THE WITT GROUP OF LAURENT POLYNOMIALS

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Abstract. We give a direct, self-contained proof of the fact that for a large class of rings A, in particular for all regular rings with involution, $W(A[t,1/t]) = W(A) \oplus W(A)$.

1. Introduction

The purpose of this note is to give a short direct proof of two fundamental theorems on the Witt group of polynomials and Laurent extensions of a ring A. These theorems were proved independently by M. Karoubi [3] and by A. Ranicki [5]. We will state them under the most general conditions on A and for their proofs we will use nothing more than a general result on the K-theory of Laurent polynomials. In the last section we will show, by two counterexamples, that the assumptions we make on A are necessary.

We begin by briefly recalling some definitions. We refer to [4] for a more detailed exposition and for the proofs of the few basic results that we will use.

Let A be an associative ring with an involution denoted by $a \mapsto a^\circ$. Except in §2 we will always assume that 2 is invertible in A. If M is a right A-module we denote by M^* its dual $\mathrm{Hom}_A(M, A)$ endowed with the right action of A given by $fa(x) = a^\circ f(x)$ for any $f : M \to A$ and $a \in A$. If P is a finitely generated projective right A-module we identify it with P^{**} through the canonical isomorphism mapping $x \in P$ to $\widehat{x} : P^* \to A$ defined by $\widehat{x}(f) = f(x)$.

Let ϵ be 1 or -1. An *ϵ-hermitian space* over A is a pair (P, α) consisting of a finitely generated projective right A-module P and an

A-isomorphism $\alpha : P \to P^*$ satisfying $\alpha = \epsilon\alpha^*$. For brevity ϵ-hermitian spaces will be called *spaces*. A 1-hermitian space (over a commutative ring A) is also called *quadratic space*.

Two spaces (P, α) and (Q, β) are *isometric* if there exists an A-isomorphism $\varphi : P \to Q$ such that the square

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ P^* & \xleftarrow[\varphi^*]{} & Q^* \end{array}$$

commutes. A space is *hyperbolic* if it is isometric to a space of the form

$$H(P) = \left(P \oplus P^*, \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}\right) .$$

The *orthogonal sum* of two spaces (P, α) and (Q, β) is the space

$$(P, \alpha) \perp (Q, \beta) = (P \oplus Q, \alpha \oplus \beta) .$$

If (P, α) is a space and M a submodule of P we denote by $M^\perp$ the orthogonal of M, defined by the exact sequence

$$0 \longrightarrow M^\perp \longrightarrow P \xrightarrow{i^* \circ \alpha} M^* ,$$

where i^* is the dual of the inclusion $i : M \to P$. A submodule M of P is *totally isotropic* if $M \subseteq M^\perp$. A *sublagrangian* of a space (P, α) is a totally isotropic direct factor of P. A *lagrangian* of (P, α) is a sublagrangian L such that $L = L^\perp$. For instance, P and P^* are lagrangians of $H(P)$.

The Witt group $W(A)$ of ϵ-hermitian spaces over A is the quotient of the Grothendieck group of ϵ-hermitian spaces with respect to orthogonal sums, by the subgroup generated by all hyperbolic spaces. We say that two spaces are *Witt equivalent* if they represent the same element of $W(A)$.

Consider now the rings $A[t]$ and $A[t, t^{-1}]$, endowed with the involution that fixes t and maps $a \in A$ to a°. For the ring $A[t, t^{-1}]$ we introduce a variant $W'(A[t, t^{-1}])$ of the Witt group. We first consider the Grothendieck group Q of ϵ-hermitian spaces over $A[t, t^{-1}]$ which are extended from A as $A[t, t^{-1}]$-modules, and its subgroup N generated by the hyperbolic spaces $H(P)$ where P is extended from A. We then define $W'(A[t, t^{-1}])$ as Q/N. Clearly $W'(A[t, t^{-1}])$ maps canonically to $W(A[t, t^{-1}])$. Here are our results.

A (Theorem 3.1). *Let A be an associative ring with involution, in which 2 is invertible. The canonical homomorphism*

$$W(A) \to W(A[t])$$

is an isomorphism.

B (Theorem 5.1). *Let A be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi : W(A) \oplus W(A) \to W'(A[t, t^{-1}])$$

mapping (ξ, η) to $\xi + t\eta$ is an isomorphism.

C (Theorem 7.1). *Let A be an associative ring with involution, in which 2 is invertible. Let*

$$\varphi : W'(A[t, t^{-1}]) \to W(A[t, t^{-1}])$$

be the canonical homomorphism.

(a) *If $H^2(\mathbb{Z}/2, K_{-1}(A)) = 0$, then φ is surjective.*

(b) *If $K_0(A) = K_0(A[t]) = K_0(A[t, t^{-1}])$, then φ is an isomorphism.*

Two examples will be constructed in §8 to show that the assumptions in (a) and in (b) cannot be omitted.

An amusing application of **B** is the following result:

D (Proposition 6.8). *Let A be a commutative semilocal ring in which 2 is invertible. Let (P, α) be a quadratic space over A. If (P, α) is isometric to $(P, t \cdot \alpha)$ over $A[t, t^{-1}]$, then (P, α) is hyperbolic.*

We remark that in general, even for a commutative local ring, there is no residue map

$$Res : W(A[t, t^{-1}]) \to W(A)$$

satisfying the following two properties:

- For any constant space $\xi \in W(A) \subset W(A[t, t^{-1}])$, $Res(\xi) = 0$.
- For any constant space $\xi \in W(A) \subset W(A[t, t^{-1}])$, $Res(t \cdot \xi) = \xi$.

In fact, the existence of such a residue map immediately implies the injectivity of

$$\varphi \circ \psi : W(A) \oplus W(A) \to W(A[t, t^{-1}]) ,$$

which may fail, as in Example 8.1. However, there exists a residue map $Res : W'(A[t,t^{-1}]) \to W(A)$ (Proposition 5.2) which yields the injectivity of ψ.

We now recall three elementary, well-known facts about hermitian spaces.

PROPOSITION 1.5. *Let (P,α) be any space. Then*

1. *The space $(P,\alpha) \perp (P,-\alpha)$ is hyperbolic.*

2. *If L is a lagrangian of (P,α), then (P,α) is isometric to $H(L)$.*

3. *If M is a sublagrangian of (P,α), then the map α induces on $M^{\perp}/M$ a natural structure of hermitian space that makes it Witt equivalent to (P,α).*

2. K-theoretic preliminaries

We recall a few results proved in the twelfth chapter of Bass'book [1]. For any ring A we denote by $K_0(A)$ the Grothendieck group of finitely generated projective right A-modules and by $K_1(A)$ the abelianized general linear group of $A : K_1(A) = GL(A)/[GL(A), GL(A)]$. By Whitehead's lemma $K_1(A)$ is also the quotient of $GL(A)$ by the subgroup $E(A)$ generated by all elementary matrices over A.

For any functor F from rings to abelian groups we denote by $N_+F(A)$ the kernel of the map $F(A[t]) \to F(A)$ obtained by putting $t = 0$. Similarly, we denote by $N_-F(A)$ the kernel of $F(A[t^{-1}]) \to F(A)$ obtained by putting $t^{-1} = 0$. The inclusions of $A[t]$ and $A[t^{-1}]$ into $A[t,t^{-1}]$ define a map

$$N_+F(A) \oplus N_-F(A) \longrightarrow F(A[t,t^{-1}])$$

whose cokernel will be denoted by $LF(A)$. The functor LK_1 turns out to be naturally isomorphic to K_0, hence we will denote LK_i by K_{i-1} for $i = 1$ and also for $i = 0$.

