodes B_k's, first decode the corresponding message W_2, and then encode it again and send *exactly* r_2 linearly independent equations describing W_2. The encoding scheme at B_k can be chosen such that the message received at A_{k+1} gets more interference, or (a part of) its interference get neutralized. We choose r_2 nodes among the top β_k available nodes for transmission, opportunistically, such that the message can be decoded at B_{k+1} and the *desired* interfering situation happens at A_{k+1}.

Also the relay nodes A_k transmit *exactly* r_1 equations, where some of them may only involve bits from W_1 and the others involve bits of both W_1 and W_2. However, the equations are chosen such that the induced equations on each of W_1 and W_2 are linearly independent, *i.e.*, it transmits a vector

$$X_1^{(k)} = \left[\begin{array}{c|c} U & Q \\ \hline V & 0 \end{array}\right]\left[\begin{array}{c} W_1 \\ \hline W_2 \end{array}\right] \tag{12}$$

where the matrices Q and $\begin{bmatrix} U^T & V^T \end{bmatrix}^T$ are full-rank.

Transmission of W_2 from S_2 to D_2 through B_k needs to only send linearly r_2 independent equations since no interference can affect the message. We call a linear equation *pure* if it only involves the bits of W_1 as the unknown variables. Let p_k denote the number of linearly independent pure equations received at the relay node A_k for $k = 1, \ldots, N-1$. Similarly $p_0 = r_1$ and p_N denote the number of pure equations at S_1 and D_1, respectively. In fact A_k has p_k pure equations and $r_1 - p_k$ *mixed* equations involving the unknown bits of both W_1 and W_2. It may also have some equations which only involve unknown bits of W_2. Such equations can be used for interference suppression as well as interference neutralization.

The value of p_k depends on both the number of pure equation in the previous layer, p_{k-1}, as well the encoding strategy used at the relay nodes A_{k-1} and B_{k-1}. Therefore, even for a fixed p_{k-1}, different values for p_k can be obtained using different

coding strategies. In the following we will study the evolution of the number of pure equations and show that if $(r_1, r_2) \in \mathcal{R}^U$ then there exist coding strategies used at the relays such that one can obtain $p_N = r_1$ pure equations at D_1. It is clear that having r_1 linearly independent equations, D_1 can reconstruct the bits of W_1.

Define $\mathcal{P}_k$ as the set of all possible number of pure equations at the k-th layer of the network. We define $M_k = \max \mathcal{P}_k$ and $m_k = \min \mathcal{P}_k$ as the largest and smallest elements of $\mathcal{P}_k$, respectively. It is clear that $M_k \leq r_1$ since the number of linearly independent equations cannot exceed the number of variables. Also note that we never send more than r_2 equations from B_{k-1}, and therefore the number of mixed equations cannot exceed r_2. Therefore, we have $r_1 - p_k \geq r_2$ and therefore $m_k \geq (r_1 - r_2)^+$.

The following example illustrates the concept of the number of pure equations and role of the encoding scheme.

Example 1: Consider Z-chain network which is used for transmission at rate pair $(r_1, r_2) = (5, 3)$. Assume that $q = 5$, and the k-th layer of the network has parameters $\alpha_k = 5$, $\beta_k = 4$, and $\gamma_k = 2$, as shown in Fig. 4. The relay node B_{k-1} receives r_2 linearly independent equations involving bits of W_2, and therefore has access to $Y_2^{(k-1)}$. Assume that the relay node A_{k-1} has received $Y_2^{(k-1)}$ as

$$Y_1^{(k-1)} = \begin{bmatrix} b_1(1) + b_1(3) + b_1(4) \\ b_1(1) + b_1(2) \\ b_1(2) + b_2(1) \\ b_1(4) + b_1(5) + b_2(2) + b_2(3) \\ b_1(5) + b_2(2) \end{bmatrix}, \qquad Y_2^{(k-1)} = \begin{bmatrix} b_2(1) \\ b_2(2) \\ b_2(3) \\ 0 \\ 0 \end{bmatrix},$$
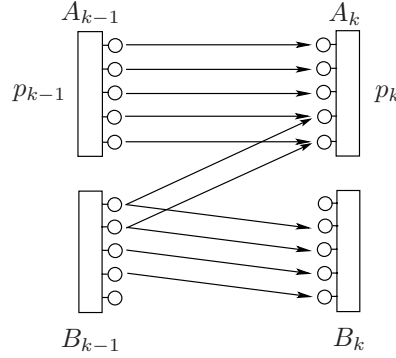
which means $p_{k-1} = 2$ is achieved.

Fig. 4: A single layer of the Z chain in Example 1.

