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Symplectic bundles over affine surfaces

M. Ojanguren, R. Parimala and R. Sridharan

Introduction

Let A be a real affine algebra of dimension 2 and $V = \operatorname{spec} A$. In [10], Pardon relates the structure of the Witt group $W^{-1}(A)$ of skew-symmetric forms over A to the group $A_0(V)$ of zero cycles of V modulo rational equivalence. He proves [10, Th. B, p 262] that if $\operatorname{Pic} V$ is trivial and V is smooth, $W^{-1}(A) \otimes \mathbb{Z}/2 \xrightarrow{\sim} A_0(V) \otimes \mathbb{Z}/2$. In this paper, by what we believe to be a more direct and elementary approach, we prove that for a real affine surface $V = \operatorname{spec} A$, not necessarily smooth, $W^{-1}(A) \otimes \mathbb{Z}/2 \xrightarrow{\sim} SK_0(A)/\operatorname{tr} \tilde{K}_0(A)$, $\operatorname{tr} \tilde{K}_0(A)$ denoting the subgroup of $SK_0(A)$ generated by all elements of the form $P \oplus P^*$. If $\operatorname{Pic} A$ is trivial, $\operatorname{tr} \tilde{K}_0(A) = 2SK_0(A)$ and our result extends Pardon's theorem. Our method of proof uses Vaserstein's symbol on unimodular rows of length three and a construction of certain generic rank 2 symplectic bundles which generalise the classical Hopf bundles over the real sphere.

The description of $W^{-1}(A)$ in terms of linear data raises the following natural question: for a projective module P over a ring A, on what conditions is the map $\det : \operatorname{Aut} P \to A^*$ surjective? This map, in general, is not surjective [8, §4 ex. 2]. We prove however, that the map det is surjective if, for instance, P is a rank d projective A-module where A is an affine algebra of dimension d over an algebraically closed field of characteristic 0.

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§1. Witt group of skew-symmetric forms

Let A be a commutative ring. A *skew-symmetric* space over A is a pair (P, s) where P is a finitely generated projective A-module and $s : P \times P \to A$ a skew-symmetric bilinear form which induces an isomorphism $s_* : P \xrightarrow{\sim} P^*$. An *isometry* of skew-symmetric spaces is an isomorphism of the underlying modules which preserves the forms. Any finitely generated projective module P gives rise to a skew-symmetric space, called the *hyperbolic space*, denoted by $H(P)$: its underlying module is $P \oplus P^*$ and the form is given by $((x, f), (x', f')) \mapsto f(x') - f'(x)$. The *orthogonal sum of* two skew-symmetric spaces (P, s) and (P', s'), denoted by $(P, s) \perp (P', s')$, is the space $(P \oplus P', t)$ where $t((v, w), (v', w')) = s(v, v') + s'(w, w')$: $v, v' \in P$, $w, w' \in P'$. For any skew-symmetric space (P, s), we have $(P, s) \perp (P, -s) \xrightarrow{\sim} H(P)$. We say that two spaces (P, s) and (P', s') are *equivalent* if $(P, s) \perp H(Q) \xrightarrow{\sim} (P', s') \perp H(Q')$ for some Q and Q'. The orthogonal sum induces a group structure on the set of equivalence classes of skew-symmetric spaces, the identity being the class of the hyperbolic spaces and the inverse of the class of (P, s) being the class of $(P, -s)$.

We denote by $K_0(A)$ the Grothendieck group of finitely generated projective A-modules, by $\operatorname{Pic} A$ the group of isomorphism classes of invertible A-modules, by $\tilde{K}_0(A)$ the kernel of the rank homomorphism and by $SK_0(A)$ the kernel of the determinant map. We cite [1] and [2] as references for these and other unexplained terms.

