A Note on the Fundamental Theorem of Projective Geometry

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Introduction

The aim of this note is to prove a generalisation to commutative rings of the classical fundamental theorem of projective geometry. In § 1, we introduce the notions of projective spaces and projectivities. In § 2, we prove the main theorem. The method of proof is similar to the proof of the theorem in the classical case as found for example in ARTIN [1]. The proof, as in the classical case, is elementary, but is trickier. In § 3, we give an example to show that a bijection between projective spaces of the same dimension which preserves collinear points is not necessarily a projectivity. This is in contrast to what happens in the case of projective spaces over fields.

§ 1. Projective Spaces and Projectivities

Let A be a cummutative ring with 1 and let M be a free A-module. Let $P(M)$ denote the set of all A-free direct summands of rank 1 of M. This set is called the *projective space associated to* M. Clearly, any element of $P(M)$ is of the form Ae where e is a unimodular element of M, i.e. there exists a linear form $g:M\to A$ with $g(e)=1$. If $(e_1,\ldots,e_n)$ is a basis for the A-module M and $e=\sum a_i e_i$, then we note that e is unimodular if and only if $\sum_{1\leqslant i\leqslant n} Ae_i=A$. If the ring A is such that every projective module of rank 1 is free, then $P(M)$ coincides with the usual projective space of algebraic geometry [2, p. 13].

DEFINITION. *Let M and N be free modules over commutative rings A and B respectively. A map $\alpha:P(M)\to P(N)$ is called a projectivity if α is bijective and for $p_1, p_2, p_3\in P(M)$, we have $\alpha p_1\subset\alpha p_2+\alpha p_3$ in N if and only if $p_1\subset p_2+p_3$ in M.*

This definition generalises the classical notion of projectivity between projective spaces over fields.

We note that by the very definition, $\alpha^{-1}:P(N)\to P(M)$ is also a projectivity. For later purposes, we need the following

LEMMA 1. *With the notation above, if $e_1,\ldots,e_n$ is a basis of M and $e\in M$ a unimodular element such that $Ae\subset\sum_{1\leqslant i\leqslant k} Ae_i$, then $\alpha Ae\subset\sum_{1\leqslant i\leqslant k}\alpha Ae_i$.*

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Proof. We prove the lemma by induction on k. Let $e=\sum_{1\leqslant i\leqslant k} a_i e_i$. Then $e'=\sum_{1\leqslant i\leqslant k-1} a_i e_i+e_k$ is unimodular and $Ae\subset Ae'+Ae_k$. By definition this implies that $\alpha Ae\subset\alpha Ae'+\alpha Ae_k$. Let $e''=\sum_{1\leqslant i\leqslant k-2} a_i e_i+e_k$. Since $e'\in Ae''+Ae_{k-1}$, we again have $\alpha Ae'\subset\alpha Ae''+\alpha Ae_{k-1}$. We thus have $\alpha Ae\subset\alpha Ae''+\alpha Ae_{k-1}+\alpha Ae_k$. By induction,

$$\alpha A e''\subset \sum_{1\leqslant i\leqslant k-2}\alpha A e_i+\alpha A e_k \quad\text{and hence}\quad \alpha A e\subset \sum_{1\leqslant i\leqslant k}\alpha A e_i.$$

Let A and B be rings and $\sigma: A\to B$ a homomorphism. If M and N are modules over A and B respectively, then a map $\Phi: M\to N$ is called *σ-semilinear* if Φ is additive and $\Phi(am)=\sigma(a)\,\Phi(m)$ for all $a\in A$, $m\in M$. If M and N are free modules over A and B of the same rank and $\Phi: M\to N$ a σ-semilinear map which takes a basis $(e_1,\ldots,e_n)$ of M into a basis of N, then if $e=\sum a_i e_i$ is a unimodular element of M, then $\Phi(e)=\sum\sigma(a_i)\,\Phi(e_i)$ is unimodular in N. For, if $\sum\lambda_i a_i=1$, $\lambda_i\in A$, we have $\sum\sigma(\lambda_i)\,\sigma(a_i)=1$ which implies $\Phi(e)=\sum\sigma(a_i)\,\Phi(e_i)$ is unimodular. It is clear that we have an induced map $P(\Phi): P(M)\to P(N)$ by setting for any unimodular element e of M, $P(\Phi)\,(Ae)=B\Phi(e)$. We then have the following rather obvious

PROPOSITION 1: *With the same notation as above, for any* $p_1, p_2, p_3\in P(M)$ with $p_1\subset p_2+p_3$, $P(\Phi)\,p_1\subset P(\Phi)\,p_2+P(\Phi)\,p_3$. If σ is an isomorphism, then $P(\Phi)$ is a projectivity.

