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A splitting theorem for quadratic forms

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1. Introduction

Our main result (Theorem 12) is a quadratic analogue of Serre's theorem on projective modules: a locally hyperbolic space over a ring A has a hyperbolic orthogonal summand if its rank is larger than twice the dimension of A. From this we deduce that if A is regular of dimension 3 and K is its field of quotients, the homomorphism $W(A) \to W(K)$ of Witt rings is injective (Theorem 24). This result has also been proved by Pardon [17], using quite different methods.

The validity of a quadratic analogue of Serre's theorem has been surmised by Bass [5]. In the proofs we use a patching technique that appears, in various disguises, in the work of Karoubi [9], Lindel [13], Landsburg [12] and presumably others.

2. Preliminaries

We recall here a few known theorems that we shall need. A good reference for standard results and terminology is [10].

Throughout this paper, A denotes a commutative noetherian ring with 1, in which 2 is invertible. Unless otherwise indicated, tensor products are over A. For any quadratic space M over A and any A-algebra B, we denote by M_B the quadratic space $B \otimes M$. Similarly, for any $s \in A$, we denote by M_s the quadratic space $A[1/s] \otimes M$ over A_s.

For any projective A-module P (of finite rank), $H(P)$ denotes the A-module $P \oplus \mathrm{Hom}_A(P, A)$ equipped with the bilinear form $\langle p \oplus f,\ q \oplus g \rangle = f(q) + g(p)$. A quadratic space is said to be hyperbolic if it is isometric to some $H(P)$. For any A-linear automorphism α of P we define an isometry $H(\alpha)$ of $H(P)$ by $H(\alpha)(p \oplus f) = \alpha(p) \oplus f \circ \alpha^{-1}$. Thus, for $P = A^n$, H defines a homomorphism of $GL_n(A)$ into the group $O_{2n}(A)$ of all isometries of $H(A^n)$. We denote by $E_n(A)$ the subgroup of $GL_n(A)$ generated by the elementary matrices $E_{ij}(a) = 1 + ae_{ij}$, where $a \in A$ and, for $i \neq j$, e_{ij} is the $n \times n$ matrix with a 1 in the (i, j)th place and zeroes elsewhere.

Let $e_1, \ldots, e_n$ be the canonical basis of A^n and $e_{n+1}, \ldots, e_{2n}$ its dual basis. For any integer i between 1 and $2n$ put $i' = i+n$ if $i \leqslant n$ and $i' = i-n$ if $i > n$. For $i \neq j$ and any $a \in A$ the matrix $H_{ij}(a) = 1 + a(e_{ij} - e_{j'i'})$ is orthogonal. Notice that, for any $a, b \in A$, $H_{ij}(a+b) = H_{ij}(a)H_{ij}(b)$. The elementary orthogonal group $EO_{2n}(A)$ is the subgroup of $O_{2n}(A)$ generated by all the $H_{ij}(a)$.

The following theorem is a basic result of Vaserstein.

THEOREM 1. *Let A be a commutative ring with noetherian maximal ideal spectrum of dimension d. Then, for any integer $r \geqslant d+1$, $O_{2r}(A) = O_{2d+2}(A)EO_{2r}(A)$.*

Proof. See [19], Theorem 2.8.

If M is a quadratic space of rank r over A, $\overset{r}{\wedge} M$ can be given a quadratic structure in a natural way ([10] IV.3). With this structure, $\overset{r}{\wedge} M$ is called the (signed) discriminant of M and is denoted by $d(M)$. A space of rank 1 is said to be trivial if it is isometric to the quadratic space A with the bilinear form $\langle a, b\rangle = ab$. The discriminant of a hyperbolic module is trivial.

THEOREM 2. *Assume that A is semilocal and let M be a quadratic space of rank 2 over A. If the discriminant of M is trivial, M is hyperbolic.*

Proof. By [10] II.3, Cor., M can be diagonalised: $M = Au \perp Av$, where $\langle u, u\rangle = a$ and $\langle v, v\rangle = b$ are units of A. If $d(M)$ is trivial, $ab = -c^2$ for some $c \in A$, hence $M = Ae \oplus Af$, where $e = u + cb^{-1}v$, $f = 1/(2a)(u - cb^{-1}v)$. Since $\langle e, e\rangle = \langle f, f\rangle = 0$ and $\langle e, f\rangle = 1$, $M = H(A)$.

