An Alternative to the Youla Parameterization for H_∞ Controller Design

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***Abstract*— All $\mathscr{H}_\infty$ controllers of a SISO LTI system are parameterized thanks to the relation between the Bounded Real Lemma and the Positive Real Lemma. This new parameterization shares the features of Youla parameterization, namely the convexity of $\mathscr{H}_\infty$ norm constraints for the closed-loop transfer functions. However, by contrast to Youla parameterization, it can deal with any controller order and any controller structure, e.g. PID controllers. Moreover, it can be easily used for systems with polytopic uncertainty. For polytopic systems, the proposed parameterization provides a convex inner approximation of the set of all $\mathscr{H}_\infty$ controllers, which can be enlarged by increasing the controller order. The effectiveness of the proposed method is shown via simulation results.**

I. Introduction

The Youla parameterization [1] is probably the most well-known controller parameterization, which parameterizes all stabilizing controllers of a system, over an infinite dimensional space. The main advantage of this parameterization is that all closed-loop sensitivity functions are affine w.r.t. the so-called Q parameter and hence it can be employed for $\mathscr{H}_\infty$ controller design in a convex optimization problem. The Youla parameterization has two major disadvantages. First, it cannot deal with fixed-order controllers or enforce a prescribed controller structure such as PID. Second, it depends on system parameters and therefore it cannot be used for systems with parametric uncertainty such as polytopic systems.

Polytopic uncertainty is one of the most general ways of capturing physical parameter uncertainty, multi-model systems and the well-known interval systems, and has attracted many robust controller designers recently. In [2] a convex parameterization of fixed-order controllers for polytopic systems is given based on the polynomial positivity. It gives a stabilizing controller (a feasible point), which closely depends on a so-called central polynomial. This method is used for convex parameterization of $\mathscr{H}_\infty$ controllers in [3], however, again the solution depends on the choice of the central polynomial. The effect of this choice on the convex set of stabilizing fixed-order controllers is studied in [4]. In a similar approach, it is shown that the dependence on the central polynomial can be relaxed by increasing the controller order [5]. This parameterization covers all stabilizing controllers in an infinite dimensional space, whose inner approximation for fixed-order controllers coincides with the results of [2] and [4]. The advantage of this parameterization w.r.t. Youla parameterization is that it parameterizes all stabilizing controllers of a polytopic system and, in addition, it can deal with any controller structure and a prescribed order. However, in contrary to Youla parameterization, it does not lead to affine closed-loop transfer functions w.r.t. the controller parameters.

An alternative to Youla parameterization of all $\mathscr{H}_\infty$ controllers is given in this paper. The main advantage of this new parameterization is that it can enforce any controller structure and order. Moreover, similar to Youla parameterization and in contrary to the parameterization of [5], all closed-loop transfer functions become affine w.r.t. the variables, which enables a convex parameterization of all $\mathscr{H}_\infty$ controllers. Furthermore, it can be easily employed for polytopic systems to give a convex inner approximation of all $\mathscr{H}_\infty$ controllers. The problem considered in this paper is very similar to that of [3], where a convex parameterization of fixed-order $\mathscr{H}_\infty$ controllers is given using the properties of positive polynomials. However, in this paper a different approach based on the strict positive realness of two transfer functions with the same Lyapunov matrix in the inequality of the Kalman-Yakubovic-Popov lemma is employed. Moreover, it is shown that by increasing the controller order the dependence on the central polynomial can be relaxed. In addition, for continuous-time systems it is shown that it is possible to minimize the desired $\mathscr{H}_\infty$ norm, whereas in [3], the iterative bisection algorithm should be used to find the minimum value of the desired $\mathscr{H}_\infty$ norm.

The rest of the paper is organized as follows. Some notations and preliminaries are briefly recalled in Section II. Main results are presented in Section III, where the new convex constraints that satisfy $\mathscr{H}_\infty$ performance specifications are introduced. The simulation results in Section IV illustrate the effectiveness of the proposed synthesis method and concluding remarks are given in Section V. Finally, it is shown in Appendix that for continuous-time systems, it is possible to minimize the desired $\mathscr{H}_\infty$ norm in a convex optimization problem instead of using the iterative bisection algorithm.

II. Preliminaries

The goal is to give a convex parameterization of all $\mathscr{H}_\infty$ controllers for a known system and then to extend the method for systems with polytopic uncertainty.