The following different coding schemes used at A_{k-1} and B_{k-1} result in different values for p_k.

- If the relay nodes encode their message and send the vectors

$$X_1^{(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} Y_1^{(k-1)}, \qquad X_2^{(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} Y_2^{(k-1)},$$

  then the received signals at the next layer would be $Y_1^{(k)} = [b_1(1) + b_1(3) + b_1(4), b_1(1) + b_1(2), b_1(2) + b_2(1), b_1(4) + b_2(1) + b_2(3), b_1(5) + b_2(2)]^T$ and $Y_2^{(k)} = [0, b_2(1), 0, b_2(2), b_3(2)]^T$. It is clear that p_k, the number of linearly independent equations involving only the bits of W_1, equals 2.

- If the first relay simply forward the vector $X_1^{(k)} = y_1^{(k-1)}$, and the second sends an encoded message

$$X_2^{(k)} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

  the relay nodes at the next layer will receive vectors $Y_1^{(k)} = [b_1(1) + b_1(3) + b_1(4), b_1(1) + b_1(2), b_1(2) + b_2(1), b_1(4) + b_1(5), b_1(5) + b_2(1) + b_2(2)]^T$ and $Y_2^{(k)} = [0, b_2(2) + b_2(3), b_2(1), b_2(2), 0]^T$. Therefore we have $p_k = 3$.

- By re-encoding the received sequence at A_{k-1} and B_{k-1} and sending $X_1^{(k)} = Y_1^{(k-1)}$ and

$$X_2^{(k)} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} Y_2^{(k-1)},$$

the relay nodes A_k and B_k will receive $Y_1^{(k)} = [b_1(1)+b_1(3)+b_1(4), b_1(1)+b_1(2), b_1(2)+b_2(1), b_1(4)+b_1(5), b_1(5)+b_2(1)]^T$ and $Y_2^{(k)} = [0, b_2(2)+b_2(3), b_2(2), b_2(1)+b_2(2)+b_2(3), 0]^T$, which corresponds to $p_k = 4$.

It is clear through the above specific encoding strategies that if $3 \in \mathcal{P}_{k-1}$ then $\{2,3,4\} \subseteq \mathcal{P}_k$ for these special channel parameters. However it is impossible to obtain $p_k = 5$ from $p_{k-1} = 2$, since there among three mixed equation available at A_{k-1}, at most two of them can become pure by interference neutralization from the cross link. ∎

In the following subsection we will investigate how the p_k changes from one layer of the network to the next one.

A. Evolution of the number of pure equations

Assume we are at the encoding part of the k-th layer of the network with parameters $(\alpha_k, \beta_k, \gamma_k)$, and have p_{k-1} pure and $r_1 - p_{k-1}$ mixed equations. We need to find m_k and M_k that can be achieved for the next layer.

Lemma 1: Given p_{k-1} linearly independent equations at the relay node A_{k-1}, the minimum number of pure equations achievable at A_k is

$$\min p_{k|p_{k-1}} = \max\{0, r_1 - r_2, p_{k-1} - \gamma_k\}.$$

Proof: Minimizing the number of pure equations is equivalent to maximizing the number of mixed equations. A mixed equations at A_k can be obtained by either receiving a mixed equations from A_{k-1} whose interference is not neutralized by the new interference, or combination of a pure equation from A_{k-1} and an interference from B_{k-1}. Note that we have $r_1 - p_{k-1}$ mixed equations, and among the p_{k-1} pure equations at most γ_k of them can become mixed in the next layer. However, since W_2 has only r_2 bits, at most r_2 equations can be affected by the interference. Therefore, the maximum number of mixed equations would be

$$\max(r_1 - p_k) = \min\left\{r_1 - p_{k-1} + \min\{p_{k-1}, \gamma_k\}, r_2\right\}$$
$$= \min\{r_1, r_1 - p_{k-1} + \gamma_k . r_2\}.$$

Hence $\min p_{k|p_{k-1}} = \max\{0, r_1 - r_2, p_{k-1} - \gamma_k\}$. ∎

Lemma 2: If the relay node A_{k-1} sends p_{k-1} pure equations, then maximum achievable number of pure equations in the next layer's relay, A_k is

$$\max p_{k|p_{k-1}} = \min\{r_1, \Psi_k - r_2 + r_1 - \gamma_k - p_{k-1}, p_{k-1} + \gamma_k\}.$$

Proof: A pure equation at the next layer A_k can be obtained by either receiving a pure equation from A_{k-1} at a sub-node which is not affected by the message from B_{k-1}, or a mixed message from A_{k-1} whose interference is neutralized[3] by another equation received from B_{k-1}. We denote the number of these two sets of equations by $p_k(P)$ and $p_k(N)$, respectively.

