Ketu* and the second invariant of a quadratic space

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Abstract. Using the Chern classes defined by Grothendieck, we map a K-theoretic invariant of invertible symmetric matrices defined by Giffen to the Clifford invariant.

Key words. Chern classes, quadratic forms, Clifford invariant.

1. Introduction

An invariant for quadratic forms over a commutative ring $A$ with values in a quotient of $K_2(A)$ has been defined by Giffen [6]. He also verified that if $A$ is a field, his invariant maps to the Clifford invariant of the quadratic form under the well-known symbol map to the Brauer group.

Using Grothendieck’s theory of equivariant Chern classes [7], Shekhtman [13] and Soulé [14] defined homomorphisms $c_{ji}: K_j(A) \to H^{2i-j}_{\text{ét}}(\text{Spec}A, \mu_m^\otimes)$ for all possible $i, j$ and $m$. We show that Giffen’s invariant maps under $c_{22}$ to the Clifford invariant in $H^2_{\text{ét}}(\text{Spec}A, \mu_2)$, defined by the connecting map associated to the sequence

$$1 \to \mu_2 \to \text{Spin} \to \text{SO} \to 1.$$ 

The method of proof involves basically two steps. The first is a general reduction to rank 2 quadratic spaces, via some calculations of étale cohomology groups of affine quadrics. The second is an explicit calculation of the invariants for rank 2 quadratic spaces over smooth affine curves. In this calculation we use Suslin’s results on $K$-cohomology of smooth affine varieties. Since no explicit description of $c_{22}$ is known, except on elements generated by symbols, our results can be interpreted as a computation of $c_{22}$ on that part of $K_2$ arising from invariants of quadratic forms.

In what follows $A$ will denote a commutative ring in which 2 is invertible.

2. The Giffen invariant

Let $A$ be a commutative ring in which 2 is invertible, $WG(A)$ the Witt-Grothendieck group of $A$ (see for instance [9]) and $W_0(A)$ its quotient by the subgroup generated by the hyperbolic plane $H(A)$. We denote by $W_1(A)$ the kernel of the forgetful map

$$s_0 : W_0(A) \to K_0(A)/2\mathbb{Z}.$$ 

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*Ketu may not be connected with $K_2$ but, according to Herrmann Graßman’s Wörterbuch zum Rig-veda, “es bezeichnet das, was sich sichtbar oder kenntlich macht...”*
Every class of $W_1(A)$ is represented by a symmetric matrix of even rank. We define a homomorphism (called discriminant)

$$s_1 : W_1(A) \longrightarrow K_1(A)/\text{Tr}(K_1(A)),$$

where $\text{Tr}(\alpha) = \alpha + \alpha^t$. Let $S$ be a symmetric $2n \times 2n$ matrix representing an element of $W_1(A)$. We set $s_1(S) = (-1)^n(\text{class of } S)$. Let $W_2(A)$ be the kernel of $s_1$. Every class of $W_2(A)$ can be represented by an elementary symmetric matrix whose rank is a multiple of 4. Let $ES(A)$ be the subset of $E(A)$ consisting of symmetric matrices of rank divisible by 4 and trivial discriminant.

We say that $\alpha$ and $\beta$ in $ES(A)$ are equivalent if there exists an elementary matrix $\epsilon$ and integers $m, n,$ such that

$$\epsilon^t \begin{pmatrix} \alpha & 0 \\ 0 & J_{2m} \end{pmatrix} \epsilon = \begin{pmatrix} \beta & 0 \\ 0 & J_{2n} \end{pmatrix},$$

where $J_k$ is the matrix with $k$ diagonal blocks equal to $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

These equivalence classes of elementary matrices form an abelian group $EW_2(A)$ in which the class of $J_{2m}$ is the identity and the class of $-\alpha$ is the inverse of the class of $\alpha$. Clearly, $W_2$ is a quotient of $EW_2(A)$. We define a map

$$s_2 : EW_2(A) \longrightarrow K_2(A)/\text{Tr}(K_2(A)).$$

This time, Tr is the map $\alpha \mapsto \alpha - \alpha^t$, where $\alpha \mapsto \alpha^t$ is the involution induced on $K_2(A)$ by the unique involution of the Steinberg group $\text{St}(A)$ that maps each generator $x_{ij}(\lambda)$ to $x_{ji}(\lambda)$.

Let $\alpha \in ES(A) \cap \text{GL}_{4n}(A)$ represent an element of $EW_2(A)$ and $\tilde{\alpha}$ denote a lift of $\alpha$ in $\text{St}(A)$. We set

$$s_2(\alpha) = n(-1, -1) + (\text{class of } \tilde{\alpha}^{-1} \tilde{\alpha}^t).$$

Clearly this respects the equivalence in $ES(A)$ and passes down to a homomorphism $s_2$.

**Remark.** The Steinberg group is written multiplicatively, but $K_2$ additively.

### 3. Catch 22 and transposition

We recall a few facts about equivariant Chern classes, from [7], [13] and [14].

Let $X$ be a $G$-scheme, i.e. a scheme $X$ with a discrete group $G$ operating on it. A sheaf $\mathcal{F}$ over $X$ is a $G$-sheaf if for every $g \in G$ there is a morphism of sheaves $\varphi_g : g^*\mathcal{F} \rightarrow \mathcal{F}$ such that $\varphi_{gh} = \varphi_h \circ h^*(\varphi_g)$ for all $g, h \in G$ and $\varphi_e = \text{Id}_X$. A global section $s$ of $\mathcal{F}$ is invariant if $\varphi_g(s) = s$ for every $g$. This condition makes sense because $g^*\mathcal{F}(X) = \mathcal{F}(g^{-1}(X)) = \mathcal{F}(X)$. Associating to every abelian $G$-sheaf the group of its global invariant sections we get a functor whose $n$-th derived functor is, by definition, the $n$-th equivariant cohomology group $H^n(X, G, \mathcal{F})$. 