Consider a SISO LTI plant represented by a finite order rational transfer function G in discrete- or in continuous-time. Assume that N and M are the coprime factors of G such that

$$G = NM^{-1}, \qquad N, M \in \mathscr{RH}_\infty \tag{1}$$

where $\mathscr{RH}_\infty$ is the set of proper stable rational transfer functions with bounded infinity norm. It is shown in [5] that the set of all stabilizing controllers of (1) is given by :

$$\mathscr{K}_S : \{K = XY^{-1} \,|\, MY + NX \in \mathscr{S}\} \tag{2}$$

where $X, Y \in \mathscr{RH}_\infty$ and $\mathscr{S}$ is the convex set of all Strictly Positive Real (SPR) transfer functions. Using the following lemma, this parameterization can be represented as LMIs :

Lemma 1: (the KYP lemma for discrete-time systems [6]) A biproper transfer function $H(z) = C\,(zI - A)^{-1}B + D$ is SPR

(ESPR in [7]) if and only if there exists a matrix $P = P^T > 0$ such that :

$$\begin{bmatrix} A^TPA - P & A^TPB - C^T \\ B^TPA - C & B^TPB - D - D^T \end{bmatrix} < 0 \quad (3)$$

It is evident that A and B (with a controllable canonical realization) are only related to the denominator of $H(z)$ and are therefore fixed in case $H = MY + NX$. The plant parameters and the controller parameters (the optimization variables) appear linearly in C and D. Hence, the above inequality becomes an LMI because the variables P,C and D appear affinely in it.

The advantage of this parameterization is that it can be easily applied to polytopic and multi-model systems because, by contrast to the Youla parameterization, the controller parameterization does not depend on the system parameters.

Consider a polytopic system with q vertices such that the i-th vertex consists of the parameters of the model $G_i = N_i M_i^{-1}$, where N_i and $M_i \in \mathscr{RH}_\infty$ are the coprime factors of G_i. The set of all models in this polytopic system can be represented by:

$$\mathscr{G} : \{G = NM^{-1} \,|\, N = \sum_{i=1}^{q} \lambda_i N_i, M = \sum_{i=1}^{q} \lambda_i M_i\} \quad (4)$$

where $\lambda_i \geq 0$ and $\sum_{i=1}^{q} \lambda_i = 1$. The set of all stabilizing controllers for this polytopic system is given by [5] :

$$\mathscr{K}_p : \{K = XY^{-1} \,|\, M_iY + N_iX \in \mathscr{S}, \quad i = 1, \ldots, q\} \quad (5)$$

where $X, Y \in \mathscr{RH}_\infty$. The main drawback of this approach is that the norm constraints on sensitivity functions are not convex with respect to X and Y, for example in the following sensitivity function:

$$S = (1 + KG)^{-1} = \frac{MY}{MY + NX}$$

. A solution to this problem is proposed in [5] by putting the constraints on numerator and denominator of S separately. This leads to the following optimization problem for a polytopic system :

$$\begin{aligned} &\text{Minimize } \max_i \gamma_i \\ &\text{Subject to:} \\ &M_iY + N_iX \in \mathscr{S} && \text{for } i = 1, \ldots, q \\ &\|M_iY + N_iX - 1\|_\infty < \gamma_i && \text{for } i = 1, \ldots, q \\ &\|W_1M_iY\| < 1 - \gamma_i && \text{for } i = 1, \ldots, q \end{aligned} \quad (6)$$

where the second and the third constraints together present convexified approximation of the $\mathscr{H}_\infty$ constraints $\|W_1S_i\| < 1$ for $i = 1, \ldots, q$.

However, this approach is not able to minimize the $\mathscr{H}_\infty$ performance index $\|W_1S_i\|_\infty$, i.e. we can just ensure that $\|W_1S_i\|_\infty < 1, i = 1, \ldots, q$. Furthermore, since the desired $\mathscr{H}_\infty$ norm is not related to the cost function, minimizing this cost function generally leads to a higher $\|W_1S_i\|_\infty$. To overcome these drawbacks, a new convex representation of the desired $\mathscr{H}_\infty$ norm is proposed in the next section.

