A purity theorem for the Witt group

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1. Introduction

We briefly review the definitions of quadratic spaces and Witt groups. A very detailed exposition of these topics may be found in [8] and in [9].

Let $X$ be a scheme such that 2 is invertible in $\Gamma(O_X)$. A quadratic space over $X$ is a pair $q = (E, q)$ consisting of a locally free coherent sheaf (we also say “vector bundle”) $E$ and a symmetric isomorphism $q : E \to E^* = \text{Hom}_{O_X}(E, O_X)$: this means that, after identifying $E$ with $E^{**}$ in the usual way, it satisfies $q = q^*$. An isometry $\varphi : q \to q'$ is an isomorphism $\varphi : E \to E'$ such that the square

$$
\begin{array}{ccc}
E & \xrightarrow{\varphi} & E' \\
q \downarrow & & \downarrow q' \\
E^* & \xrightarrow{\varphi^*} & E'^*
\end{array}
$$

commutes.

The orthogonal sum of $q$ and $q'$ is the space $q \perp q' = (E \oplus E', q \oplus q')$.

Let $q = (E, q)$ be a quadratic space over $X$ and $F$ a subsheaf of $E$. The orthogonal $F^\perp$ of $F$ is the kernel of $q \circ i^*$, where $i$ denotes the inclusion of $F$ into $E$.

A subbundle $L$ of $E$ is a sublagrangian of $q$ if $L \subseteq L^\perp$, and it is a lagrangian if $L = L^\perp$. Note that lagrangians and sublagrangians are subbundles, i.e. locally direct factors, not just subsheaves. A space $q = (E, q)$ is said to be metabolic if it has a lagrangian.

Let $GW(X)$ denote the Grothendieck group of quadratic spaces over $X$ with respect to the orthogonal sum. Let $M$ be the subgroup of $GW(X)$ generated by metabolic spaces. The Witt group of $X$ is the quotient $W(X) = GW(X)/M$. If $f : X \to Y$ is a map of schemes and $q = (E, q)$ is space over $Y$, the pair $f^*q = (f^*E, f^*q)$ is a quadratic space over $X$. It is easily seen that $f^*$ respects orthogonal sums and maps metabolic spaces to metabolic spaces, thus $f$ induces a group homomorphism $W(f) : W(Y) \to W(X)$ and $W$ turns out to be a contravariant functor from the category of schemes to the category of abelian groups.

If $X = \text{Spec}(A)$ is affine, a quadratic space over $X$ is the same as a pair $(P, q)$ consisting of a finitely projective $A$-module $P$ and an $A$-linear isomorphism $q : P \to P^*$ such that $q = q^*$. In this case a space $(P, q)$ is metabolic if and only if it is isometric to a space of the form $(L \oplus L^*, (0^1_{10}))$.

For an affine scheme $X = \text{Spec}(A)$ we denote $W(X)$ by $W(A)$.

Let now $X$ be an integral scheme and $K = k(X)$ its field of rational functions. By the functoriality of $W$ there is a canonical map $W(X) \to W(K)$ and, for every point $x \in X$, a canonical map $W(O_{X,x}) \to W(K)$. We say that an element $\xi \in W(K)$ is unramified
at $x$ if $\xi$ is in the image of $W(O_{X,x})$. We say that an element $\xi \in W(K)$ is unramified (over $X$) if it is unramified at every height one point $x \in X$. We say that purity holds for $X$ if every unramified element of $W(K)$ belongs to the image of $W(X)$ in $W(K)$.

Purity is known to hold for every regular integral noetherian scheme of dimension at most two [3] and for every regular integral noetherian affine scheme of dimension three [14].

The main result of this paper is the following purity theorem (§7).

**Theorem A.** Purity holds for any regular local ring containing a field of characteristic $\neq 2$.

Theorem A will be deduced from the same statement for essentially smooth local algebras over a field, using a well-known result of Dorin Popescu.

Further, using essentially the same methods, we prove (§8)

**Theorem B.** Let $A$ be a regular local ring containing a field of characteristic $\neq 2$ and $K$ the field of fractions of $A$. Let $f$ be a regular parameter of $A$. The natural homomorphism $W(A_f) \to W(K)$ is injective.

From this, using a result of Piotr Jaworski for 2-dimensional regular rings, we deduce (§9)

**Theorem C.** Let $A$ be a regular local ring containing a field of characteristic $\neq 2$ and $f$ a regular parameter of $A$. There is a short exact sequence

$$0 \to W(A) \to W(A_f) \xrightarrow{\delta} W(A/AF) \to 0,$$

where $\delta$ is the restriction to $W(A_f)$ of the second residue homomorphism $\partial_f$ at the height one prime $p = Af$.

Let $A((t)) = A[[t]]$, be the ring of formal Laurent series over $A$. As a special case of Theorem C we can formulate (§9)

**Theorem D.** Let $A$ be a regular local ring containing a field of characteristic $\neq 2$. There exists a split short exact sequence

$$0 \to W(A) \to W(A((t))) \to W(A) \to 0.$$

**Remark.** The method used for proving purity for an essentially smooth local $k$-algebra $A$ also yields a new proof of the injectivity of $W(A)$ into the Witt group $W(K)$ of its field of fractions. Since this result is well-known and not very difficult (see for instance [13]) we use it whenever it is convenient, without proving it again.

Our proof has been inspired by Vladimir Voevodsky’s work [19] and makes essential use of a non-degenerate trace form for finite extensions of smooth algebras, which was discovered by Leonhard Euler in a special case. We recall its definition and main properties in §§2 and 3.

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2. The Euler trace

Let $k$ be any field and $A \hookrightarrow B$ a finite extension of smooth, purely $d$-dimensional $k$-algebras. Let $\Omega_A$ and $\Omega_B$ be the modules of Kähler differentials of $A$ and $B$ over $k$ and let $\Omega_{B/A}$ be the module of relative differentials of $B$ over $A$. Let $\omega_A = \bigwedge^d \Omega_A$, $\omega_B = \bigwedge^d \Omega_B$.