THEOREM 3. *Assume that A is a domain. Let M be a quadratic space of rank 2 over A. If the discriminant of M is trivial, M is hyperbolic.*

Proof. Let $\mathfrak{p}$ be any prime ideal of A. By Theorem 2, $M_{\mathfrak{p}}$ is of the form $A_{\mathfrak{p}}e(\mathfrak{p}) \oplus A_{\mathfrak{p}}f(\mathfrak{p})$, where $\langle e(\mathfrak{p}), e(\mathfrak{p})\rangle = \langle f(\mathfrak{p}), f(\mathfrak{p})\rangle = 0$ and $\langle e(\mathfrak{p}), f(\mathfrak{p})\rangle = 1$. In particular $M_K = Ke \oplus Kf$, where $e = e((0))$, $f = f((0))$. Since every isotropic vector of $M_{\mathfrak{p}}$ is a multiple of $e(\mathfrak{p})$ or $f(\mathfrak{p})$, we may assume that $(M \cap Ke)_{\mathfrak{p}} = M_{\mathfrak{p}} \cap Ke = A_{\mathfrak{p}}e(\mathfrak{p})$ for every $\mathfrak{p}$. Hence $I = M \cap Ke$ is a maximal totally isotropic direct summand of M and $M = H(I)$.

We recall that, for any quadratic space M of even rank over A, the Witt invariant of M is the class $w(M)$ of the Clifford algebra of M in the Brauer group of A.

The next result is a special case of a classification theorem proved in [11].

THEOREM 4. *Assume that A is a domain. Let M be a quadratic space of rank 4 over A. Assume that $d(M)$ and $w(M)$ are both trivial. Then there exist two projective modules P, Q of rank 2 over A and an isomorphism $\varepsilon : \overset{2}{\wedge} P \otimes \overset{2}{\wedge} Q \xrightarrow{\sim} A$ such that M is isometric to the quadratic space $P \otimes Q$ with the bilinear form $\langle p \otimes q, p' \otimes q' \rangle = \varepsilon(p \wedge p' \otimes q \wedge q')$.*

Proof. Let $\mathcal{N}$ be the reduced norm functor constructed in [11]. By [11], Theorem 4.6, there are a quaternion A-algebra Λ, a projective left Λ-module Γ of rank 1 and a generator u of the rank 1 A-module $\mathcal{N}(\Gamma)$ such that M is isometric to the A-module Γ equipped with the following quadratic form $q : \Gamma \to A$: identify any $\gamma \in \Gamma$ with the Λ-homomorphism $\Lambda \to \Gamma$ that maps 1 to γ and choose $q(\gamma) \in A$ such that $\mathcal{N}(\gamma)(x) = q(\gamma)xu$ for all $x \in A = \mathcal{N}(\Lambda)$. By [11], Proposition 4.1, $w(M)$ is the class of Λ in $\mathrm{Br}(A)$, hence, by the assumption $w(M) = 0$, $\Lambda = \mathrm{End}_A P$ for some projective A-module P of rank 2. By Morita theory Γ is of the form $P \otimes Q$, where Q is also of rank 2. By [11], Theorem 2.1, $\mathcal{N}(\Gamma) = \overset{2}{\wedge} P \otimes \overset{2}{\wedge} Q$ and for any A-homomorphism $\phi : P^* \to Q$ the norm of the Λ-isomorphism $1 \otimes \phi : \Lambda = P \otimes P^* \to \Gamma = P \otimes Q$ is $1_{\overset{2}{\wedge} P} \otimes \overset{2}{\wedge} \phi : A \to \overset{2}{\wedge} P \otimes \overset{2}{\wedge} Q$. If $\gamma = \sum p_i \otimes q_i \in \Gamma = P \otimes Q$, the Λ-homomorphism $\Lambda \to \Gamma$ that maps 1 to γ can be written in the form $1 \otimes \phi$, where $\phi : P^* \to Q$. It is easily checked by localization that $(1 \otimes \overset{2}{\wedge} \phi)(1) = \sum p_i \wedge p_j \otimes q_i \wedge q_j$; hence, if $\varepsilon : \overset{2}{\wedge} P \otimes \overset{2}{\wedge} Q \to A$ maps u to 1, $q(\gamma) = \varepsilon(\sum p_i \wedge p_j \otimes q_i \wedge q_j)$.

THEOREM 5. *Let I be an ideal of A and assume that A is I-adically complete. For every quadratic space M over A/I there exists a quadratic space M_1 over A such that $A/I \otimes M_1 \cong M$. Given two quadratic spaces M_1, M_2 over A, any isometry $\phi : A/I \otimes M_1 \xrightarrow{\sim} A/I \otimes M_2$ can be lifted to an isometry $\tilde{\phi} : M_1 \xrightarrow{\sim} M_2$.*

Proof. See [20], Theorem 2.

The next result is certainly well-known, but we did not find a suitable reference.

THEOREM 6. *Let A be a noetherian ring and s an element of A. Let $\hat{A}$ be the s-adic completion of A. The dimension of $\hat{A}$ is not larger than that of A.*

Proof. By [8], 7.2.3, the dimension of the power series ring $A[[X]]$ is $\dim A + 1$. By [14] §23, Cor. 5, $\hat{A} = A[[X]]/(X - s)$. Hence, if we show that $X - s$ is not a zero divisor in $A[[X]]$, we can conclude by [1], Cor. 11.9. Now, $(X - s)\sum_{i \geq m} a_i X^i = 0$ implies $sa_m = 0$, $a_m = sa_{m+1}$, $a_{m+1} = sa_{m+2}$, etc. Hence $a_m \in \bigcap_{i \geq 0} As^i$ and, by Krull's intersection theorem, $(1 + as)a_m = 0$ for some $a \in A$. Thus, since $sa_m = 0$, $a_m = 0$ and, by induction, $a_n = 0$ for all n.

