Witt groups of the punctured spectrum of a 3-dimensional regular local ring and a purity theorem

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Introduction

Let A be a regular local ring with quotient field K. Assume that 2 is invertible in A. Let $W(A) \to W(K)$ be the homomorphism induced by the inclusion $A \hookrightarrow K$, where $W(\ )$ denotes the Witt group of quadratic forms. If $\dim A \leq 4$, it is known that this map is injective ([**6**], [**7**]). A natural question is to characterise the image of $W(A)$ in $W(K)$. Let $\mathrm{Spec}^1(A)$ be the set of prime ideals of A of height 1. For $P \in \mathrm{Spec}^1(A)$, let π_P be a parameter of the discrete valuation ring A_P and $k(P) = A_P/PA_P$. For this choice of a parameter π_P, one has the *second residue homomorphism* $\partial_P : W(K) \to W(k(P))$ ([**9**], p. 209). Though the homomorphism ∂_P depends on the choice of the parameter π_P, its kernel and cokernel do not. We have a homomorphism

$$\partial = (\partial_P) : W(K) \to \bigoplus_{P \in \mathrm{Spec}^1(A)} W(k(P)).$$

A part of the so-called Gersten conjecture is the following question on "purity":
Is the sequence

$$W(A) \to W(K) \xrightarrow{\partial} \bigoplus_{P \in \mathrm{Spec}^1(A)} W(k(P))$$

exact ?
The above question has an affirmative answer for $\dim(A) \leq 2$ ([**3**], p.277,

[**1**]). There have been speculations by Pardon and Barge-Sansuc-Vogel on the question of purity. However, in the literature, there is no proof for purity even for $\dim(A) = 3$. One of the consequences of the main result of this paper is an affirmative answer to the purity question for $\dim(A) = 3$.

We briefly outline our main result. For any scheme X let $W^\epsilon(X)$ denote the Witt group of ϵ-symmetric spaces on X, $\epsilon = \pm 1$ ($W^{+1}(X) = W(X)$ being the usual Witt group of symmetric spaces over X). Let A be a regular local ring of dimension 3 with maximal ideal m and $Y = \mathrm{Spec}(A) \setminus \{m\}$. We associate (§2) to an ϵ-symmetric space over Y a $(-\epsilon)$-symmetric space over a finite length A-module. This assignment leads to a homomorphism $W^\epsilon(Y) \to W_{fl}^{-\epsilon}(A)$, where $W_{fl}^\epsilon(A)$ is the Witt group of ϵ-symmetric spaces of finite length A-modules (cf §0). Then we prove (§3) that the sequence

$$0 \to W^\epsilon(A) \to W^\epsilon(Y) \to W_{fl}^{-\epsilon}(A) \to 0$$

is exact, where the map $W^\epsilon(A) \to W^\epsilon(Y)$ is induced by the restriction. Since $W_{fl}^\epsilon(A) \simeq W^\epsilon(A/m)$, it follows that $W_{fl}^{-1}(A) = 0$. Thus the map $W(A) \to W(Y)$ is an isomorphism. This leads to the purity theorem for the Witt groups. On the other hand, since every skew-symmetric space over A is hyperbolic, $W^{-1}(A) = 0$ and we get an isomorphism $W^{-1}(Y) \simeq W(A/m)$. We observe the curious fact that if A is complete, $W^{\pm 1}(Y)$ is isomorphic to $W(A/m)$.

A crucial result used in our proof of the main theorem is a theorem of Horrocks ([**2**]) on vector bundles on the punctured spectrum $Y = \mathrm{Spec}(A) \setminus \{m\}$, where A is a regular local ring of dimension 3 and m its maximal ideal. We use his theorem on the equivalence of the category of "Φ-equivalence" classes of vector bundles on Y with the category of finite length A-modules.

We would like to remark parenthetically that purity for dimension 3 was used in [**8**] while establishing the equivalence of finite generation of Witt groups of affine real 3-folds and the finite generation of Chow groups of codimension 2 cycles modulo 2.

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0. ϵ-symmetric spaces reminisced

Let A be a regular local ring of dimension three in which 2 is invertible. We recall the definition of ϵ-symmetric spaces on finite length A-modules and their Witt groups. For A-modules M, N and $i \geq 0$, let $\mathrm{Ext}^i(M,N)$ denote the group of congruence classes of i-fold extensions of N by M ([**4**], p. 84). For any homomorphism $f : M \to M'$ of A-modules, let $\mathrm{Ext}^i(N,f) : \mathrm{Ext}^i(N,M) \to \mathrm{Ext}^i(N,M')$ be the induced homomorphisms defined as follows: Let

$$\zeta = 0 \to M \xrightarrow{\alpha} Z_i \xrightarrow{\partial_i} Z_{i-1} \to \cdots \to Z_2 \xrightarrow{\partial_2} Z_1 \xrightarrow{\beta} N \to 0$$

be an i-fold extension of N by M. Let $Z = (Z_i \oplus M')/(\{(\alpha(x), f(x)) \mid x \in M\})$ be the push-out of the diagram ([**4**])

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & Z_i \\ \downarrow f & & \\ M' & & . \end{array}$$

Then

$$\mathrm{Ext}^i(N,f)(\zeta) = 0 \to M' \xrightarrow{\alpha'} Z \xrightarrow{\partial'} Z_{i-1} \xrightarrow{\partial_{i-1}} \cdots \to Z_2 \xrightarrow{\partial_2} Z_1 \xrightarrow{\beta} N \to 0,$$

where α' and ∂' are the natural homomorphisms induced by the push-out. Similarly, we define $\mathrm{Ext}^i(f,N)$ as the pull-back under f of an i-fold extension of N by M'. Let M be a finite length A-module and $M^\vee = \mathrm{Ext}^3(M,A)$. If M, M' are two finite length A-modules and $f : M \to M'$ an A-linear map, then we denote $\mathrm{Ext}^3(f,A)$ by $f^\vee$. Let

$$\mathcal{P} = 0 \to P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\theta} M \to 0$$

be a projective resolution of M. Since $\mathrm{Ext}^i(M,A) = 0$ for $i = 0,1,2$ ([**5**], Th. 18.1), by dualising the above exact sequence we see that

$$\mathcal{P}^* = 0 \to P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\partial_2^*} P_2^* \xrightarrow{\partial_3^*} P_3^* \xrightarrow{\theta'} M^\vee \to 0$$

is a projective resolution of $M^\vee$, where $P_i^* = \mathrm{Hom}_A(P_i, A)$, ∂_i^* is induced by ∂_i and for any $f \in P_3^*$,

$$\theta'(f) = \mathrm{Ext}^3(f,M)(\mathcal{P}) \in M^\vee.$$

Throughout this paper, for any surjection $\theta : P_0 \to M$ as above, θ' denotes the map defined as above. We define a canonical homomorphism $\mathcal{C} : M \to M^{\vee\vee}$ as follows : Let $x \in M$. Choose $y \in P_0$ such that $\theta(y) = x$. We define

$\mathcal{C}(x) = \mathrm{Ext}^3(-e_y, M^\vee)(\mathcal{P}^*) \in M^{\vee\vee}$, where, for $f \in P_0^*$, $e_y(f) = f(y)$. Then it is easy to see that $\mathcal{C}(x)$ is independent of the choice of y and the diagram

$$\begin{array}{ccccccccccccc} 0 & \to & P_3 & \xrightarrow{\partial_3} & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\theta} & M & \to & 0 \\ & & \downarrow\mathcal{C} & & \downarrow\mathcal{C} & & \downarrow\mathcal{C} & & \downarrow\mathcal{C} & & \downarrow -\mathcal{C} & & \\ 0 & \to & P_3^{**} & \xrightarrow{\partial_3^{**}} & P_2^{**} & \xrightarrow{\partial_2^{**}} & P_1^{**} & \xrightarrow{\partial_1^{**}} & P_0^{**} & \xrightarrow{\theta''} & M^{\vee\vee} & \to & 0 \end{array} \qquad \dagger$$

is commutative, where $\mathcal{C} : P_i \to P_i^{**}$ are the canonical isomorphisms. Thus $\mathcal{C} : M \to M^{\vee\vee}$ is an isomorphism and it is obvious that it is independent of the choice of the projective resolution. We use this isomorphism to identify M with $M^{\vee\vee}$. The choice of the *negative* sign at e_y in the definition of $\mathcal{C}$ is explained in the following : Let $m = (x_1, x_2, x_3)$ be the maximal ideal of A and

$$\zeta = 0 \to A \xrightarrow{\delta_3} A^3 \xrightarrow{\delta_2} A^3 \xrightarrow{\delta_1} A \xrightarrow{\eta} A/m \to 0$$

be the Koszul resolution of A/m with respect to (x_1, x_2, x_3). With respect to the standard basis $\{e_1, e_2, e_3\}$ of A^3, we have

$$\delta_1 = (\, x_1 \quad x_2 \quad x_3 \,), \quad \delta_2 = \begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix}$$

and $\eta : A \to A/m$ is the natural homomorphism. Let M be a finite dimensional vector space over A/m. Then M is a finite length A-module. Let $\widetilde{M} = \mathrm{Hom}(M, A/m)$. The assignment $f \mapsto \mathrm{Ext}^3(f, A)(\zeta) \in M^\vee$ induces a homomorphism

$$\Phi_M : \widetilde{M} \to M^\vee.$$

The following lemmas are well known, but for the sake of completeness we will give their proofs here.