There is an involution σ on $K_0(A)$ which maps the class of P to the class of P^*. For any $x \in \tilde{K}_0(A)$, we have, $x + \sigma(x) \in SK_0(A)$ and we denote by $\operatorname{tr}(\tilde{K}_0(A))$ the subgroup of $SK_0(A)$ consisting of all elements of the form $x + \sigma(x)$, $x \in \tilde{K}_0(A)$.

We record here some stability results on skew-symmetric spaces which will be used in sequel.

THEOREM 1.1 ([2, 4.11.2]). *Let (P, s) be a skew symmetric space over A. If P has a unimodular element, then $(P, s) \xrightarrow{\sim} (P', s') \perp H(A)$. If A is a noetherian ring of dimension d, any skew symmetric space over A splits as $(P, s) \perp H(A^n)$ with* $\operatorname{rank} P \leq d$.

THEOREM 1.2 ([2, 4.16]). *Let A be a noetherian ring of dimension ≤ 2. If $(P, s) \perp (Q, t) \xrightarrow{\sim} (P', s') \perp (Q, t)$, then $(P, s) \xrightarrow{\sim} (P', s')$.*

(One should note that in the proof of (4.16) of [2], the reference should be to (4.14) instead of (4.15).)

COROLLARY 1.3. *Let A be a noetherian ring of dimension ≤ 2. Then every class in $W^{-1}(A)$ has a representative (P, s) with* $\operatorname{rank} P = 2$.

Let P be a projective module of rank 2. Any nonsingular skew symmetric form s on P induces an isomorphism $\Lambda^2 P \xrightarrow{\sim} A$. Conversely any isomorphism $\Lambda^2 P \xrightarrow{\sim} A$ gives rise to a skew symmetric structure on P. Thus, any rank 2 projective module with trivial determinant carries a skew symmetric structure which is unique up to units of A. If (P, s) is a rank 2 skew-symmetric space and u

a unit of A, then (P, s) and (P, us) are isometric if and only there exists an automorphism α of P with $\det \alpha = u$.

Let A be a noetherian ring of dimension 2 and (P, s) a skew-symmetric space over A. By (1.3),

$$(P, s) \perp H(Q) \cong (P', s') \perp H(Q'),$$

where rank $P' = 2$ for suitable Q and Q'. Taking determinants, we get, $\det P \cong \det P' \cong A$. Thus associating to each skew-symmetric space its underlying module, we obtain a homomorphism

$$\Phi : W^{-1}(A) \to SK_0(A)/\mathrm{tr}\, \tilde{K}_0(A).$$

The map Φ is surjective since every element of $SK_0(A)$ can be represented by a rank 2 projective module with trivial determinant, which as we saw above, carries a skew-symmetric structure. Since $2SK_0(A) \subset \mathrm{tr}\, \tilde{K}_0(A)$, Φ induces a homomorphism

$$\varphi : W^{-1}(A)/2W^{-1}(A) \to SK_0(A)/\mathrm{tr}\, \tilde{K}_0(A).$$

We shall show that this map φ is an isomorphism for a certain class of 2-dimensional affine algebras. To do this, we begin with some preliminary results.

Let $Um_3(A)$ denote the set of unimodular rows of length 3 over A. For $\alpha = (a, b, c) \in Um_3(A)$, let $\xi = (x, y, z) \in Um_3(A)$ be such that $ax + by + cz = 1$. Let

$$S(\alpha, \xi) = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & -c & b \\ y & c & 0 & -a \\ z & -b & a & 0 \end{pmatrix}.$$