Theorem 2.1. *Let A be any associative ring.*

(a) *For $i=0$ or 1 there exists a natural embedding*

$$\lambda_i : K_{i-1}(A) \longrightarrow K_i(A[t,t^{-1}])$$

such that the composite

$$K_{i-1}(A) \xrightarrow{\lambda_i} K_i(A[t,t^{-1}]) \longrightarrow LK_i(A) = K_{i-1}(A)$$

is the identity.

(b) *The embedding λ_i and the canonical homomorphism*

$$N_{\pm}K_i(A) \to K_i(A[t,t^{-1}])$$

yield canonical decompositions

$$K_1(A[t,t^{-1}]) = K_1(A) \oplus N_+K_1(A) \oplus N_-K_1(A) \oplus K_0(A)$$

and

$$K_0(A[t,t^{-1}]) = K_0(A) \oplus N_+K_0(A) \oplus N_-K_0(A) \oplus K_{-1}(A)\ .$$

Proof. See [1], Theorem 7.4 of chapter XII. □

We will also use the following well-known result.

Proposition 2.2. *If 2 is invertible in A, the groups $N_{\pm}K_1(A)$ are uniquely divisible by 2.*

Proof. By [1], XII, 5.3 every element of $N_+K_1(A)$ can be represented by a matrix $\alpha = 1+\nu t$, with ν a nilpotent matrix of $M_n(A)$. Let

$$P(X) = \sum_0^{\infty} \binom{1/2}{n} X^n \in \mathbb{Z}[1/2][X]\ .$$

Then $P(\nu t) \in M_n(A[t])$ and $(P(\nu t))^2 = 1+\nu t$. This shows that $N_+K_1(A)$ is divisible by 2. To show uniqueness it suffices to show that $N_+K_1(A)$ has no 2-torsion. Take $\alpha = 1+\nu t$ as before and suppose that $\alpha^2 \in E(A[t])$. Put $s = t(2+\nu t)$, so that $\alpha^2 = 1+\nu s$. Since

$$t = \sum_1^{\infty} \binom{1/2}{n} \nu^{n-1} s^n$$

we have $M_n(A)[t] = M_n(A)[s]$. If $\alpha^2 = 1+\nu s \in E(A[s]) = E(M_n(A)[s])$ we clearly also have $\alpha = 1+\nu t \in E(M_n(A)[t])$. □

Corollary 2.3. *If* 2 *is invertible in* A, *the groups* $N_\pm K_0(A)$ *are uniquely divisible by* 2.

Proof. $K_0(A)$ is a direct factor of $K_1(A[X,X^{-1}])$, hence $N_\pm K_0(A)$ is a direct factor of $N_\pm K_1(A[X,X^{-1}])$. □

Assume now that A has an involution. Associating to any projective module its dual and to any matrix its conjugate transpose yields actions of $\mathbb{Z}/2$ on K_0 and K_1 which are compatible with the decompositions of Theorem 2.1. From Corollary 2.3 we immediately deduce

Corollary 2.4. *Suppose that* A *is a ring with involution, in which* 2 *is invertible. Then*

$$H^2(\mathbb{Z}/2, K_0(A[t,t^{-1}])/K_0(A)) = H^2(\mathbb{Z}/2, K_{-1}(A)) .$$

3. The Witt group of polynomial rings

Theorem 3.1. *Let* A *be an associative ring with involution, in which* 2 *is invertible. Let* ϵ *be* 1 *or* -1 *and let* W *be the Witt group functor of* ϵ*-hermitian spaces. The natural homomorphism*

$$W(A) \longrightarrow W(A[t])$$

is an isomorphism.

Proof. It suffices to show that the homomorphism $W(A[t]) \to W(A)$ given by the evaluation at $t=0$ is an isomorphism. Surjectivity is obvious. To prove injectivity let (P,α) be a space over $A[t]$ and $(P(0),\alpha(0))$ its reduction modulo t. Suppose that $(P(0),\alpha(0))$ is isometric to some hyperbolic space $H(Q)$. Choosing a projective module Q' such that $Q \oplus Q'$ is free and adding to (P,α) the space $H(Q'[t])$ we may assume that $P(0)$ is the hyperbolic space over a free module. The class of P in $K_0(A[t])/K_0(A) = N_+(A)$ is a symmetric element. By Corollary 2.4 it can be written as $a + a^*$, hence, adding to (P,α) a suitable free hyperbolic space, we may assume that (P,α) is of the form

$$H(A^n[t]) \perp (R \oplus R^*, \beta) .$$

Let R' be an $A[t]$-module such that $R \oplus R'$ is free. Adding to (P,α) the hyperbolic space $H(R')$ we are reduced to the case in which P is free and α is an invertible ϵ-hermitian matrix with entries in $A[t]$.

LEMMA 3.2. *Let $\alpha = \epsilon\alpha^* \in \mathrm{M}_n(A[t])$ be any ϵ-hermitian matrix. There exist an integer m and a matrix $\tau \in \mathrm{GL}_{n+2m}(A[t])$ (actually in $\mathrm{E}_{n+2m}(A[t])$) such that*

$$\tau^* \begin{pmatrix} \alpha & 0 \\ 0 & \chi \end{pmatrix} \tau = \alpha_0 + t\alpha_1 \ ,$$

where α_0 and α_1 are constant matrices and χ is a sum of hyperbolic blocs $\left(\begin{smallmatrix} 0 & 1 \\ \epsilon 1 & 0 \end{smallmatrix}\right)$ of various sizes.

Proof of the lemma. Write $\alpha = \gamma + \delta t^N$, where δ is constant and γ of degree less than N. Assume that N is at least 2. Since δ is ϵ-hermitian and 2 is invertible in A we can write $\delta = \sigma + \epsilon\sigma^*$. Then

$$\begin{pmatrix} 1 & t & -\sigma^* t^{N-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma + \sigma t^N + \epsilon\sigma^* t^N & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \epsilon & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\sigma t^{N-1} & 0 & 1 \end{pmatrix}$$

is of degree $\leq N-1$ and after $N-1$ such transformations we get a linear matrix. □

Writing $\alpha = \alpha_0 + t\alpha_1$ as $\alpha_0(1+\nu t)$ we see immediately that, α being invertible, ν is nilpotent. The formal power series

$$\tau = (1+\nu t)^{-1/2} = \sum \binom{-1/2}{k} (\nu t)^k$$

is a polynomial. From $\alpha = \epsilon\alpha^*$ we get $\alpha_0^* = \epsilon\alpha_0$ and $\nu^*\alpha_0^* = \epsilon\alpha_0\nu$. This implies that $\tau^*\alpha_0^* = \epsilon\alpha_0\tau$ and therefore

$$\tau^*\alpha\tau = \tau^*\alpha_0(1+\nu t)\tau = \alpha_0\tau(1+\nu t)\tau = \alpha_0 \ .$$

This proves that (P, α) is Witt equivalent to $(P(0), \alpha(0))$ and is, therefore, hyperbolic. □

4. THE WITT GROUP OF TORSION MODULES

Let M be a finitely generated right $A[t]$-module and suppose that it is a t-torsion module and that it is projective as an A-module. Obviously, it will be finitely generated over A. We denote by $M^\sharp$ the left $A[t]$-module $\mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t])$ and we consider it as a right module through the involution on $A[t]$.