THEOREM 7. *Let A be a Dedekind domain, K its field of fractions and M a quadratic space over A. If M_K is isotropic, M splits as $M' \perp H(I)$, where I is a nonzero ideal of A. If M_K is hyperbolic, so is M.*

Proof. See [12], IV, Corollary 3.3.

3. Patching

Let $\varepsilon : A \to B$ be a homomorphism of noetherian rings and s an element of A such that $\varepsilon(s)$ is not a zero divisor in B. Assume that ε induces an isomorphism $A/As \xrightarrow{\sim} B/B\varepsilon(s)$. Under these assumptions we call the diagram

$$(*) \qquad \begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ A_s & \to & B_s \end{array}$$

a patching diagram. Since in most applications A is a subring of B, we shall omit ε and write a instead of $\varepsilon(a)$. Let now P be a quadratic space over A_s, Q a quadratic space over B and $\sigma : P_B \to Q_s$ an isometry of B_s-spaces. We put $\mathcal{M}(P, \sigma, Q) = \{(x, y) \in P \times Q \mid \sigma(x \otimes 1) = y/1\}$.

THEOREM 8. *Let P, Q and σ be as above and $M = \mathcal{M}(P, \sigma, Q)$. If $(*)$ is a patching diagram, then M is a quadratic space over A and the projections of $P \times Q$ onto its factors induce isometries $M_s \xrightarrow{\sim} P$ and $M_B \xrightarrow{\sim} Q$. Furthermore, if N is a quadratic space over A and $\chi : (N_s)_B \xrightarrow{\sim} (N_B)_s$ the canonical isometry, then $N = \mathcal{M}(N_s, \chi, N_B)$.*

Proof. The first assertion is proved in [16], Theorem 1. The second one follows from the fact that a patching diagram is cartesian. It is also worth noticing that the injectivity of $B \to B_s$ implies that of $A \to A_s$.

THEOREM 9. *Let $(*)$ be a patching diagram and P, Q as in Theorem 8. Suppose that there are isometries $\phi : P \xrightarrow{\sim} H(A_s^n)$ and $\psi : Q_s \xrightarrow{\sim} H(B_s^n)$. For any $\sigma \in O_{2n}(B_s)$ and any $\varepsilon \in EO_{2n}(B_s)$ the quadratic spaces $\mathcal{M}(P, \psi^{-1}\sigma\phi_B, Q)$ and $\mathcal{M}(P, \psi^{-1}\sigma\varepsilon\phi_B, Q)$ are isometric.*

Proof. It suffices to prove the theorem for $\varepsilon = H_{ij}(b)$, where $b \in B_s$. The fact that $A/As \xrightarrow{\sim} B/Bs$ implies that, for any integer k, b can be written as $a + s^k c$, where $a \in A_s$ and $c \in B$. Then, $\varepsilon = H_{ij}(s^k c) H_{ij}(a)$. If k is large enough, $\rho =$

$(\psi^{-1}\sigma)H_{ij}(s^kc)(\psi^{-1}\sigma)^{-1}$ is an isometry of Q. On the other hand $\tau = \phi^{-1}H_{ij}(a)\phi$ is an isometry of P and it is easy to check that $\tau \times \rho^{-1}$ induces an isometry of $\mathcal{M}(P, \psi^{-1}\sigma\varepsilon\phi_B, Q)$ onto $\mathcal{M}(P, \psi^{-1}\sigma\phi_B, Q)$.

COROLLARY 10. *Suppose that* $P = P' \perp H(A_s^{n-m})$, $Q = Q' \perp H(B^{n-m})$ *and that there are isometries* $\phi' : P' \xrightarrow{\sim} H(A_s^m)$ *and* $\psi' : Q'_s \xrightarrow{\sim} H(B_s^m)$. *Let* $\chi : P_B \xrightarrow{\sim} Q_s$ *be any isometry. If every* $\sigma \in O_{2n}(B_s)$ *can be written as* $\sigma'\varepsilon$, *where* $\varepsilon \in EO_{2n}(B_s)$ *and* $\sigma' \in O_{2m}(B_s)$ *(embedded in* $O_{2n}(B_s)$ *as* $O_{2m}(B_s) \perp \mathrm{id}$*), then* $\mathcal{M}(P, \chi, Q)$ *is of the form* $N \perp H(A^{n-m})$.