We note that $S(\alpha, \xi)$ is the most general skew-symmetric matrix with Pfaffian $Pf(S(\alpha, \xi)) = ax + by + cz = 1$. If $\xi' = (x', y', z')$ also satisfies $ax' + by' + cz' = 1$, then there exists $U \in GL_4(A)$ such that $S(\alpha, \xi') = US(\alpha, \xi)U^t$ [12, (5.1)]. For $V \in SL_3(A)$, if $\alpha' = \alpha V$, and $\xi' = \xi(V^t)^{-1}$, then, there exists $U \in GL_4(A)$ such that $S(\alpha', \xi') = US(\alpha, \xi)U^t$ [12, (5.2)]. Thus the isometry class of the skew symmetric space $(A^4, S(\alpha, \xi))$ is uniquely determined by the class of α in $Um_3(A)/SL_3(A)$. We denote this isometry class by $\Sigma(\alpha)$. We remark that any rank 4 skew-symmetric space whose underlying module is free is in $\Sigma(\alpha)$ for some $\alpha \in Um_3(A)$; in fact, for any $T \in GL_4(A)$ and any skew-symmetric matrix

$S \in GL_4(A)$, $Pf(TST^t) = Pf(S) \det T$. We have a map $w : Um_3(A)/SL_3(A) \to W^{-1}(A)$ which sends the class of α to the class of $\Sigma(\alpha)$.

PROPOSITION 1.4. *The image of w is the kernel of Φ. In particular, if $SL_3(A)$ acts transitively on unimodular rows, then Φ is an isomorphism.*

Proof. The underlying module of any skew-symmetric space (P, s) whose class is in $\ker \Phi$ is of the form $Q \oplus Q^*$ for some projective module Q. Let Q' be such that $Q \oplus Q'$ is free. Then $(P, s) \perp H(Q')$ is free. By (1.3), this space is isometric to $(P', s') \perp H(A^n)$ with rank $P' = 2$ and P' stably free. The class of (P, s) in $W^{-1}(A)$ is the class of $(P', s') \perp H(A)$. By a well-known cancellation theorem for projective modules, [1, p 172], $P' \oplus A^2$ is free so that by our earlier remarks, $(P', s') \perp H(A)$ is in $\Sigma(\alpha)$ for some $\alpha \in Um_3(A)$.

COROLLARY 1.5. *Let A be an affine algebra of dimension 2 over a field K. Suppose that one of the following conditions is satisfied:*

1) *K is algebraically closed.*
2) *K is finite.*
3) *K is real closed and the set of K-rational points of $\operatorname{spec} A$ lies in a closed subscheme of dimension ≤ 1.*

Then $\Phi : W^{-1}(A) \to SK_0(A)/\operatorname{tr} \tilde{K}_0(A)$ is an isomorphism.

Proof. In each of these cases, $SL_3(A)$ acts transitively on $Um_3(A)$ (See [7, Theorem 1] and [12, Corollary 17.3])

COROLLARY 1.6. *If A is a regular affine algebra of dimension 2 over an algebraically closed field, then $W^{-1}(A) = 0$.*

Proof. In view of [7, Theorem 3], $SK_0(A) = 2SK_0(A) \subset \operatorname{tr} \tilde{K}_0(A)$ and the result follows from (1.5).

§2. Real surfaces and generic Hopf bundles

Throughout this section, R denotes a real closed field and A denotes an affine algebra over R of dimension 2.

PROPOSITION 2.1. *Every element of $Um_3(A)/SL_3(A)$ can be represented by $\xi = (x, y, z) \in Um(A)$ such that $ax + y^2 + cz = 1$ for some a, $c \in A$.*

Proof. Let $\xi = (x, y, z) \in Um_3(A)$. Operating on ξ by elementary transformations, we may, in view of [3, §3, Lemma 2], assume that $I = Ax + Az$ has height 2. Let $a, b, c \in A$ be such that $ax + by + cz = 1$. The ring A/I, modulo its radical is a finite product of copies of R or C, C denoting the algebraic closure of R. Hence any square in A/I is a fourth power. Let $\bar{b}^2 = \bar{t}^4$ and let $t \in A$ be a lift of $\bar{t}$. Since $t^4y^2 \equiv 1 \bmod I$, there exist a', $c' \in A$ such that $a'x + (t^2y)^2 + c'z = 1$. To complete the proof of the proposition, it suffices to show that there exists an element of $SL_3(A)$ which maps (x, y, z) to (x, t^2y, z). This is achieved by the following