III. Main Result

To ensure the robust performance, we want to parameterize all stabilizing controllers that satisfy some $\mathscr{H}_\infty$ norm bounds on some weighted transfer functions of the closed-loop system. However, for the simplicity reasons, we demonstrate the method with $\mathscr{H}_\infty$ norm bound on only one sensitivity function. Thus, without loss of generality, suppose that it is desired to have :

$$\|W_1S\|_\infty = \left\| \frac{W_1MY}{MY + NX} \right\|_\infty < \gamma \quad (7)$$

for a given γ. It is well known that an infinity norm constraint could be presented as LMIs via Bounded Real Lemma, if the denominator of its argument is fixed [8]. However, in this case, the controller parameters appear both in numerator and denominator of W_1S, which results in a Bilinear Matrix Inequality (BMI) problem. In order to convexify this performance constraint, the relation between the Bounded Real Lemma and the Positive Real Lemma is employed. It is well known that (7) is equivalent to the SPRness of [9] :

$$\frac{(MY + NX) - \gamma^{-1}W_1MY}{(MY + NX) + \gamma^{-1}W_1MY} \quad (8)$$

Therefore, the set of all controllers that result in a closed-loop system with $\|W_1S\|_\infty < \gamma$ for a system G defined in (1), is given by :

$$\mathscr{K}_\infty : \{K = XY^{-1} \,| \\ \frac{(MY + NX) - \gamma^{-1}W_1MY}{(MY + NX) + \gamma^{-1}W_1MY} \in \mathscr{S}\} \quad (9)$$

where $X, Y \in \mathscr{RH}_\infty$. Using the KYP lemma (Lemma 1 for discrete-time systems), the SPRness of a transfer function with fixed denominator can be represented via LMIs. However, in (9), both numerator and denominator contain optimization variables and hence, the set is not convex in this form.

In the sequel, (9) is represented via LMIs. Moreover, it is shown that the resulting LMIs give the complete set of all stabilizing $\mathscr{H}_\infty$ controllers.

A. LMI representation

The following definitions and lemmas are required to proceed.

Definition 1: A matrix A is called the state space matrix of a monic polynomial p, if A is the controllable canonical state space matrix of the transfer function $1/p$.

Definition 2: Consider two equal-order monic polynomials p_1 and p_2 and their state space matrices A_1 and A_2, respectively. Then, p_1 and p_2 (also A_1 and A_2) are called Common Lyapunov stable, or CL-stable, if A_1 and A_2 satisfy a Lyapunov inequality with the same Lyapunov matrix P, namely for discrete-time systems $\exists\ P = P^T > 0$ such that :

$$A_1^TPA_1 - P < 0 \quad \text{and} \quad A_2^TPA_2 - P < 0$$

Lemma 2: [7] A transfer function H is SPR if and only if its numerator and denominator are CL-stable.

Definition 3: Two equal-order SPR transfer functions H_1 and H_2 with controllable canonical state space realizations

(A_1,B_1,C_1,D_1) and (A_2,B_2,C_2,D_2) are called Common Lyapunov Strictly Positive Real, or CL-SPR, if both satisfy the inequality of the KYP lemma (Inequality (3) for discrete-time systems) with the same Lyapunov matrix P.

Remark : A very simple consequence of the above definition is that an SPR transfer function H_1 is CL-SPR with all positive fixed transfer functions such as $H_2 = 1$.

Lemma 3: An SPR transfer function H and its inverse H^{-1} are CL-SPR.

Proof: Using Schur complement on (3) for both H and its inverse, the proof is obtained easily. Furthermore, the proof is similar for continuous-time systems. ■

Since the inequality of the KYP lemma contains the Lyapunov stability constraint in its first block, Lemma 3 covers Lemma 2.

The following corollary shows how a CL-SPR constraint results in a CL-stability constraint :

Corollary 1: If two transfer functions H_1 and H_2 are CL-SPR then all of their numerator and denominator polynomials are CL-stable.

Proof: Suppose that there exists $P = P^T > 0$ satisfying the inequality of the KYP lemma for both transfer functions H_1 and H_2. Since the first block of this LMI is the same as the Lyapunov stability criterion, the denominators of these two transfer functions are CL-stable. Furthermore, according to Lemma 3, the same matrix P satisfies the LMI of the KYP lemma for H_1^{-1} and H_2^{-1}, which means that the same P satisfies Lyapunov stability criterion for the numerators of H_1 and H_2. Hence, all of the four polynomials are CL-stable with the same matrix P. ■

Remark : Any $P = P^T > 0$ that reveals the CL-stability of two polynomials, does not necessarily satisfy the LMI of the KYP lemma for their SPR ratio. Hence, the above lemma could be proved just in the mentioned direction.