Proof. If we put $\phi = \phi' \perp \mathrm{id} : P \xrightarrow{\sim} H(A_s^n)$ and $\psi = \psi' \perp \mathrm{id} : Q'_s \xrightarrow{\sim} H(B_s^n)$, we get $\mathcal{M}(P, \chi, Q) = \mathcal{M}(P, \psi^{-1}\sigma\phi_B, Q)$ for some $\sigma \in O_{2n}(B_s)$. By assumption $\sigma = \sigma'\varepsilon$, where $\varepsilon \in EO_{2n}(B_s)$ and $\sigma' \in O_{2m}(B_s)$. By Theorem 9, $\mathcal{M}(P, \psi^{-1}\sigma\phi_B, Q) \cong \mathcal{M}(P, \psi^{-1}\sigma'\phi_B, Q) = \mathcal{M}(P', (\psi')^{-1}\sigma'\phi'_B, Q') \perp \mathcal{M}(H(A_s^{n-m}), \mathrm{id}, H(B^{n-m})) = N \perp H(A^{n-m})$.

THEOREM 11. *Suppose that* (*) *is a patching diagram and that the maximal spectrum of* B_s *has dimension* d. *Let* $Q = Q' \perp H(B^{n-m})$ *be a quadratic space over* B, $P = H(A_s^n)$ *and* $\chi : P_B \xrightarrow{\sim} Q_s$ *an isometry. Suppose that* $Q'_s \cong H(B_s^m)$, *where* $m \geq d+1$. *Then* $\mathcal{M}(P, \chi, Q)$ *is of the form* $N \perp H(A^{n-m})$.

Proof. By Theorem 1, $O_{2n}(B_s) = O_{2m}(B_s)EO_{2n}(B_s)$. Hence the assumptions of Corollary 10 are satisfied.

4. A splitting theorem

Let M be a quadratic space over A. We say that M is locally hyperbolic if, for any maximal ideal $\mathfrak{m}$ of A, the localization $M_{\mathfrak{m}}$ is hyperbolic over $A_{\mathfrak{m}}$.

THEOREM 12. *Let* A *be a noetherian ring of Krull dimension* d *and* M *a locally hyperbolic quadratic space of rank* $2n \geq 2d$ *over* A. *Then* M *is of the form* $N \perp H(A^{n-d})$.

Proof. We prove the theorem by induction on d. By Theorem 5 we may assume that A is reduced. If $d = 0$, A is a product of fields and the assertion is true. Suppose that $d > 0$. Since A is reduced, $(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal primes of A. Let S denote the multiplicative set $A \setminus \bigcup \mathfrak{p}_i$. Then $S^{-1}A = \prod A_{\mathfrak{p}_i}$ where each $A_{\mathfrak{p}_i}$ is a field. By assumption $M_{\mathfrak{m}}$ is hyperbolic for any maximal $\mathfrak{m}$, hence $M_{\mathfrak{p}_i}$ is hyperbolic. This is clearly the same as saying that $S^{-1}M$

is hyperbolic and hence, for some $s \in S$ there is an isometry $\phi : M_s \xrightarrow{\sim} H(A_s^n)$. Since the zero ideal of A has no embedded components the set of zero divisors of A is $\bigcup \mathfrak{p}_i$, hence s is not a zero divisor. Let $\hat{A}$ be the s-adic completion of A. Since $\hat{A}$ is A-flat, s is not a zero divisor of $\hat{A}$. Furthermore, $\hat{A}/\hat{A}s = A/As$, hence

$$(**)\qquad \begin{array}{ccc} A & \rightarrow & \hat{A} \\ \downarrow & & \downarrow \\ A_s & \rightarrow & \hat{A}_s \end{array}$$

is a patching diagram. The quadratic space M/sM is locally hyperbolic and $\dim A/As \leqslant d-1$. By the induction assumption M/sM splits as $\bar{N} \perp H(A/As)^{n-d+1})$. Since $\hat{M} = \hat{A} \otimes_A M$ is s-adically complete, by Theorem 5 we can lift the decomposition of M/Ms to $\hat{M} = \tilde{N} \perp H(\hat{A}^{n-d+1})$. Writing $P' = H(A_s^d)$ and $Q' = \tilde{N} \perp H(\hat{A})$, from $(\hat{M})_s \cong (M_s)_{\hat{A}}$ we get $P'_{\hat{A}} \perp H(\hat{A}_s^{n-d}) \cong Q'_s \perp H(\hat{A}_s^{n-d})$. By Theorem 6, the dimension of $\hat{A}$ is not larger than that of A and $s \in \operatorname{rad} \hat{A}$; hence the dimension of $\hat{A}_s$ is less then the hyperbolic rank of $P_{\hat{A}}$. By a well known cancellation theorem (see for instance [18], Theorem 7.2), the isomorphism above implies that $Q'_s \cong H(\hat{A}_s^d)$. Thus we are in the situation of Theorem 11 and we conclude that $M = N \perp H(A^{n-d})$.

5. Rings of dimension 2

The following result is a special case of a theorem of Bialynicki-Birula ([6], Theorem 1).