Recall that, as an A-module, the quotient $A[t,t^{-1}]/A[t]$ can be written as a direct sum

$$A[t,t^{-1}]/A[t] = At^{-1} \oplus At^{-2} \oplus \cdots \ .$$

Thus, to any $f \in \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t])$ we can associate an A-linear map $f_{-1} : M \to A$, which is defined as the composite of f with the projection onto At^{-1}.

Proposition 4.1. *The map*

$$\partial = \partial_M : M^\sharp = \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t]) \longrightarrow \mathrm{Hom}_A(M,A) = M^*$$

obtained by associating f_{-1} to f is a functorial A-linear isomorphism.

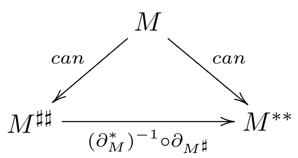
Proof. It is clear that ∂ is A-linear. To show that it is bijective we construct its inverse. Given any $g \in M^*$ define $\widetilde{g}$ by the (finite!) sum

$$\widetilde{g}(x) = t^{-1}g(x) + t^{-2}g(tx) + t^{-3}g(t^2x) + \cdots \ .$$

It is easy to check that $\widetilde{g} \in M^\sharp$, $(\widetilde{g})_{-1} = g$ and $\widetilde{f_{-1}} = f$. Functoriality is clear. □

Corollary 4.2. *For any finitely generated t-torsion module M which is projective as an A-module the canonical homomorphism $M \to M^{\sharp\sharp}$ is an isomorphism.*

Proof. It suffices to remark that the diagram



commutes and that $M \xrightarrow{can} M^{**}$ is an isomorpism. □

An *ϵ-hermitian t-torsion space* (or, briefly, a *t-torsion space*) is a pair $(M, <\,,>)$ consisting of a finitely generated t-torsion right $A[t]$-module M which is projective as an A-module, and a perfect ϵ-hermitian pairing $<\,,>: M \times M \to A[t,t^{-1}]/A[t]$. Giving $<\,,>$ is the same, of course, as giving its adjoint $\varphi : M \to M^\sharp$ defined by $\varphi(a)(b) = <a,b>$.

Isometries and orthogonal sums are defined in the obvious way. For any subset $X \subset M$ we define its orthogonal as

$$X^{\perp} = \{ y \in M \mid <x, y>= 0 \;\; \forall x \in X \} \; .$$

A *sublagrangian* of (M, φ) is an $A[t]$-submodule L of M which satisfies the following two conditions:

(1) It is contained in its own orthogonal: $L \subseteq L^{\perp}$.

(2) The quotient M/L is projective over A (which is the same as saying that L, as an A-module, is a direct factor of M).

A sublagrangian L is a *lagrangian* if $L = L^{\perp}$. A t-torsion space is *metabolic* if it has a lagrangian. The Witt group of t-torsion spaces is the quotient of the Grothendieck group of t-torsion spaces with respect to orthogonal sums, modulo the subgroup generated by the metabolic spaces. We will denote it by $W_{tors}(A[t])$. Lemma 4.6 below will show that the opposite of the class of (M, φ) is the class of $(M, -\varphi)$.

Lemma 4.3. *Let M and N be finitely generated t-torsion modules and $i : N \to M$ an $A[t]$-linear homomorphism. Assume that as A-modules M and N are projective. Then the map $i^{\sharp} : M^{\sharp} \to N^{\sharp}$ is surjective (respectively injective) if and only if $i^* : M^* \to N^*$ is surjective (respectively injective).*

Proof. Look:

$$\begin{array}{ccc} M^{\sharp} & \xrightarrow{\;i^{\sharp}\;} & N^{\sharp} \\ {\scriptstyle \partial_M}\Big\downarrow & & \Big\downarrow{\scriptstyle \partial_N} \\ M^{*} & \xrightarrow{\;i^{*}\;} & N^{*} \end{array} \; .$$

□

Proposition 4.4. *Let (M, φ) be a t-torsion space and L an $A[t]$-submodule of M. If M/L is projective over A, then $L = L^{\perp\perp}$ and $L^{\perp}$ is a direct factor of M as an A-module.*

Proof. First observe that as an A-module L is finitely generated and projective. Let $i : L \to M$ be the natural injection. By Lemma 4.3 the map $i^{\sharp} \circ \varphi$ is surjective, thus the sequence

$$0 \longrightarrow L^{\perp} \xrightarrow{j} M \xrightarrow{i^{\sharp} \circ \varphi} L^{\sharp} \longrightarrow 0$$

is exact. Hence $L^{\perp}$ is a direct factor of M as an A-module; in particular it is A-projective. Identifying L with $L^{\sharp\sharp}$ we can write the dual sequence as

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{j^\sharp \circ \varphi^\sharp} (L^\perp)^\sharp \longrightarrow 0\ .$$

Notice that it is exact by Lemma 4.3. Again by Lemma 4.3 the sequence

$$0 \longrightarrow L^{\perp\perp} \to M \xrightarrow{j^\sharp \circ \varphi} (L^\perp)^\sharp \longrightarrow 0$$

is exact because $L^\perp$ is a direct factor of M as an A-module. Since $\varphi^\sharp = \pm\varphi$, comparing the last two sequences we get the result. □

We now prove a fundamental result on the equivalence of t-torsion spaces.

THEOREM 4.5. *Let (M, φ) be an ϵ-hermitian t-torsion space and L a sublagrangian of (M, φ). The quotient $L^\perp/L$ carries a natural structure of t-torsion ϵ-hermitian space and its class in $W_{tors}(A[t])$ is the same as that of (M, φ).*

Proof. We first prove the following lemma.

LEMMA 4.6. *Let (M, φ) be any ϵ-hermitian t-torsion space. The space $(M, \varphi) \perp (M, -\varphi)$ is metabolic.*

Proof of Lemma 4.6. We show that the image $L = \Delta(M)$ of the diagonal map $M \xrightarrow{\Delta} M \oplus M$ is a lagrangian. The condition $L \subseteq L^\perp$ is immediately verified. The quotient $(M \oplus M)/L$ is isomorphic to M, hence it is projective over A. It remains to see that $L^\perp \subseteq L$. If $(a, b) \in L^\perp$ we have $0 = < (a, b), (x, x) > = < a - b, x >$ for any $x \in M$. Since the pairing $< , >$ is perfect, this implies $a = b$, i.e. $(a, b) \in L$. □

We now prove the theorem. By Proposition 4.4, $L^\perp$ is a direct factor of M as an A-module. Since $L \subseteq L^\perp$ is also a direct factor of M, the quotient $L^\perp/L$ is projective. Denoting by $\overline{a}, \overline{b}$ the classes modulo L of two elements $a, b \in L$, we define the hermitian structure of $L^\perp/L$ by $<\overline{a}, \overline{b}> = <a, b>$. It is clear that $<a, b>$ only depends on $\overline{a}$ and $\overline{b}$. We first check that this pairing defines a t-torsion space. It is clearly ϵ-hermitian. The injectivity of the adjoint map $L^\perp/L \to (L^\perp/L)^\sharp$ follows immediately from Proposition 4.4. To show surjectivity consider any $A[t]$-linear map $f : L^\perp \to A[t, t^{-1}]/A[t]$. Since $L^\perp$ is a direct factor of M as an A-module, f, by Lemma 4.3, extends to an $A[t]$-linear map $\widetilde{f} : M \to A[t, t^{-1}]/A[t]$. Choose an $m \in M$ for which $\widetilde{f} = <m, \cdot>$. If $\widetilde{f}$ vanishes on L, then m is in $L^\perp$. This proves that $L^\perp/L$ is a t-torsion space.