**Proposition 2.1.** There exists an isomorphism of $\omega$-algebras. Let $\Omega$ be any field and $\omega$ be the modules of Kähler differentials of $B$ over $A$. Then, for every $n$, we deduce, taking maximal exterior powers, that

\[
(\omega_B \otimes_B \bigwedge^n (I/I^2)) \cong B \otimes_A \omega_A.
\]

On the other hand, from the canonical exact sequence of projective $B$-modules (see [1], VII, Theorem 5.8)

\[
0 \to I/I^2 \to B \otimes_R \Omega_R \to \Omega_B \to 0,
\]

we deduce, taking maximal exterior powers, that

\[
\omega_B \cong (B \otimes_A \omega_A) \otimes_B \text{Hom}_B \left( \bigwedge^n (I/I^2), B \right) \cong (B \otimes_A \omega_A) \otimes_B \bigwedge^n (\text{Hom}_B(I/I^2), B)
\]

and then, from $(\ast)$,

\[
(B \otimes_A \omega_A) \otimes_B \bigwedge^n (\text{Hom}_B(I/I^2), B) \cong \omega_A \otimes_A \text{Hom}_A(B, A) \cong \text{Hom}_A(B, \omega_A).
\]

**Corollary 2.2.** If $\omega_A$ and $\omega_B$ are trivial there exists an isomorphism of $B$-modules

\[
\lambda : B \cong \text{Hom}_A(B, A).
\]

The isomorphism $\lambda$ induces an $A$-linear map

\[
\epsilon : B \to A
\]

defined by $\epsilon(x) = \lambda(1)(x)$. We call it an Euler trace, because Euler discovered a special case of it (see [5] and also [16], Chap. III). Conversely, from $\epsilon$ we get back $\lambda$ as $\lambda(x)(y) = \epsilon(xy)$.

In the next proposition we record, without proof, a few obvious properties of $\epsilon$ and $\lambda$.

**Proposition 2.3.** Let $B$ be a finite locally free $A$-algebra and $\epsilon : B \to A$ an $A$-linear map such that the bilinear map

\[
\lambda : B \to \text{Hom}_A(B, A) \quad \text{given by } \lambda(x)(y) = \epsilon(xy)
\]

is an isomorphism. Then, for every $A \to A'$, we have an $A'$-linear map

\[
\epsilon' = \epsilon \otimes_A A' : B' = B \otimes_A A' \to A'
\]

such that the associated $\lambda' : B' \to \text{Hom}_A(B', A')$ is an isomorphism of $B'$-modules. If $B = B_1 \times B_2$, $\lambda$ decomposes as $\lambda_1 \times \lambda_2$, where $\lambda_i : B_i \to \text{Hom}_A(B_i, A)$ is the map associated to $\epsilon|_{B_i}$. In particular, if $B = B_1 \times A$ the map $\lambda_2 : A \to A$ is the multiplication by a unit of $A$. 


3. Traces and quadratic spaces

Let \( A \hookrightarrow B \) be a finite flat extension of commutative rings. Let \( \epsilon : B \to A \) be an \( A \)-linear map such that the associated \( \lambda : B \to \text{Hom}_A(B, A) \) is an isomorphism. To every quadratic space \( q = (P, q) \) over \( B \) we associate the bilinear form \( \text{Tr}^\epsilon(q) = (P_A, \epsilon \circ q) \), where \( P_A \) denotes \( P \) considered as an \( A \)-module. This bilinear form is in fact a quadratic space and it is easy to check (see [9], I, §7) that \( \text{Tr} \) has the following properties:

1. \( \text{Tr}^\epsilon(q \perp q') = \text{Tr}^\epsilon(q) \perp \text{Tr}^\epsilon(q') \).
2. If \( q \) is hyperbolic, \( \text{Tr}^\epsilon(q) \) is hyperbolic.
3. For any homomorphism of commutative rings \( A \to A' \) we have
   \[ \text{Tr}^{\epsilon'}(q \otimes_A A') = \text{Tr}^\epsilon(q) \otimes_A A', \]
   where \( \epsilon' = \epsilon \otimes_A A' \).
4. If, as at the end of §2, \( B = B_1 \times B_2 \) and \( \epsilon_i = \epsilon|_{B_i} \),
   \[ \text{Tr}^\epsilon(q) = \text{Tr}^{\epsilon_1}(q_1) \perp \text{Tr}^{\epsilon_2}(q_2), \]
   where \( q_i = q \otimes_B B_i \).
5. If, as in (4), \( B = B_1 \times B_2 \) but \( B_2 = A \), then \( \epsilon_2 \) is the multiplication by a unit \( u \in A^* \) and thus, for any quadratic space \( q \),
   \[ \text{Tr}^{\epsilon_2}(q_2) = u \cdot q_2. \]

If \( f : A \to A' \) is a ring homomorphism and \( B' = B \otimes_A A' \), clearly \( B' = B'_1 \times B'_2 \) with \( B'_2 = A' \) and \( \epsilon'_2 \) is the multiplication by \( f(u) \).

6. Suppose that the map \( f : A \to A' \) considered in (5) has a section \( s : A' \to A \) and that \( B \otimes_A A' = B' = B'_1 \times B'_2 \) with \( B'_2 = A' \). Then, by (5), \( \epsilon'_2 \) is the multiplication by a unit \( u' \) of \( A' \). Replacing \( \epsilon \) by \( s(u')^{-1} \epsilon \) we get a new Euler map \( \epsilon : B \to A \) for which \( \epsilon'_2 = \text{id}_{A'} \) and for any ring homomorphism \( A' \to A'' \) we have \( B'' = B'_1 \times B''_2 \) with \( B''_2 = A'' \) and \( \epsilon''_2 = \text{id}_{A''} \). Thus, for any quadratic space \( q'' \) over \( B'' \),
   \[ \text{Tr}^{\epsilon''_2}(q''_2) = q''_2. \]

7. The linear map \( \epsilon : B \to A \) induces a homomorphism of Witt groups
   \[ \text{Tr}^\epsilon : W(B) \to W(A). \]

8. If \( B \) is of the form \( A[t]/(f) = A[\tau] \), where \( f \) is a monic polynomial of odd degree and \( \tau \) the class of \( t \), we can define an Euler map by
   \[ \epsilon(\tau^i) = \begin{cases} 0 & \text{if } i < n - 1, \\ 1 & \text{if } i = n - 1. \end{cases} \]

In this case a direct computation shows that the composite homomorphism
   \[ W(A) \to W(B) \to W(A) \]
   is the identity of \( W(A) \).
4. Reduction of purity to infinite base fields