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Suppose now that \( \mathcal{E} \) is a \( G \)-vector bundle of rank \( r \) over \( X \). The projective bundle \( \mathbb{P}(\mathcal{E}) \) associated to \( E \) is a \( G \)-scheme. For every integer \( m \) invertible in \( \mathcal{O}_X \) let \( \mu_m \) be the sheaf of \( m \)-th roots of unity on \( X \) or \( \mathbb{P}(\mathcal{E}) \). The sheaf \( \mathcal{E} \) determines a canonical element \([\mathcal{O}_E(1)] \in H^1(\mathbb{P}(\mathcal{E}), G, \mu_m)\) which maps, under the connecting map \( \partial \) associated to the Kummer sequence

\[
1 \longrightarrow \mu_2 \longrightarrow G_m \longrightarrow \mathbb{G}_m \longrightarrow 1,
\]
to an element \( \xi(\mathcal{E}) = \partial[\mathcal{O}_E(1)] \in H^2(\mathbb{P}(\mathcal{E}), G, \mu_m) \). By \([7]\),

\[
H^*(\mathbb{P}(\mathcal{E}), G, \mu_m^{\otimes r}) = \bigoplus_{0 \leq j \leq r-1} H^*(X, G, \mu_m^{\otimes(r-j)}) \xi^j
\]

where \( \xi^j \) is the \( j \)-th cup-product power of \( \xi = \xi(\mathcal{E}) \). Thus, \( \xi^r \) is a linear combination of its lower powers and there is a relation

\[
\xi^r + c_1(\mathcal{E})\xi^{r-1} + \cdots + c_r(\mathcal{E}) = 0,
\]

where \( c_i(\mathcal{E}) \in H^{2i}(X, G, \mu_m^{\otimes i}) \).

If \( G \) operates trivially on \( X \), the Künneth formula gives, for any \( j \), a natural homomorphism

\[
H^{2j}(X, G, \mu_m^{\otimes j}) \longrightarrow \bigoplus_{0 \leq i \leq 2j} \text{Hom}(H_i(G, \mathbb{Z}), H^{2j-i}(X, \mu_m^{\otimes j}))
\]

which, for \( i = j = 2 \), maps \( c_2(\mathcal{E}) \) to a homomorphism

\[
\varphi_{22}(\mathcal{E}) : H_2(G, \mathbb{Z}) \longrightarrow H^2(X, \mu_m^{\otimes 2}).
\]

Taking now \( X = \text{Spec}A, \mathcal{E} = A^r, m = 2 \) and \( G = \text{GL}_r(A) \) with the trivial action on \( X \) and the natural action on \( \mathcal{E} \), we get, for any \( r \), homomorphisms

\[
\varphi^{(r)}_{22} : H_2(\text{GL}_r(A), \mathbb{Z}) \longrightarrow H^2(\text{Spec}A, \mu_2)
\]

which are compatible with the inclusions \( \text{GL}_r \subset \text{GL}_{r+1} \). We therefore have a homomorphism

\[
c_{22} : K_2(A) = H_2(E(A), \mathbb{Z}) \longrightarrow H^2(\text{Spec}A, \mu_2).
\]

**Proposition 3.1.** For any \( \alpha \in K_2(A) \), \( c_{22}(\alpha) = c_{22}(\alpha^r) \).

**Proof.** We denote by \( \# \) the inverse of the transpose of matrices and of elements of \( K_1, K_2 \), etc. On \( K_2(A) = H_2(E(A), \mathbb{Z}) \) the involution \( \# \) is induced by \( \# \) on \( E(A) \), hence we only have to show that \( \varphi^{(r)}_{22}(\alpha^r) = \varphi^{(r)}_{22}(\alpha) \) for any \( \alpha \in H_2(\text{GL}_r(A), \mathbb{Z}) \). Let \( G = \text{GL}_r(A) \). The isomorphism \( \# : G \longrightarrow G \) induces \( \# : H^2(X, G, \mu_2) \longrightarrow H^2(X, G, \mu_2) \) with the property that, for any \( G \)-bundle \( \mathcal{E} \) on \( X \),

(1) \( \# c_i(\mathcal{E}) = c_i(\mathcal{E}^\#) \),

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where $E^#$ is $E$ with the action of $G$ twisted through # (see [7], §2.3).

For $E = A^r$, $E^#$ is the dual $\tilde{E}$ of $E$, in the category of $G$-bundles. For any scheme $Y$ with a $G$-action and any $G$-bundle $F$ on $Y$, $c_i(F) = (-1)^i c_i(\tilde{F}) = c_i(\tilde{F})$ in $H^{2i}(Y, G, \mu_2)$. (This can be proved by the splitting principle as in [7], page 247, noting that the statement is clear for line bundles with $G$-action.) From (1) we get $\# c_i(E) = c_i(E)$. Since $\# c_i(E)$ induces the map $\phi^{(r)}_{ij} \circ \#$ from $H^j(GL_r(A), Z)$ to $H^{2i-j}(\text{Spec} A, \mu_2)$, it follows that $c_{22}(\alpha) = c_{22}(\alpha^#) = -c_{22}(\alpha^t) = c_{22}(\alpha^t)$ and the proposition is proved.

4. An interlude

**Theorem 4.1.** Let $X$ be a noetherian reduced scheme and $U$ an open subscheme containing all the singular points and all generic points of $X$. Then the restriction map

$$H^2(X, G_m) \longrightarrow H^2(U, G_m)$$

is injective.

**Proof.** Let $i : U \longrightarrow X$ be the inclusion. Since $U$ contains the generic points of $X$, the sheaf $G_m$ on $X$ injects into $i_*G_m$. Let the sheaf $D$ be defined by the exact sequence

$$1 \longrightarrow G_m \longrightarrow i_*G_m \longrightarrow D \longrightarrow 0 .$$

Associating to a rational function its order at a codimension 1 regular point induces a map of (étale) sheaves

$$i_*G_m \longrightarrow \bigoplus_{x \in X^{(1)} \setminus U} i_*Z,$$

where $i_x : x \longrightarrow X$ is the inclusion of the point $x = \text{Spec} k(x)$ into $X$. This map vanishes on $G_m$ and induces a surjective map

$$d : D \longrightarrow \bigoplus_{x \in X^{(1)} \setminus U} i_*Z .$$

We show that $d$ is injective; let $V$ be an étale neighborhood of a point $x \in X$. Suppose that $f \in i_*G_m(V)$ maps to zero in $i_*Z$ for every $x \in X^{(1)} \setminus U$. By definition $f$ is a unit at every point of $V \times_X U$. Every point $y \in V$ outside $V \times_X U$ is regular and $f$ has order zero at every codimension 1 generalization of $y$ belonging to $V \setminus V \times_X U$ because it maps to zero under $d$. On the other hand, $f$ has order zero at the generizations of $y$ belonging to $V \times_X U$ because $f$ is a unit on $V \times_X U$. Hence $f$ is a unit at $y$ and $d$ is an isomorphism.