THEOREM 13. *Let A be a semilocal 1-dimensional domain, K its field of quotients and M a quadratic space over A. Assume that M has trivial discriminant, that M_K is hyperbolic and that $M/\mathfrak{m}M$ is hyperbolic for every maximal ideal $\mathfrak{m}$ of A. Then M is hyperbolic.*

Proof. We translate the proof of [6] into our lingo. Let s be a nonzero element of $\operatorname{rad} A$ and $\hat{A}$ the s-adic completion of A. By assumption $M/\operatorname{rad} A \cdot M$ is hyperbolic. The kernel of $A/As \rightarrow A/\operatorname{rad} A$ is nilpotent, hence, by Theorem 5, M/sM is also hyperbolic and, again by the same theorem, this implies that $\hat{M}$ is hyperbolic. Observing that $\dim \hat{A}_s = 0$, from Theorem 11 applied to (**), we get $M = N \perp H(A^{n-1})$, where N is of rank 2 (or zero). By assumption the discriminant of M and, hence, that of N, is trivial. By Theorem 3, $N = H(A)$.

THEOREM 14. *Let A be a semilocal normal domain of dimension 2, K its* *field of quotients and M a quadratic space over A. Assume that M_K is hyperbolic and that $M/\mathfrak{m}M$ is hyperbolic for every singular maximal ideal $\mathfrak{m}$ of A. Then M is of the form $N \perp H(A^k)$, where N is of rank $\leqslant 4$. If the Witt invariant of M is zero, M is hyperbolic.*

Proof. It is known (see [7], Proposition 2.1) that if $\mathfrak{m}$ is a regular maximal ideal and M_K is hyperbolic, then $M/\mathfrak{m}M$ is hyperbolic. Hence we may assume that $M/\mathfrak{m}M$ is hyperbolic for every maximal ideal of A. The discriminant of M_K is locally trivial because K is integrally closed. By a result of Bass ([4], Proposition 2.6.2), this implies that the discriminant of M is trivial. Let s be a nonzero element of rad A. The ring A_s is a Dedeking domain and therefore, by Theorem 7, M_s is hyperbolic. We claim that M/Ms is hyperbolic. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal associated primes of As. By Krull's theorem they are of height one and, A being normal, for every $\mathfrak{p} = \mathfrak{p}_i$, $A_{\mathfrak{p}}$ is a discrete valuation ring. Again by Theorem 7, $M_{\mathfrak{p}}$ is hyperbolic. Put $\bar{A} = A/\mathfrak{p}$, $\bar{K} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ and $\bar{M} = M/\mathfrak{p}M$. $\bar{M}_{\bar{K}} = M_{\mathfrak{p}} \otimes \bar{K}$ is hyperbolic and $\bar{M}/\bar{\mathfrak{m}}\bar{M}$ is hyperbolic for any maximal ideal $\bar{\mathfrak{m}}$ of $\bar{A}$. By Theorem 13, $\bar{M}$ is hyperbolic. To show that M/sM is hyperbolic it suffices to show that M_B is hyperbolic, where $B = A/\bigcap \mathfrak{p}_i$ is the quotient of A/As by its nilpotent radical. Consider the cartesian diagram

$$\begin{array}{ccc} B & \rightarrow & \tilde{B} \\ \downarrow & & \downarrow \\ B/\mathfrak{c} & \rightarrow & \tilde{B}/\mathfrak{c} \end{array}$$

where $\tilde{B} = \Pi(A/\mathfrak{p}_i)$ and $\mathfrak{c}$ is the conductor of $\tilde{B}$ in B. $M_{\tilde{B}}$ is hyperbolic because every $M/\mathfrak{p}_iM$ is hyperbolic. The conductor contains all the intersections $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{i-1} \cap \mathfrak{p}_{i+1} \cap \cdots \cap \mathfrak{p}_r$; hence its image in any $A/\mathfrak{p}_i$ is not zero. This shows that $\tilde{B}/\mathfrak{c}$ and $B/\mathfrak{c}$ are artinian. In particular, $B/\mathfrak{c}$ is complete and therefore $(B/\mathfrak{c}) \otimes M$ is hyperbolic. By the quadratic version of Milnor's construction of projective modules ([3], Theorem 2.2), $M_B = \mathcal{M}(H(B/\mathfrak{c})^n), \sigma, H(\tilde{B}^n))$, where $\sigma \in O_{2n}(\tilde{B}/\mathfrak{c})$. Since $\tilde{B}/\mathfrak{c}$ is zero-dimensional, σ can be written as $\varepsilon\sigma_1$, where $\varepsilon \in EO_{2n}(\tilde{B}/\ )$ and $\sigma_1 \in O_2(\tilde{B}/\mathfrak{c})$. The isometry ε can be lifted to $\tilde{\varepsilon} \in EO_{2n}(\tilde{B})$. The map $\mathrm{id} \times \tilde{\varepsilon}^{-1}$ of $H(B/\mathfrak{c})^n) \times H(\tilde{B}^n)$ into itself induces an isometry of M_B onto $\mathcal{M}(H((B/\mathfrak{c})^n), \sigma_1, H(\tilde{B}^n)) = \bar{N} \perp H(B^{n-1})$, where $\bar{N}$ is of rank 2. But the discriminant of $\bar{N}$ is trivial, hence, by Theorem 2, $\bar{N} = H(B)$. We have thus proved that M/sM is hyperbolic and this implies that the s-adic completion $\hat{M}$ of M is also hyperbolic. Let $\phi : M_s \xrightarrow{\sim} H(A_s^n)$ and $\psi : \hat{M} \xrightarrow{\sim} H(\hat{A}^n)$ be isometries, Φ and Ψ their extension to $\hat{M}_s$ and $\chi = \Psi\Phi^{-1}$. Then $M = \mathcal{M}(H(A_s^n), \chi, H(\hat{A}^n))$ and by Theorem 11, $M = N \perp H(A^{n-2})$. This proves the first assertion of the theorem.