Let $\mathbb{F}$ be a finite field of odd characteristic $p$ and $A$ a local, essentially smooth $\mathbb{F}$-algebra with maximal ideal $\mathfrak{m}$. Suppose that purity holds for essentially smooth local algebras over any infinite field $k$. Let $K$ be the field of fractions of $A$ and $\xi$ an unramified element of $W(K)$. Let $p^m$ be the cardinality of $A/\mathfrak{m}$ and $s$ an odd integer greater than 2 and prime to $m$. For any $i$ let $k_i$ be the field (in some fixed algebraic closure of $\mathbb{F}$) of degree $s^i$ over $\mathbb{F}$. Let $k$ be the union of all $k_i$. Since $k \otimes_{\mathbb{F}} (A/\mathfrak{m})$ is still a field, $B = k \otimes_{\mathbb{F}} A$ is a local, essentially smooth algebra over the infinite field $k$. Let $L = k \otimes_{\mathbb{F}} K$ be its field of fractions. The image $\xi_L$ of $\xi$ in $W(L)$ is unramified. In fact, let $q$ be a height one prime of $B$ and $p = A \cap q$. By assumption $\xi \in W(A_p)$ and since $A_p \rightarrow L$ factors through $B_q$ the class $\xi_L$ is in $W(B_q)$ for every $q$. Since purity holds for $B$, $\xi_L \in W(B)$. We can now find a finite subfield $\mathbb{F}'$ of $k$ and, for $A' = \mathbb{F}' \otimes_{\mathbb{F}} A$ a $\xi' \in W(A')$ which maps to $\xi_L$. Let $K'$ be the field of fractions of $A'$. Further enlarging $\mathbb{F}'$ we may assume that the images of $\xi$ and $\xi'$ in $W(K')$ coincide. Consider now the diagram

$$
\begin{array}{ccc}
W(A) & \longrightarrow & W(A') \\
\downarrow & & \downarrow \\
W(K) & \longrightarrow & W(K')
\end{array}
\xrightarrow{\text{Tr}_{\mathbb{F}'}} \begin{array}{c}
W(A) \\
W(K)
\end{array}
$$

where $\epsilon$ has been chosen as in §3 (8). Since the composition of the horizontal maps is the identity, we have $\alpha \circ \text{Tr}(\xi') = \xi$ in $W(K')$. Thus $\xi$ is indeed in the image of $W(A)$.

5. The geometric presentation lemma

We state and prove a lemma that will play a crucial role in the sequel. In geometrical disguise it sounds like this:

Lemma 5.1. Let $A$ be a local ring of a smooth variety over an infinite field $k$. Let $U = \text{Spec}(A)$ and let $u$ be the closed point of $U$. Let $p : \mathcal{X} \rightarrow U$ be an affine $U$-scheme, essentially smooth over $k$. Let $f$ be a regular element of $k[\mathcal{X}]$ such that $k[\mathcal{X}]/(f)$ is finite over $A$. We denote by $\mathcal{X}_f$ the principal open set defined by $f \neq 0$. Assume that there exists a finite surjective morphism $\mathcal{X} \rightarrow U \times \mathbb{A}^1_k$ of $U$-schemes and that there exists a section $\Delta : U \rightarrow \mathcal{X}$ of $p$ such that $p$ is smooth along $\Delta(U)$.

Then there exists a finite surjective morphism

$$
\pi : \mathcal{X} \rightarrow U \times \mathbb{A}^1_k
$$

of $U$-schemes with the following properties:

(a) $\pi^{-1}(U \times \{1\})$ is in $\mathcal{X}_f$.

(b) $\pi^{-1}(U \times \{0\}) = \Delta(U) \amalg \mathcal{D}$, where $\mathcal{D} \subset \mathcal{X}_f$.

Clearly the statement above is equivalent to the following, purely algebraic one.

Lemma 5.2. Let $A$ be a local essentially smooth algebra over an infinite field $k$, $\mathfrak{m}$ its maximal ideal and $R$ an essentially smooth $k$-algebra, which is finite over the polynomial algebra $A[\epsilon]$. Suppose that $\epsilon : R \rightarrow A$ is an $A$-augmentation and let $I = \ker(\epsilon)$. Assume that $R$ is smooth over $A$ at every prime containing $I$. Given $f \in R$ such that $R/Rf$ is finite over $A$ we can find an $s \in R$ such that
(1) $R$ is finite over $A[s]$.
(2) $R/Rs = R/I \times R/J$ for some ideal $J$ of $R$.
(3) $J + Rf = R$.
(4) $R(s - 1) + Rf = R$.

**Proof.** Replacing $t$ by $t - \epsilon(t)$ we may assume that $t \in I$. We denote by “bar” the reduction modulo $m$. By the assumptions made on $R$ the quotient $\overline{R}$ is smooth over $\overline{A}$ at its maximal ideal $\overline{I}$. Choose an $\alpha \in R$ such that $\overline{\alpha}$ is a local parameter of the localization $\overline{R}_\alpha$ of $\overline{R}$ at $\overline{I}$. By the chinese remainders’ theorem we may assume that $\overline{\alpha}$ does not vanish at the zeros of $\overline{I}$ different from $\overline{I}$. Without changing $\overline{\alpha}$ we may replace $\alpha$ by $\alpha - \epsilon(\alpha)$ and assume that $\alpha \in I$. Since $R$ is integral over $A[t]$ there exists a relation of integral dependence

$$
\alpha^n + p_1(t)\alpha^{n-1} + \ldots + p_n(t) = 0.
$$

For any $r \in k^*$ and any $N$ larger than the degree of each $p_i(t)$, putting $s = \alpha - rt^N$ we see that from the equation above that $t$ is integral over $A[s]$. Hence $R$, which is integral over $A[t]$, is integral over $A[s]$. Clearly $s \in I$. To insure that $\overline{\alpha}$ is also a local parameter of $\overline{R}_\alpha$ it suffices to take $n \geq 2$. By assumption $R$ and $A[s]$ are both regular and since $R$ is finite over $A[s]$, $R$ is locally free over $A[s]$ (see for instance Corollary 18.17 of [4]) and hence $R/Rs$ is free over $A$. Since $\overline{\alpha}$ is a local parameter of $\overline{R}_\alpha$, $R/\overline{sR}$ is étale over $\overline{A}$ at the augmentation ideal $\overline{I}$ and so we can find a $g \notin I + mR$ such that $(R/Rs)_g$ is étale over $A$. By the next sublemma $R/Rs$ splits as in (2).

**Sublemma 5.3.** Let $B$ be a commutative ring, $\gamma : B \to C$ a finite commutative $B$-algebra and $\lambda : C \to B$ an augmentation with augmentation ideal $I$. Let $h \in C$ be such that

(a) $C_h$ is étale over $B$.
(b) $\lambda(h)$ is invertible in $B$.

Then $C$ splits as $C/I \times C/J$ for some ideal $J$ of $C$.

**Proof.** Since $B \to C_h$ is étale and the composite map

$$
B \xrightarrow{\gamma} C_h \xrightarrow{\lambda} B
$$

is the identity of $B$, by Prop. 4.7 of [1], $C_h \to B$ is étale. But $C \to C_h$ is étale, hence $\lambda : C \to B$ is étale and in particular it induces an open morphism $\lambda^* : \text{Spec}(B) \to \text{Spec}(C)$. Its image $\lambda^*(\text{Spec}(B) = \text{Spec}(C/I)$ is therefore open and since it is also closed, $C$ splits as claimed.