By Lemma 1.9 of [8] we have $H^1(X, i_*Z) = 0$ and hence $H^1(X, D) = 0$. We therefore have an injection

$$H^2(X, G_m) \longrightarrow H^2(X, i_*G_m) .$$
There is a spectral sequence

\[ H^p(X, R^q i_* G_m) \implies H^n(U, G_m) . \]

Since the étale sheaves \( R^1 i_* G_m \) and \( R^2 i_* G_m \) are zero we have an isomorphism

\[ H^2(X, i_* G_m) \cong H^2(U, G_m) . \]

**Remark.** It follows easily from Theorem 4.1 that for any commutative noetherian ring \( A \) such that \( \text{Spec} A \) has finitely many singularities the Giffen invariant and the Clifford invariant have the same image in the Brauer group of \( A \). To see this, it suffices to replace \( A \) by its semilocalization at the singular primes and use the fact that, for a semilocal ring, \( K_2(A) \) is generated by symbols. This was our first approach to the comparison of the two invariants. For the result of the last section Theorem 4.1 is only needed in the well-known special case of a regular scheme, but since it does not seem to appear anywhere in print, we decided to record it here.

## 5. Quadrics

Let \( \alpha = (a_{ij}) \in \text{GL}_n(A) \) be a symmetric matrix, \( u \in G_m(A) \), \( q(X_1, \ldots, X_n) = \sum a_{ij} X_i X_j \),

\[ C = \frac{A[X_1, \ldots, X_n]}{(q(X_1, \ldots, X_n) - u)} \]

and \( B = C_{X_1} \). For any \( A \)-algebra \( A' \), we write \( B_{A'}, C_{A'} \) for \( B \otimes_A A', C \otimes_A A' \). Let \( \pi : \text{Spec} B \to \text{Spec} A \) be the structure map.

**Proposition 5.1.** Let \( A \) be reduced and \( n \geq 3 \). The homomorphisms \( \mathbb{Z} \to \pi_* G_m \) given by \( n \mapsto X^n \) and \( G_m \to \pi_* G_m \) induce an isomorphism of étale sheaves

\[ G_m \times \mathbb{Z} \xrightarrow{\sim} \pi_* G_m . \]

**Proof.** Since \( G_m \) commutes with direct limits we may assume that \( A \) is noetherian.

Suppose first that \( A \) is a separably closed field. If \( n \geq 4 \), \( X_1 \) is a prime in \( C \) and \( G_m(C) = G_m(A) \), hence \( G_m(B) = G_m(A) \times \mathbb{Z} \). If \( n = 3 \), by Proposition 5.3, either \( X_1 \) is a prime or \( B \simeq (A[X_1, X_2, X_3]/(X_1 X_2 - X_2^2 - 1))_{X_1} \simeq A[X_1, X_1^{-1}, X_3] \), so that \( G_m(B) \) is again as above. If \( A \) is any field and \( \overline{K} \) its separable closure, since \( \overline{K} \cap B = A \), we have again \( G_m(B) = G_m(A) \times \mathbb{Z} \).

Suppose \( A \) is any domain and \( K \) its field of fractions. Since \( B \) is faithfully flat over \( A \), \( G_m(B) \cap K = G_m(A) \), hence

\[ G_m(A) \times \mathbb{Z} \subseteq G_m(B) \subseteq G_m(B) \cap G_m(B_K) = G_m(B) \cap (G_m(K) \times \mathbb{Z}) = G_m(A) \times \mathbb{Z} . \]
Suppose now that $A$ is strictly henselian with maximal ideal $\mathfrak{m}$ and minimal prime ideals $p_1, \ldots, p_s$. Any unit $u$ of $B$ can be written, in $B_{A/p_i}$, as $v_iX_1^{n_i}$, with $v_i \in G_m(A/p_i)$. If $\pi = wX_1^m$ in $B_{A/m}$ with $w \in G_m(A/m)$, then $n_i = m$ for all $i$ and thus $ux_1^{-m}$ maps into $G_m(A/p_i)$ for each $i$. Since $A$ is strictly henselian, there is a retraction $\eta : B \rightarrow A$. The element $uX_1^{-m} - \eta(ux_1^{-m})$ maps to zero in $B_{A/p_1} \times \cdots \times B_{A/p_s}$ and, $A$ being reduced, the map $B \rightarrow B_{A/p_1} \times \cdots \times B_{A/p_s}$ is injective. Thus $ux_1^{-m} \in G_m(A)$.

The map $G_m \times \mathbb{Z} \rightarrow \pi_*G_m$ is an isomorphism on stalks because the strict henselization of a reduced local ring is reduced. Hence this map is an isomorphism.

**Corollary 5.2.** With the same notation as above, $\mu_2 \rightarrow \pi_*\mu_2$ is an isomorphism.

**Proof.** Restrict the isomorphism of the above proposition to the 2-torsion subsheaves.

**Proposition 5.3.** Let $A$ be a local domain and $C = A[X, Y, Z]/(XY - Z^2 - u)$, $u$ a unit of $A$. Let $\ell = aX + bY + cZ$ with $Aa + Ab + Ac = A$. If $\ell$ is not a prime in $C$, $C_\ell \simeq A[X_1, X_1^{-1}, Z]$.

**Proof.** If $a$ is invertible in $A$, we may assume that $a = 1$ and take $\ell = X + bY + cZ = X_1$ as a variable, so that

$$C = \frac{A[X_1, Y, Z_1]}{(X_1Y - b_1Y^2 - Z_1^2 - u)}$$

with $Z_1 = Z + \frac{1}{2}cY$ and $b_1 = b - \frac{1}{2}c$.