To show that $L^\perp/L$ is equivalent to (M,φ) we check that the image of the diagonal map $\Delta : L^\perp \to M \oplus L^\perp/L$ is a lagrangian of $(M,-\varphi) \perp L^\perp/L$ which is, therefore, metabolic. It is easy to check that $\Delta(L^\perp)$ is contained in its own orthogonal. Conversely, if $(a,\overline{b}) \in M \oplus L^\perp/L$ is orthogonal to every $(x,\overline{x})$, then $< a-b, x >= 0$ for every $x \in L^\perp$. This means that $a-b$ is in $L^{\perp\perp}$, which by Proposition 4.4 coincides with L. We thus have $(a,\overline{b}) = (a,\overline{a}) \in \Delta(L^\perp)$. □

The next proposition connects the Witt group of t-torsion spaces with the Witt group of A.

Proposition 4.7. *The isomorphisms*

$$\partial_M : \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t]) \to \mathrm{Hom}_A(M,A)$$

induce a surjective homomorphism

$$\partial^W : W_{tors}(A[t]) \to W(A) .$$

Proof. Associating to any t-torsion space (M,φ) the hermitian space $(M, \partial_M \circ \varphi)$ preserves isometries and orthogonal sums and, by Lemma 4.3, transforms metabolic t-torsion spaces into hyperbolic spaces (with the same lagrangian). Therefore it induces a homomorphism

$$\partial^W : W_{tors}(A[t]) \to W(A) .$$

To find a preimage (M,φ) of a space (M,α) over A consider M as an $A[t]$-module annihilated by t and replace $\alpha : M \to M^*$ by $\varphi = \partial_M^{-1} \circ \alpha$. □

5. The Witt group of extended spaces

Let $W'(A[t,t^{-1}])$ be the group defined in the introduction.

Theorem 5.1. *Let A be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi : W(A) \oplus W(A) \to W'(A[t,t^{-1}])$$

mapping (ξ,η) to $\xi + t\eta$ is an isomorphism.

Proof. The injectivity of ψ is based on the following result, whose proof will be given in §6.

PROPOSITION 5.2. *There exists a homomorphism*

$$Res : W'(A[t,t^{-1}]) \to W(A)$$

with the following properties:

R_1: *For any constant space* $\xi \in W(A) \subset W'(A[t,t^{-1}])$, $Res(\xi) = 0$.

R_2: *For any constant space* $\xi \in W(A) \subset W'(A[t,t^{-1}])$, $Res(t \cdot \xi) = \xi$.

Proof. See Theorem 6.7. □

Assuming this proposition, suppose that for two elements $\xi, \eta \in W(A)$ we have $\xi + t \cdot \eta = 0$. Then $0 = Res(\xi + t \cdot \eta) = \eta$ and hence $\xi = 0$.

We now turn to the surjectivity of ψ. We have to show that every hermitian space (P, α) over $A[t,t^{-1}]$ with $P = P_0[t,t^{-1}]$ is Witt equivalent to a space of the form $(Q_0[t,t^{-1}], \alpha_0) \perp (Q_1[t,t^{-1}], t\alpha_1)$. Let P_1 be a projective A-module such that $P_0 \oplus P_1 = A^n$ for some n. Replacing (P, α) by

$$(P_0[t,t^{-1}], \alpha) \perp (P_0[t,t^{-1}], -\alpha(1)) \perp H(P_1[t,t^{-1}])$$

we may assume that P_0 is free. Replacing α by $t^{2N}\alpha$ with a suitable N, we may also assume that α maps $P_0[t]$ into $P_0^*[t]$. By Lemma 3.2 we are reduced to the case where $\alpha = \alpha_0 + t\alpha_1$ for some ϵ-hermitian maps $\alpha_0, \alpha_1 : P_0 \to P_0^*$.

LEMMA 5.3. *If, for a constant matrix* β,

$$\alpha = 1 + (t-1)\beta \in \mathrm{GL}_n(A[t,t^{-1}]) \cap \mathrm{M}_n(A[t]) ,$$

then there exists an N *such that* $(1-\beta)^N \beta^N = 0$.

Proof. This is Corollary 2.4 of [2]. For the convenience of the reader we reprove it here.

Writing the inverse of α as a Laurent polynomial and equating coefficients in the identity

$$1 = \alpha\alpha^{-1} = (1 - \beta + t\beta)(\gamma_{-q}t^{-q} + \cdots + \gamma_{-1}t^{-1} + \gamma_0 + \gamma_1 t + \cdots + \gamma_p t^p)$$

we get

$$(1-\beta)\gamma_{-q} = 0,\ (1-\beta)\gamma_{-q+1} + \beta\gamma_{-q} = 0,\ \ldots,\ (1-\beta)\gamma_{-1} + \beta\gamma_{-2} = 0 ,$$

$$(1-\beta)\gamma_0 + \beta\gamma_{-1} = 1$$

and

$$(1-\beta)\gamma_1+\beta\gamma_0=0,\ \ldots,\ (1-\beta)\gamma_p+\beta\gamma_{p-1}=0,\ \beta\gamma_p=0\ .$$

From the first line we get $(1-\beta)^q\gamma_{-1}=0$, from the third $\beta^{p+1}\gamma_0=0$ and then from the middle one $\beta^{p+1}(1-\beta)^q=0$. □

We put $\beta=\alpha(1)^{-1}\alpha_1 : P_0\to P_0$, so that

$$\alpha(1)^{-1}\alpha=1+(t-1)\beta\ .$$

We will repeatedly use the fact that β is adjoint with respect to $\alpha,\alpha(1),\alpha_0,\alpha_1$, by which we mean that $\alpha\beta=\beta^*\alpha$, and so on. The same clearly holds for any polynomial in β with integral coefficients.

By Lemma 5.3 we can find an integer N such that $\beta^N(1-\beta)^N=0$. Denoting by $\mathbb{Z}[\beta]$ the subring of $\mathrm{End}_A(P_0)$ generated by β we can write $\mathbb{Z}[\beta]=\mathbb{Z}[\beta]e\times\mathbb{Z}[\beta](1-e)$, where e is an idempotent of the form $\beta+\nu$ and ν is a nilpotent matrix. Note that e and ν are polynomials in β and therefore they commute with β and with each other. If we decompose P_0 as $eP_0+(1-e)P_0$ and represent A-linear endomorphisms of P_0 as 2×2 block matrices, we have

$$e=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \beta=\begin{pmatrix}1+\nu_1&0\\0&\nu_2\end{pmatrix}$$

and

$$\alpha=\begin{pmatrix}\alpha_{11}&\alpha_{12}\\ \epsilon\alpha_{12}^*&\alpha_{22}\end{pmatrix}(1+(t-1)\beta)\ .$$

Computing the product we see that the condition $\alpha^*=\epsilon\alpha$ implies

$$\alpha_{12}(1-\nu_2)=-\nu_1^*\alpha_{12}\ ,\quad \alpha_{11}^*=\epsilon\alpha_{11}\quad\text{and}\quad\alpha_{22}^*=\epsilon\alpha_{22}\ .$$

From this we immediately deduce

$$\alpha_{12}(1-\nu_2)^k=(-\nu_1^*)^k\alpha_{12}$$

for any natural integer k. Since ν_1 and ν_2 are nilpotent, this implies that $\alpha_{12}=0$. Thus α is of the form

$$\begin{pmatrix}\alpha_{11}t(1+\nu_1)-\alpha_{11}\nu_1&0\\0&\alpha_{22}(1+(t-1)\nu_2)\end{pmatrix}$$

and $(P_0[t,t^{-1}],\alpha)$ splits as a hermitian space.