To finish the proof of Lemma 5.2 we still have to choose $r \in k^*$ so that conditions (3) and (4) are satisfied. Since $R/Rf$ is semilocal, there are only finitely many maximal ideals of $R$ containing $f$. We denote by $m_1, \ldots, m_p$ those which, in case $f \in I + mR$, are different from $I + mR$. Recalling that $\alpha$ was chosen outside $m_1 \cup \ldots \cup m_p$, we have $s \notin m_1 \cup \ldots \cup m_p$ for almost any choice of $r \in k^*$. To see that condition (3) is satisfied it suffices to show that $J \not\subset m_i$ for $1 \leq i \leq p$ and that $J \not\subset mR + I$. The first assertion is clear because $s \in J \setminus m_i$ for $1 \leq i \leq p$. For the second one note that, since $R/Rs = R/I \times R/J$, we have $I + J = R$ and therefore $J \not\subset mR + I$. It remains to satisfy (4). Since $R/Rf$ is semilocal there exists a $\lambda \in k$ such that $s - \lambda$ is invertible in $R/Rf$. Without perturbing conditions (1), (2) and (3) we may replace $s$ by $\frac{1}{\lambda}s$ and thus satisfy (4) as well.
6. A commutative diagram for relative curves

Lemma 6.1. With the notation and the hypotheses of Lemma 5.2, let \( U = \text{Spec}(A) \) and \( X = \text{Spec}(R) \). Let \( p : X \rightarrow U \) be the structural morphism and \( \Delta : U \rightarrow X \) the morphism corresponding to the augmentation \( e : R \rightarrow A \). Let \( Z \subset X \) be a closed set of codimension at least 2, contained in vanishing locus of \( f \). Suppose that \( \omega_{X/k} \) is trivial. There exists a nonzero element \( g \in A \) such that \( X_g \subseteq X \setminus Z \) and for any such \( g \) there exists a commutative diagram

\[
\begin{array}{ccc}
W(X \setminus Z) & \xrightarrow{\psi} & W(U) \\
W(i) \downarrow & & \downarrow W(i) \\
W(X_g \setminus Z_g) = W(X_g) & \xrightarrow{\psi} & W(U_g),
\end{array}
\]

where \( i : U_g \rightarrow U \) and \( j : X_g \rightarrow X \setminus Z \) are the inclusions.

Proof. By Lemma 5.2 there exists an element \( s \in R \) satisfying the conditions (1) to (4). The evaluation in \( s \) defines a finite surjective morphism \( \pi : X \rightarrow U \times \mathbb{A}_k^1 \) of \( U \)-schemes such that \( \pi^{-1}(U \times \{0\}) = \Delta(U) \amalg D_0 \) with \( D_0 \subset X_f \). Since \( \omega_{U \times \mathbb{A}_k^1/k} \) is obviously trivial and \( \omega_{X/k} \) is trivial by assumption, we can use Corollary 2.2 to find an Euler trace map \( e : B \rightarrow A[t] \) such that the associated map \( \lambda : B \rightarrow \text{Hom}_{A[t]}(B, A[t]) \) is an isomorphism. We can then choose a trace map \( \text{Tr} : W(X) \rightarrow W(U \times \mathbb{A}_k^1) \) as in §3. Restricting \( \text{Tr} \) to \( W(\pi^{-1}(U \times \{0\})) \) yields a homomorphism \( W(\pi^{-1}(U \times \{0\})) \rightarrow W(U \times \{0\}) \). Since the natural embedding \( A \hookrightarrow A[t] \) is a section of the evaluation at \( t = 0 \), by (6) of §3 we may choose the Euler trace map \( e : B \rightarrow A[t] \) such that \( \text{Tr}|_{W(\Delta(U))} = W(\Delta) \).

Having fixed \( e \) and \( \text{Tr} \) in this way, restricting \( e \) to \( D_i \), \( i = 1, 2 \), we get trace maps \( \text{Tr}_i : W(D_i) \rightarrow W(U) \). Let \( \varphi_i : D_i \rightarrow X \setminus Z \) be the inclusion. We put

\[
\psi = \text{Tr}_1 \circ W(\varphi_1) - \text{Tr}_0 \circ W(\varphi_0).
\]

Since \( Z \) is of codimension \( \geq 2 \) in \( X \) and \( \pi : X \rightarrow U \times \mathbb{A}_k^1 \) is finite, the image of \( Z \) in \( U \) under the structural map is contained in the vanishing locus of some non zero \( g \in A \). Making now the base change of \( e \) by means of the inclusion \( i : U_g \rightarrow U \) we get \( e_g \) and \( \text{Tr}_g \) such that we still have \( \text{Tr}_g|_{W(\Delta(U_g))} = W(\Delta_g) \) (see (6) of §3). Further restricting \( e_g \) to \( D_i g, \ i = 1, 2 \), we get trace maps \( \text{Tr}_i g : W(D_i g) \rightarrow W(U_g) \). Let \( \varphi_i g : D_i g \rightarrow X_g \setminus Z_g = X_g, \ i = 1, 2 \), be the inclusions. We put

\[
\psi_g = \text{Tr}_1 g \circ W(\varphi_{1 g}) - \text{Tr}_0 g \circ W(\varphi_{0 g}).
\]

Clearly properties (3) and (4) of §3 imply the relation \( W(i) \circ \psi = \psi_g \circ W(j) \). Thus, to complete the proof of the lemma, it suffices to check the relation \( \psi_g = W(\Delta_g) \). For this take any \( \xi \) in \( W(X_g) \) and write a chain of relations

\[
\text{Tr}_g(\xi)|_{U_g \times \{1\}} - \text{Tr}_g(\xi)|_{U_g \times \{0\}} = \text{Tr}_{1 g}(\xi|_{D_{1 g}}) - \text{Tr}_{0 g}(\xi|_{D_{0 g}}) - \text{Tr}_g(\xi|_{\Delta(U_g)}) = \psi_g(\xi) - W(\Delta_g)(\xi).
\]

A well-known theorem of Max Karoubi (see [9], VII, §4) asserts that for any affine \( k \)-scheme \( S \) the canonical homomorphism \( W(S) \rightarrow W(S \times \mathbb{A}_k^1) \) is an isomorphism, and therefore the left hand side of the relation above is zero. This proves the relation \( \psi_g = W(\Delta_g) \), whence the commutativity of the diagram.
7. Purity

Theorem 7.1. Let \(A\) be a local, essentially smooth algebra over an infinite field \(k\) and let \(K\) be its field of fractions. Every unramified element of \(W(K)\) belongs to \(W(A)\).