According to Lemma 2, we need to have CL-stability between numerator and denominator polynomials of (9) to prove its SPRness. However, the SPRness constraint of (9) becomes a non-convex inequality due to the existence of variables in its denominator, which causes multiplication of variables in the first block of the inequality of the KYP lemma. Taking into account Corollary 1, it is possible to impose a CL-SPR constraint on the transfer functions of numerator and denominator of (9) instead of a CL-stability constraint on its numerator and denominator polynomials. This CL-SPR constraint brings conservatism if the denominators of the mentioned transfer functions are fixed. However, this conservatism can be removed by letting the order of the controller be increased, which is the same idea used in [5] in order to remove the conservatism of (2). The following theorem states the main result of this paper for a single system.

Theorem 1: Consider the numerator and the denominator transfer functions of (9) :

$$(MY+NX) - \gamma^{-1} W_1 MY \tag{10}$$

$$(MY+NX) + \gamma^{-1} W_1 MY \tag{11}$$

Then, the set of all stabilizing controllers that result in a closed-loop system with $\|W_1 S\|_\infty < \gamma$ for a system G defined in (1), is given by :

$$\mathscr{K}_\infty : \{K = XY^{-1} \mid \;\; (10) \text{ and } (11) \text{ be CL-SPR}\} \tag{12}$$

where $X, Y \in \mathscr{RH}_\infty$.

Proof: Sufficiency: It should be shown that any controller satisfying (12), satisfies the norm constraint (7) and stabilizes the closed-loop system too. Taking into account Corollary 1, the numerators of (10) and (11) are CL-stable because the controller $K = XY^{-1}$ satisfies the CL-SPR constraint on (10) and (11). Thus, based on Lemma 2, this controller belongs to the set represented in (9), which means that it satisfies the $\mathscr{H}_\infty$ constraint in (7). Furthermore, since the SPRness is a convex constraint, having two SPR transfer functions (10) and (11), results in the SPRness of their sum, which means that $MY + NX$ is SPR. Therefore, K belongs to the set of all stabilizing controllers $\mathscr{K}_s$ in (2).

Necessity: It should be shown that if $K_0 = X_0 Y_0^{-1}$ is a stabilizing controller that satisfies (7), then it belongs to $\mathscr{K}_\infty$ in (12). Suppose that $X_0, Y_0 \in \mathscr{RH}_\infty$ are coprime factors of K_0. Then,

$$\frac{(MY_0 + NX_0) - \gamma^{-1} W_1 MY_0}{(MY_0 + NX_0) + \gamma^{-1} W_1 MY_0} \tag{13}$$

is SPR, but $(MY_0 + NX_0) + \gamma^{-1} W_1 MY_0$ and $(MY_0 + NX_0) - \gamma^{-1} W_1 MY_0$ are not CL-SPR. We should show that there exists always a transfer function F such that (10) and (11) become CL-SPR with $X = X_0 F$ and $Y = Y_0 F$, which means that $K_0 = (X_0 F)(Y_0 F)^{-1}$ belongs to $\mathscr{K}_\infty$ in (12). By taking $F = (MY_0 + NX_0) + \gamma^{-1} W_1 MY_0$, (10) and (11) respectively become equal to the SPR transfer functions (13) and 1, which are CL-SPR according to the remark after Definition 3. Hence, K_0 belongs to $\mathscr{K}_\infty$ in (12) with $X = X_0 F$ and $Y = Y_0 F$. ■

To design a fixed-order controller, a fixed polynomial should be chosen for the denominators of (10) and (11). It is clear that by fixing these denominators, the convex feasibility set of the CL-SPR constraint of (12) would be an inner approximation of the non-convex set of all $\mathscr{H}_\infty$ stabilizing controllers of the desired order. An unsuitable choice of this polynomial may cause a null feasibility set. This conservatism can be removed by letting the order of X and Y be increased. By increasing the order of X and Y, not only some $\mathscr{H}_\infty$ stabilizing controllers of the new orders are included in the feasible set of the problem, but also more controllers of lower orders enter in the feasible set as can be seen by the above proof. To develop a simulation program, X and Y can be approximated using different types of orthonormal basis functions. For instance, consider that X and Y are linearly parameterized by :

$$X = \sum_{i=0}^{m} x_i \beta_i \qquad ; \qquad Y = \sum_{i=0}^{m} y_i \beta_i \tag{14}$$

where $\beta_i = 1/(z-\zeta)^i, i = 0, \ldots, m$ are the basis functions. As a result, the CL-SPR constraint in (12) becomes linear in the parameters of X and Y and can be represented by LMIs thanks to the KYP lemma :