Since $A$ is a domain and $X_1$ is not a prime, $b_1 = 0$ and $B \simeq A[X_1, X_1^{-1}, Z]$.

Suppose $a$ and $b$ are both non units. Then $c$ is a unit and we may assume $c = 1$. Let $Z_1 = Z + aX + bY$. Then

$$C/(Z_1) = A[X, Y, Z_1]/(XY - (aX + bY)^2 - u, Z_1) = A[X, Y]/(XY - (aX + bY)^2 - u)$$

is a domain since $4ab \neq 1$, contradicting the assumption that $\ell$ is not a prime.

**Proposition 5.4.** Let $q = (a_{ij})$ be a symmetric stably elementary matrix of size $n \geq 3$. Let $Q = \text{Spec} B$ where $B = C_{X_1}$ and $C = A[X_1, \ldots, X_n]/(\sum a_{ij}X_iX_j \pm 1)$. Let $\pi : Q \rightarrow \text{Spec}A$ be the structure map. The induced homomorphism

$$H^2(A, \mu_2) \rightarrow H^2(B, \mu_2)$$

is injective.

We postpone the proof of this proposition to the end of the section.

**Lemma 5.5.** Suppose that $A$ is local and reduced and that $q$ is split. For $n \geq 2$, Pic$B$ has no 2-torsion.

**Proof.** Suppose that $A$ is a normal domain. If $n \geq 4$, $X_1$ generates a prime ideal in $C$ and Pic$B$ injects into Cl$(B)/\text{Cl}(A)$ which in turn injects into Cl$(B_K) = \text{Cl}(C_K) = \text{Pic}C_K = 0$, $K$ being the field of fractions of $A$. 

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If \( n = 3 \) and \( X_1 \) generates a prime ideal, arguing as above we get that \( \text{Pic} B \) injects into \( \text{Pic} C_K = \mathbb{Z} \). If \( X_1 \) is not a prime, by Proposition 5.3, \( B \cong A[X_1, X_1^{-1}, X_2] \), hence \( \text{Pic} B = \text{Pic} A = 0 \). If \( n = 2 \), \( B \cong A[X_1, X_1^{-1}, \ell^{-1}] \), where \( \ell = aX_1 + b \), with \( a, b \in A \) and \( Aa + Ab = A \). In particular, \( B \) is faithfully flat over \( A \). An invertible \( B \)-module is equivalent to the image of a divisorial ideal \( J \) of \( A \). Since \( B \) is faithfully flat over \( A \), \( J \) must be projective over \( A \), hence free. Thus \( \text{Pic} B = 0 \).

To deal with the general case, we may assume that \( A \) is essentially of finite type over \( \mathbb{Z} \) and proceed by induction on its dimension. The zero-dimensional case is already proved. Since \( A \) is essentially of finite type over \( \mathbb{Z} \), its integral closure \( \overline{A} \) is a finite \( A \)-module. Let \( c \) be the conductor of \( \overline{A} \) in \( A \). The cartesian square

\[
\begin{array}{ccc}
B & \longrightarrow & \overline{B} \\
\downarrow & & \downarrow \\
B/cB & \longrightarrow & \overline{B}/c\overline{B}
\end{array}
\]

gives an exact sequence ([2], IX, 5.4)

\[
\mathbf{G}_m(B/cB) \times \mathbf{G}_m(\overline{B}) \longrightarrow \mathbf{G}_m(\overline{B}/c\overline{B}) \longrightarrow \text{Pic} B \longrightarrow \text{Pic}(B/cB) \times \text{Pic}\overline{B},
\]

where \( \overline{B} = B \otimes_A \overline{A} \). Since \( \dim A/c < \dim A \), by induction, \( \text{Pic}(B/cB) \) has no 2-torsion. Since \( \overline{A} \) is a product of normal domains, the argument above shows that \( \text{Pic} \overline{B} \) has no 2-torsion. It remains to show that \( \text{Coker} \alpha \) has no 2-torsion.

Let \( A_0 = (A/c)_{\text{red}} \), \( B_0 = B \otimes_A A_0 \), \( \overline{A}_0 = (\overline{A}/c)_{\text{red}} \), \( \overline{B}_0 = B \otimes \overline{A}_0 \).

Let \( u \) be a unit in \( \mathbf{G}_m(\overline{B}/c\overline{B}) \) which represents a 2-torsion element in \( \text{Coker} \alpha \). Its image \( v \) in \( \mathbf{G}_m(\overline{B}_0) \) can be lifted to an element of \( \mathbf{G}_m(B_0) \times \mathbf{G}_m(\overline{B}) \). In fact, by 5.1, on each connected component of \( \text{Spec} \overline{B}_0 \), \( v \) is of the form \( v_0X_1^m \), \( v_0 \in \mathbf{G}_m(\overline{A}_0) \). The map \( \mathbf{G}_m(\overline{A}/c) \longrightarrow \mathbf{G}_m(\overline{A}_0) \) is surjective and, \( A/c \) being local, the map \( \mathbf{G}_m(A/c) \times \mathbf{G}_m(\overline{A}) \longrightarrow \mathbf{G}_m(\overline{A}/c) \) is also surjective. On each component, \( X_1^m \) lifts to a unit of \( B/cB \), thus, modifying \( u \) by an element coming from \( \mathbf{G}_m(B/cB) \times \mathbf{G}_m(\overline{B}) \), we may assume that \( u \) maps to 1 in \( \mathbf{G}_m(\overline{B}_0) \), so that \( u \in 1 + \text{nil}(\overline{B}/c\overline{B}) \). Since this group has a finite filtration whose quotients, being isomorphic to \((\text{nil}(\overline{B}/c\overline{B}))^1 / (\text{nil}(\overline{B}/c\overline{B}))^{i+1}\), are uniquely 2-divisible, it follows that the image of \( u \) in \( \text{Coker} \alpha \) is zero. Thus the lemma is proved.

**Corollary 5.6.** Let \( n \geq 3 \). The sheaf \( \mathcal{R}^1\pi_*\mu_2 \) is constant, isomorphic to \( \mathbb{Z}/2 \), generated by the image of \( X_1 \).