We first enumerate the first kind of such messages. The maximum number of sub-nodes in A_k which can receive message from A_{k-1} and have not occupied by the signal received from B_{k-1} can be found as illustrated in Fig. 5. The relay node B_{k-1} chooses r_2 sub-nodes among its top β_k sub-nodes to transmit its message to B_k, and at least $r_2 - (\beta_k - \gamma_k)^+$ of them would be among the top γ_k sub-nodes, whose message will be also observed by A_k. Among them, at most $(\gamma_k - \alpha_k)$ are out of the range of A_{k-1}, but the remaining will have overlap with the sub-nodes in range of A_{k-1}. Therefore, at least $[r_2 - (\beta_k - \gamma_k)^+ - (\gamma_k - \alpha_k)^+]^+$ sub-nodes among the α_k sub-nodes of A_k get interfered. Hence, the maximum number of pure equations of the first kind would be the minimum of the number of available pure equations, and the number of non-occupied sub-nodes, which equals to

$$p_k(P) = \min\left\{p_{k-1}, \alpha_k - [r_2 - (\beta_k - \gamma_k)^+ - (\gamma_k - \alpha_k)^+]^+\right\}$$
$$= \min\left\{p_{k-1}, \alpha_k - [r_2 - \max(\alpha_k, \gamma_k) + \alpha_k - \max(\beta_k, \gamma_k) + \gamma_k]^+\right\}$$
$$= \min\left\{p_{k-1}, \alpha_k - [r_2 + \alpha_k + \gamma_k - \Psi_k]^+\right\} \tag{13}$$

On the other hand, we have $r_1 - p_{k-1}$ mixed equations, where at most γ_k of them can be neutralized by the message from B_{k-1}. Thus, the maximum number of the second class of pure equations at the k-th layer would be

$$p_k(N) = \min\{r_1 - p_{k-1}, \gamma_k\}. \tag{14}$$

---

[3]This is done by pre-coding at B_{k-1}. Since it can decode the whole vector $\mathbf{X}_2$, it can encode it again such that a desired number of mixed equations get neutralized.
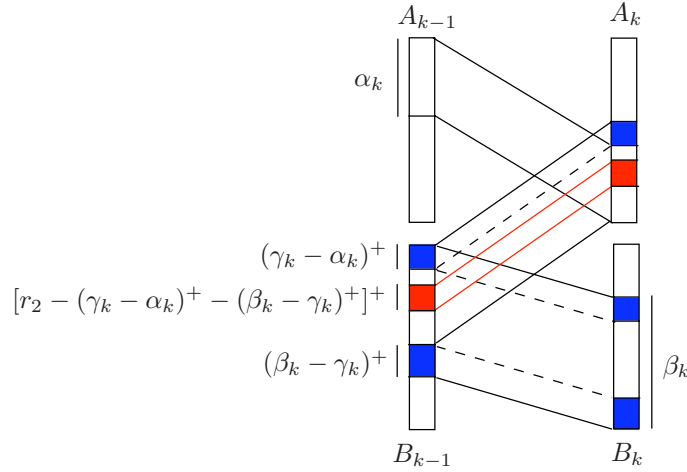
Fig. 5: The maximum number of nodes in A_k which are not affected by interference.

By adding up (13) and (14) we have

$$\max p_{k|p_{k-1}} = \min\left\{p_{k-1}, \alpha_k - [r_2 + \alpha_k + \gamma_k - \Psi_k]^+\right\} + \min\{r_1 - p_{k-1}, \gamma_k\}$$

$$\overset{(a)}{=} \min\{p_{k-1}, \alpha_k - [r_2 + \alpha_k + \gamma_k - \Psi_k]\} + \min\{r_1 - p_{k-1}, \gamma_k\}$$

$$= \min\{r_1, \Psi_k - r_2 + r_1 - \gamma_k - p_{k-1}, p_{k-1} + \gamma_k, \Psi_k - r_2\}$$

$$\overset{(b)}{=} \min\{r_1, \Psi_k - r_2 + r_1 - \gamma_k - p_{k-1}, p_{k-1} + \gamma_k\} \tag{15}$$

where in (a) we have used the fact that $p_{k-1} \leq r_1 \leq \alpha_k$, and (b) is due to the inequality $\Psi_k \geq \alpha_k + \beta_k \geq r_1 + r_2$ which shows that dropping the last term does not change the minimum.

Note that this value is always achievable by choosing the transmitting nodes of B_{k-1} and the equations they send properly, such that the required number of nodes A_k do not affected by interference and a specific number of the rest get neutralized. ∎

The following theorem summarizes the discussion of this subsection.