Proof. Let \(U = \text{Spec}(A)\) and let \(\xi\) be an unramified element of \(W(K)\). By assumption there exist a smooth \(d\)-dimensional \(k\)-algebra \(R = k[t_1, \ldots, t_n]\) and a prime ideal \(p\) of \(R\) such that \(A = R_p\). We first reduce the proof to the case in which \(p\) is maximal. To do this, choose a maximal ideal \(m\) containing \(p\). Since \(k\) is infinite, by a standard general position argument we can find \(d\) algebraically independent elements \(X_1, \ldots, X_d\) such that \(R\) is finite over \(k[X_1, \ldots, X_d]\) and étale at \(m\). After a linear change of coordinates we may assume that \(R/p\) is finite over \(B = k[X_1, \ldots, X_m]\), where \(m\) is the dimension of \(R/p\). Clearly \(R\) is smooth over \(B\) at \(m\) and thus, for some \(h \in R \setminus m\), the localization \(R_h\) is smooth over \(B\). Let \(S\) be the set of nonzero elements of \(B\), \(k' = S^{-1}B\) the field of fractions of \(B\) and \(R' = S^{-1}R_h\). The prime ideal \(p' = S^{-1}p_h\) is maximal in \(R'\), the \(k'\)-algebra \(R'\) is smooth and \(A = R'_p\).

From now on we assume that \(A = O_{X,x}\) is the local ring of a closed point \(x\) of a smooth \(d\)-dimensional affine variety \(X\) over \(k\).

Replacing \(X\) by a sufficiently small affine neighbourhood of \(x\) we may assume that \(\omega_{X/k}\) is trivial. By Proposition 2.4 of [3] we may assume that \(\xi\) is defined on the complement of a closed set \(Z\) of codimension at least 2 in \(X\). Let \(f \neq 0\) be a regular function on \(X\) which vanishes on a closed set \(Y\) containing \(Z\). By Quillen’s trick (see [15], Lemma 5.12) we can find a morphism \(q : X \to \mathbb{A}^{d-1}_k\) with the following properties:

1. \(q\) is smooth at \(x\).
2. \(q|_Y : Y \to \mathbb{A}^{d-1}_k\) is finite.
3. \(q\) factors as

\[
\begin{array}{ccc}
X & \xrightarrow{q_1} & \mathbb{A}^d_k \\
\downarrow q & & \downarrow p \\
\mathbb{A}^{d-1}_k & \xleftarrow{pr} & \mathbb{A}^{d-1}_k
\end{array}
\]

with \(q_1\) finite and surjective.

Consider the cartesian square

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{p_X} & X \\
\downarrow p & & \downarrow q \\
U & \xrightarrow{r} & \mathbb{A}^{d-1}_k
\end{array}
\]

where \(U = \text{Spec}(O_{X,x})\), \(r = q|_U\), \(\mathcal{X} = U \times \mathbb{A}^{d-1}_k\), \(p\) is the first projection and \(\Delta : U \to \mathcal{X}\) the diagonal. Denote again by \(f\) the composition of \(f\) with \(p_X\).

Since \(r\) is smooth, \(p_X\) is also smooth and since \(X\) is smooth over \(k\), so is \(\mathcal{X}\). By base change, condition (3) implies that \(\mathcal{X}\) is an affine relative curve over \(U\). Since \(U\) is local and \(q\) is smooth at \(x\), \(p\) is smooth along \(\Delta(U)\). From (3), by base change via
$r : U \to \mathbb{A}^{d-1}_k$, we get a commutative triangle

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{p_1} & U \times \mathbb{A}^d_k \\
\downarrow p & & \downarrow \pi \\
U & & \\
\end{array}
$$

with $p_1$ finite. Again by the same base change we see that $k[\mathcal{X}]/(f)$ is finite over $A$. Thus all the hypotheses of Lemma 5.1 are satisfied and we can find a $U$-morphism $\pi : \mathcal{X} \to U \times \mathbb{A}^d_k$ satisfying conditions (a) and (b).

We further claim that $\omega_{\mathcal{X}}$ is trivial. To see this observe that

$$
\omega_{\mathcal{X}/k} \simeq p_1^* (\omega_{\mathcal{X}/k}) \otimes_{\mathcal{O}_{\mathcal{X}}} \omega_{\mathcal{X}/X}
$$

and that $\omega_{\mathcal{X}/X} \simeq p^* \omega_{U/\mathbb{A}^{d-1}_k}$. Since $U$ is essentially smooth over $\mathbb{A}^{d-1}_k$, $\omega_{U/\mathbb{A}^{d-1}_k}$ is locally free of rank one, hence trivial because $U$ is local. Thus $p^* \omega_{U/\mathbb{A}^{d-1}_k}$ is trivial and, since $\omega_{\mathcal{X}/k}$ is trivial by assumption, we conclude that $\omega_{\mathcal{X}/k}$ is trivial.

We can now apply Lemma 6.1 with $Z = U \times \mathbb{A}^{d-1}_k Z \subset \mathcal{X}$. We define $\eta = \psi(W(p_X)(\xi))$ and claim that $\eta$ is an extension of $\xi$ to $U$. In fact, choosing $g \in A$ as in 6.1 and denoting by $i : U_g \to U$, $i' : U_g \to \mathcal{X} \setminus Z$ and $j : \mathcal{X}_g \to \mathcal{X} \setminus Z$ the inclusions, we have

$$
W(i)\eta = W(i) \circ \psi \circ W(p_X)\xi = W(\Delta_g) \circ W(j) \circ W(p_X)\xi = W(p_X \circ j \circ \Delta_g)\xi = W(i')\xi.
$$

This completes the proof of Theorem 7.1.

To prove Theorem A we now recall a celebrated result of Dorin Popescu (see [10], [11] and [12] or [2] or, for a self-contained proof, [18]).

Let $k$ be a field and $R$ a local $k$-algebra. We say that $R$ is geometrically regular if $k' \otimes_k R$ is regular for any finite extension $k'$ of $k$. A ring homomorphism $A \to R$ is called geometrically regular if it is flat and for each prime ideal $q$ of $R$ lying over $p$, $R_q/pR_q = k(p) \otimes_A R_q$ is geometrically regular over $k(p) = A_p/p_p$.

Observe that any regular local ring containing a field $k$ is geometrically regular over the prime field of $k$.