LEMMA 2.2. *Let A be any ring of dimension 2 and $x, y, z, t \in A$ such that (x, t^2y, z) is unimodular. Then there exists $\alpha \in SL_3(A)$ such that $(x, t^2y, z)\alpha = (x, y, z)$.*

Proof. Since $\dim A = 2$, for $r \geq 4$, $E_r(A)$ acts transitively on the set $Um_r(A)$ of unimodular rows of length r. In view of [12, Theorem 5.2], $(x, t^2y, z) \sim (x, y, z)$ under the action of $SL_3(A)$ if and only if

$$\Sigma(x, t^2y, z) \perp H(A^r) \cong \Sigma(x, y, z) \perp H(A^r)$$

for some r. Since $\dim A = 2$, by (1.3), this happens if and only if $\Sigma(x, t^2y, z) \cong \Sigma(x, y, z)$. By [12, Theorem 5.2], if $px + qy + rz = 1$, then

$$\Sigma(x, y, z) \perp \Sigma(x, t^2, z) \cong \Sigma(x, t^2y - rz, (t^2 + q)z) \perp H(A^2).$$

Denoting by $\sim_E$ the equivalence under the action of $E_3(A)$, we have, (cf. [15, p 380])

$$\begin{aligned}(x, t^2y - rz, (t^2 + q)z) &= (x, t^2y - 1 + px + qy, (t^2 + q)z)\\ &\sim_E (x, (t^2 + q)y - 1, (t^2 + q)z)\\ &\sim_E (x, (t^2 + q)y - 1, (t^2 + q)^2z)\\ &\sim_E (x, (t^2 + q)y - 1, z)\\ &\sim_E (x, t^2y, z).\end{aligned}$$

Thus in view of [12, Theorem 5.2],

$$\Sigma(x, y, z) \perp \Sigma(x, t^2, z) \cong \Sigma(x, t^2y, z) \perp H(A^2).$$

Since (x, t^2, y) is completable in $SL_3(A)$ (see [14, Theorem 2.1]), $\Sigma\,(x, t^2, y) \simeq H(A^2)$ and by (1.3), $\Sigma\,(x, y, z) \simeq \Sigma\,(x, t^2y, z)$.

Let S, S' be two 4×4 skew symmetric matrices with S' nonsingular. Then $S'^{-1}S$ satisfies [9, Lemma 3.5] the quadratic equation $Pf(S - S't) = (Pf(S))t^2 - Pf(S, S')t + Pf(S') = 0$ where $Pf(S, S')$ is the bilinear form associated to the quadratic form $S \mapsto Pf(S)$. Let

$$S = S((a, y, c), (x, y, z)) = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & -c & y \\ y & c & 0 & -a \\ z & -y & a & 0 \end{pmatrix}$$

be the generic skew symmetric matrix defined over the commutative R-algebra B generated by x, y, z, a, c with relation $ax + y^2 + cz = 1$. Choosing

$$S' = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

we see that $Pf(S, S') = 0$ and $Pf(S') = -1$ so that $U = S'^{-1}S$ has square 1. Let

$$E = \tfrac{1}{2}(1 + U) = \tfrac{1}{2}\begin{pmatrix} 1+y & c & 0 & -a \\ z & 1-y & a & 0 \\ 0 & x & 1+y & z \\ -x & 0 & c & 1-y \end{pmatrix}.$$

Then $E^2 = E$. Let $\mathscr{H}$ be the projective module EB^4. If we specialise $a = x$, $c = z$, we recover the Hopf bundle on the 2-sphere [5]. Let $\mathscr{H}' = (1 - E)B^4$. Computations reveal that $B^4 = \mathscr{H} \oplus \mathscr{H}'$ is an orthogonal decomposition for both the structures (B^4, S) and $(B^4, h) \simeq H(B^2)$, where

$$h = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

PROPOSITION 2.3. *The class of* (B^4, S) *in* $W^{-1}(B)$ *belongs to* $2W^{-1}(B)$.