Since α, α_{11} and α_{22} are symmetric, evaluating the above matrix at $t=1$ we see that

$$\alpha_{11}\nu_1=\nu_1^*\alpha_{11}\quad\text{and}\quad\alpha_{22}\nu_1=\nu_2^*\alpha_{22}\ .$$

The first block can be written as

$$\sigma_1 = \alpha_{11} t(1 + \nu_1 - t^{-1}\nu_1) = \alpha_{11} t(1 + (1 - t^{-1})\nu_1)\ .$$

Since $(1 - t^{-1})\nu_1$ is nilpotent, the formal power series

$$\tau_1 = (1 + (1 - t^{-1})\nu_1)^{-1/2} = \sum \binom{-1/2}{k} ((1 - t^{-1})\nu_1)^k$$

is a Laurent polynomial and we can replace the first block by $\tau_1^* \sigma_1 \tau_1 = \alpha_{11} t$. Similarly, the power series

$$\tau_2 = (1 + (t - 1)\nu_2)^{-1/2} = \sum \binom{-1/2}{k} ((t - 1)\nu_2)^k$$

is a Laurent polynomial and we can replace the second block by $\tau_2^* \sigma_2 \tau_2 = \alpha_{22}$.

This shows that

$$(P_0[t, t^{-1}], \alpha) \simeq (P_0 e[t, t^{-1}], t\alpha_{11}) \perp (P_0(1 - e)[t, t^{-1}], \alpha_{22})\ ,$$

thus proving the surjectivity of ψ. □

6. The residue

In this section we construct a residue map

$$Res : W'(A[t, t^{-1}]) \to W(A)$$

satisfying R_1 and R_2 of § 5.

The definition of Res will be preceded by a few preliminaries.

Lemma 6.1. *Let P_0 be a (finitely generated) projective A-module and define $M(\alpha)$ by the exact sequence*

$$0 \longrightarrow P_0[t] \xrightarrow{\alpha} P_0^*[t] \longrightarrow M(\alpha) \longrightarrow 0\ , \tag{2}$$

where α is $A[t]$-linear. Suppose that its localization $\alpha_t : P_0[t, t^{-1}] \to P_0[t, t^{-1}]$ is an isomorphism. Then, as an A-module, $M(\alpha)$ is finitely generated and projective.

Proof. Decompose $P_0[t, t^{-1}]$ as a direct sum $P_0[t] \oplus t^{-1}P_0[t^{-1}]$ of A-modules. Let π be the projection onto the first summand. Then $\beta = \pi \circ \alpha_t^{-1}|_{P_0^*[t]}$ is an A-linear splitting of α. Hence $M(\alpha)$ is A-projective. It is also finitely generated as an $A[t]$-module, hence, being annihilated by a power of t, it is finitely generated as an A-module. □

Let $M = M(\alpha)$ be as in the previous lemma. Assume that α is ϵ-symmetric. We define a pairing

$$M \times M \to A[t,t^{-1}]/A[t]$$

by $<\overline{a},\overline{b}> = a(\alpha_t^{-1}(b))$, where a and b are representatives in $P_0^*[t]$ of $\overline{a},\overline{b} \in M$.

Lemma 6.2. *If α is ϵ-hermitian, then $<,>$ is a perfect ϵ-hermitian pairing.*

Proof. Since α_t is ϵ-hermitian, denoting by $x \mapsto x^\circ$ the involution on A we have

$$<\overline{a},\overline{b}> = a(\alpha_t^{-1}(b)) = \epsilon(b(\alpha_t^{-1}(a)))^\circ = \epsilon <\overline{b},\overline{a}>^\circ \quad .$$

This proves the first assertion.

We now check that the adjoint of $<,>$

$$\chi : M \to \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t]) \ ,$$

defined as $\chi(\overline{a})(\overline{b}) = <\overline{a},\overline{b}>$, is an isomorphism. We first prove injectivity. Suppose that, for some a and every x in M, $\chi(\overline{a})(\overline{x}) = 0$. This means that $a(\alpha_t^{-1}(x)) \in A[t]$ for every $x \in P_0^*[t]$. We only have to show that $\alpha_t^{-1}(a) \in P_0[t]$. Consider the diagram

$$\begin{array}{ccc}
P_0[t] & \xrightarrow{\sim} & \mathrm{Hom}_{A[t]}(P_0^*[t], A[t]) \\
\downarrow & & \downarrow \\
P_0[t,t^{-1}] & \xrightarrow{\sim} & \mathrm{Hom}_{A[t]}(P_0^*[t], A[t,t^{-1}])
\end{array}$$

where the horizontal arrows are the canonical ones. Since $P_0[t]$ is projective (and finitely generated!) over $A[t]$, they both are isomorphisms. Therefore an element $b \in P_0[t,t^{-1}]$ is in $P_0[t]$ if and only if, for any $x \in P_0^*[t]$, $x(b)$ is in $A[t]$. This is indeed the case for $b = \alpha_t^{-1}(a)$ because $x(\alpha_t^{-1}(a)) = \epsilon(a(\alpha_t^{-1}(x)))^\circ \in A[t]$ by the very assumption on a. Thus injectivity is proved. We now check that χ is surjective. Let $\overline{f} : M \to A[t,t^{-1}]/A[t]$ be an $A[t]$-linear map. Since $P_0[t]^*$ is projective, there exits an f which makes the right hand square of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_0[t] & \xrightarrow{\alpha} & P_0[t]^* & \xrightarrow{p} & M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \overline{f}} & & \\
0 & \longrightarrow & A[t] & \longrightarrow & A[t,t^{-1}] & \xrightarrow{q} & A[t,t^{-1}]/A[t] & \longrightarrow & 0
\end{array}$$

commute, p and q being the canonical surjections. Clearly $q \circ f \circ \alpha = 0$, hence there exists an $A[t]$-linear map $a : P_0[t] \to A[t]$ such $f \circ \alpha = i \circ a$, i being the inclusion $A[t] \to A[t, t^{-1}]$. We claim that $\chi(a) = \overline{f}$. For this it suffices to show that for any $b \in P_0[t]^*$ we have $a(\alpha_t^{-1}(b)) \equiv f(b)$ modulo $A[t]$. We denote by a_t the localization of a at t and by $f_t : P_0[t, t^{-1}]^* \to A[t, t^{-1}]$ the unique $A[t, t^{-1}]$-linear extension of f. Observing that $\alpha_t^{-1}(a) = a_t \circ \alpha_t^{-1}$ we get the following relations:

$$a(\alpha_t^{-1}(b)) = (a_t \circ \alpha_t^{-1})(b) = f_t(b) = f(b) \, .$$

This proves that χ is surjective. $\square$

Let now $(P_0[t, t^{-1}], \alpha)$ be an ϵ-hermitian space. For any natural integer n for which $t^{2n}\alpha(P_0[t]) \subseteq P_0[t]^*$ we define $M(\alpha, n)$ by the exact sequence