Lemma 0.1 The homomorphism Φ_M is an isomorphism and the diagram

$$\begin{array}{ccc} M & \xrightarrow{\mathcal{C}} & M^{\vee\vee} \\ \downarrow\iota & & \downarrow\Phi_M^\vee \\ \widetilde{\widetilde{M}} & \xrightarrow{\Phi_{\widetilde{M}}} & (\widetilde{M})^\vee \end{array}$$

is commutative, where $\iota : M \to \widetilde{\widetilde{M}}$ is the canonical isomorphism.

Proof. Since $M \simeq \oplus_1^n A/m$, $M^\vee \simeq \oplus_1^n (A/m)^\vee$ and $\widetilde{M} \simeq \oplus_1^n \widetilde{A/m}$, it is enough to prove the lemma in the case when $M = A/m$. In this case it is easy to see that $\Phi_M \neq 0$. Since $M^\vee \simeq A/m$ ([**5**], Th 18.1) and $\widetilde{M} \simeq A/m$, Φ_M is an isomorphism. We now prove the commutativity of the diagram. For all $x \in M$ and $f \in \widetilde{M}$ we have $\iota(x)(f) = f(x)$ and

$$\Phi_{\widetilde{M}}(\iota(x)) = 0 \to A \xrightarrow{\delta_3} A^3 \xrightarrow{\delta_2} A^3 \xrightarrow{\delta_1} A \xrightarrow{\iota(x)^{-1}\eta} A/m \to 0.$$

Let $y \in A$ be such that $\eta(y) = x$. Then we have

$$\mathcal{C}(x) = 0 \to A \xrightarrow{-\delta_1^* e_y^{-1}} A^{3*} \xrightarrow{\delta_2^*} A^{3*} \xrightarrow{\delta_3^*} A^* \xrightarrow{\eta'} (A/m)^\vee \to 0.$$

Since Φ_M is an isomorphism, we have

$$\Phi_M^\vee(\mathcal{C}(x)) = 0 \to A \xrightarrow{-\delta_1^* e_y^{-1}} A^{3*} \xrightarrow{\delta_2^*} A^{3*} \xrightarrow{\delta_3^*} A^* \xrightarrow{\Phi_M^{-1}\eta'} (A/m)^\vee \to 0.$$

Let $\{e_1^*, e_2^*, e_3^*\}$ be the dual basis of A^{3*}. For $i = 1, 2$, let $\theta_i : A^3 \to A^{3*}$ be given by the following matrices, with respect to the bases $\{e_1, e_2, e_3\}$ and $\{e_1^*, e_2^*, e_3^*\}$.

$$\theta_1 = \begin{pmatrix} 0 & 0 & -y^{-1} \\ 0 & y^{-1} & 0 \\ -y^{-1} & 0 & 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & 0 & y^{-1} \\ 0 & -y^{-1} & 0 \\ y^{-1} & 0 & 0 \end{pmatrix}.$$

Let $\theta_3 : A \to A^*$ be the homomorphism defined by $\theta_3(1) = l_y$, where $l_y(a) = ay$ for all $a \in A$. It is easy to see that the following diagram

$$\begin{array}{ccccccccccc} 0\to & A & \xrightarrow{\delta_3} & A^3 & \xrightarrow{\delta_2} & A^3 & \xrightarrow{\delta_1} & A & \xrightarrow{\iota(x)^{-1}\eta} & (A/m)^\vee & \to 0 \\ & \downarrow id & & \downarrow\theta_1 & & \downarrow\theta_2 & & \downarrow\theta_3 & & \downarrow id & \\ 0\to & A & \xrightarrow{-\delta_1^* e_y^{-1}} & A^{3*} & \xrightarrow{\delta_2^*} & A^{3*} & \xrightarrow{\delta_3^*} & A^* & \xrightarrow{\Phi_M^{-1}\eta'} & (A/m)^\vee & \to 0 \end{array}$$

is commutative. Thus $\Phi_{\widetilde{M}}\ \iota = \Phi_M^\vee \mathcal{C}$. □

Lemma 0.2 Let $\psi : M \to \widetilde{M}$ be a homomorphism and $\widetilde{\psi} : \widetilde{\widetilde{M}} \to \widetilde{M}$ the induced homomorphism. Then the following diagram

$$\begin{array}{ccc} \widetilde{\widetilde{M}} & \xrightarrow{\Phi_{\widetilde{M}}} & (\widetilde{M})^\vee \\ \downarrow\widetilde{\psi} & & \downarrow\psi^\vee \\ \widetilde{M} & \xrightarrow{\Phi_M} & M^\vee \end{array}$$

is commutative.

Proof. Let $f \in \widetilde{\widetilde{M}}$. Then $\Phi_{\widetilde{M}}(f)$ is the pull-back of the Koszul resolution ζ under $f : \widetilde{M} \rightarrow A/m$ and $\psi^\vee(\Phi_{\widetilde{M}}(f))$ is the pull-back of the extension $\Phi_{\widetilde{M}}(f)$ under ψ. Thus $\psi^\vee(\Phi_{\widetilde{M}}(f))$ is the pull-back of the Koszul resolution under the homomorphism $f\psi : M \rightarrow A/m$. Since $\widetilde{\psi}(f) = f\psi$, $\Phi_M(\widetilde{\psi})(f)$ is the pull-back of the Koszul resolution under $f\psi$. Thus $\Phi_M \psi^\vee = \psi^\vee \Phi_{\widetilde{M}}$. $\square$

The above lemmas enable us to embed the category of ϵ-symmetric spaces on finite dimensional A/m-vector spaces into the category of ϵ-symmetric spaces on finite length A-modules (cf (0.3)).

Let $\epsilon = \pm 1$. An *ϵ-symmetric space of finite length* is a pair (M, ψ) where M is a finite length A-module and $\psi : M \rightarrow M^\vee = \mathrm{Ext}^3(M, A)$ is an isomorphism with $\psi^\vee\ \mathcal{C} = \epsilon\psi$. Let ψ_1 and ψ_2 be two ϵ-symmetric spaces on finite length A-modules M_1 and M_2 respectively. We say that ψ_1 is *isometric to* ψ_2, if there exists a homomorphism $\theta : M_1 \rightarrow M_2$ such that $\psi_1 = \theta^\vee \psi_2 \theta$. An ϵ-symmetric space ψ on M is called *metabolic* if there exists a submodule N of M such that

$$0 \rightarrow N \overset{i}{\rightarrow} M \overset{i^\vee \psi}{\rightarrow} N^\vee \rightarrow 0$$

is exact, where $i : N \rightarrow M$ is the inclusion. The *Witt group* of ϵ-symmetric spaces of finite length A-modules is defined as the quotient of the Grothendieck group of isometry classes of ϵ-symmetric spaces with the orthogonal sum as addition, modulo the subgroup generated by metabolic spaces. It is denoted by $W^\epsilon_{\mathrm{fl}}(A)$.

Corollary 0.3 Let M be a finite dimensional vector space over A/m. Let $\psi : M \rightarrow \widetilde{M}$ be an ϵ-symmetric space, i.e., $\widetilde{\psi}\iota = \psi$ and ψ is an isomorphism. Then $\Phi_M \psi : M \rightarrow M^\vee$ is an ϵ-symmetric space.

Proof. By (0.1), we have $(\Phi_M \psi)^\vee \mathcal{C} = \psi^\vee \Phi_M^\vee \mathcal{C} = \psi^\vee \Phi_{\widetilde{M}}\ \iota$. Using (0.2), we get that $(\Phi_M \psi)^\vee \mathcal{C} = \Phi_M \widetilde{\psi} \iota = \epsilon \Phi_M \psi$. Thus $\Phi_M \psi$ is an ϵ-symmetric space. $\square$

We need the following

Lemma 0.4 Let M be a finite length A-module and $\psi : M \rightarrow M^\vee$ an ϵ-symmetric space. If (M, ψ) is stably metabolic, then it is metabolic.