$$\mathscr{K}_\infty : \{K = XY^{-1} \mid \; P = P^T > 0, \tag{15}$$

$$\begin{bmatrix} A^T PA - P & A^T PB - C_1^T \\ B^T PA - C_1 & B^T PB - D_1 - D_1^T \end{bmatrix} < 0 \; , \tag{16}$$

$$\begin{bmatrix} A^T PA - P & A^T PB - C_2^T \\ B^T PA - C_2 & B^T PB - D_2 - D_2^T \end{bmatrix} < 0\} \tag{17}$$

where (A,B,C_1,D_1) and (A,B,C_2,D_2) are the controllable canonical state space realizations of (10) and (11), respectively. The state matrix A is assumed to be identical for both realizations because the denominators of both transfer functions are the same. Besides, B is always the same for controllable canonical realizations.

Using the above parameterization, any controller structure and order can be enforced, whereas in Youla parameterization it is not possible. Moreover, another important feature of this parameterization is that it can be easily applied to systems with polytopic uncertainty. The following theorem extends this method for polytopic systems.

Theorem 2: Consider the transfer functions of numerator and denominator of (9) for each vertices of the system polytope defined in (4) :

$$(M_iY + N_iX) - \gamma^{-1} W_1 M_i Y \tag{18}$$

$$(M_iY + N_iX) + \gamma^{-1} W_1 M_i Y \tag{19}$$

Then, any controller belonging to the convex set :

$$\mathscr{K}_{p_\infty} : \{K = XY^{-1} \mid \text{(18) and (19) be CL-SPR}, i = 1, \dots, q\} \tag{20}$$

where $X, Y \in \mathscr{RH}_\infty$, stabilizes the polytopic system and results in a closed-loop polytopic system with $\|W_1 S\|_\infty < \gamma$ for all of its members.

Proof: It should be shown that if $(M_iY + N_iX) - \gamma^{-1}W_1M_iY$ and $(M_iY + N_iX) - \gamma^{-1}W_1M_iY$ are CL-SPR for $i = 1, \dots, q$, then the controller $K = XY^{-1}$ stabilizes the whole system polytope and in addition, satisfies $\|W_1S\|_\infty < \gamma$ for all members of the polytopic system $\mathscr{G}$ defined in (4). For each vertices of $\mathscr{G}$, the sum of two SPR transfer functions (18) and (19) results in the SPRness of $M_iY + N_iX, \; i = 1, \dots, q$. In Theorem 2 of [5], it is proved that such a controller stabilizes all members of $\mathscr{G}$. Next, we should prove robust performance with this controller, i.e. that it satisfies the $\mathscr{H}_\infty$ norm constraint for all members of $\mathscr{G}$. This is shown more easily via the LMI representation of (20) :

$$\mathscr{K}_{p_\infty} : \{K = XY^{-1} \mid \; P_i = P_i^T > 0, \; i = 1, \dots, q$$

$$\begin{bmatrix} A^T P_i A - P_i & A^T P_i B - C_{i_1}^T \\ B^T P_i A - C_{i_1} & B^T P_i B - D_{i_1} - D_{i_1}^T \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} A^T P_i A - P_i & A^T P_i B - C_{i_2}^T \\ B^T P_i A - C_{i_2} & B^T P_i B - D_{i_2} - D_{i_2}^T \end{bmatrix} < 0\} \tag{22}$$

where (A,B,C_{i_1},D_{i_1}) and (A,B,C_{i_2},D_{i_2}) are the controllable canonical state space realizations of (18) and (19) respectively. A is assumed to be identical for all transfer functions (18) and (19), because of their identical denominators and B is fixed because of the realization. Since all these constraints, i.e. (21) and (22), are linear w.r.t. the parameters of the system vertices, it is evident that any member of $\mathscr{G}$, such that $M = \sum_{i=1}^{q} \lambda_i M_i$ and $N = \sum_{i=1}^{q} \lambda_i N_i$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{q} \lambda_i = 1$, satisfies the LMIs (21) and (22) with $P = \sum_{i=1}^{q} \lambda_i P_i$ and therefore, satisfies $\|W_1S\|_\infty < \gamma$ too. ■

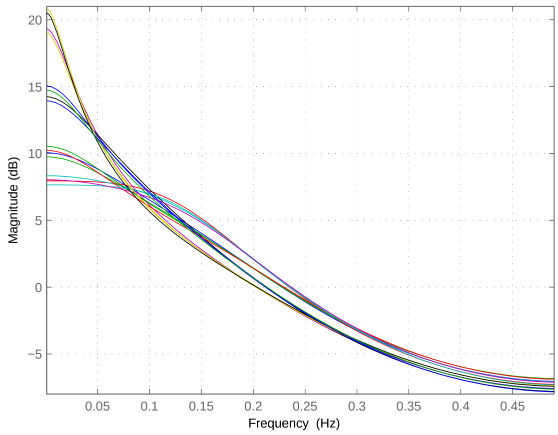

Fig. 1. Magnitude Bode diagrams of all vertices of the polytopic system.