**Proof.** Let \( s \) be a global section of \( \mathcal{R}^1\pi_*\mu_2 \), represented on an étale open set \( \text{Spec} A' \) by an element \( \zeta \in H^1(B_{A'}, \mu_2) \). In view of 5.5 and 5.1 we may assume, by shrinking \( \text{Spec} A' \), that \( \zeta = X_1^s \).

**Corollary 5.7.** Let \( n \geq 3 \). If \( A \) is connected, the group \( H^0(A, \mathcal{R}^1\pi_*\mu_2) \) is generated by \( X_1 \).
Proof of Proposition 5.4. Since \( \mu_2(R) = \mu_2(R_{\text{red}}) \) for every ring \( R \) in which 2 is invertible, we may assume that \( A \) is reduced. The Leray spectral sequence for \( \pi : Q \to \text{Spec}A \) gives an exact sequence

\[
H^1(B, \mu_2) \xrightarrow{\alpha} H^0(A, R^1\pi_*\mu_2) \to H^2(A, \pi_*\mu_2) \xrightarrow{\beta} H^2(B, \mu_2) .
\]

By Corollary 5.7, \( \alpha \) is surjective and hence \( \beta \) is injective. Since \( A \) is reduced, it follows from Corollary 5.2 that \( \mu_2 \to \pi_*\mu_2 \) is an isomorphism and this completes the proof of the proposition.

6. The generalized Rees ring of an invertible module

Let \( I \) be an invertible \( A \)-module and \( L = \bigoplus_{n \in \mathbb{Z}} I^n \) where \( I^0 = A \), \( I^n \) is the \( n \)-fold tensor product \( I \otimes_A \cdots \otimes_A I \) if \( n \) is positive and \( (I^{-n})^{-1} \) if \( n \) is negative. The canonical isomorphism \( I^n \otimes_A I^n \to I^{n+n} \) defines on \( L \) the structure of a graded \( A \)-algebra, usually called the \textbf{generalized Rees ring of} \( I \).

**Proposition 6.1.** Let \( A \) be reduced. The \( A \)-algebra \( L \) is faithfully flat and the kernel of the map \( \text{Pic}A \to \text{Pic}L \) is the cyclic group generated by the class of \( I \).

**Proof.** Since locally \( I \) is free and \( L \) isomorphic to \( A[t, t^{-1}] \), \( L \) is faithfully flat over \( A \). Clearly, \( L \otimes_A I \simeq L \) as \( L \)-modules. Let \( P \) be an invertible \( A \)-module and \( \phi : L \to L \otimes_A P \) an isomorphism of \( L \)-modules. The fact that \( \phi(1) \) belongs to some homogeneous component \( I^n \otimes_A P \) and that \( \phi \) induces an isomorphism \( A \simeq I^n \otimes_A P \) may be verified locally on \( A \), using the fact that \( A \) is reduced (compare with [4], III.3). Thus \( P \simeq I^{-n} \) for some \( n \).

**Proposition 6.2.** Let \( A \) be reduced. The kernel of \( H^2(A, \mu_2) \to H^2(L, \mu_2) \) is generated by \( \partial[I] \), where \( [I] \) is the class of \( I \) in \( H^1(A, \mathbb{G}_m) \) and \( \partial \) is the connecting homomorphism of the Kummer exact sequence.

**Proof.** Let \( \pi : \text{Spec}L \to \text{Spec}A \) be the structure map. The natural map \( \mu_2 \to \pi_*\mu_2 \) is an isomorphism since, for a local ring \( A \), the ring \( L \) is just \( A[t, t^{-1}] \). Hence the Leray spectral sequence gives an exact sequence

\[
H^1(A, \mu_2) \xrightarrow{H^1(\pi)} H^1(L, \mu_2) \xrightarrow{\delta} H^0(A, R^1\pi_*\mu_2) \to H^2(A, \mu_2) \to H^2(L, \mu_2) .
\]

For any open set \( \text{Spec}A_f \) over which \( I \) is free there is a split exact sequence

\[
0 \to H^1(A_f, \mu_2) \to H^1(L_f, \mu_2) \xrightarrow{\delta_f} H^0(A_f, R^1\pi_*\mu_2) \to 0
\]

with \( L_f \simeq A_f[t, t^{-1}] \). We claim that \( H^0(A_f, R^1\pi_*\mu_2) = \mathbb{Z}/2 \). Let \( s \) be a global section of \( R^1\pi_*\mu_2 \), \( x \in \text{Spec}A_f \) and \( A_x \) the strict henselisation of \( A \) at \( x \). On \( \text{Spec}A_x \), \( s \)
is represented by an element $\xi \in H^1(L_{A_x}, \mu_2) \simeq L_{A_x}^* / (L_{A_x}^*)^2 \times \text{Pic} L_{A_x}$. Since $L_{A_x} \simeq A_x[t, t^{-1}]$ and $A_x$ is reduced, $L_{A_x}^* \simeq A_x^* \times \mathbb{Z}$. Further, by 5.5, $\text{Pic} L_{A_x} = 0$, so that $\xi = t^\epsilon$, $\epsilon = 0$ or 1. Thus $H^0(A_f, R^1\pi_*\mu_2) = \mathbb{Z}/2$, generated by $t$ and hence $H^0(A, R^1\pi_*\mu_2)$ is either $\mathbb{Z}/2$ or trivial.

Suppose $I$ is not a square in Pic$A$. Then, under the connecting map $\partial$ of the Kummer sequence, $\partial[I]$ is a nontrivial element in $\ker (H^2(A, \mu_2) \longrightarrow H^2(L, \mu_2))$ by Proposition 6.1 and hence it generates the whole kernel.

Suppose $I = J^2$ for some $J \in \text{Pic} A$. If $\{\text{Spec} A_f, f \in S\}$ is a covering which trivializes $J$, we can choose local generators $t_f$ for $I$ such that $t_f = u^2_{fg} t_g$, $u_{fg} \in A^*_g$ denoting the cocycle associated to $J$. The set $\{t_f, f \in S\}$ represents a nonzero section of $R^1\pi_*\mu_2$ since it is nonzero even locally on $A$. This section is the image under $\delta$ of the discriminant module $JL \otimes L J \simeq J^2 L = IL \simeq L$ over $L$. In this case $H^2(A, \mu_2) \longrightarrow H^2(L, \mu_2)$ is injective.