Proof. If M is an A/m-module, then the result follows from the corresponding result for ϵ-symmetric spaces over the field A/m. We reduce the general case to the above case by induction on the length of M. Assume that the length of M is at least 1. Let V be a maximal submodule of M which is an A/m-module. Suppose that ψ restricted to V is singular. Then there exists a non-zero submodule L of V such that $L \subset L^{\perp} = \ker(M \stackrel{i^{\vee}\psi}{\to} L^{\vee})$ and ψ induces an ϵ-symmetric form $\overline{\psi}$ on $L^{\perp}/L$ which is Witt equivalent to (M, ψ). Suppose that (M, ψ) is stably metabolic. Then $(L^{\perp}/L, \overline{\psi})$ is stably metabolic. By induction there exists a submodule N_1 of $L^{\perp}/L$ such that

$$0 \to N_1 \stackrel{i}{\to} L^{\perp}/L \stackrel{i^{\vee}\overline{\psi}}{\to} N_1^{\vee} \to 0$$

is exact. Let N be the submodule of M containing L such that $N/L = N_1$. Then it is easy to see that the sequence

$$0 \to N \stackrel{i}{\to} M \stackrel{i^{\vee}\psi}{\to} N^{\vee} \to 0$$

is exact and (M, ψ) is metabolic. We may therefore assume that ψ restricted to V is non-singular. Then $(M, \psi) \simeq (V, \psi \mid_V) \perp (M_1, \psi_1)$. If $M_1 \neq 0$, then M_1 contains a non-zero submodule which is an A/m-module, contradicting the maximality of V. Thus $M_1 = 0$ and $M = V$ is an A/m-module. This completes the proof of the lemma. $\square$

Let X be a scheme such that 2 is invertible in $\Gamma(X)$. Let $\mathcal{E}$ be a vector bundle over X of finite rank. An *ϵ-symmetric space* on $\mathcal{E}$ is an isomorphism $\mathbf{q} : \mathcal{E} \to \mathcal{E}^* = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ such that $\mathbf{q}^*\mathcal{C} = \epsilon\mathbf{q}$, where $\mathcal{C} : \mathcal{E} \to \mathcal{E}^{**}$ is the canonical identification. Let $W^{\epsilon}(X)$ be the Witt group of ϵ-symmetric spaces on vector bundles over X ([**3**], p. 144). If $X = \mathrm{Spec}(A)$, then we denote $W^{\epsilon}(X)$ by $W^{\epsilon}(A)$.

Throughout this paper, by an A-module we mean a finitely generated A-module. We call an ϵ-symmetric space simply a *quadratic space* if $\epsilon = +1$ and a *symplectic space* if $\epsilon = -1$. We also denote $W^{+1}(X)$ by $W(X)$. For a vector bundle $\mathcal{E}$ over X, we denote the hyperbolic space on $\mathcal{E}$ by $\mathbb{H}(\mathcal{E})$ ([**3**], p. 130).

1. Reflexive modules

Let A be a regular local ring of dimension three with 2 invertible. An A-module E is said to be *reflexive* if it is finitely generated and the canonical homomorphism $E \rightarrow E^{**}$ is an isomorphism. For a reflexive A-module E we use the canonical isomorphism to identify E^{**} with E. It is well-known that a reflexive module over a regular ring of dimension 3 has projective dimension at most 1. Let E be a reflexive A-module and $M = \mathrm{Ext}^1(E^*, A)$, where $E^* = \mathrm{Hom}_A(E, A)$. Since reflexive modules over regular rings of dimension at most 2 are projective, M is a finite length A-module. We define a homomorphism $\beta_E : \mathrm{Ext}^1(E, A) \rightarrow M^\vee = \mathrm{Ext}^3(M, A)$ as follows. Let

$$0 \rightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} E^* \rightarrow 0 \qquad (\star)$$

be a projective resolution of E^*. Then by dualising, we get an exact sequence

$$0 \rightarrow E \xrightarrow{\partial_0^*} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \mathrm{Ext}^1(E^*, A) = M \rightarrow 0,$$

where δ is defined by push-outs. We have the following

Lemma 1.1 The Yoneda composition ([**4**], p. 82) $\beta_E : \mathrm{Ext}^1(E, A) \rightarrow M^\vee$ given by

$$\beta_E(0 \rightarrow A \xrightarrow{\eta} Z \xrightarrow{\eta'} E \rightarrow 0) = (0 \rightarrow A \xrightarrow{\eta} Z \xrightarrow{\partial_0^*\eta'} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} M \rightarrow 0)$$

is an isomorphism and is independent of the choice of the projective resolution $(\star)$ of E^*.

Proof. Consider the long exact sequence of cohomology associated to the short exact sequences

$$0 \rightarrow E \xrightarrow{\partial_0^*} P_0^* \xrightarrow{\partial_1^*} \ker(\delta) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \ker(\delta) \hookrightarrow P_1^* \xrightarrow{\delta} M \rightarrow 0.$$

Since $\mathrm{Ext}^i(M, A) = 0$ for $i \leq 2$ and P_1^* is a projective module, $\mathrm{Ext}^1(\ker(\delta), A) = 0$ and the connecting homomorphisms $\mathrm{Ext}^1(E, A) \rightarrow \mathrm{Ext}^2(\ker(\delta), A)$ and $\mathrm{Ext}^2(\ker(\delta), A) \rightarrow \mathrm{Ext}^3(M, A)$ induced by the above short exact sequences are isomorphisms. Since β_E, up to sign, is the composition of these two connecting homomorphisms ([**4**], p. 97, Th 9.1), β is an isomorphism.

Suppose that

$$0\to F_1 \stackrel{\partial_1'}{\to} F_0 \stackrel{\partial_0'}{\to} E^*\to 0$$

is another projective resolution of E^*. Then by lifting the identity map on E^*, we get homomorphisms $P_i\to F_i$, i = 0,1, such that the diagram

$$\begin{array}{cccccccc} 0\to & P_1 & \stackrel{\partial_1}{\to} & P_0 & \stackrel{\partial_0}{\to} & E^* & \to 0 \\ & \downarrow & & \downarrow & & \downarrow id & \\ 0\to & F_1 & \stackrel{\partial_1'}{\to} & F_0 & \stackrel{\partial_0'}{\to} & E^* & \to 0 \end{array}$$

is commutative. By dualising this diagram we get a commutative diagram

$$\begin{array}{cccccccccc} 0\to & E & \stackrel{\partial_0'^*}{\to} & F_0^* & \stackrel{\partial_1'^*}{\to} & F_1^* & \stackrel{\delta'}{\to} & M & \to 0 \\ & \downarrow id & & \downarrow & & \downarrow & & \downarrow id & \\ 0\to & E & \stackrel{\partial_0^*}{\to} & P_0^* & \stackrel{\partial_1^*}{\to} & P_1^* & \stackrel{\delta}{\to} & M & \to 0, \end{array}$$

where δ' is defined by push-outs. This implies that

$$(0\to E \stackrel{\partial_0'^*}{\to} F_0^* \stackrel{\partial_1'^*}{\to} F_1^* \stackrel{\delta'}{\to} M\to 0) \quad = \quad (0\to E \stackrel{\partial_0^*}{\to} P_0^* \stackrel{\partial_1^*}{\to} P_1^* \stackrel{\delta}{\to} M\to 0)$$

in $\mathrm{Ext}^2(M, E)$. Thus the homomorphism β_E is independent of the choice of the projective resolution of E^*. □

Lemma 1.2 1) For any reflexive A-module E we have

$$\beta_{E^*} = -\beta_E^\vee \mathcal{C}.$$

2) Let E and E' be reflexive A-modules. Then, for any isomorphism $f : E\to E'$, we have

$$\mathrm{Ext}^1(f^*)^\vee \beta_{E'} = \beta_E \mathrm{Ext}^1(f).$$

Proof. Let $0\to P_1 \stackrel{\partial_1}{\to} P_0 \stackrel{\partial_0}{\to} E^*\to 0$ and $0\to F_1 \stackrel{\partial_1'}{\to} F_0 \stackrel{\partial_0'}{\to} E\to 0$ be projective resolutions of E^* and E respectively. By dualising these exact sequences, we get exact sequences

$$0\to E \stackrel{\partial_0^*}{\to} P_0^* \stackrel{\partial_1^*}{\to} P_1^* \stackrel{\delta}{\to} \mathrm{Ext}^1(E^*, A)\to 0$$

and

$$0\to E^* \stackrel{\partial_0'^*}{\to} F_0^* \stackrel{\partial_1'^*}{\to} F_1^* \stackrel{\delta_1}{\to} \mathrm{Ext}^1(E,A)\to 0.$$