**Popescu’s theorem.** A homomorphism $A \to R$ of noetherian rings is geometrically regular if and only if $R$ is a filtered direct limit of smooth $A$-algebras.

**Proof of Theorem A.** Let $R$ be a regular local ring containing a field. Let $k$ be the prime field of $R$. By Popescu’s theorem $R = \varinjlim A_\alpha$, where the $A_\alpha$’s are smooth $k$-algebras. We first observe that we may replace the direct system of the $A_\alpha$’s by a system of essentially smooth local $k$-algebras. In fact, if $m$ is the maximal ideal of $R$, we can replace each $A_\alpha$ by $(A_\alpha)_{p_\alpha}$, where $p_\alpha = m \cap A_\alpha$. Note that in this case the canonical morphisms $\varphi_\alpha : A_\alpha \to R$ are local and every $A_\alpha$ is a regular local ring, in particular a factorial ring.

Let now $L$ be the field of fractions of $R$ and, for each $\alpha$ let $K_\alpha$ be the field of fractions of $A_\alpha$. Let $\xi$ be an unramified element of $W(L)$. We may represent $\xi$ by a diagonal matrix $q = \text{diag}(r_1, \ldots, r_n)$ with $r_1, \ldots, r_n$ in $R$. Let $\Sigma$ be the (finite) set of height one primes of $R$ which divide at least one of the $r_i$. For every $p \in \Sigma$ we can find a matrix
\( \sigma(p) \in \text{GL}_n(L) \) that transforms \( q \) into a diagonal form \( \text{diag}(u_1(p), \ldots, u_n(p)) \) with every \( u_i(p) \in R \setminus p \). Clearing denominators we may assume that \( \sigma(p) \in M_n(R) \) and that

\[
\sigma(p)^T q \sigma(p) = \text{diag}(u_1(p), \ldots, u_n(p))(d(p))^2
\]

for some \( d(p) \in R \). We can now choose an index \( \alpha \) such that, for every \( p \in \Sigma \), \( A_\alpha \) contains preimages \( \tilde{r}_1, \ldots, \tilde{r}_n, \tilde{u}_1(p), \ldots, \tilde{u}_n(p) \), \( \tilde{d}(p) \) and \( \tilde{\sigma}_{ij}(p) \) of the elements \( r_1, \ldots, r_n, u_1(p), \ldots, u_n(p), d(p) \) and of the coefficients \( \sigma_{ij}(p) \) of \( \sigma(p) \). Having chosen these preimages consider the relations

\[
(\ast) \quad \tilde{\sigma}(p)^T \tilde{q} \tilde{\sigma}(p) = \text{diag}(\tilde{u}_1(p), \ldots, \tilde{u}_n(p))(\tilde{d}(p))^2
\]

where \( \tilde{q} = \text{diag}(\tilde{r}_1, \ldots, \tilde{r}_n) \) and \( \tilde{\sigma}(p) \) is the matrix \( (\tilde{\sigma}_{ij}(p)) \). Since they hold over \( R \), we may assume, after replacing \( \alpha \) by some larger index, that they hold over \( A_\alpha \). We claim that the class of \( \tilde{q} \) (which we still denote by \( \tilde{q} \)) is an unramified element of \( W(K_\alpha) \). To show this suppose that \( \tilde{q} \) is ramified at a height one prime ideal \( pA_\alpha \). Then \( p \) divides some \( \tilde{r}_i \). Any height one prime \( p \) of \( R \) containing \( pR \) also contains \( r_i \) and thus belongs to \( \Sigma \). Since \( u_i(p) \in R \setminus p \) we have \( \tilde{u}_i(p) \in A_\alpha \setminus pA_\alpha \) and thus the relation \((\ast)\) shows that \( \tilde{q} \) is unramified at \( pA_\alpha \). By purity for \( A_\alpha \) there exists a \( \xi_\alpha \in W(A_\alpha) \) that coincides with \( \tilde{q} \) in \( W(K_\alpha) \). The ideal \( r = \ker(\varphi_\alpha) \) is prime and does not contain any \( \tilde{r}_i \). Hence \( \tilde{q} \) is a quadratic space over the essentially smooth local algebra \( B_\alpha = (A_\alpha)_r \). Since \( \tilde{q} \) and \( \xi_\alpha \) coincide in \( W(K_\alpha) \) they already coincide in \( W(B_\alpha) \) because \( W(B_\alpha) \to W(K_\alpha) \) is injective. The commutative diagram of ring homomorphisms

\[
\begin{array}{ccc}
A_\alpha & \xrightarrow{\varphi_\alpha} & R \\
\downarrow & & \downarrow \\
B_\alpha & \longrightarrow & L \\
\end{array}
\]

shows that \( W(\varphi_\alpha)(\xi_\alpha) = q \) in \( W(L) \). This proves that \( q \) is indeed in \( W(R) \).

8. An injectivity theorem

If \( A \) is a regular ring of dimension greater than 3 and \( K \) its field of fractions, the canonical homomorphism \( W(A) \to W(K) \) need not be injective. In this section we prove the following injectivity result, from which we shall deduce Theorem C.

**Theorem 8.1.** Let \( A \) be a local, essentially smooth algebra over an infinite field \( k \) of characteristic \( \neq 2 \). Let \( K \) be the field of fractions of \( A \) and \( f \) a regular parameter of \( A \). The canonical homomorphism \( W(A_f) \to W(K) \) is injective.

The proof of this theorem is similar to that of Theorem 7.1. As we did there, we assume, without loss of generality, that \( A \) is the local ring of a closed point \( x \) of a smooth affine variety \( X \). If \( A \) is 1-dimensional \( A_f = K \) and there is nothing to prove, so we assume that \( A \) is at least 2-dimensional. We need the following variant of Quillen’s trick.

**Lemma 8.2.** Let \( X \) be an irreducible affine smooth variety over an infinite field \( k \) and \( x \) a closed point of \( X \). Let \( A \) be the local ring of \( x \), \( f \in k[X] \) a regular function on \( X \) which is a regular parameter of \( A \) and \( g \in k[X] \), \( g \) prime to \( f \). Denote by \( Y \) the vanishing
locus of $f$ and by $Z$ the vanishing locus of $g$. There exists a morphism $q : X \to \mathbb{A}^{d}_{k}$ with the following properties:

(1) $q$ is smooth at $x$.
(2) $q|_{Y \cap Z} : Y \cap Z \to \mathbb{A}^{d}_{k}$ is finite.
(3) $q$ factors as

$$
\begin{array}{c}
X \xrightarrow{q_1} \mathbb{A}^{d}_{k} \xrightarrow{pr} \mathbb{A}^{d-1}_{k}
\end{array}
$$

with $q_1$ finite and surjective.
(4) $q(Y) = \{0\} \times \mathbb{A}^{d-2}_{k}$.
(5) $q^{-1}(\{0\} \times \mathbb{A}^{d-2}_{k}) = Y \cup Y'$ for some closed set $Y' \subset X$ which avoids $x$.