Assume now that the Witt invariant of M, and, hence, of N, is zero. We may use Theorem 4 because, as already remarked, the discriminant of N is trivial. Since A is semilocal, projective modules of constant rank (2 in our case) are free and it is easily checked that, in this situation, the quadratic space $P\otimes Q$ of Theorem 4 is isometric to $H(A^2)$.

THEOREM 15. *Let A be a normal domain of dimension 2, K its field of quotients and M a quadratic space over A. Assume that M_K is hyperbolic and that, for every singular maximal ideal $\mathfrak{m}$ of A, $M/\mathfrak{m}M$ is hyperbolic. Then, if the Witt invariant of M is zero, M is stably hyperbolic.*

Proof. We may assume that the rank of M is at least 4. By Theorem 14, M is locally hyperbolic. By Theorem 12, $M=N\perp H(A^{n-2})$, where N is of rank 4. Since the discriminant of N is trivial, Theorem 4 tells us that there are two projective A-modules P, Q of rank 2 and an isomorphism $\varepsilon:\overset{2}{\wedge}P\otimes\overset{2}{\wedge}Q\xrightarrow{\sim}A$ such that $N\cong(P\otimes Q)_\varepsilon$, the quadratic structure on $(P\otimes Q)_\varepsilon$ being defined by $\langle p\otimes q, p'\otimes q'\rangle=\varepsilon(p\wedge p'\otimes q\wedge q')$. It is enough to show that $(P\otimes Q)_\varepsilon$ is stably hyperbolic. To do this, we define, for any projective A-module T of rank 4 and any isomorphism $\phi:\overset{2}{\wedge}T\xrightarrow{\sim}A$, a quadratic space $(\overset{2}{\wedge}T)_\phi$ by $\langle x,y\rangle=\phi(x\wedge y)$. We then need the following results.

THEOREM 16. *Let P, Q and R be projective A-modules of rank, respectively, 2, 2 and 3. Suppose that there are isomorphisms $\varepsilon:\overset{2}{\wedge}P\otimes\overset{2}{\wedge}Q\xrightarrow{\sim}A$ and $\phi:\overset{2}{\wedge}(R\oplus A)\xrightarrow{\sim}A$. Then*

$$(P\otimes Q)_{-\varepsilon}\perp H(\overset{2}{\wedge}P)\cong(\overset{2}{\wedge}(P\oplus Q))_\varepsilon$$

and

$$(\overset{2}{\wedge}(R\oplus A))_\phi\cong H(R).$$

Proof. the obvious maps are isometries.

We now finish the proof of Theorem 15. By Serre's theorem, $P\oplus Q=R\oplus A$. By Theorem 16, $(\overset{2}{\wedge}(R\oplus A))_\varepsilon$ is hyperbolic. Hence, by the first isomorphism of Theorem 16, $(P\otimes Q)_{-\varepsilon}$ is stably hyperbolic and so is N.

As an immediate corollary of Theorem 15, we obtain the following result, proved by Pardon ([17], Theorem 5) with different methods.

THEOREM 17. *Let A be a regular domain of dimension 2 and K its field of quotients. The homomorphism of Witt rings $W(A)\to W(K)$ is injective.*

Proof. Let M be a quadratic space over A such that M_K is hyperbolic. The Witt invariant of M is zero because, by [2], Theorem 7.2, $\mathrm{Br}(A) \to \mathrm{Br}(K)$ is injective. By Theorem 15, M is stably hyperbolic.

6. Some cohomological results

In this section all quadratic spaces are assumed to have trivial discriminant. A reference for unexplained terms is [4].

The sequence of $\mathbb{Z}[\frac{1}{2}]$-group schemes

$$1 \to \mu_2 \to \mathrm{Spin}_{2n} \to SO_{2n} \to 1 \tag{6.1}$$

is known to be exact in the étale topology. We shall use some properties of the corresponding étale cohomology sequence

$$H^1(A, \mathrm{Spin}_{2n}) \to H^1(A, SO_{2n}) \xrightarrow{\partial} H^2(A, \mu_2).$$

We state them in the next four theorems. For any quadratic space M of rank $2n$ over A (with trivial discriminant), we shall denote by $[M]$ its class in $H^1(A, SO_{2n})$.