Proof. Let s, s' be the restrictions of (B^4, S) to $\mathscr{H}$ and $\mathscr{H}'$ respectively. Since the only units B are non-zero elements of R, the restrictions of h to $\mathscr{H}$ and $\mathscr{H}'$ are respectively εs, $\varepsilon' s'$ where ε, ε' are ± 1. We have isometries

$$(B^4, S) \xrightarrow{\sim} (\mathscr{H}, s) \perp (\mathscr{H}', s')$$
$$(B^4, h) \to (\mathscr{H}, \varepsilon s) \perp (\mathscr{H}', \varepsilon' s').$$

Adding these equations in $W^{-1}(B)$, we see that the class of (B^4, S) in $W^{-1}(B)$ belongs to $2W^{-1}(B)$.

THEOREM 2.4. *Let R be a real closed field and A a 2-dimensional affine algebra over R. Then the map $\varphi : W^{-1}(A)/2W^{-1}(A) \to SK_0(A)/\mathrm{tr}\,\tilde{K}_0(A)$ is an isomorphism.*

Proof. We have seen earlier that Φ is surjective and that its kernel is generated by the classes $\Sigma(\alpha)$, $\alpha \in Um_3(A)$. By (2.1), we may assume that $\alpha = (a, y, c)$ with $ax + y^2 + cz = 1$ for some $x, z \in A$. By (2.3), the class of $\Sigma(\alpha)$ in $W^{-1}(A)$ belongs to $2W^{-1}(A)$.

COROLLARY 2.5. *Let A be a regular affine algebra of dimension 2 over R. Suppose* $\mathrm{Pic}\,A$ *is trivial. If $V = \mathrm{spec}\,A$, we have an isomorphism $W^{-1}(A)/2W^{-1}(A) \xrightarrow{\sim} A_0(V)/2A_0(V)$, where $A_0(V)$ denotes the group of zero cycles of V modulo rational equivalence.*

Proof. For a smooth affine surface $V = \mathrm{spec}\,A$, $SK_0(A) \xrightarrow{\sim} A_0(V)$ [6, p 298]. Since $\mathrm{Pic}\,A = 0$, $\mathrm{tr}\,\tilde{K}_0(A) = 2SK_0(A)$.

Remark. The group $A_0(V)/2A_0(V)$ can be computed using results of Colliot-Thélène and Ischebeck [4]. If V has no R-rational points at infinity, then, $A_0(V)/2A_0(V) \xrightarrow{\sim} (\mathbb{Z}/2)^s$ where s is the number of algebraic real components of V [10, 3.2].

Remark. Pardon, in [10], raises the question whether the condition $\mathrm{Pic}\,A = 0$ is necessary to conclude that $W^{-1}(A)/2W^{-1}(A) \xrightarrow{\sim} A_0(V)/2A_0(V)$. The following example, suggested by Mohan Kumar, shows that this condition is indeed necessary.

EXAMPLE. Let $\mathrm{Spec}\,A = \mathbb{P}^2_{\mathbb{R}} - S$ where S is the curve $x^2 + y^2 + z^2 = 0$. Then $\mathrm{Pic}\,A \xrightarrow{\sim} \mathbb{Z}/2$, generated by the restriction L of $\mathcal{O}(1)$ on $\mathbb{P}^2_{\mathbb{R}}$ to A and $SK_0(A) \xrightarrow{\sim} \mathbb{Z}/2$, generated by $L \oplus L^*$. Thus $SK_0(A) = \mathrm{tr}\,\tilde{K}_0(A)$ and $2\ SK_0(A) = 0$. Then we have $W^{-1}(A)/2\ W^{-1}(A) \xrightarrow{\sim} SK_0(A)/\mathrm{tr}\ \tilde{K}_0(A) = 0$ (2.4) whereas $A_0(V)/2\ A_0(V) \xrightarrow{\sim} SK_0(A)/2\ SK_0(A) \xrightarrow{\sim} \mathbb{Z}/2$.