Let $\zeta = (0\to A \stackrel{\alpha}{\to} Z \stackrel{\beta}{\to} E^*\to 0) \in \mathrm{Ext}^1(E^*,A)$. Since P_0 and P_1 are projective, there exist homomorphisms $f : P_1\to A$ and $g : P_0\to Z$ such that the diagram

$$\begin{array}{cccccccc} 0\to & P_1 & \stackrel{\partial_1}{\to} & P_0 & \stackrel{\partial_0}{\to} & E^* & \to 0 \\ & \downarrow f & & \downarrow g & & \downarrow id & \\ 0\to & A & \stackrel{\alpha}{\to} & Z & \stackrel{\beta}{\to} & E^* & \to 0 \end{array}$$

is commutative. By the definition of δ we have $\delta(f) = \zeta$. Since

$$0\to F_1 \stackrel{\partial_1'}{\to} F_0 \stackrel{\partial_0^*\partial_0'}{\to} P_0^* \stackrel{\partial_1^*}{\to} P_1^* \stackrel{\delta}{\to} \mathrm{Ext}^1(E^*,A)\to 0$$

is a projective resolution of $\mathrm{Ext}^1(E^*,A)$, by dualising it we get an exact sequence

$$0\to P_1 \stackrel{\partial_1}{\to} P_0 \stackrel{\partial_0'^*\partial_0}{\to} F_0^* \stackrel{\partial_1'^*}{\to} F_1^* \stackrel{\delta_1'}{\to} \mathrm{Ext}^1(E^*,A)^\vee\to 0.$$

Thus $\mathcal{C}(\zeta) = -\xi$, where

$$\xi = (0\to A \stackrel{\alpha}{\to} Z \stackrel{\partial_0'^*\beta}{\to} F_0^* \stackrel{\partial_1'^*}{\to} F_1^* \stackrel{\delta_1'}{\to} \mathrm{Ext}^1(E^*,A)^\vee\to 0).$$

From the definitions of δ, δ' and β_E, it follows that the diagram

$$\begin{array}{cccccccccc} 0\to & E^* & \stackrel{\partial_0'^*}{\to} & F_0^* & \stackrel{\partial_1'^*}{\to} & F_1^* & \stackrel{\delta_1}{\to} & \mathrm{Ext}^1(E,A) & \to 0 \\ & \downarrow id & & \downarrow id & & \downarrow id & & \downarrow \beta_E & \\ 0\to & E^* & \stackrel{\partial_0'^*}{\to} & F_0^* & \stackrel{\partial_1'^*}{\to} & F_1^* & \stackrel{\delta_1'}{\to} & \mathrm{Ext}^1(E^*,A)^\vee & \to 0 \end{array}$$

is commutative. It follows from the definition of $\beta_E^\vee$ that

$$\beta_E^\vee(\xi) = (0\to A \stackrel{\alpha}{\to} Z \stackrel{\partial_0'^*\beta}{\to} F_0^* \stackrel{\partial_1'^*}{\to} F_1^* \stackrel{\delta_1}{\to} \mathrm{Ext}^1(E,A)\to 0).$$

On the other hand we have

$$\beta_{E^*}(\zeta) = (0\to A \stackrel{\alpha}{\to} Z \stackrel{\partial_0'^*\beta}{\to} F_0^* \stackrel{\partial_1'^*}{\to} F_1^* \stackrel{\delta_1}{\to} \mathrm{Ext}^1(E,A)\to 0) = \beta_E^\vee(\xi).$$

Thus $-\beta_E^\vee\mathcal{C} = \beta_{E^*}$.

Let $f : E \to E'$ be an isomorphism. Then $0 \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{f^{*-1}\partial_0} E'^* \to 0$ is a projective resolution of E'^*. By dualising it we get an exact sequence

$$0 \to E' \xrightarrow{\partial_0^* f^{-1}} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta_2} \mathrm{Ext}^1(E'^*, A) \to 0,$$

with $\delta_2 = \mathrm{Ext}^1(f^*)\delta$. Let $\zeta = (0 \to A \xrightarrow{\alpha} Z \xrightarrow{\beta} E' \to 0) \in \mathrm{Ext}^1(E', A)$. Then

$$\beta_{E'}(\zeta) = (0 \to A \xrightarrow{\alpha} Z \xrightarrow{\partial_0^* f^{-1}\beta} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta_2} \mathrm{Ext}^1(E'^*, A) \to 0)$$

and

$$\mathrm{Ext}^1(f^*)^\vee \beta_{E'}(\zeta) = (0 \to A \xrightarrow{\alpha} Z \xrightarrow{\partial_0^* f^{-1}\beta} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \mathrm{Ext}^1(E^*, A) \to 0),$$

since $\delta = \mathrm{Ext}^1(f^*)^{-1}\delta_2$. On the other hand we have

$$\mathrm{Ext}^1(f)(\zeta) = (0 \to A \xrightarrow{\alpha} Z \xrightarrow{f^{-1}\beta} E \to 0)$$

and

$$\begin{array}{rcl} \beta_E \mathrm{Ext}^1(f)(\zeta) & = & (0 \to A \xrightarrow{\alpha} Z \xrightarrow{\partial_0^* f^{-1}\beta} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \mathrm{Ext}^1(E^*, A) \to 0) \\ & = & \mathrm{Ext}^1(f^*)^\vee \beta_{E'}(\zeta). \end{array}$$

This proves the lemma. □

Let A be any local ring in which 2 is invertible and m its maximal ideal. Let E be a reflexive A-module. By an *ϵ-symmetric space* on E we mean an isomorphism $q : E \to E^*$ such that $q^*\mathcal{C} = \epsilon q$, where $\mathcal{C} : E \to E^{**}$ is the canonical isomorphism.

Let V be an A-module. By a *unimodular* element of V we mean an element $x \in V$ such that $f(x) = 1$ for some A-linear map $f : V \to A$. For example, an element $(a_1, \cdots, a_n) \in A^n$ is unimodular if and only if $a_i \notin m$ for some i. Thus, if an A-module V has no unimodular elements and $\eta : V \to A^n$ is an A-linear map, then $\eta(V) \subset mA^n$.

Lemma 1.3 Let E be a reflexive A-module and q an ϵ-symmetric space on E. Suppose that $E = E_0 \oplus A^n$ with E_0 having no unimodular elements. Then there exist ϵ-symmetric spaces q_1 and q_2 over E_0 and A^n respectively such that

$$(E, q) \simeq (E_0, q_1) \perp (A^n, q_2).$$

Proof. Let $E = E_0 \oplus A^n$ be such that E_0 has no unimodular elements. Then

$$q = \begin{pmatrix} q_1 & \eta \\ \epsilon\eta^* & q_1' \end{pmatrix}$$

for some $q_1 : E_0 \to E_0^*$, $q_1' : A^n \to A^{n*}$ and $\eta : A^n \to E_0^*$. Since E_0 has no unimodular elements, $\eta^*(E_0) \subset mA^{n*}$ and hence $\eta(A^n) \subset mE_0^*$. This implies that

$$q \equiv \begin{pmatrix} q_1 & 0 \\ 0 & q_1' \end{pmatrix} \bmod mE^*.$$

Since q is an isomorphism, q_1 and q_1' are isomorphisms. We have

$$\begin{pmatrix} 1 & 0 \\ -\epsilon\eta^* q_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} q_1 & \eta \\ \epsilon\eta^* & q_1' \end{pmatrix} \begin{pmatrix} 1 & -q_1^{-1}\eta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q_1 & 0 \\ 0 & -\eta^* q_1^{-1}\eta + q_1' \end{pmatrix}.$$

Let $q_2 = -\eta^* q_1^{-1}\eta + q_1' : A^n \to A^{n*}$. Since $q_1^* = \epsilon q_1$, $(E, q) \simeq (E_0, q_1) \perp (A^n, q_2)$. □