$$0 \longrightarrow P_0[t] \xrightarrow{t^{2n}\alpha} P_0^*[t] \longrightarrow M(\alpha, n) \longrightarrow 0$$

and equip it with the ϵ-hermitian structure defined above:

$$<\overline{a}, \overline{b}> = a((t^{2n}\alpha_t)^{-1}(b)) \, .$$

Lemma 6.3. *Let $\psi : (P_0[t, t^{-1}], \alpha) \to (Q_0[t, t^{-1}], \beta)$ be an isometry and assume that $\psi(P_0[t]) \subseteq Q_0[t]$, $\alpha(P_0[t]) \subseteq P_0[t]^*$ and $\beta(Q_0[t]) \subseteq Q_0[t]^*$. Then $M(\alpha)$ and $M(\beta)$ are Witt equivalent t-torsion spaces.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & 0 & & K & & & & \\
 & & \downarrow & & \uparrow{\scriptstyle \hat{q}} & & & & \\
0 & \longrightarrow & P_0[t] & \xrightarrow{\alpha} & P_0[t]^* & \xrightarrow{q_\alpha} & M(\alpha) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \psi^*} & & & & \\
0 & \longrightarrow & Q_0[t] & \xrightarrow{\beta} & Q_0[t]^* & \xrightarrow{q_\beta} & M(\beta) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q} & & \uparrow & & & & \\
 & & L & & 0 & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & & .
\end{array}$$

By Lemma 6.1 the module L, viewed as an A-module, is finitely generated and projective. The map ψ^* is obtained from the map ψ by dualizing over $A[t]$. We denote the cokernel of ψ^* by K and we denote the canonical map $P_0[t]^* \to K$ by $\hat{q}$. One may observe that K is isomorphic to $L^\sharp$ (see §4 for the notation) but we will not use this observation.

The $A[t]$-linear map $\theta = q_\alpha \circ \psi^* : Q_0[t]^* \to M(\alpha)$ induces a map $\overline{\theta} : M(\beta) \to \theta(Q_0[t]^*)/\theta(\beta(Q_0[t]))$. The statement will be deduced from the following claims.

(1) The map $\overline{\theta}$ is an $A[t]$-linear isomorphism.

(2) The map $\hat{q}$ induces an $A[t]$-linear isomorphism

$$\rho : M(\alpha)/\theta(Q_0[t]^*) \to K \, .$$

(3) $\theta(\beta(Q_0[t]))$ is a sublagrangian of $M(\alpha)$.

(4) $(\theta(\beta(Q_0[t]))^\perp = \theta(Q_0[t]^*)$.

(5) The map $\overline{\theta}$ is an isometry of t-torsion spaces.

In fact, by (4), (5) and Theorem 4.5, $M(\beta)$ is Witt equivalent to $M(\alpha)$.

We now prove the claims. The surjectivity of $\overline{\theta}$ is clear. To show injectivity, suppose that $x \in \ker(\theta)$. Choose a lift $\widetilde{x} \in Q_0[t]^*$ of x. There exist a $y \in Q_0[t]$ and a $z \in P_0[t]$ such that $\psi^*(\beta(y) - \widetilde{x}) = \alpha(z)$. Replacing α by $\psi^* \circ \beta \circ \psi$ we get $\psi^*(\widetilde{x}) = \psi^*(\beta(y - \psi(z)))$. Since ψ^* is injective, this shows that $\widetilde{x} \in \operatorname{Im}(\beta)$ and hence $x = 0$.

To prove (2) observe that, since $\hat{q} \circ \alpha = \hat{q} \circ \psi^* \circ \beta \circ \psi = 0$, $\hat{q}$ induces a surjective map $\rho : M(\alpha)/\theta(Q_0[t]^*) \to K$. Injectivity is also clear.

To prove (3) we first observe that $\theta(\beta(Q_0[t]))$ is a direct factor (as an A-module) of $M(\alpha)$. In fact, by (2), $\theta(Q_0[t]^*)$ is a direct factor (as an A-module) of $M(\alpha)$ and, by (1), $\theta(\beta(Q_0[t]))$ is a direct factor of $\theta(Q_0[t]^*)$. For any two elements $a, b \in P_0[t]^*$ let us denote by $<a, b>_\alpha$ the element $a(\alpha_t^{-1}(b))$, and similarly for $<a, b>_\beta$. We then have

$$<a, b>_\beta = <\psi^*(a), \psi^*(b)>_\alpha$$

because ψ_t is an isometry. Let now $\overline{a}, \overline{b} \in \theta(\beta(Q_0[t]))$ and $x, y \in Q_0[t]$ such that $a = \psi^*(\beta(x))$ and $b = \psi^*(\beta(y))$ are preimages of a and b. We have to check that $<\overline{a}, \overline{b}> = 0$. This is the same as saying that $<a, b>_\alpha$ is in $A[t]$. This is indeed the case because

$$<a, b>_\alpha = <\psi^*(\beta(x)), \psi^*(\beta(y))>_\alpha = <\beta(x), \beta(y)>_\beta = \beta(x)(y) \in A[t] \, .$$

We now prove (4). For any $\overline{a} \in \theta(\beta(Q_0[t]))$ and any $\overline{b} \in M(\alpha)$ we choose preimages a and b of the form $a = \psi^*(\beta(x))$ and $b = \psi_t^*(y)$ with $x \in Q_0[t]$ and $y \in Q_0[t, t^{-1}]^*$. Then we have

$$< a, b >_\alpha = < \psi^*(\beta(x)), \psi_t^*(y) >_\alpha = < \beta(x), y >_\beta = \epsilon \cdot y(x)^\circ ,$$

which shows that, for any $y \in Q_0[t, t^{-1}]^*$, $< \psi^*(\beta(Q_0[t])), b >_\alpha$ is in $A[t]$ if and only if $y \in Q_0[t]^*$, which is equivalent to $\overline{b} \in \theta(Q_0[t]^*)$.

We now prove (5). We already know that $\overline{\theta}$ is an $A[t]$-linear isomorphism. A computation like the one above proves that it is an isometry. □

Corollary 6.4. *Let $(P_0[t, t^{-1}], \alpha)$ be an ϵ-hermitian space. Let n be such that $t^{2n}\alpha(P_0[t]) \subseteq P_0[t]^*$. Then the class of $M(\alpha, n)$ in $W_{tors}(A[t])$ does not depend on the choice of n.*

Corollary 6.5. *Let $(P_0[t, t^{-1}], \alpha)$ and $(P_0[t, t^{-1}], \beta)$ be isometric spaces and assume that for some natural integers m and n $t^{2m}\alpha(P_0[t]) \subseteq P_0[t]^*$ and $t^{2n}\beta(P_0[t]) \subseteq P_0[t]^*$. Then $M(\alpha, m)$ and $M(\beta, n)$ are Witt equivalent t-torsion spaces.*

Proof. Let $\psi : (P_0[t, t^{-1}], t^{2m}\alpha) \to (P_0[t, t^{-1}], t^{2n}\beta)$ be an isometry and let k be a natural integer such that $t^k\psi(P_0[t]) \subseteq P_0[t]^*$. Then $t^k\psi : (P_0[t, t^{-1}], t^{2m}\alpha) \to (P_0[t, t^{-1}], t^{2n+2k}\beta)$ is an isometry and, by Lemma 6.3, $M(\alpha, m)$ and $M(\beta, n + k)$ are Witt equivalent. Hence, by Corollary 6.4, $M(\alpha, m)$ and $M(\beta, n)$ are Witt equivalent as well. □