In other words, the proposed method ensures robust performance in addition to robust stability for the polytopic system.

Remark :

- Although the above theorem gives a sufficient condition and not a necessary and sufficient one, it is evident that by increasing the controller order, not only some controllers of new orders are included in $\mathscr{K}_{p_\infty}$, but also some controller of lower orders fall inside $\mathscr{K}_{p_\infty}$ by non-coprime X and Y.
- Since we have not a frequency interpretation for the CL-SPR constraint, the frequency gridding method of [5] is not applicable in this paper and only the LMI formulation can be used.

IV. Simulation Results

Consider the problem of robust controller design for the following third-order system [5] :

$$G(z) = \frac{z + a}{z^3 + bz^2 + cz + d} \qquad T_s = 1\text{s}$$

with $a = 0.2, \; b = -1.2, \; c = 0.5$ and $d = -0.1$, where all the parameters are uncertain up to $\pm 7\%$ of their nominal values, resulting in a four-dimensional hypercube with $2^4 = 16$ vertices. The magnitude Bode diagrams of all the 16 vertices of this polytope are depicted in Fig. 1. Large uncertainty in low frequencies shows that this is a tough system for robust control methods. Assume that the goal is to design a stabilizing controller that contains an integrator, and $\|W_1S\|_\infty$ should be minimized over all members of the polytopic system. The

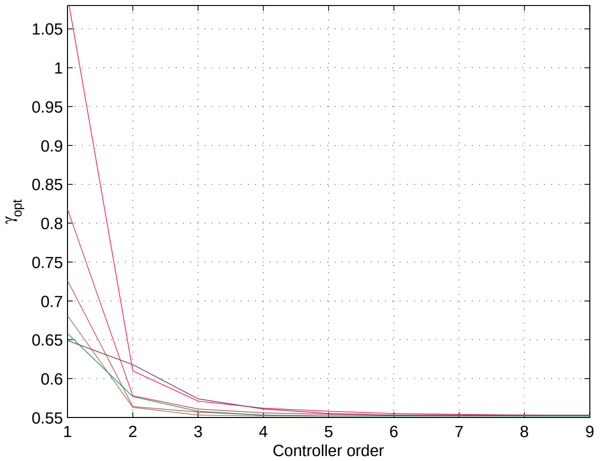

Fig. 2. γ_{opt} for the system 26 versus the order of the controller for different basis functions. Looking to the starting point from the highest curve, the basis function are chosen to have $\zeta = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ respectively.

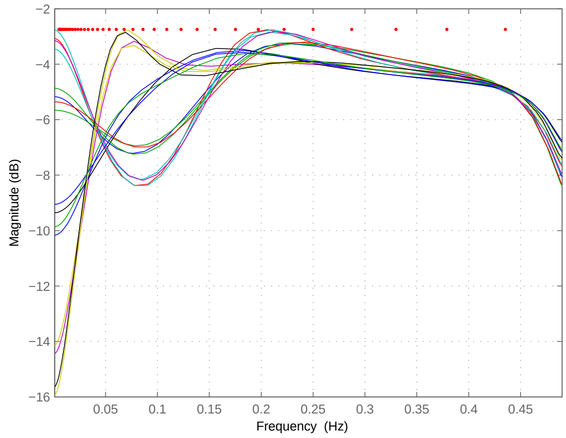

Fig. 3. Bode magnitude diagram of $W_1 S_i$ of all vertices of the polytopic system with the second-order controller (27) (Solid) and γ_{opt} (Dotted).

weighting function $W_1(z)$ is chosen to be the same as in [5] :

$$W_1(z) = \frac{0.4902(z^2 - 1.0431z + 0.3263)}{z^2 - 1.282z + 0.282} \tag{23}$$

which is a low-pass weighting filter based on the inverse of the desired sensitivity function. The same coprime factorization of [5] is used for the nominal plant model:

$$N = \frac{z + 0.2}{(z - 0.1)(z^2 - 1.0431z + 0.3263)} \tag{24}$$

$$M = \frac{z^3 - 1.2z^2 + 0.5z - 0.1}{(z - 0.1)(z^2 - 1.0432z + 0.3263)} \tag{25}$$

Note that the denominator of all coprime factors are identical for all models in the polytopic system and $\zeta = 0.1$ as in [5].