2. Spaces over the punctured spectrum and on finite length modules

We begin by recalling from a paper of Horrocks ([**2**]) an equivalence between the categories $\Phi\mathcal{P}$ of Φ-equivalence classes of vector bundles on the punctured spectrum of a regular local ring A of dimension 3 and the category $\mathcal{M}$ of finite length A-modules. Let m be the maximal ideal of A and $Y = \mathrm{Spec}(A) \setminus \{m\}$. Let $\mathcal{E}$ be a vector bundle over Y and $E = \Gamma(\mathcal{E})$ the module of sections of $\mathcal{E}$. Then E is a reflexive A-module ([**2**], Th.4.1) and $M = \mathrm{Ext}^1(E^*, A)$, which is isomorphic to $H^1(Y, \mathcal{E})$ ([**2**], §5), is a finite length A-module ([**2**], Cor.7.2.5). The functor $T : \Phi\mathcal{P} \to \mathcal{M}$ given by $T(\mathcal{E}) = \mathrm{Ext}^1(E^*, A)$ is an equivalence of categories ([**2**], Cor. 7.2.5). Let M be a finite length A-module. The construction below gives a vector bundle $\mathcal{E}$ on Y such that $T(\mathcal{E}) = M$. Let, in fact

$$0 \to P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\eta} M \to 0$$

be a projective resolution of M. Let $E = \ker(\partial_1)$. Then E is an A-module of projective dimension at most 1 and $\mathrm{Ext}^1(E^*, A) = M$. Since M is a finite length module, for any prime ideal p of A, $p \neq m$, $M_p = 0$ and hence E_p is free. Thus $E = \Gamma(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on Y ([**2**], Th. 4.1).

Let A be a regular local ring of dimension 3 in which 2 is invertible. Let $\mathcal{E}$ be a vector bundle and $\mathbf{q}$ an ϵ-symmetric space on $\mathcal{E}$. We associate to $(\mathcal{E}, \mathbf{q})$ a $(-\epsilon)$-symmetric space $\rho(\mathbf{q})$ of finite length. The ϵ-symmetric space $\mathbf{q}$ on $\mathcal{E}$ gives rise to an ϵ-symmetric space (E, q), where $E = \Gamma(\mathcal{E})$. Then $M = \mathrm{Ext}^1(E^*, A)$ is a finite length A-module. The isomorphism $q : E \to E^*$ induces an isomorphism $\mathrm{Ext}^1(q) : M = \mathrm{Ext}^1(E^*, A) \to \mathrm{Ext}^1(E, A)$. Let $\rho(\mathbf{q}) = \beta_E \mathrm{Ext}^1(q)$. We have the following

Lemma 2.1 $\rho(\mathbf{q}) : M \to M^\vee$ is a $(-\epsilon)$-symmetric space.

Proof. In the following diagram

$$\begin{array}{ccccc}
M = \mathrm{Ext}^1(E^*, A) & \stackrel{\mathrm{Ext}^1(q)}{\to} & \mathrm{Ext}^1(E, A) & \stackrel{\beta_E}{\to} & \mathrm{Ext}^1(E^*, A)^\vee = M^\vee \\
\downarrow \beta_{E^*} & & \downarrow \beta_E & & \downarrow \mathrm{id} \\
\mathrm{Ext}^1(E, A)^\vee & \stackrel{\epsilon \mathrm{Ext}^1(q)^\vee}{\to} & \mathrm{Ext}^1(E^*, A)^\vee & \stackrel{id}{\to} & \mathrm{Ext}^1(E^*, A)^\vee \\
\downarrow \beta_E^{\vee -1} & & \downarrow \epsilon \mathrm{Ext}^1(q)^{\vee -1} & & \downarrow \mathrm{id} \\
\mathrm{Ext}^1(E^*, A)^{\vee\vee} & \stackrel{\beta_E^\vee}{\to} & \mathrm{Ext}^1(E, A)^\vee & \stackrel{\epsilon \mathrm{Ext}^1(q)^\vee}{\to} & \mathrm{Ext}^1(E^*, A)^\vee
\end{array}$$

clearly all the squares except perhaps the top left one commutes. Since $q^* = \epsilon q$, by (1.2) this square also commutes. By (1.2), the composition of maps on the first column is equal to $-\mathcal{C}$. Thus

$$\rho(q)^\vee \mathcal{C} = \mathrm{Ext}^1(q)^\vee \beta_E^\vee \mathcal{C} = -\epsilon \beta_E \mathrm{Ext}^1(q) = -\epsilon \rho(q).$$

$\square$

Lemma 2.2 Let M be a finite length A-module and ψ an ϵ-symmetric form on M. Suppose that there exists an exact sequence

$$N \xrightarrow{f} M \xrightarrow{f^\vee \psi} N^\vee$$

of finite length A-modules. Then (M, ψ) is metabolic.

Proof. Since the map f factors as $N \to N/\mathrm{ker(f)} \xrightarrow{\overline{f}} M$, we have an exact sequence

$$0 \to N/\mathrm{ker(f)} \xrightarrow{\overline{f}} M \xrightarrow{\overline{f}^\vee \psi} (N/\mathrm{ker(f)})^\vee.$$

Since, dimension of A being 3, $\mathrm{Ext}^4(L, A) = 0$, the map $\overline{f}^\vee \psi$ is surjective and hence (M, ψ) is metabolic. $\square$

Lemma 2.3 If $(\mathcal{E}, \mathbf{q})$ is metabolic, then $(M, \rho(\mathbf{q}))$ is metabolic.

Proof. Suppose that $(\mathcal{E}, \mathbf{q})$ is metabolic. Let $\mathcal{F}$ be a subbundle of $\mathcal{E}$ such that the sequence

$$0 \rightarrow \mathcal{F} \xrightarrow{i} \mathcal{E} \xrightarrow{i^* \mathbf{q}} \mathcal{F}^* \rightarrow 0$$

is exact, where $\mathcal{F} \xrightarrow{i} \mathcal{E}$ is the inclusion. By taking global sections and then applying the Ext functor to the following exact sequence of bundles

$$0 \rightarrow \mathcal{F} \xrightarrow{\mathbf{q}i} \mathcal{E}^* \xrightarrow{i^*} \mathcal{F}^* \rightarrow 0$$

we get an exact sequence

$$\mathrm{Ext}^1(F^*, A) \rightarrow \mathrm{Ext}^1(E^*, A) \rightarrow \mathrm{Ext}^1(F, A)$$

of finite length modules, where $F = \Gamma(\mathcal{F})$. Let $N = \mathrm{Ext}^1(F^*, A)$. Then the canonical identification of $\mathrm{Ext}^1(F, A)$ with $N^\vee$ gives an exact sequence

$$N \xrightarrow{f} M \xrightarrow{f^\vee \rho(\mathbf{q})} N^\vee.$$

Now the lemma follows from (1.2). $\square$

Lemma 2.4 The assignment $(\mathcal{E}, \mathbf{q}) \mapsto (M, \rho(\mathbf{q}))$ induces a homomorphism

$$\rho : W^\epsilon(Y) \rightarrow W_{fl}^{-\epsilon}(A).$$

Proof. Since ρ is clearly additive, it is enough to show that ρ takes stably metabolic spaces to metabolic spaces. Let $(\mathcal{E}, \mathbf{q})$ be an ϵ-symmetric spaces over Y which is stably metabolic. Then there exists a metabolic space $(\mathcal{E}_1, \mathbf{q}_1)$ such that $(\mathcal{E}, \mathbf{q}) \perp (\mathcal{E}_1, \mathbf{q}_1)$ is metabolic. By (2.3), $\rho(\mathbf{q}_1)$ and $\rho(\mathbf{q} \perp \mathbf{q}_1) = \rho(\mathbf{q}) \perp \rho(\mathbf{q}_1)$ are metabolic. Thus $\rho(\mathbf{q})$ is stably metabolic. $\square$

We note that if $\mathcal{E}$ is a trivial bundle then $M = 0$. Thus, if $(\mathcal{E}, \mathbf{q})$ comes from an ϵ-symmetric space on A, then $\rho(\mathbf{q}) = 0$.

The proof of the following lemma is by straightforward verification, hence we omit it.