THEOREM 18. *Let M and N be quadratic spaces over A, of rank $2m$ and $2n$ respectively. Then $\partial[M \perp N] = \partial[M] \cdot \partial[N]$.*

THEOREM 19. *For any invertible A-module I, $\partial[H(I)] = \delta[I]$, where $[I]$ is the class of I in $H^1(A, G_m) = \mathrm{Pic}\, A$ and $\delta : H^1(A, G_m) \to H^2(A, \mu_2)$ is the coboundary map corresponding to the exact sequence of $\mathbb{Z}[\frac{1}{2}]$-group schemes*

$$1 \to \mu_2 \to G_m \xrightarrow{2} G_m \to 1. \tag{6.2}$$

THEOREM 20. *The diagram*

$$\begin{tikzcd} H^1(A, SO_{2n}) \arrow[r] \arrow[dr, "w"'] & H^2(A, \mu_2) \arrow[d] \\ & H^2(A, G_m), \end{tikzcd}$$

where w is the Witt invariant, commutes.

Let e_1, e_2, e_3, e_4 be the canonical basis of A^4 and let $\varepsilon : \overset{4}{\wedge} A^4 \xrightarrow{\sim} A$ be the isomorphism that maps $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ to 1. We denote by $(\overset{2}{\wedge} A^4)_\varepsilon$ the quadratic space defined by $\langle x, y\rangle = \varepsilon\langle x \wedge y\rangle$, $x, y \in \overset{2}{\wedge} A^4$. This space is nothing but $H(A^3)$ and we can map SL_4 to SO_6 by sending any $\alpha \in SL_4(A)$ to its second exterior power $\overset{2}{\wedge}\alpha$.

THEOREM 21. *There is an isomorphism of group schemes* $\phi : SL_4 \xrightarrow{\sim} \mathrm{Spin}_6$ *such that*

$$\begin{tikzcd} \mathrm{Spin}_6 \arrow[r] & SO_6 \\ & SL_4 \arrow[ul, "\phi"] \arrow[u, "\overset{2}{\wedge}"'] \end{tikzcd}$$

commutes.

The first three theorem are easily proved by an explicit computation with cocycles. We leave the details to the readers (if any). To prove Theorem 21, it is necessary to describe ϕ explicitly. To do this, we identify the Clifford algebra $C(A)$ of $H(A^3)$ with $M_8(A)$, the gradation being given by

$$C_0 = \begin{pmatrix} M_4(A) & 0 \\ 0 & M_4(A) \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 0 & M_4(A) \\ M_4(A) & 0 \end{pmatrix}.$$

$H(A^3)$ can then be identified with the submodule of all matrices $\begin{pmatrix} 0 & \xi \\ \xi^* & 0 \end{pmatrix}$ where

$$\xi = \begin{pmatrix} x & y & z & 0 \\ a & c & 0 & z \\ -b & 0 & c & -y \\ 0 & -b & -a & x \end{pmatrix} \quad \text{and} \quad \xi^* = \begin{pmatrix} c & -y & -z & 0 \\ -a & x & 0 & -z \\ b & 0 & x & y \\ 0 & b & a & c \end{pmatrix}.$$

The canonical involution on $C(A)$ is then given by

$$\begin{Bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{Bmatrix} \to \begin{Bmatrix} \tilde\alpha_{44} & \tilde\alpha_{34} & -\tilde\alpha_{24} & -\tilde\alpha_{14} \\ \tilde\alpha_{43} & \tilde\alpha_{33} & -\tilde\alpha_{23} & -\tilde\alpha_{13} \\ -\tilde\alpha_{42} & -\tilde\alpha_{32} & \tilde\alpha_{22} & \tilde\alpha_{12} \\ -\tilde\alpha_{41} & -\tilde\alpha_{31} & \tilde\alpha_{21} & \tilde\alpha_{11} \end{Bmatrix}$$

where, for any

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \tilde{\alpha} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For $\sigma \in M_4(A)$, let ${}^t\sigma$ be the transpose of σ.

The map ϕ is defined by

$$\phi(\sigma) = \begin{pmatrix} {}^t\sigma^{-1} & 0 \\ 0 & \omega\sigma\omega^{-1} \end{pmatrix},$$

where

$$\omega = \begin{pmatrix} 0 & \chi \\ -\chi & 0 \end{pmatrix} \quad \text{and} \quad \chi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A computation shows that $\phi(\sigma) \in \mathrm{Spin}_6(A)$. Clearly ϕ is an isomorphism of SL_4 onto a closed subscheme of Spin_6. Since Spin_6 is irreducible and of the same dimension as SL_4, ϕ is surjective. But SL_4 is reduced, hence ϕ is an isomorphism.