We first recall an auxiliary result, which has been proved in slightly different versions by several authors.

**Lemma 8.3.** Under the assumptions of Lemma 8.2 there exists a morphism $q_2 : X \to \mathbb{A}^{d}_{k}$ such that

(i) $q_2$ is finite.
(ii) $q_2$ is étale at $x$.
(iii) $k(x) = k(q_2(x))$.
(iv) $Y \cap q_2^{-1}(q_2(x)) = \{x\}$.

**Proof.** Suppose that $X$ is a closed set of $\mathbb{A}^{N}_{k} \subset \mathbb{P}^{N}$ and let $\overline{X}$ be its closure in $\mathbb{P}^{N}$. To prove Lemma 8.3 we will take for $q_2$ the projection from a suitable linear subspace $L$ at infinity. Let $\overline{k}$ be an algebraic closure of $k$ and $\varphi : \overline{k} \otimes_{k} X \to X$ the canonical projection. Then $\varphi^{-1}(x)$ is a finite set of closed points $\{x_1, \ldots, x_n\}$ of $\overline{k} \otimes_{k} X$. Choose an $N - d - 1$-dimensional linear subspace $L$ in $\mathbb{P}^{N} \setminus \mathbb{A}^{N}_{k}$ with the following properties:

(a) $L$ is defined over $k$.
(b) $L$ does not intersect $\overline{k} \otimes_{k} \overline{X}$.
(c) $L$ does not intersect the tangent planes of $\overline{k} \otimes_{k} \overline{X}$ at $x_1, \ldots, x_n$.
(d) For $i \neq j$ we have $q_2(x_i) \neq q_2(x_j)$.
(e) $L$ does not intersect the closures of the cones with vertices $x_1, \ldots, x_n$ and base $\overline{k} \otimes_{k} Y$.

Dimension considerations show the existence of infinitely many such linear spaces. Condition (a) insures that $q_2$ is defined over $k$. Condition (b) insures that $q_2 : X \to \mathbb{A}^{d}_{k}$ is finite. Condition (c) insures that $q_2$ is étale at $x$. Since the group $Aut_k(\overline{k})$ acts transitively on $\{x_1, \ldots, x_n\}$, by condition (d) it acts transitively on $\{q_2(x_1), \ldots, q_2(x_n)\}$ as well. This shows that the separability degree of $k(q_2(x))$ over $k$ is the same as that of $k(x)$. But $q_2$ is étale at $x$, hence the extension $k(x)/k(q_2(x))$, being separable, must be of degree one. Thus condition (iii) is satisfied. Finally, condition (iv) follows from (e).

**Proof of Lemma 8.2.** We choose $q_2$ as in the previous lemma. We put $B = k[\mathbb{A}^{d}_{k}]$ and $C = k[X]$. The map $q_2$ induces an inclusion $\iota : B \hookrightarrow C$ and $C$ is a finite $B$-module. The images of the closed subschemes $Y = \{f = 0\}$ and $Z = \{g = 0\}$ of $X$ are two closed closed subschemes of $\mathbb{A}^{d}_{k}$ defined, respectively, by $f_0 = 0$ and $g_0 = 0$ for some $f_0, g_0 \in k[\mathbb{A}^{d}_{k}]$. The inclusion $\iota$ induces a finite map $B/Bf_0 \to C/Cf$. Let $\mathfrak{m}$ be the maximal ideal of $B$ corresponding to the closed point $q_2(x)$. Since $x$ is the unique closed
point of $Y$ lying over $q_2(x)$, the localization $(C/Cf)_m = B_m \otimes_B (C/Cf)$ is local and finite over $(B/Bf_0)_m$. By condition (iii) these two local rings have the same residue field, hence by Nakayama’s lemma they are isomorphic. This shows in particular that $f_0$ is a regular parameter of $B$ at $q_2(x)$. On the other hand, since $C$ is étale over $B$ at $x$, $f_0$ is also a regular parameter of $C$ at $x$.

We now have two polynomials $f_0$ and $g_0$ in $B = k[X_1, \ldots, X_d]$ which we may assume monic in, say, $X_1$. The map $k[Y_1, \ldots, Y_d] \to k[X_1, \ldots, X_d]$ defined by $Y_1 \mapsto f_0$ and $Y_i \mapsto X_i$ for $i \neq 1$ induces a finite morphism $q_3 : A_k^d \to A_k^d$. Composing $q_2$ with $q_3$ we obtain a finite map $q_1 = q_3 \circ q_2 : X \to A_k^d$. This map is smooth at $x$ because $q_2$ is étale at $x$ and $f_0$ is a regular parameter at $q_2(x)$. It maps $Y$ onto the hyperplane $Y_1 = 0$ and $Z$ onto some closed set $\{g_1 = 0\}$. Since $q_1$ is a local-étale isomorphism at $x$, $Y_1$ does not divide $g_1$. Thus $q_1(Y \cap Z)$ is a proper closed subset of the hyperplane $Y_1 = 0$. We may therefore assume, after a linear change of coordinates involving only $Y_2, \ldots, Y_d$, that the projection $pr$ onto $Y_2 = 0$ is finite on $q_1(Y \cap Z)$. We now take $q = pr \circ q_1$.

Since $q^{-1}(\{0\} \times A_k^{d-2})$ is smooth at $x$, it contains only one component —namely $Y$— that passes through $x$, whence (5).

**Proof of Theorem 8.1.** Let $\xi$ be an element in the kernel of $W(A_f) \to W(K)$. There is a $g \in A$, which we may suppose prime to $f$, such that $\xi \in \ker(W(A_f) \to W(A_{fg}))$. We may represent $\xi$ by a quadratic space $q$ defined over $A_f$ which becomes hyperbolic over $A_{fg}$. Patching $q$ over $\text{Spec}(A_f)$ with a suitable hyperbolic space over $\text{Spec}(A_g)$ we get a space over the complement of the closed set $W = Y \cap Z$, where $Y = \{f = 0\}$ and $Z = \{g = 0\}$. Applying Lemma 8.2 we get a map $q : X \to A_k^{d-1}$ satisfying properties (1) to (5). Let $h \in k[X]$ be an element which vanishes identically on $W$ and such that $q$ is finite on the closed subscheme defined by $\{h = 0\}$. As in the proof of Theorem 7.1, but with $h$ instead of $f$ and $W$ instead of $Z$, we get a commutative square

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{p_X} & X \\
p & \downarrow{\Delta} & \downarrow{q} \\
U & \xrightarrow{r} & A_k^{d-1}
\end{array}
$$

where $U = \text{Spec}(\mathcal{O}_{X,x})$, $r = q|_U$, $\mathcal{X} = U \times_{A_k^{d-1}} X$, $p$ is the first projection and $\Delta : U \to \mathcal{X}$ the diagonal. We denote again by $h$ the composition of $h$ with $p_X$ and we put $W = U \times_{A_k^{d-1}} W$. As in the proof of 7.1, we assume that $X$ has been so chosen that $\omega_X$ is trivial.