Lemma 2.5 Let R be a ring. Let $0 \to N \xrightarrow{i} M \xrightarrow{j} N' \to 0$ be an exact sequence of R-modules. Assume that the projective dimensions of N and N' are finite. Let

$$0 \to P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\alpha} N \to 0$$

and

$$0 \to Q_n \xrightarrow{\partial'_n} Q_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \to Q_1 \xrightarrow{\partial'_1} Q_0 \xrightarrow{\beta} N' \to 0$$

be projective resolutions of N and N' respectively. Let, for $l \geq 1$, $\phi_l : Q_l \to P_{l-1}$ and $\theta : Q_0 \to M$ be R-linear homomorphisms. Let

$$\delta_l = \begin{pmatrix} \partial_l & (-1)^l \phi_l \\ 0 & \partial'_l \end{pmatrix}.$$

Then the following diagram

$$\begin{array}{cccccccccccc}
0 \to & P_n & \xrightarrow{\partial_n} & P_{n-1} & \cdots & \to & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\alpha} & N & \to 0 \\
 & \downarrow & & \downarrow & & & \downarrow & & \downarrow & & \downarrow i & \\
0 \to & P_n \oplus Q_n & \xrightarrow{\delta_n} & P_{n-1} \oplus Q_{n-1} & \cdots & \to & P_1 \oplus Q_1 & \xrightarrow{\delta_1} & P_0 \oplus Q_0 & \xrightarrow{(i\alpha,\theta)} & M & \to 0 \\
 & \downarrow & & \downarrow & & & \downarrow & & \downarrow & & \downarrow j & \\
0 \to & Q_n & \xrightarrow{\partial'_n} & Q_{n-1} & \cdots & \to & Q_1 & \xrightarrow{\partial'_1} & Q_0 & \xrightarrow{\beta} & N' & \to 0
\end{array}$$

is commutative if and only if the following diagram

$$\begin{array}{cccccccccccccccc}
 & 0 & \to & Q_n & \xrightarrow{\partial'_n} & Q_{n-1} & \xrightarrow{\partial'_{n-1}} & & \cdots & \to & Q_2 & \xrightarrow{\partial'_2} & Q_1 & \xrightarrow{\partial'_1} & Q_0 & \xrightarrow{\beta} & N' & \to 0 \\
 & \downarrow & & \downarrow \phi_n & & \downarrow \phi_{n-1} & & & & & \downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \theta & & \downarrow id & \\
0 \to & P_n & \xrightarrow{\partial_n} & P_{n-1} & \xrightarrow{\partial_{n-1}} & P_{n-2} & \xrightarrow{\partial_{n-2}} & & \cdots & \to & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{i\alpha} & M & \xrightarrow{j} & N' & \to 0
\end{array}$$

is commutative.

Proposition 2.6 Let $(\mathcal{E}, \mathbf{q})$ be an ϵ-symmetric space over Y. Suppose that $E = \Gamma(\mathcal{E})$ has no unimodular elements and that $\rho(\mathbf{q})$ is metabolic. Then there exist ϵ-symmetric spaces $\mathbf{q}_1$ and $\mathbf{q}_2$ on $\mathcal{E}$ and ${\mathcal{O}_Y}^n$ respectively such that $\mathbf{q}_1 \perp \mathbf{q}_2$ is metabolic and $\rho(\mathbf{q}) \simeq \rho(\mathbf{q}_1)$.

Proof. Let $M = \mathrm{Ext}^1(E^*, A)$ and $\rho(\mathbf{q})$ be the ϵ-symmetric space on M. Since $\rho(\mathbf{q})$ is metabolic, there exists an exact sequence

$$0 \to N \xrightarrow{i} M \xrightarrow{i^\vee \rho(q)} N^\vee \to 0.$$

Let

$$0 \to Q_3 \xrightarrow{\partial_3} Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\eta} N \to 0$$

be a projective resolution of N. By dualising this resolution, we get a projective resolution

$$0 \to Q_0^* \xrightarrow{\partial_1^*} Q_1^* \xrightarrow{\partial_2^*} Q_2^* \xrightarrow{\partial_3^*} Q_3^* \xrightarrow{\eta'} N^\vee \to 0$$

of $N^\vee$. By lifting the identity map of $N^\vee$, we obtain a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \to & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \to 0 & \\
\downarrow & & \downarrow\phi_3 & & \downarrow\phi_2 & & \downarrow\phi_1 & & \downarrow\theta & & \downarrow\mathrm{id} & & (\star) \\
Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{i\eta} & M & \xrightarrow{i^\vee\rho(q)} & N^\vee & \to 0. &
\end{array}$$

Let

$$\delta_1 = \begin{pmatrix} \partial_1 & -\phi_1 \\ 0 & \partial_3^* \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} \partial_2 & \phi_2 \\ 0 & \partial_2^* \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} \partial_3 & -\phi_3 \\ 0 & \partial_1^* \end{pmatrix}.$$

By (2.5), the diagram

$$\begin{array}{cccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\eta} & N & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow i & \\
0 \to & Q_3 \oplus Q_0^* & \xrightarrow{\delta_3} & Q_2 \oplus Q_1^* & \xrightarrow{\delta_2} & Q_1 \oplus Q_2^* & \xrightarrow{\delta_1} & Q_0 \oplus Q_3^* & \xrightarrow{(i\eta,\theta)} & M & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow i^\vee\rho(q) & \\
0 \to & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

is commutative. Since the first row, the last rows and all the columns in the above diagram are exact and the diagram is commutative, from the long exact homology sequence ([**4**], Th. 4.1 p. 45) we get that the middle row is also exact. By dualising this diagram, we get the commutative diagram

$$
\begin{array}{cccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0\to & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\eta''} & N^{\vee\vee} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\rho(q)^\vee i^{\vee\vee} & \\
0\to & Q_3\oplus Q_0^* & \xrightarrow{\delta_1^*} & Q_2\oplus Q_1^* & \xrightarrow{\delta_2^*} & Q_1\oplus Q_2^* & \xrightarrow{\delta_3^*} & Q_0\oplus Q_3^* & \xrightarrow{(\mu,\nu)} & M^\vee & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow i^\vee & \\
0\to & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

with exact rows and columns. By (2.5), the diagram

$$
\begin{array}{cccccccccccc}
0 & \to & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \to 0 \\
\downarrow & & \downarrow\phi_1^* & & \downarrow\phi_2^* & & \downarrow\phi_3^* & & \downarrow\nu & & \downarrow\mathrm{id} & \\
Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\rho(q)^\vee i^{\vee\vee}\eta''} & M^\vee & \xrightarrow{i^\vee} & N^\vee & \to 0
\end{array}
$$

is commutative. From the definition of η'' and $\mathcal{C} : N\to N^{\vee\vee}$ (cf diagram † in §0) it follows that $\mathcal{C}\eta = -\eta''$. Since $\rho(q)^\vee\mathcal{C} = -\epsilon\rho(q)$, we have the commutative diagram

$$
\begin{array}{cccccccccccc}
0 & \to & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \to 0 \\
\downarrow & & \downarrow\phi_1^* & & \downarrow\phi_2^* & & \downarrow\phi_3^* & & \downarrow\nu' & & \downarrow\epsilon\mathrm{id} & \\
Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{i\eta} & M & \xrightarrow{i^\vee\rho(q)} & N^\vee & \to 0
\end{array}
$$

where $\nu' = \epsilon\rho(q)^{-1}\nu$. From $(\star)$ and the above commutative diagram we get maps $s_1 : Q_2^*\to Q_1$ and $s_2 : Q_1^*\to Q_2$ such that $\phi_2 - \epsilon\phi_2^* = \partial_2 s_2 - s_1\partial_2^*$. Let $\phi = \phi_2 - \epsilon\partial_2 s_1^*$. Then we have

$$
\begin{aligned}
\partial_1\phi &= \partial_1\phi_2 \\
&= \partial_1(\phi_2 - \epsilon\phi_2^*) \\
&= \partial_1(\partial_2 s_2 - s_1\partial_2^*) + \epsilon\partial_1\phi_2^* \\
&= \epsilon\partial_1(\phi_2^* - \epsilon_1\partial_2^*) \\
&= \epsilon\partial_1\phi^*.
\end{aligned}
$$

Let

$$\delta = \begin{pmatrix} \partial_2 & \frac{\phi+\epsilon\phi^*}{2} \\ 0 & \epsilon\partial_2^* \end{pmatrix}.$$