§3. Surjectivity of the determinant map

We prove in this section, the following

THEOREM 3.1. *Let A be an affine algebra of dimension d over a field K. Suppose one of the following two conditions holds.*

1) *K is algebraically closed of characteristic prime to d.*
2) *K is real closed and the set of K-rational points of A lies in a closed subscheme of dimension $\leq d-1$.*

Then for any projective module P over A of rank $\geq d$, the map $\det: \mathrm{Aut}\, P \to A^*$ *is surjective.*

For the proof of this theorem, we need the following result which is a minor variation of a theorem of Suslin [11].

THEOREM 3.2. *Let A and P be as in* (3.1). *Then $SL(A \oplus P)$ acts transitively on the set of unimodular elements of $A \oplus P$.*

Sketch of a proof. If rank $P > d$, by Serre's theorem, P contains a free direct summand and the theorem is immediate. We therefore assume that rank $P = d$. Let $(a, v) \in A \oplus P$ be a unimodular element. Let J be the intersection of all the maximal ideals $\mathfrak{m}$ of A such that $A/\mathfrak{m}$ is real. By our assumption, $\dim A/J \leq d-1$. By a version of Bertini's theorem given in [13, Theorem 1.4], there exists a finite subset $T \subset P$ such that for a generic linear combination w of elements of T, $I = 0(v + aw)$, has the property that $\dim A/I = 0$. Since $\dim A/J \leq d-1$, there exists a finite subset $S \subset \bar{P}$, bar denoting modulo J, such that for a generic linear combination $\bar{w}$ of elements of S, $\bar{v} + \bar{a}\bar{w}$ is unimodular. By enlarging T if necessary, we assume that the image of T in A/J contains S so that, for a generic linear combination w of elements of T, $\dim A/I = 0$ and $I + J = A$. Since A/I, modulo its radical, is a product of algebraically closed fields, $\tilde{a} = b'^d$ for some $b' \in A/I$, d being invertible in A/I, $\sim$ denoting reduction modulo I. Let $b \in A$ be a lift of b'. Then there exists an elementary transformation of $A \oplus P$ which maps (a, v) to (b^d, v') for some $v' \in P$. A unimodular element of the form (b^d, v') can be mapped to $(1, 0)$ by an element of $SL(A \oplus P)$. This follows from steps 6 and 7 of the proof of [11, Theorem].

Remark. If A is reduced, the assumption on the characteristic of K in the above theorem can be dropped.

Proof of Theorem 3.1. Let u be a unit of A. By (3.2), there exists an automorphism

$$\begin{pmatrix} \theta & p \\ a & u^{-1} \end{pmatrix}$$

of $P \oplus A$ mapping $(0, u)$ to $(0, 1)$ with determinant 1. We have $p = 0$ and θ is an automorphism of P with det $\theta = u$.

COROLLARY 3.3. *Let A be as in* (3.1). *If* $\dim A = 2$, *every* rank 2 *projective module P over A with trivial determinant carries a skew-symmetric structure s which is unique up to isometry. The map which sends the class of P in $SK_0(A)$ to the class of (P, s) in $W^{-1}(A)$ yields a homomorphism $SK_0(A) \to W^{-1}(A)$ which in turn induces a homomorphism $SK_0(A)/\mathrm{tr}\, \tilde{K}_0(A) \to W^{-1}(A)$ which is inverse to Φ.*

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Institut de Mathématiques
Université de Lausanne
1015 Lausanne-Dorigny
Switzerland

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Colaba, Bombay 400 005
India

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