Theorem 2: The set of all achievable numbers of linearly independent pure equations at the k-th layer of the network is

$$\mathcal{P}_k = \{p \in \mathbb{Z}^+ : m_k \leq p \leq M_k\} \tag{16}$$

where

$$m_k = \max\{0, r_1 - r_2, r_1 - \Gamma_k\}, \qquad k = 1, 2, \ldots, N, \tag{17}$$

and M_k is obtained using the recurrence relations

$$M_k = \min\{r_1, \Psi_k + \Gamma_k - 2\gamma_k - r_2, \gamma_k + M_{k-1}\}, \tag{18}$$

for $k = 1, 2, \ldots, N$ and with the initial condition $M_0 = r_1$.

Proof: The Lemmas 1 and 2 determine the minimum and maximum achievable p_k provided that the relay node A_{k-1} sends p_{k-1} pure equations. A similar argument shows that the extreme values in both lemmas are in fact achievable. Not surprisingly, it can shown that if $p_k = u_1$ and $p_k = u_2$ are achievable for $u_1 < u_2$, then any integer $u \in [u_1, u_2]$ is also achievable.

The minimum and maximum values obtained in Lemmas 1 and 2 depend on p_{k-1}. However depending on the required p_k at the k-layer, one can choose any $p_{k-1} \in \mathcal{P}_{k-1}$ for encoding at A_{k-1}. We will prove (17) using induction over k. For $k = 1$, the claim is just rewriting Lemma 1 since $m_0 = r_1$. Assuming (17) for $k - 1$, we have

$$m_k = \min_{p_{k-1} \in \mathcal{P}_{k-1}} \max\{0, r_1 - r_2, p_{k-1} - \gamma_k\}$$

$$= \max\{0, r_1 - r_2, m_{k-1} - \gamma_k\}$$

$$= \max\{0, r_1 - r_2, \max\{0, r_1 - r_2, r_1 - \Gamma_{k-1}\} - \gamma_k\}$$

$$= \max\{0, r_1 - r_2, r_1 - \Gamma_k\}.$$

Similarly, using (15) M_k can be obtained as the maximum achievable p_k is the maximum value that the RHS of (15) can get for different value of p_{k-1}.

$$M_k = \max_{p_{k-1} \in \mathcal{P}_{k-1}} \min\{r_1, r_1 + \Psi_k - r_2 - \gamma_k - p_{k-1}, \gamma_k + p_{k-1}\}$$

$$= \min\{r_1, r_1 + \Psi_k - r_2 - \gamma_k - m_{k-1}, \gamma_k + M_{k-1}\}. \tag{19}$$

Replacing m_k from (17), we get

$$r_1 + \Psi_k - r_2 - \gamma_k - m_{k-1} = r_1 + \Psi_k - r_2 - \gamma_k - \max\{0, r_1 - r_2, r_1 - \Gamma_{k-1}\}$$

$$= r_1 + \Psi_k - r_2 - \gamma_k + \min\{0, r_2 - r_1, \Gamma_{k-1} - r_1\}$$

$$= \min\{r_1 + \Psi_k - r_2 - \gamma_k, \Psi_k - \gamma_k, \Psi_k + \Gamma_k - 2\gamma_k - r_2\}$$

where the first two terms are not less that r_1. and do not affect the minimization in (19). Therefore,

$$M_k = \min\{r_1, \Psi_k + \Gamma_k - 2\gamma_k - r_2, \gamma_k + M_{k-1}\} \tag{20}$$

However, solving the last recursive relation and evaluating M_k is not easy and we leave it as an optimization expression. ∎

B. An achievable path for $p_N = r_1$

Next we show implicitly that for (r_1, r_2) satisfying the outer bound, $r_1 \in \mathcal{P}_N$, using the recursive form of the evolution of pure equations obtained in the last subsection. Therefore, this shows that rate (r_1, r_2) is indeed achievable.

Lemma 3: Let

$$p_k^* = \min\left(r_1, \min_{1 \le \ell \le k}\{\Psi_\ell + \Gamma_k - 2\gamma_\ell - r_2\}\right). \tag{21}$$

Then $p_k^* \in \mathcal{P}_k$ for $k = 1, 2, \ldots, N$.