It is easy to see that the diagram

$$\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0\to & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\eta} & N & \to 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow i & & \\
0\to & Q_3 \oplus Q_0^* & \xrightarrow{\delta_1^*} & Q_2 \oplus Q_1^* & \xrightarrow{\delta} & Q_1 \oplus Q_2^* & \xrightarrow{\delta_1} & Q_0 \oplus Q_3^* & \xrightarrow{(\eta,\theta)} & M & \to 0 & (\star\star) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow i^\vee\rho(q) & & \\
0\to & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\epsilon\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \to 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

commutes. Since the first row, the last row and all the columns are exact, the middle row is also exact. Let $E' = \ker(\delta_1)$. Since $\delta^*\mathcal{C} = \epsilon\delta$, from the middle row of the diagram $(\star\star)$ it is easy to see that δ induces an ϵ-symmetric isomorphism $q' : E' \to E'^*$. Let $(\mathcal{E}', \mathbf{q}')$ be the ϵ-symmetric space over Y with $\Gamma(\mathcal{E}') = E'$ and $\Gamma(\mathbf{q}') = q'$. Since $\mathrm{Ext}^1(E', A) \simeq M = \mathrm{Ext}^1(E^*, A)$ and E has no unimodular elements, by Cor. 7.2.5 and Lemma 7.1 of [**2**] we have $E' = E \oplus A^n$. Then by (1.3), $(E', q') \simeq (E, q_1) \perp (A^n, q_2)$ for some ϵ-symmetric spaces q_1 and q_2 on E and A^n respectively. Let $\mathbf{q_1}$ be the ϵ-symmetric space on $\mathcal{E}$ such that $\Gamma(\mathbf{q_1}) = q_1$. Let $F = \ker(\partial_1)$ and $\mathcal{F}$ the vector bundle over Y with $\Gamma(\mathcal{F}) = F$. Then using $(\star\star)$, it is easy to see that $\mathcal{F}$ is a Lagrangian for $(\mathcal{E}', \mathbf{q}') \simeq (\mathcal{E}, \mathbf{q_1}) \perp ({\mathcal{O}_Y}^n, \mathbf{q_2})$, where $\Gamma(\mathbf{q_2}) = q_2$. Since the map $E' \to F^*$ induced by the diagram $(\star\star)$, induces $i^\vee\rho(\mathbf{q}) : M \to N^\vee$, we have $i^\vee\rho(\mathbf{q}) = i^\vee\rho(\mathbf{q_1})$. Thus, by the lemma (2.7) below, we have $\rho(\mathbf{q}) \simeq \rho(\mathbf{q_1})$. $\square$

Lemma 2.7 Let ψ_1 and ψ_2 be two ϵ-symmetric spaces on M. Suppose there exists a submodule N such that

$$0 \to N \xrightarrow{i} M \xrightarrow{i^\vee\psi_1} N^\vee \to 0$$

is exact and $i^\vee\psi_1 = i^\vee\psi_2$. Then $\psi_1 \simeq \psi_2$.

Proof. Since $i^\vee\psi_1 = i^\vee\psi_2$, there exists a map $\theta : M \to N$ such that $\psi_1^{-1}\psi_2 - 1 = i\theta$, i.e., $\psi_1 i\theta = \psi_2 - \psi_1 = \theta^\vee i^\vee \psi_1$. We have

$$\begin{array}{rcl} \frac{(1+\theta^\vee i^\vee)}{2}\psi_1\frac{(1+i\theta)}{2} & = & \frac{(\psi_1+\theta^\vee i^\vee\psi_1)}{2}\frac{(1+i\theta)}{2} \\ & = & \psi_1 + \psi_1\frac{i\theta}{2} + \frac{\theta^\vee i^\vee}{2}\psi_1 + \frac{\theta^\vee i^\vee}{2}\psi_1\frac{i\theta}{2} \\ & = & \psi_1 + \frac{\psi_2-\psi_1}{2} + \frac{\psi_2-\psi_1}{2} \\ & = & \psi_2 \end{array}$$

□

3. The Witt groups of the punctured spectrum and purity

Let A be a regular local ring of dimension three with 2 invertible. Let $Y = \mathrm{Spec}(A) \setminus \{m\}$, where m is the maximal ideal of A.

Proposition 3.1. Let $\mathcal{E}$ be a vector bundle on Y and $\mathbf{q} : \mathcal{E} \to \mathcal{E}^*$ an ϵ-symmetric isomorphism. Suppose that $\Gamma(\mathcal{E})$ has no unimodular elements. If $\rho(\mathbf{q})$ is isomorphic to a hyperbolic space, then $\mathbf{q}$ is in the image of $W^\epsilon(A)$.

Proof. Let N be a finite length A-module such that $(M, \rho(\mathbf{q}))$ is isomorphic to the hyperbolic space $I\!H(N)$. Let $\mathcal{F}$ be the vector bundle on Y with $\Gamma(\mathcal{F})$ having no unimodular elements and such that $H^1(Y, \mathcal{F}) \simeq N$ (cf §2). Since $H^1(Y, \mathcal{E}) \simeq N \oplus N^\vee \simeq H^1(Y, \mathcal{F} \oplus \mathcal{F}^*)$ with $\Gamma(\mathcal{E})$ and $\Gamma(\mathcal{F} \oplus \mathcal{F}^*)$ admitting no unimodular elements, by Lemma 7.1 and Cor.7.2.5 of [**2**] we can and do identify $\mathcal{E}$ with $\mathcal{F} \oplus \mathcal{F}^*$. Let $\tilde{\psi}$ be an isometry of $\rho(\mathbf{q})$ with $\rho(\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix})$. Then by [**2**], Cor.7.2.5 there exists an automorphism ψ of $\mathcal{E}$ such that $H^1(\psi) = \tilde{\psi}$. By the definition of ρ we have $\rho(\psi^*\mathbf{q}\psi) = \beta_E \mathrm{Ext}^1(\Gamma(\psi^*)q\Gamma(\psi))$ $= \beta_E \mathrm{Ext}^1(\Gamma(\psi^*)\mathrm{Ext}^1(q)\mathrm{Ext}^1(\Gamma(\psi))$, where $E = \Gamma(\mathcal{E})$ and $q = \Gamma(\mathbf{q})$. By (1.2 (2)), we have $\beta_E \mathrm{Ext}^1(\Gamma(\psi^*)) = \mathrm{Ext}^1(\Gamma(\psi))^\vee \beta_E$, so that $\rho(\psi^*\mathbf{q}\psi) = \mathrm{Ext}^1(\Gamma(\psi))^\vee \beta_E \mathrm{Ext}^1(q)\mathrm{Ext}^1(\Gamma(\psi)) = \mathrm{Ext}^1(\Gamma(\psi))^\vee \rho(\mathbf{q})\mathrm{Ext}^1(\Gamma(\psi))$ $= H^1(\psi)^\vee \rho(\mathbf{q}) H^1(\psi) = \tilde{\psi}\rho(\mathbf{q})\tilde{\psi}$. We replace $\mathbf{q}$ by $\psi^*\mathbf{q}\psi$ and assume that $\rho(\mathbf{q}) = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}$. Let $\mathbf{q} = \begin{pmatrix} \alpha & \delta \\ \epsilon\delta^* & \beta \end{pmatrix}$ with $\alpha : \mathcal{F} \to \mathcal{F}^*$, $\beta : \mathcal{F}^* \to \mathcal{F}$, $\delta : \mathcal{F}^* \to \mathcal{F}^*$ maps such that $\alpha^* = \epsilon\alpha$, $\beta^* = \epsilon\beta$. Since $\rho(\mathbf{q}) = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}$,

$H^1(\alpha) = 0$ and $H^1(\beta) = 0$. Therefore, by ([H], Cor.7.2.5), there exist $f_1 : \mathcal{F} \to \mathcal{O}_Y^n$, $f_2 : \mathcal{F}^* \to \mathcal{O}_Y^n$, $g_1 : \mathcal{O}_Y^n \to \mathcal{F}^*$ and $g_2 : \mathcal{O}_Y^n \to \mathcal{F}$ such that $\alpha = g_1 f_1$ and $\beta = g_2 f_2$. Let us consider the automorphism

$$\psi = \begin{pmatrix} 1 & 0 & -g_1/2 & f_1 * \\ 0 & 1 & -g_2/2 & f_2^* \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of $\mathcal{F} \oplus \mathcal{F}^* \oplus {\mathcal{O}_Y}^n \oplus {\mathcal{O}_Y}^{n*}$. We have

$$\mathbf{q}' = \psi \begin{pmatrix} \alpha & \delta & 0 & 0 \\ \epsilon\delta^* & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \epsilon & 0 \end{pmatrix} \psi^* = \begin{pmatrix} 0 & X & \epsilon f_1^* & -g_1/2 \\ \epsilon X^* & 0 & \epsilon f_2^* & -g_2/2 \\ f_1 & f_2 & 0 & 1 \\ -\epsilon g_1^*/2 & -\epsilon g_2^*/2 & \epsilon & 0 \end{pmatrix}$$