Applying the geometric presentation lemma we find a map $\pi : \mathcal{X} \to U \times A_k^1$ of $U$-schemes such that $\pi^{-1}(U \times \{1\}) = D_1$ is in $\mathcal{X}_h$ and $\pi^{-1}(U \times \{0\}) = \Delta(U) \cap D_0$, where $D_0 \subset \mathcal{X}_h$. Put for simplicity $s = Y_2$. By condition (5) we have $W \subset \mathcal{X} \setminus \mathcal{X}_s$ and hence, by Lemma 6.1, there exists a commutative square

$$
\begin{array}{ccc}
W(\mathcal{X} \setminus W) & \xrightarrow{\psi} & W(U) \\
W(\mathcal{X}_s) & \xrightarrow{W(\psi)} & W(U_s) \\
W(U_s) & \xrightarrow{W(\iota)} & W(U_i)
\end{array}
$$

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where \( i : U_s \rightarrow U \) and \( j : X_s \rightarrow X \setminus W \) are the inclusions. Repeating the argument of the proof of Theorem 7.1, we define \( \eta = \psi(W(p_X)(\xi)) \in W(A) \) and get \( \eta_s = \xi_s \). By condition (5), \( A_s = A_f \) and since \( W(A) \rightarrow W(K) \) is injective and \( \xi \) vanishes on \( W(K) \) we get \( \eta = 0 \). This shows that \( \xi = 0 \) as well.

**Proof of Theorem C.** We first want to show that

This homomorphism fits into the exact sequence

where

and

with filtered direct limits, we have

This is even simpler than for Theorem A: we find a sufficiently large odd degree extension \( \mathbb{F}' \) of the finite base field \( \mathbb{F} \) such that \( A' = \mathbb{F}' \otimes_{\mathbb{F}} A \) is still a local ring and \( \xi_{\mathbb{F}'} = 0 \) in \( W(A') \). Then, choosing \( \epsilon \) as in §3, (8), we see that \( \xi = \text{Tr}^\epsilon(\xi_{\mathbb{F}'}) = 0 \).

We now prove Theorem B. Let \( R \) be a regular local ring containing a field and let \( L \) be the field of fractions of \( R \). Let \( k \) be the prime field of \( R \). As in the proof of Theorem A, \( R = \lim A_\alpha \), where the \( A_\alpha \)'s are essentially smooth local \( k \)-algebras. Let \( f \) be a regular parameter of \( R \) and \( \xi \) an element in the kernel of \( W(R_f) \rightarrow W(L) \). There exists a \( g \in R \) such that \( \xi \) vanishes in \( W(R_{fg}) \). For a suitable index \( \alpha \) choose lifts \( f_\alpha \) and \( g_\alpha \) of \( f \) and \( g \) in \( A_\alpha \). We may replace the filtered direct system of the \( A_\alpha \)'s by the subsystem of all \( A_\beta \) with \( \beta \geq \alpha \). Clearly we still have \( R = \lim A_\beta \). We put, for every \( \beta \geq \alpha \),

and \( g_\beta = \varphi_{\beta\alpha}(g_\alpha) \) where the \( \varphi_{\beta\alpha} : A_\alpha \rightarrow A_\beta \) are the transition homomorphisms. It is easy to see that \( \lim \alpha (A_\beta f_\alpha) = R_f \) and \( \lim (A_\beta f_{\beta g_\alpha}) = R_{fg} \). Since the functor \( W \) commutes with filtered direct limits, we have

Since \( \varphi_{\beta} : A_\beta \rightarrow R \) is local, \( f_\beta \) is a regular parameter of \( A_\beta \). Hence the left hand side vanishes and, in particular, \( \xi = 0 \). This proves Theorem B.

9. **A short exact sequence**

Let \( B \) be a discrete valuation ring, \( p = Bp \) its maximal ideal and \( L \) its field of fractions. Let \( v : L^* \rightarrow \mathbb{Z} \) be the corresponding valuation of \( L \). Recall that there is a homomorphism (which depends on the choice of the local parameter \( p \) ) \( \partial_p : W(L) \rightarrow W(B/p) \) called second residue and defined on rank one forms \( < up^m > \) with \( u \in B^* \) by

\[
\partial_p(< up^m >) = \begin{cases} 
0 & \text{if } m \text{ is even} \\
< \overline{u} > & \text{if } m \text{ is odd}
\end{cases}
\]

where \( \overline{u} \) is the image of \( u \) in \( B/p \).

This homomorphism fits into the exact sequence

\[
0 \rightarrow W(B) \rightarrow W(L) \xrightarrow{\partial_p} W(B/p) \rightarrow 0
\]

**Proof of Theorem C.** We have a commutative diagram

of solid arrows in which the bottom line is exact. We first want to show that

\[
\partial_f \circ \beta(W(A_f)) \subseteq W(A/Af)
\]
and then check that the top line is exact.

For the first assertion it suffices to show, by purity, that, for any \( \xi \in W(Af) \), \( \partial_f \circ \beta(\xi) \) is unramified over \( A/Af \). Let \( q/Af \) be a prime of height one of \( A/Af \). We want to show that \( \partial_f \circ \beta(\xi) \) is in the image of \( W(Aq/Af) \). For this, after replacing \( A \) by \( A_q \) in the diagram above, we may assume that \( A \) is a local regular ring of dimension 2. But in this case the assertion is precisely Theorem 3 of [7].

Exactness left and right is obvious. Let \( \xi \) be an element of \( \ker(\delta) \). Since \( \beta \) is injective, we may consider \( \xi \) as an element of \( W(K) \). From the exactness of the bottom line we see that \( \xi \) is in the image of \( W(A_p) \). Since it also belongs to \( \beta(W(Af)) \), it is unramified and by purity it comes from \( W(A) \).

**Proof of Theorem D.** Apply Theorem C to the local ring \( A[[t]] \), taking \( t \) as regular parameter and using the fact that \( W(A[[t]]) = W(A) \).

**References**


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