Proposition 6.6. *Associating to any space $(P_0[t, t^{-1}], \alpha)$ the torsion space $M(\alpha, n)$ (for a suitable n) yields a homomorphism*

$$res : W'(A[t, t^{-1}]) \to W_{tors}(A[t]) .$$

Proof. By Corollary 6.5, associating to the isometry class of a space $(P_0[t, t^{-1}], \alpha)$ the Witt class of the t-torsion space $M(\alpha, n)$ for some suitable n is a well defined map. It is obvious that the orthogonal sum of two spaces is mapped to the corresponding sum of t-torsion spaces, hence this map induces a homomorphism $\omega : K_H \to W_{tors}(A[t])$, where K_H is the Grothendieck group of ϵ-hermitian spaces of the form $(P_0[t, t^{-1}], \alpha)$. It is clear from the definition of $M(\alpha, n)$ that a standard hyperbolic space $H(Q_0[t, t^{-1}])$ is mapped to zero, hence ω induces a homomorphism $res : W'(A[t, t^{-1}]) \to W_{tors}(A[t])$. □

If we compose res with $\partial^W : W_{tors}(A[t]) \to W(A)$ we get a homomorphism

$$Res = \partial^W \circ res : W'(A[t, t^{-1}]) \to W(A)$$

which we call *residue*.

THEOREM 6.7. *The residue*

$$Res : W'(A[t, t^{-1}]) \to W(A)$$

satisfies the following two properties:

R_1 *: For any constant space* $\xi \in W(A) \subset W(A[t, t^{-1}])$ *,* $Res(\xi) = 0$ *.*

R_2 *: For any constant space* $\xi \in W(A)$ *,* $Res(t \cdot \xi) = \xi$ *.*

Proof. The two properties immediately follow from the construction of res. □

An amusing application of the existence of Res is the following result.

PROPOSITION 6.8. *Let* A *be a commutative semilocal ring in which* 2 *is invertible. Let* (P, α) *be a quadratic space over* A *. If* (P, α) *is isometric to* $(P, t \cdot \alpha)$ *over* $A[t, t^{-1}]$ *, then* (P, α) *is hyperbolic.*

Proof. Let ξ be the class of (P, α) in $W(A)$. In $W'(A[t])$ we have $\xi = t \cdot \xi$. Applying Res to both sides we obtain $\xi = 0$. Since A is semilocal, by Witt's cancelletion theorem we conclude that (P, α) is hyperbolic. □

7. THE WITT GROUP OF LAURENT POLYNOMIALS

Let $W'(A[t, t^{-1}])$ be the group defined in the introduction.

THEOREM 7.1. *Let* A *be an associative ring with involution in which* 2 *is invertible. Let*

$$\varphi : W'(A[t, t^{-1}]) \to W(A[t, t^{-1}])$$

be the canonical homomorphism.

(a) *If* $H^2(\mathbb{Z}/2, K_{-1}(A)) = 0$ *, then* φ *is surjective.*

(b) *If* $K_0(A) = K_0(A[t]) = K_0(A[t, t^{-1}])$ *, then* φ *is an isomorphism.*

Proof of (a). Corollary 2.4 implies that

$$H^2(\mathbb{Z}/2, K_0(A[t,t^{-1}])/K_0(A)) = 0\ .$$

This means that every projective $A[t,t^{-1}]$-module P is in the same class as some projective module of the form

$$P_0[t,t^{-1}] \oplus Q \oplus Q^*\ ,$$

where P_0 is a projective A-module. Therefore, adding to a space (P,α) a hyperbolic space $H(Q')$ with $Q \oplus Q'$ free, we may assume that P is of the form $P_0[t,t^{-1}]$. This means precisely that the class of (P,α) is in the image of $W'(A[t,t^{-1}])$. □

Proof of (b). Surjectivity is obvious, because by assumption every projective $A[t,t^{-1}]$-module is stably extended from A. Suppose that the class of a space $(P_0[t,t^{-1}],\alpha)$ vanishes in $W(A[t,t^{-1}])$. This means that, for some Q and R, there exists an isometry

$$(P_0[t,t^{-1}],\alpha) \perp H(Q) \simeq H(R)\ .$$

Adding to both sides a suitable $H(A[t,t^{-1}]^n)$ we may replace Q and R by extended modules $Q_0[t,t^{-1}]$ and $R_0[t,t^{-1}]$. Then the isometry means precisely that the class of $(P_0[t,t^{-1}],\alpha)$ vanishes in $W'(A[t,t^{-1}])$. □

We can restate assertion (b) of Theorem 7.1 as follows.

THEOREM 7.2. *Let A be an associative ring with involution, in which 2 is invertible. Assume that $K_0(A) = K_0(A[t]) = K_0(A[t,t^{-1}])$. Then there exists a natural homomorphism Res such that the sequence*

$$0 \longrightarrow W(A) \longrightarrow W(A[t,t^{-1}]) \xrightarrow{Res} W(A) \longrightarrow 0$$

is split exact. The homomorphism Res restricts to an isomorphism of $t \cdot W(A)$ onto $W(A)$.

8. Two counterexamples

In this section we show that in general the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ is neither surjective nor injective.

EXAMPLE 8.1. We first recall the Mayer-Vietoris sequence associated to a cartesian sqare of commutative rings (see [1], Ch. IX, Corollary 5.12). Let

$$\begin{array}{ccc} R & \longrightarrow & S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \overline{R} & \longrightarrow & \overline{S} \end{array}$$

be a cartesian dagram of commutative rings, with f or g surjective. Denote by $\widetilde{K_0}$ the kernel of the rank function on K_0. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} K_1(\overline{R}) \times K_1(S) & \longrightarrow & K_1(\overline{S}) & \longrightarrow & \widetilde{K_0}(R) & \longrightarrow & \widetilde{K_0}(\overline{R}) \times \widetilde{K_0}(S) & \longrightarrow & \widetilde{K_0}(\overline{S}) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \bigwedge^{\max}} & & \downarrow{\scriptstyle \bigwedge^{\max}} & & \downarrow{\scriptstyle \bigwedge^{\max}} \\ \mathbb{G}_m(\overline{R}) \times \mathbb{G}_m(S) & \longrightarrow & \mathbb{G}_m(\overline{S}) & \longrightarrow & Pic(R) & \longrightarrow & Pic(\overline{R}) \times Pic(S) & \longrightarrow & Pic(\overline{S}) \, . \end{array}$$

Let A be the local ring at the origin of the complex plane curve $Y^2 = X^2 - X^3$, $\widetilde{A}$ the normalisation of A and $\mathfrak{c}$ the conductor of $\widetilde{A}$ in A. Applying the big diagram above to the cartesian squares

$$\begin{array}{ccc} A & \longrightarrow & \widetilde{A} \\ \downarrow & & \downarrow \\ (A/\mathfrak{c}) & \longrightarrow & (\widetilde{A}/\mathfrak{c}) \end{array} \quad \text{and} \quad \begin{array}{ccc} A[t,t^{-1}] & \longrightarrow & \widetilde{A}[t,t^{-1}] \\ \downarrow & & \downarrow \\ (A/\mathfrak{c})[t,t^{-1}] & \longrightarrow & (\widetilde{A}/\mathfrak{c})[t,t^{-1}] \, , \end{array}$$

it is easy to see that $\widetilde{K_0}(A[t,t^{-1}]) = \mathbb{C}^* \oplus \mathbb{Z} = Pic(A[t,t^{-1}])$. This shows that a projective $A[t,t^{-1}]$-module P is stably free if and only if its maximal exterior power $\bigwedge^{\max}(P)$ is isomorphic to $A[t,t^{-1}]$.