The pointed set $H^1(A, SL_4)$ classifies pairs (T, ε) where T is a projective module of rank 4 over A and ε is an isomorphism $\overset{4}{\wedge} T \xrightarrow{\sim} A$. Hence, by the theorem above, the map $H^1(A, SL_4) \to H^1(A, SO_6)$ induced by ϕ associates to (T, ε) the quadratic space $(\overset{2}{\wedge} T)_\varepsilon$ defined by $\langle x, y\rangle = \varepsilon(x \wedge y)$. This proves the next theorem.

THEOREM 22. *If N is a quadratic space of rank 6 over A such that $d(N)$ is trivial and $\partial[N] = 1$, then N is of the form $(\overset{2}{\wedge} T)_\varepsilon$.*

7. Rings of dimension 3

THEOREM 23. *Let A be a local 3-dimensional regular domain, K its field of quotients and M a quadratic space over A. If M_K is hyperbolic, M is hyperbolic.*

Proof. By Witt's cancellation theorem we may assume that M is of rank $\geqslant 6$. Let $\mathfrak{m}$ be the maximal ideal of A and s a regular parameter of $\mathfrak{m}$. Then A/As is a regular local ring of dimension 2. Put $S = A \setminus As$. Then $S^{-1}A$ is a Dedekind domain and, by Theorem 7, $S^{-1}M$ is hyperbolic. This implies that $S^{-1}(M/sM)$ is hyperbolic and therefore, by Theorem 17, M/sM is hyperbolic. On the other hand, A_s is regular and 2-dimensional, hence M_s is stably hyperbolic. By the cancellation theorem already quoted ([18], Theorem 7.2), M_s is hyperbolic, i.e. of the form $H(P)$ where P is a projective module of rank 3. Since A is regular, $K_0(A) \to K_0(A_s)$ is surjective. But A is local, hence $K_0(A_s) = K_0(A) = \mathbb{Z}$. This shows that P is stably free and, since rank $P > \dim A_s$, P is free by the well-known cancellation theorem of Bass–Schanuel. Hence, $M_s = H(A_s^3)$ and, applying Theorem 11 to (**), we obtain that $M = N \perp H(A^n)$, where N is of rank 6. The discriminant of N is trivial, hence N is represented by an element $[N]$ of the étale cohomology set $H^1(A, SO_6)$. To simplify the notations, we write $H^i(G)$ instead of $H^i(A, G)$. The exact sequences (6.1) and (6.2) give, respectively, the horizontal and the vertical exact sequence of the diagram

$$\begin{tikzcd} & & H^1(G_m) \arrow[d, "2"] \\ & & H^1(G_m) \arrow[d] \\ H^1(SL_4) \arrow[r] & H^1(SO_6) \arrow[r] & H^2(\mu_2) \arrow[d] \\ & & H^2(G_m). \end{tikzcd}$$

Since the homomorphism of Brauer groups $\mathrm{Br}(A) \to \mathrm{Br}(K)$ is injective ([2], Theorem 7.2), $[N]$ maps to zero in $H^2(G_m)$. But $H^1(G_m) = \mathrm{Pic}\, A = 0$ because A is local, hence $[N]$ maps to zero in $H^2(\mu_2)$ and is, therefore, in the image of $H^1(SL_4)$. Since A is local, $H^1(SL_4) = 0$. Hence $[N] = 0$ and N is hyperbolic.

THEOREM 24. *Let A be a regular 3-dimensional domain, K its fields of quotients, M a quadratic space over A such that M_K is hyperbolic. Then M is stably hyperbolic.*

Proof. We may assume that M is of rank $\geqslant 6$. By Theorem 23, M is locally hyperbolic. By Theorem 12, M is of the form $N \perp H(A^k)$, where N is of rank 6. Using the diagram above we see, as in the proof of Theorem 23, that $[N]$ maps to zero in $H^2(G_m)$. Hence $\partial[N] = \delta[I]$ for some $[I] \in \mathrm{Pic}\, A$. By Theorem 19, $\delta[I] = \partial[H(I)]$ and, by Theorem 18, $\partial[N \perp H(I)] = 1$. By Theorem 12, $N \perp H(I) = N' \perp H(A)$ and, by Theorem 18 again, $\partial[N'] = 1$. It suffices to show that N' is stably hyperbolic. By the horizontal exact sequence of the diagram above, $[N']$ comes from $H^1(SL_4)$, hence, by Theorem 22, $N' \cong (\overset{2}{\wedge} T)_\varepsilon$. By Serre's theorem on projective modules, $T = R \oplus A$ and, by Theorem 16, N is hyperbolic.

Remark. Theorem 24 is false for 4-dimensional regular rings. In fact, as noticed by M.-A. Knus, replacing A by $\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ in the example of [11], §5, yields an example of a regular 4-dimensional affine $\mathbb{R}$-algebra $A = \Lambda \otimes_{\mathbb{R}} \Lambda$ for which the homomorphism $W(A) \to W(K)$ is not injective.

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