7. Rank 2 forms over an affine curve

Assume that $A$ is a smooth affine algebra of dimension 1 over a field of characteristic not equal to 2 and let $H(I)$ be the hyperbolic plane over an invertible $A$-module $I$. We can represent $H(I)$ by a symmetric matrix

$$\alpha = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

with $b^2 - ac = 1$. We assume that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is stably elementary, so that it represents a class in EW$_2(A)$. In this case, since $\text{dim} A = 1$, this matrix is in fact elementary.

**Proposition 7.1.** $c_{22} \circ s_2 \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = 0$ if and only if the class of $I$ is a square in Pic$A$.

**Proof.** By 4.2.1 of [15] the map $c_{22} : K_2(A) \longrightarrow H^2(A, \mu_2)$ factors through $H^0(X, K_2)$, where $K_2$ is the Zariski sheaf associated to the presheaf $U \mapsto K_2(U)$. Since the transition on $K_2$ coincides locally with the inverse, this induces a factoring of $c_{22}$ as a composite

$$K_2(A)/\text{Tr}(K_2(A)) \longrightarrow H^0(X, K_2)/2 \longrightarrow H^2(A, \mu_2).$$

Since $A$ is smooth, by 4.2.1 and Theorem 4.3 of [15] the second map is injective. Thus, $c_{22} \circ s_2(q) = 0$ if and only if the image of $s_2(q)$ in $H^0(X, K_2)/2$ is zero, $q$ denoting the class of $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in EW$_2(A)$. Let $U = \{U_i, 1 \leq i \leq N\}$ be an affine open covering of $X = \text{Spec} A$ which trivializes $I$ and let $\{u_{ij} \in \mathbb{G}_m(\mathcal{O}_X(U_i \cap U_j)), 1 \leq i, j \leq N\}$ be a cocycle representing $I$. Let $\tilde{\alpha}$ be a lift $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in $\text{St}(A)$. Then $s_2(q) = \text{class of } (-1, -1) + \tilde{\alpha}^{-1} \tilde{\alpha}^t$ in $K_2(A)/\text{Tr}(K_2(A))$. 

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By shrinking \( U \), we may assume that

\[
\begin{pmatrix} \alpha & 0 \\ 0 & J \end{pmatrix} = \beta_i J_2 \beta_i^t
\]
on \( U_i \), with \( \beta_i \in E_4(O_X(U_i)) \). Let \( \tilde{\beta}_i \) be a lift of \( \beta_i \) in \( St(O_X(U_i)) \). Let

\[
\tilde{J} = w_{21}(-1)w_{43}(-1)h_{13}(-1)
\]
where \( w_{ij}(\lambda) = x_{ij}(\lambda)x_{ji}(-\lambda^{-1})x_{ij}(\lambda) \), \( h_{ij}(\lambda) = w_{ij}(\lambda)w_{ij}(-1) \) (cf [10] page 71). Then \( \tilde{J} \) is a lift of \( J_2 \) and \( \tilde{J}^{-1}\tilde{J}^t = (-1, -1) \). On \( U_i \), \( \tilde{\alpha} = c_i \tilde{\beta}_i \tilde{J}_i \tilde{\beta}_i^t \) with \( c_i \in K_2(O_X(U_i)) \). By further shrinking \( U \), we may assume that each \( c_i \) is a sum of symbols, so that \( -c_i = c_i^t \). Then

\[
(-1, -1) + \tilde{\alpha}^{-1}\tilde{\alpha}^t = -2c_i
\]
on \( U_i \) so that \( 2c_i = 2c_j \) on \( U_i \cap U_j \). The image of \( s_2(q) \) in \( H^0(X, K_2) \) is represented by the cocycle \( \{-2c_i, 1 \leq i \leq N\} \). Since \( 2(c_i - c_j) = 0 \) on \( U_i \cap U_j \), \( \{c_i - c_j, 1 \leq i, j \leq N\} \) defines a 1-cocycle on \( SpecA \) with values in \( 2K_2 \).

**Lemma 7.2.** The class of the 1-cocycle \( \{c_i - c_j, 1 \leq i, j \leq N\} \) in \( H^1(X, 2K_2) \) is equal to the class of the 1-cocycle \( \{(-1, u_{ij}), 1 \leq i, j \leq N\} \).

We grant this lemma. Suppose the image of \( s_2(q) \) in \( H^0(X, K_2)/2 \) is zero. After possibly shrinking \( U \), we may assume that there exists a cocycle \( \{c_i', 1 \leq i \leq N\} \) in \( H^0(X, K_2) \) such that \( 2c_i = 2c_j \) on \( U_i \). Then \( 2(c_i - c_j') = 0 \) and \( c_i - c_j = (c_i - c_i') - (c_j - c_j') \), which shows that the 1-cocycle \( \{c_i - c_j\} \) with values in \( 2K_2 \) is a coboundary. By Lemma 7.2, the cocycle \( \{(-1, u_{ij})\} \) is trivial in \( H^1(X, 2K_2) \).

**Lemma 7.3.** Let \( X \) be a smooth curve over a field \( k \) of characteristic different from 2. There is an isomorphism \( PicX/2 \xrightarrow{} H^1(X, 2K_2) \), which, in terms of Čech cocycles, is given by mapping the class of \( \{f_{ij}\} \) to that of \( \{(-1, f_{ij})\} \).

By Lemma 7.3, \( [I] \) is trivial in \( PicA/2 \). Conversely, if the class of \( I \) in \( PicA/2 \) is trivial, reversing the above arguments, we see that the image of \( s_2(q) \) in \( H^0(X, K_2)/2 \) is zero. This proves Proposition 7.1, provided we prove the two lemmas.