Let I be an ideal representing $(1,1)$ in $\mathbb{C}^* \oplus \mathbb{Z} = Pic(A[t,t^{-1}])$. The module underlying the space $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ is free. In fact it is stably free because its determinant is trivial, hence, by a well-known cancellation theorem it is free. This shows that $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ is a quadratic space of the form $(P_0[t,t^{-1}], \alpha)$ with P_0 free of rank 6 over A. Clearly this space represents the zero element of $W(A[t,t^{-1}])$. We claim that its class in $W'(A[t,t^{-1}])$ is not trivial.

Since A is local, projective modules extended from A are free. If $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ were hyperbolic in $W'(A[t,t^{-1}])$ it would be stably isometric to $H(A[t,t^{-1}] \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ and hence, by the quadratic cancellation theorem (see [4],VI, 6.2.5) it would be isometric to it. Recall that, for any commutative ring R in which 2 is invertible and any finitely generated projective R-module P, the even Clifford algebra C_0 of $H(P)$ is of the form

$$C_0 = \mathrm{End}_R(\textstyle\bigwedge^{even}(P)) \times \mathrm{End}_R(\textstyle\bigwedge^{odd}(P)) \ ,$$

where $\bigwedge^{even}(P)$ (respectively $\bigwedge^{odd}(P)$)) is the even (respectively odd) part of the exterior algebra of P. In the case $P = I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}]$ we have

$$C_0 = \mathrm{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2) \times \mathrm{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2) \ .$$

Suppose now that $H(I \oplus A[t,t^{-1}]^2)$ and $H(A[t,t^{-1}]^3)$ are isometric. In this case their even Clifford algebras would be isomorphic, hence the algebra $\mathrm{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2)$ would be a 4×4 matrix algebra. By Morita theory the module $A[t,t^{-1}]^2 \oplus I^2$ would be of the form J^4 for some invertible ideal J. Taking the fourth exterior power of both sides we would have $I^2 = J^4$, which is impossible because I represents $(1,1)$ in $\mathbb{C}^* \oplus \mathbb{Z}$.

This shows that, even for a one-dimensional local domain, the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ may fail to be injective.

Example 8.2. We define a commutative ring A by the cartesian diagram of real algebras

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{R}[X,Y] \\ \downarrow & & \downarrow \pi \\ \mathbb{R} & \xrightarrow{\ \iota\ } & C \ , \end{array} \tag{2}$$

where $C = \mathbb{R}[x,y] = \mathbb{R}[X,Y]/(X^2+Y^2-1)$, π is the canonical projection and ι the canonical injection. Then $C \oplus C$ is the direct sum of its two submodules

$$P = C\tfrac{1}{2}(y+1,-x)+C\tfrac{1}{2}(-x,1-y) \quad \text{and} \quad P' = C\tfrac{1}{2}(1-y,x)+C\tfrac{1}{2}(x,1+y)$$

and we can define an automorphism α of $C[t,t^{-1}] \oplus C[t,t^{-1}]$ as the identity on P' and multiplication by t on P. With respect to the canonical basis of $C[t,t^{-1}] \oplus C[t,t^{-1}]$

$$\alpha = \tfrac{1}{2}\begin{pmatrix} t(1+y)+1-y & -tx+x \\ -tx+x & t(1-y)+1+y \end{pmatrix} \ .$$

The matrix α has determinant equal to t and thus lies in $GL_2(C[t,t^{-1}])$. According to Theorem 7.4 of [1] its class in $K_1(C[t,t^{-1}])$ is the image of

P by the canonical injection λ mentioned in §2. It is easy to see that P is not free over C. In fact it turns out to represent the non trivial class of $\mathrm{Pic}(C) = \mathbb{Z}/2$. Since the homomorphism ι in the cartesian square that defines A is surjective, tensoring the diagram with $\mathbb{R}[t, t^{-1}]$ yields a Milnor patching diagram

$$\begin{array}{ccc} A[t,t^{-1}] & \longrightarrow & \mathbb{R}[X,Y][t,t^{-1}] \\ \downarrow & & \downarrow \pi \\ \mathbb{R}[t,t^{-1}] & \xrightarrow{\ \iota\ } & C[t,t^{-1}] \ . \end{array}$$

We can use this diagram and the matrix α (see for instance [1], Chapter IX, theorem 5.1) to patch a rank 2 free module Q over $\mathbb{R}[X,Y][t,t^{-1}]$ with a rank 2 free module R over $\mathbb{R}[t,t^{-1}]$ and get a rank 2 projective module

$$M = \{(q,r) \in Q \times R \mid \alpha(\pi_*(q)) = \iota_*(r)\}$$

over $A[t,t^{-1}]$. We now equip M with a skew-symmetric structure. To do this we put on Q and on R the skew-symmetric structures defined, respectively, by the matrices

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1/t \\ -1/t & 0 \end{pmatrix} .$$

Since $\alpha^* \tau \alpha = \sigma$, the skew-symmetric structures $\sigma : Q \to Q^*$ and $\tau : R \to R^*$ are compatible with the patching and therefore they define a skew-symmetric structure $\varphi : M \to M^*$ on M.

We claim that the class of this space is not in the image of $W'([t,t^{-1}])$. Extending to K_{-1} the Mayer-Vietoris sequence associated to (2) (see [1], Chapter XII, Theorem 8.3) we get an exact sequence

$$K_0(\mathbb{R}[X,Y]) \oplus K_0(\mathbb{R}) \to K_0(C) \to K_{-1}(A) \to K_{-1}(\mathbb{R}[X,Y]) \oplus K_{-1}(\mathbb{R}) .$$

From the fact that regular rings have a vanishing K_{-1}, that $K_0(\mathbb{R}[X,Y]) = K_0(\mathbb{R}) = \mathbb{Z}$ and that $K_0(C) = \mathbb{Z} \oplus \mathbb{Z}/2$ where the element of order 2 is the class of P, we easily deduce that $K_{-1}(A) = \mathbb{Z}/2$, generated by the image of M. Thus, by Corollary 2.4, the class of M generates $H^2(\mathbb{Z}/2, K_0(A[t,t^{-1}])/K_0(A)) = \mathbb{Z}/2$. Consider now the homomorphism

$$\omega : W(A[t,t^{-1}]) \longrightarrow H^2(\mathbb{Z}/2, K_0(A[t,t^{-1}])/K_0(A))$$

obtained by associating to any space its underlying projective module. Since $\omega((M,\varphi)) \neq 0$, (M,φ) cannot be Witt equivalent to a space supported by a module extended from A. This shows that the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ is not surjective.

REMARK 8.3. We suspect that even if the assumption of (a) is satisfied the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ may not be injective, but we did not find an example to confirm our suspicion.

ACKNOWLEDGMENT. We warmly thank Paul Balmer for carefully reading various versions of this paper, dramatically reducing our output of mistakes.

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