Proof: In order to prove the lemma, we need to show $m_k \le p_k^* \le M_k$. The first inequality is straight forward and shown as follows.

$$p_k^* - m_k = \min\left(r_1, \min_{1 \le \ell \le k}\{\Psi_\ell + \Gamma_k - 2\gamma_\ell - r_2\}\right) - \max(0, r_1 - r_2, r_1 - \Gamma_k)$$

$$= \min\left(r_1, \min_{1 \le \ell \le k}\{\Psi_\ell + \Gamma_k - 2\gamma_\ell - r_2\}\right) + \min(0, r_2 - r_1, \Gamma_k - r_1)$$

$$= \min\left\{r_1, r_2, \Gamma_k, \min_{1 \le \ell \le k}\{\Psi_\ell + \Gamma_k - 2\gamma_\ell - r_2\}, \min_{1 \le \ell \le k}\{\Psi_\ell + \Gamma_k - 2\gamma_\ell - r_1\}, \min_{1 \le \ell \le k}\{\Psi_\ell + 2\Gamma_k - 2\gamma_\ell - r_1 - r_2\}\right\}$$

$$\ge 0$$

where the last inequality follows from the facts that $r_1 \le \alpha_\ell$ and $r_2 \le \beta_\ell$ for $\ell = 1, \ldots, N$.

To show the second inequality we use induction over k, namely, we show that $p_k^* \le M_k$ provided that $p_{k-1}^* \le M_{k-1}$. For $k = 1$, is claim is trivial by just comparing p_1^* and M_1 in (18). Assuming $p_{k-1}^* \le M_{k-1}$ and using (18), we have

$$M_k \ge \min\{r_1, \Psi_k + \Gamma_k - 2\gamma_k - r_2, \gamma_k + p_{k-1}^*\}$$

$$= \min\left\{r_1, \Psi_k + \Gamma_k - 2\gamma_k - r_2, \gamma_k + \min\left(r_1, \min_{1 \le \ell \le k-1}\{\Psi_\ell + \Gamma_{k-1} - 2\gamma_\ell - r_2\}\right)\right\}$$

$$= \min\left\{r_1, \Psi_k + \Gamma_k - 2\gamma_k - r_2, \min\left(r_1 + \gamma_k, \min_{1 \le \ell \le k-1}\{\Psi_\ell + \Gamma_k - 2\gamma_\ell - r_2\}\right)\right\}$$

$$= \min\left\{r_1, \Psi_k + \Gamma_k - 2\gamma_k - r_2, \min_{1 \le \ell \le k-1}\{\Psi_\ell + \Gamma_k - 2\gamma_\ell - r_2\}\right\}$$

$$= \min\left\{r_1, \min_{1 \le \ell \le k}\{\Psi_\ell + \Gamma_k - 2\gamma_\ell - r_2\}\right\}$$

$$= p_k^*.$$

which shows that p_k^* does not exceed M_k. Hence $p_k^* \in \mathcal{P}_k$. ∎

Lemma 4:

$$p_N^* = r_1. \tag{22}$$

Proof: Note that by definition

$$p_N^* = \min\left(r_1, \min_{1 \leq \ell \leq N} \{\Psi_\ell - 2\gamma_\ell + \Gamma_N - r_2\}\right).$$

In order to prove the lemma one has to show that $\Psi_\ell - 2\gamma_\ell + \Gamma_N - r_2 \geq r_1$ for $1 \leq \ell \leq N$. But such inequality always holds since (r_1, r_2) satisfy (3). ∎

Lemmas 3 and 4 together show that $r_1 \in \mathcal{P}_N$, *i.e.*, there exist encoding schemes used at the relays which can provide r_1 linearly independent pure equations for $A_N = D_1$. Then it is clear that D_1 can use such equations to solve for the bits of W_1.

V. CONCLUSION

We studied the deterministic Z-chain network and fully characterized the capacity region of the network. We believe that the analysis technique of bounding the number of pure signals (equations) would be useful for many other interference network problems.

REFERENCES

[1] T. S. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Trans. Inform. Theory*, vol. 27, no. 1, pp. 49–60, Jan. 1981.
[2] R. Etkin, D. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit," submitted to the *IEEE Trans. Inform. Theory*, Feb. 2007.
[3] S. Avestimehr, S. N. Diggavi, and D. N. C. Tse, "A Deterministic approach to wireless relay networks," *Proceedings of Allerton Conference on Communication, Control, and Computing*, Illinois, September 2007.
[4] S. Avestimehr, S. N. Diggavi, and D. N. C. Tse, "Approximate capacity of Gaussian relay networks," *Proceeding of IEEE International Symposium on Information Theory*, Toronto, July 2008.
[5] S. Mohajer, S. N. Diggavi, C. Fragouli, and D. N. C. Tse, " Transmission techniques for relay-interference networks," *Proceedings of Allerton Conference on Communication, Control, and Computing*, Illinois, September 2008.