Before dealing with the polytopic system, we want to show the main advantage of the proposed method. That is, for a system without uncertainty, the proposed method can achieve the optimal $\mathscr{H}_\infty$ norm by increasing the controller order, irrelevant to the choice of the basis functions. Using the MATLAB command *hinfsyn* for one of the system vertices

$$G_1 = \frac{z - .186}{z^3 - 1.116z^2 + 0.465z - 0.093} \tag{26}$$

a fifth-order controller with $\|W_1 S\|_\infty = 0.552$ is obtained. Fig. 2 shows that by increasing the controller order, the proposed method converges to the optimum norm bound of $\|W_1 S\|_\infty$, independent of the choice of *zeta* in the basis functions.

Next, we design a controller for the polytopic system. The problem in [5] does not become feasible for controller orders less than 5, because the second and the third line of (6) are sufficient conditions for $\|W_1 S_i\|_\infty < 1$. However, using the proposed method in this paper, it becomes feasible with a second-order controller. The controller $K = XY^{-1}$ is parameterized as follows :

$$X = \frac{x_1 z^2 + x_2 z + x_3}{(z - 0.1)^2} \quad , \quad Y = \frac{(z-1)(y_1 z + y_2)}{(z - 0.1)^2}$$

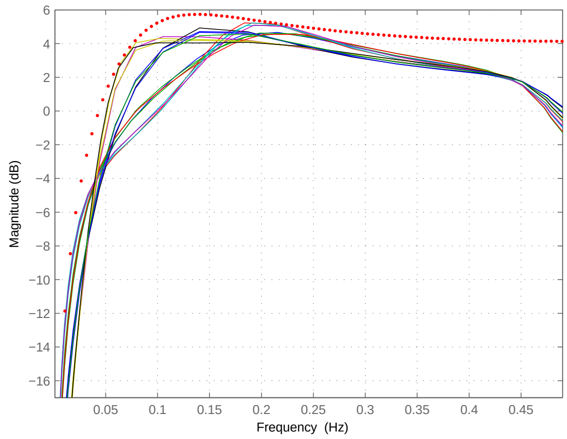

Fig. 4. Output sensitivity function of all vertices of the polytopic system with the second-order controller (27) (Solid), Bode magnitude diagram of γ_{opt}/W_1 (Dotted).

To solve the problem in MATLAB, YALMIP [10] is used as the interface and SDPT3 [11] as the solver. Using the iterative bisection algorithm, the optimal value of $\gamma_{opt.} = 0.729$ is obtained with the following controller :

$$K = \frac{0.802(z - 0.6347)(z - 0.1887)}{(z-1)(z + 1.156)} \tag{27}$$

The magnitude Bode diagrams of $W_1 S_i$ for all the 16 vertices of the polytopic system are shown in Fig. 3, where $\gamma_{opt.} = 0.729 = -2.7454dB$ is also depicted. Furthermore, their sensitivity functions are shown in Fig. 4 in addition to the Bode magnitude diagram of γ_{opt}/W_1. The maximum value of the sensitivity functions is around 5.3 dB, which is quite desirable [6].

V. Conclusions

A new convex parameterization of all $\mathscr{H}_\infty$ stabilizing controllers for SISO-LTI systems is given based on the new concept of Common Lyapunov Strictly Positive Realness. This convex parameterization provides a complete set of all $\mathscr{H}_\infty$ controllers for a single system in an infinite dimensional space. Furthermore, a convex approximation of the set of all fixed-order $\mathscr{H}_\infty$ controllers is given that depends on the choice of some basis functions. The effect of this choice can be relaxed by increasing the controller order. The proposed approach can also be applied to systems with polytopic uncertainty.