where $X = \delta - \epsilon f_1^* g_2^*/2 - g_1 f_2/2$. Since $\Gamma(\mathcal{F})$ and $\Gamma(\mathcal{F}^*)$ admit no unimodular elements, $f_1 \equiv f_2 \equiv 0 \bmod m$. Hence X is an isomorphism. Since $\mathbf{q}'$ restricted to $\mathcal{F} \oplus \mathcal{F}^*$ is

$$\begin{pmatrix} 0 & X \\ \epsilon X^* & 0 \end{pmatrix}$$

with X an isomorphism, $\mathbf{q}'$ splits as

$$\begin{pmatrix} 0 & X \\ \epsilon X^* & 0 \end{pmatrix} \perp \mathbf{q}''$$

for some $\mathbf{q}''$ supported on a bundle $\mathcal{E}''$. Since $\mathcal{E} \oplus {\mathcal{O}_Y}^n \oplus {\mathcal{O}_Y}^{n*} \simeq \mathcal{E} \oplus \mathcal{E}''$, by [**2**], Cor.7.2.5, $\mathcal{E}''$ is a trivial bundle and hence $\mathbf{q}''$ is in the image of $W^\epsilon(A)$. Since

$$\mathbf{q} \perp H({\mathcal{O}_Y}^n) \simeq \mathbf{q}' \simeq \begin{pmatrix} 0 & X \\ \epsilon X^* & 0 \end{pmatrix} \perp \mathbf{q}'',$$

it follows that $\mathbf{q}$ is in the image of $W^\epsilon(A)$. □

Lemma 3.2 Let M be a finite length A-module and $\psi : M \to M^\vee$ be an ϵ-symmetric isomorphism. Then there exists a vector bundle $\mathcal{E}$ over Y with a $(-\epsilon)$-symmetric isomorphism $\mathbf{q} : \mathcal{E} \to \mathcal{E}^*$ such that $\rho(\mathbf{q}) = \psi$.

Proof. Let $\mathcal{E}$ be a vector bundle on Y such that $H^1(Y, \mathcal{E}) = M$ (cf §2). and $E = \Gamma(\mathcal{E})$ has no unimodular elements. Then (cf §2) the projective dimension of E is less than or equal to 1 and $\mathrm{Ext}^1(E^*, A) \simeq M$. Let

$$0 \to P_1 \to P_0 \to E \to 0$$

and

$$0 \to Q_1 \to Q_0 \to E^* \to 0$$

be projective resolutions of E and E^* respectively. By dualising the projective resolution of E^* we get an exact sequence

$$0 \to E \to Q_0^* \to Q_1^* \to M \to 0.$$

By taking the Yoneda composition of this exact sequence with the projective resolution of E we get a projective resolution

$$0 \to P_1 \to P_0 \to Q_0^* \to Q_1^* \to M \to 0$$

of M. By dualising this we get a projective resolution

$$0 \to Q_1 \to Q_0 \to P_0^* \to P_1^* \to M^\vee \to 0$$

of $M^\vee$. By lifting the ϵ-symmetric isomorphism $\psi : M \to M^\vee$, we get a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \to & P_1 & \to & P_0 & \to & Q_0^* & \to & Q_1^* & \to & M & \to & 0 \\
 & & \downarrow\psi_3 & & \downarrow\psi_2 & & \downarrow\psi_1 & & \downarrow\psi_0 & & \downarrow\psi & & \\
0 & \to & Q_1 & \to & Q_0 & \to & P_0^* & \to & P_1^* & \to & M^\vee & \to & 0
\end{array}$$

of exact sequences. Since E has no unimodular elements it is easy to see, as in the proof of (2.6), that the above diagram induces a $(-\epsilon)$-symmetric space $\mathbf{q}$ on $\mathcal{E}$ such that $\rho(\mathbf{q}) = \psi$. $\square$

Theorem 3.3 Let A be a regular local ring of dimension 3 and m its maximal ideal. Assume that 2 is invertible in A. Let $Y = \mathrm{Spec}(A) \setminus \{m\}$. Then the the complex

$$0 \to W^\epsilon(A) \xrightarrow{\iota} W^\epsilon(Y) \xrightarrow{\rho} W_{fl}^{-\epsilon}(A) \to 0$$

is exact, where ι is induced by the restriction.

Proof. If $\epsilon = 1$, then the injectivity of ι follows from the injectivity of the canonical homomorphism $W(A) \to W(K)$ ([**6**],Th. 23), where K is the quotient field of A. If $\epsilon = -1$, ι is injective because $W^{-1}(A) = 0$.

We now prove the exactness in the middle. As we remarked in §2, $\rho\iota = 0$. Let $(\mathcal{E}, \mathbf{q})$ be an ϵ-symmetric space over Y such that $\rho(\mathbf{q})$ is zero in $W_{fl}^{-\epsilon}(A)$. Then, by (0.4), $\rho(\mathbf{q})$ is metabolic. We show that $(\mathcal{E}, \mathbf{q})$ is in the image of ι. In view of (1.3), we assume that $\Gamma(\mathcal{E})$ has no unimodular elements. Then, by (2.6), there exist ϵ-symmetric spaces $\mathbf{q}_1$ and $\mathbf{q}_2$ supported respectively on $\mathcal{E}$ and ${\mathcal{O}_Y}^n$ for some integer n, such that $\mathbf{q}_1 \perp \mathbf{q}_2$ is metabolic and $\rho(\mathbf{q}) \simeq \rho(\mathbf{q}_1)$. Thus $\rho(\mathbf{q} \perp -\mathbf{q}_1)$ is isomorphic to a hyperbolic space. Since $\Gamma(\mathcal{E} \oplus \mathcal{E})$ has no unimodular elements, it follows from (3.1) that $\mathbf{q} \perp -\mathbf{q}_1$ is in the image of ι. Since $\mathbf{q}_2$ is in the image of ι and $\mathbf{q}_1 \perp \mathbf{q}_2$ is metabolic, $\mathbf{q}_1$ and hence $\mathbf{q}$ is in the image of ι.

The surjectivity of ρ follows from (3.2). $\square$

Let A be any regular ring. Let $\mathrm{Spec}^1(A)$ denote the set of prime ideals of A of height 1. Then for any $P \in \mathrm{Spec}^1(A)$, the local ring A_P is a discrete valuation ring. Let $\partial_P : W(K) \to W(A_P/PA_P)$ denote the second residue homomorphism with respect to some choice of a parameter of PA_P, where K is the quotient field of A.

Corollary 3.4 Let A be a regular local ring of dimension 3, m its maximal ideal and K its quotient field. Assume that 2 is invertible in A. The sequence

$$0 \to W(A) \to W(K) \overset{\oplus\partial_P}{\to} \bigoplus_{P \in \mathrm{Spec}^1(A)} W(A_P/PA_P)$$

is exact.

Proof. The injectivity of $W(A) \to W(K)$ is proved in [**6**], Th. 23. Since $W_{fl}^{-1}(A) \simeq W^{-1}(A/m) = 0$, by (3.3) we have $W(A) \simeq W(Y)$. Thus it is enough to prove that the complex

$$W(Y) \to W(K) \overset{\oplus\partial_P}{\to} \bigoplus_{P \in \mathrm{Spec}^1(A)} W(A_P/PA_P)$$

is exact. Let q be a quadratic space over K such that $\partial_P(q) = 0$ for all height 1 prime ideals P of A. Since Y is a regular scheme of dimension 2, by [**1**]

2.5 p. 112, there exists a quadratic space $(\mathcal{E}, \mathbf{q})$ over $Y = \mathrm{Spec}(A) \setminus \{m\}$, such that its image in $W(K)$ under the restriction map, is equal to q. This completes the proof of corollary. $\square$

Using (3.4), one can prove the following theorem (cf [**8**], Prop. 2.1).

Corollary 3.5 Let A be a regular ring of dimension 3 and K its quotient field. Assume that 2 is invertible in A. The sequence

$$0 \to W(A) \to W(K) \overset{\oplus \partial_P}{\to} \bigoplus_{P \in \mathrm{Spec}^1(A)} W(A_P/PA_P)$$

is exact.

We end this paper by giving a computation of $W^{-1}(Y)$ using (3.3).

Corollary 3.6 Let A be a regular local ring of dimension 3 and m its maximal ideal. Assume that 2 is invertible in A. Let $Y = \mathrm{Spec}(A) \setminus \{m\}$. Then $W^{-1}(Y) \simeq W(A/m)$.

Proof. Since $W^{-1}(A) = 0$ and $W_{fl}(A) \simeq W(A/m)$, the result follows from (3.3). $\square$

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