**Proof of Lemma 7.2.** We can choose the trivializations \( \beta_i \) such that

\[
\beta_j^{-1}\beta_i = \begin{pmatrix} u_{ij}^{-1} & 0 \\ 0 & u_{ij} \end{pmatrix}
\]
This has a lift \( h_{12}(u_{ij}) \) in \( St(U_i \cap U_j) \). Putting \( u_{ij} = u \) we get, using the identities

\[
\begin{align*}
w_{ij}(u)^t &= w_{ij}(u) \\
h_{ij}(u)^t &= h_{ij}(-u)h_{ij}(-1)^{-1}
\end{align*}
\]
and Corollary 9.4 of [10],

\[ c_j - c_i = \tilde{f}^{-1} h_{12}(u) \tilde{h}_{12}(u)^t = \]
\[ = h_{13}(-1)^{-1} w_{43}(-1)^{-1} w_{21}(-1)^{-1} h_{12}(u) w_{21}(-1) w_{43}(-1) h_{13}(-1) h_{12}(-u) h_{13}(-1)^{-1} \]
\[ = h_{13}(-1)^{-1} h_{21}(-u) h_{21}(-1)^{-1} h_{13}(-1) h_{12}(-u) h_{12}(-1)^{-1} \]
\[ = h_{21}(u) h_{12}(-u) h_{12}(-1)^{-1} \]
\[ = h_{12}(u) h_{12}(-1)^{-1} h_{12}(-u)^{-1} = (-1, u). \]

**Proof of Lemma 7.3.** We denote by \( \mathcal{H}^n \) the Zariski sheaf on \( X \), associated to the presheaf \( U \mapsto H^n(U, \mu_2) \). By Theorem 6.1 of [3], \( \text{Pic}X/2 \cong H^1(X, \mathcal{H}^1) \) is the cokernel of the map

\[ H^1(k(X), \mu_2) = k(X)/(k(X)^*)^2 \overset{\partial}{\longrightarrow} \bigoplus_{x \in X^{(1)}} H^0(k(x), \mu_2) = \bigoplus_{x \in X^{(1)}} \mathbb{Z}/2, \]

where the map is given, at \( x \), by \( \partial(f) = (-1)^{w_x f} \). By [11], 8.7.8.(b), \( H^1(X, 2\mathcal{K}_2) \) is the cokernel of the tame symbol map

\[ 2\mathcal{K}_2(k(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} 2\mathcal{K}_1(k(x)) = \bigoplus_{x \in X^{(1)}} \mathbb{Z}/2. \]

By Theorem 1.8 of [15], mapping \( f \in k(X) \) to the symbol \((-1, f)\) yields a surjection

\[ k(X)/(k(X)^*)^2 \longrightarrow 2\mathcal{K}_2(k(X)), \]

such that the diagram

\[ \begin{array}{ccc}
\kappa(X)/(k(X)^*)^2 & \longrightarrow & \bigoplus \mathbb{Z}/2 \\
\downarrow & & \downarrow \\
2\mathcal{K}_2(k(X)) & \longrightarrow & \bigoplus \mathbb{Z}/2
\end{array} \]

is commutative. This yields an isomorphism \( \text{Pic}X/2 \cong H^1(X, 2\mathcal{K}_2) \) on the cokernels.

Since all the maps in the resolutions are explicit, it is easy to verify that, in terms of cocycles, the isomorphism is as claimed.

**Theorem 7.4.** The class of \( I \) in \( \text{Pic}A/2 \hookrightarrow H^2(A, \mu_2) \) is precisely \( c_{22} \circ s_2(\alpha_0^0) \). In particular, \( c_{22} \circ s_2(\alpha_0^0) = e_2(\alpha_0^0) \).

**Proof.** Let \( L = \bigoplus_{n \in \mathbb{Z}} I^n \). Since \( IL \) is trivial in \( \text{Pic}L \), by Proposition 7.1, \( c_{22} \circ s_2(q \otimes L) = 0 \). In view of (5.3), \( c_{22} \circ s_2(q) \) is either zero or the class of \( I \) in \( \text{Pic}A/2 \subset H^2/A, \mu_2 \). If \( I \) is not a square in \( \text{Pic}A \), by Proposition 7.1, \( c_{22} \circ s_2(q) \neq 0 \) so that \( c_{22} \circ s_2(q) = [I] \). If \( I \) is is a square in \( \text{Pic}A \), by Proposition 7.1, \( c_{22} \circ s_2(q) = [I] = 0 \). On the other hand, by [12], Theorem 20, \( e_2(q) = [I] \) and this completes the proof of the theorem.

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8. The comparison

**Lemma 8.1** (Casanova). Let \( \alpha \in \text{GL}_{4n}(A) \) represent an element of \( \text{EW}_2(A) \) and \( \theta \in \text{GL}_{4n}(A) \) a matrix such that \( \theta^t \theta \in \text{E}(A) \). Then

\[
s_2(\theta^t \alpha \theta) = s_2(\alpha) + s_2(\theta^t \theta) + n(-1,-1). 
\]

**Proof.** Let \( \tilde{\alpha} \) be a lift of \( \alpha \) and \( \tilde{\theta}^t \theta \) a lift of \( \theta^t \theta \) in \( \text{St}(A) \). For any \( \varphi \in \text{GL}(A) \), let \( x \mapsto x^\varphi \) be the unique isomorphism of \( \text{St}(A) \) lifting the conjugation by \( \varphi \) in \( \text{E}(A) \). Choosing \( \tilde{\theta}^t \theta \tilde{\alpha}^\theta \) as a lift of \( \theta^t \alpha \theta \), we get

\[
s_2(\theta^t \alpha \theta) = n(-1,-1) + \left( (\tilde{\theta}^t \theta \cdot \tilde{\alpha}^\theta)^t (\tilde{\theta}^t \theta \cdot \tilde{\alpha}^\theta)^{-1} \right) 
= s_2(\theta^t \theta) + (\tilde{\alpha}^\theta)^t (\tilde{\alpha}^{-1})^{(\theta^t \theta)}^{-1} 
= s_2(\theta^t \theta) + (\tilde{\alpha}^{-1})^{(\theta^t \theta)}^{-1}
\]

since, for any \( x \in \text{St}(A) \), \( (x^\theta)^t = (x^\theta)^{(\theta^t \theta)}^{-1} \). To show that the second term coincides with \( \tilde{\alpha}^{-1} \) and thus complete the proof, it suffices to observe that the action of any \( \varphi \in \text{GL}(A) \) on \( \text{K}_2(A) \) is trivial: indeed, by [1], \( \text{K}_2(A) = \text{H}_2(\text{E}(A), \mathbb{Z}) \) is a direct factor of \( \text{H}_2(\text{GL}(A), \mathbb{Z}) \) and an inner homomorphism of any group \( G \) induces the identity map on \( \text{H}_*(G, \mathbb{Z}) \).