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Appendix

In case of continuous-time systems, it is possible to remove the multiplication between the controller parameters and γ in the dual equations of (16) and (17) (and (21) and (22)). This way, we can minimize γ in our convex optimization problem without an iterative bisection algorithm. Let the biproper transfer functions $(MY+NX)+\gamma^{-1}W_1MY$ and $(MY+NX)-\gamma^{-1}W_1MY$ have controllable canonical state space realizations (A,B,C_1,D_1) and (A,B,C_2,D_2) respectively. Since W_1 is strictly proper [1], γ appears just in $C_i=\hat{C}_i+\gamma^{-1}\tilde{C}_i$ $i=1,2$ and not in D_1 and D_2. Obviously, $\hat{C}_1=\hat{C}_2=\hat{C}$ and $\tilde{C}_1=-\tilde{C}_2=\tilde{C}$. Moreover, for a strictly proper system $D_1=D_2=D$. Taking into account the KYP Lemma for the continuous-time systems [12], and imposing the CL-SPR constraint (12) : $\exists\ P=P^T>0$ such that

$$\begin{bmatrix} A^TP+PA & PB-(\hat{C}+\gamma^{-1}\tilde{C})^T \\ B^TP-(\hat{C}+\gamma^{-1}\tilde{C}) & -D-D^T \end{bmatrix} < 0 \qquad (28)$$

$$\begin{bmatrix} A^TP+PA & PB-(\hat{C}-\gamma^{-1}\tilde{C})^T \\ B^TP-(\hat{C}-\gamma^{-1}\tilde{C}) & -D-D^T \end{bmatrix} < 0 \qquad (29)$$

Since $(NX+MY)+\gamma^{-1}W_1MY$ is biproper, $D+D^T$ is invertible. Using the inverse of Schur complement, (28) is equivalent to :

$$A^TP+PA+(PB-\hat{C}^T-\gamma^{-1}\tilde{C}^T)(D+D^T)^{-1} (B^TP-\hat{C}-\gamma^{-1}\tilde{C})<0 \qquad (30)$$

To simplify the notations, let $Q=A^TP+PA$ and $V=PB-\hat{C}^T$ and $R=(D+D^T)$. Therefore, (30) is equivalent to :

$$Q+VR^{-1}V^T+\gamma^{-2}\tilde{C}^TR^{-1}\tilde{C}-\gamma^{-1}\tilde{C}^TR^{-1}V^T-\gamma^{-1}VR^{-1}\tilde{C}<0$$

which is equal to :

$$\tilde{Q}+\tilde{C}^T(-\gamma^{-1}R^{-1})V^T+V(-\gamma^{-1}R^{-1})\tilde{C}<0$$

where $\tilde{Q}=Q+VR^{-1}V^T+\gamma^{-2}\tilde{C}^TR^{-1}\tilde{C}$. Since R is fixed, $\gamma^{-1}R^{-1}$ contains no controller variables. Therefore, using Finsler's lemma [12], this constraint becomes equivalent to :

$$\exists\ \sigma\in\mathbb{R} \text{ s.t.}$$
$$\tilde{Q}+\sigma\tilde{C}^T\tilde{C}<0 \qquad (31)$$
$$\tilde{Q}+\sigma VV^T<0 \qquad (32)$$

This way, the multiplication of γ^{-1} with other variables can be removed from the constraints. The new constraints (31) and (32) are not convex. However, by using Schur complement three times, they become convex :

$$\begin{bmatrix} A^TP+PA & PB-\hat{C}^T & \tilde{C}^T & PB-\hat{C}^T \\ B^TP-\hat{C} & -D-D^T & 0 & 0 \\ \tilde{C} & 0 & -\eta(D+D^T) & 0 \\ B^TP-\hat{C} & 0 & 0 & -\mu I \end{bmatrix} < 0 \qquad (33)$$

$$\begin{bmatrix} A^TP+PA & PB-\hat{C}^T & \tilde{C}^T & \tilde{C}^T \\ B^TP-\hat{C} & -D-D^T & 0 & 0 \\ \tilde{C} & 0 & -\eta(D+D^T) & 0 \\ \tilde{C}^T & 0 & 0 & -\mu I \end{bmatrix} < 0 \qquad (34)$$

where $\eta=\gamma^2$ and $\mu=\sigma^{-1}$. These inequalities represent a convex version of (28). By changing the sign of $\tilde{C}$, similar LMIs can be derived for (29). Since $\tilde{C}$ appears symmetrically in (33) and (34), its sign does not change the determinant of any of the leading principal minors of these matrices and hence, it is sufficient to satisfy these two LMIs and there is no need for the other ones. Hence, the set of all controllers given by (9) can be represented by (33) and (34) and $\eta=\gamma^2$ can be minimized.