**Lemma 8.2.** Let \( \theta \in \text{SL}_2(A) \). Then \( \theta^t \theta \) is stably elementary and \( e_2 \) coincides with \( c_{22} \circ s_2 \) on

\[
\begin{pmatrix}
1_2 & 0 \\
0 & 0
\end{pmatrix}
\]

They both are \((-1) \sim (-1)\) in \( \text{H}_2(A, \mathbf{m}_2) \).

**Proof.** Since \( \theta^t = (\begin{smallmatrix}0 & 1 \\ -1 & 0\end{smallmatrix}) \theta^{-1} (\begin{smallmatrix}0 & 1 \\ -1 & 0\end{smallmatrix})^{-1} \), \( \theta^t \theta \) is a commutator and hence stably elementary. To prove the second assertion, it suffices to consider the generic case of the matrix \( \theta = (x \ y \\ z \ t) \) over the ring \( A = \mathbb{Z} \{ [1_2, x, y, z, t]/(xt - yz - 1) \}. \) Since \( x \) is a prime in \( A \) and \( A \) is regular, \( \text{Pic}(A) \simeq \text{Pic}(A) = 0 \), so that \( \text{H}_2(A, \mathbf{m}_2) \simeq 2\text{Br}(A) \). Since \( \text{Br}(A) \) injects \((\mathbb{Z}/4!)\) into \( \text{Br}(K) \), \( K \) the field of fractions of \( A \), it suffices to compare the two invariants in \( \text{Br}(K) \), which has been done in [6]. Clearly

\[
e_2 \begin{pmatrix} 1_2 & 0 \\ 0 & 0 \end{pmatrix} = e_2(1_4) = (-1) \sim (-1).
\]

**Theorem 8.3.** The Giffen invariant maps, under \( c_{22} \), to the Clifford invariant.

**Proof.** Let \( \alpha \in \text{GL}_{4n}(A) \) represent an element of \( \text{EW}_2(A) \) and let \( q \) be the corresponding quadratic form.
Replacing $A$ by the $A$-algebra $B$ of Proposition 5.4, we can make the matrix $\alpha$ elementarily equivalent to $\begin{pmatrix} \pm 1 & 0 \\ 0 & \beta \end{pmatrix}$. The comparison of the invariants in $H^2(A, \mu_2)$ reduces, by Proposition 5.4, to their comparison in $H^2(B, \mu_2)$ for the form $\begin{pmatrix} \pm 1 & 0 \\ 0 & \beta \end{pmatrix}$. Repeating this process $4n - 3$ times we are reduced to compare the invariants of the matrix

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix}.$$ 

Since $ac - b^2 = -1$, the quadratic space $(A^2, (a \ b \ b \ c))$ is isometric to $H(I)$ for some invertible module $I$. By Theorem 20 of [12], $e_2(H(I))$ is the class of $I$ in Pic$A/2 \subset H^2(A, \mu_2)$. If we extend the scalars to $L = \bigoplus_{n \in \mathbb{Z}} I^n$, $I$ becomes principal and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \gamma^t J \gamma$$

for some $\gamma \in \text{SL}_2(L)$. By Lemma 8.1,

$$s_2(\beta) = s_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + s_2 \begin{pmatrix} 12 \\ \gamma^t \gamma \end{pmatrix} + (-1, -1)$$

and by Lemma 8.2,

$$c_{22} \circ s_2 \begin{pmatrix} 12 \\ \gamma^t \gamma \end{pmatrix} = (-1) \sim (-1).$$

For a matrix of the form

$$\alpha = \begin{pmatrix} 1 \\ -a \\ -b \\ ab \end{pmatrix},$$

$s_2(\alpha) = (a, b)$ modulo Tr$(K_2(A))$, and $c_{22} \circ s_2(\alpha) = (a) \sim (b)$ (see [6]). Since

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is elementarily equivalent to

$$\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix},$$

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it follows that over $L$, $c_{22 \circ s_2}(\beta) = 0$. Therefore $c_{22 \circ s_2}(\beta)$, by Proposition 6.1, is either the class of $I$ or zero. Suppose

$$\left( \begin{array}{c} 1_{4n} \\
\beta \end{array} \right)$$

is elementary. Then this holds for the matrix

$$\beta = \left( \begin{array}{cc}
1 & 0 \\
0 & -1 \\
x & y \\
y & z
\end{array} \right)$$

over the ring

$$A = \frac{\mathbb{Z}[\frac{1}{2}, x, y, z, X_{ij}, Y_{ij}, Z_{ij}, T_{ij}][[(\det X)^{-1}, \ldots, (\det T)^{-1}]}{(xz - y^2 + 1, XYZ^{-1}Y^{-1}ZT^{-1}T^{-1} = \beta)},$$

where $X = (X_{ij}), \ldots, T = (T_{ij})$ are generic matrices and $1 \leq i, j \leq 4n + 4$. We note that if

$$\left( \begin{array}{ccc}
1 & -1 & \\
& a & b \\
& b & c
\end{array} \right) = \gamma$$

is such that $ac - b^2 = -1$ and

$$\left( \begin{array}{c} 1_{4n} \\
\gamma
\end{array} \right)$$

is elementary over a ring $B$, there is a specialization from $A$ to $B$, specializing $\beta$ to $\gamma$, since over any commutative ring, any elementary matrix is a product of two commutators ([5], Remark 21). If $c_{22 \circ s_2}(\beta) = 0$ this invariant would be zero for any matrix of the form $\beta$ over any ring. However the computations in §7 show the existence of matrices $\beta$ over Dedekind domains $C$ for which it is not zero: we may take $C = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ and

$$\beta = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
x^2 - y^2 & 0 & 2xy & y^2 - x^2 \\
0 & 2xy & y^2 - x^2 & 0
\end{array} \right).$$

The corresponding ideal $I$ is the generator of $\text{Pic}C = \mathbb{Z}/2$. This shows that the two invariants coincide.
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