§ 2 The Theorem

Our object in this section is to prove the following theorem which generalises to commutative rings the classical "Fundamental theorem of projective geometry".

THEOREM. *Let M and N be free modules of finite rank $\geqslant 3$ over commutative rings A and B respectively. If $\alpha: P(M)\to P(N)$ is a projectivity, then there exists an isomorphism $\sigma: A\to B$ and a σ-semilinear isomorphism $\Phi: M\to N$ such that $\alpha=P(\Phi)$. If $\sigma_i: A\to B$, $i=1,2$, are isomorphisms and $\Phi_i: M\to N$ are σ_i-semilinear isomorphisms such that $P(\Phi_1)=P(\Phi_2)$, then there exists a $b\in B$ such that $\Phi_1=b\cdot\Phi_2$ and $\sigma_1=\sigma_2$.*

Proof. Let $e_1,\ldots,e_n$ be a basis for M and let $\alpha Ae_i=Bf_i$, $1\leqslant i\leqslant n$. We assert that $f_1,\ldots,f_n$ generate the B-module N. Since any element of N is a linear combination of elements of a basis for N, it is enough to check that any unimodular element $f\in N$ is a linear combination of $f_1,\ldots,f_n$. If $e\in M$ is a unimodular element with $\alpha Ae=Bf$ and $e=\sum_{1\leqslant i\leqslant n} a_i e_i$, we have $Ae\subset\sum_{1\leqslant i\leqslant n} Ae_i$ and by lemma 1, we get $Bf\subset\sum_{1\leqslant i\leqslant n} Bf_i$.

This proves that $f_1,\ldots,f_n$ generate N. Since B is a commutative ring, this implies that rank $N\leqslant n$. Since α^{-1} is also a projectivity, it follows that rank $M=$rank N and $f_1,\ldots,f_n$ is a basis for N.

Let $\alpha A e_1 = Bf_1$ and $\alpha A e_2 = Bg_2$. Now $e_1 + e_2$ is unimodular and $A(e_1+e_2) \subset$ $\subset Ae_1 + Ae_2$ which implies that $\alpha A(e_1+e_2) \subset Bf_1 + Bg_2$. Hence $\alpha A(e_1+e_2) = B(b_1 f_1 +$ $+ b_2 g_2)$. Since $Ae_2 \subset Ae_1 + A(e_1+e_2)$ we have $Bg_2 \subset Bf_1 + B(b_1 f_1 + b_2 g_2)$. Thus $g_2 = bf_1 + c(b_1 f_1 + b_2 g_2)$. Since f_1, g_2 are independent, it follows that $cb_2 = 1$, i.e. b_2 is a unit in B. Similarly b_1 is also a unit. Writing $f_2 = b_1^{-1} b_2 g_2$, we see that f_2 is unimodular, $Bf_2 = Bg_2$ and $\alpha A(e_1+e_2) = B(f_1+f_2)$. Doing this for any $i > 1$, we get a basis $f_1, f_2, \ldots, f_n$ of N such that

$$\begin{aligned} \alpha A e_i &= B f_i & 1 \leqslant i \leqslant n \\ \alpha A(e_1 + e_i) &= B(f_1 + f_i) & 2 \leqslant i \leqslant n . \end{aligned} \tag{1}$$

It is clear as before that for any $a \in A$ $\alpha A(e_1 + ae_2) = B(b_1 f_1 + b_2 f_2)$ with b_1 a unit of B. Thus we can write

$$\alpha A(e_1 + a e_2) = B(f_1 + \sigma(a) f_2), \tag{2}$$

where $\sigma: A \to B$ is a well defined map. Clearly

$$\sigma(0) = 0 \quad \text{and} \quad \sigma(1) = 1 . \tag{3}$$

For any fixed $i > 2$, we can similarly define $\tau: A \to B$ by

$$\alpha A(e_1 + a e_i) = B(f_1 + \tau(a) f_i) \tag{4}$$

and we have

$$\tau(0) = 0 \quad \text{and} \quad \tau(1) = 1 . \tag{5}$$

Since $e_1 + ae_2 + a' e_i \in A(e_1 + ae_2) + Ae_i$, we have $\alpha A(e_1 + ae_2 + a' e_i) \subset B(f_1 + \sigma(a) f_2)$ $+ Bf_i$. Hence $\alpha A(e_1 + ae_2 + a' e_i) = B(b(f_1 + \sigma(a) f_2) + b' f_i)$. Similarly, $\alpha A(e_1 + ae_2 +$ $+ a' e_i) = B(c(f_1 + \tau(a') f_i) + c' f_2)$.
Combining the above equations, we find that

$$\alpha A(e_1 + a e_2 + a' e_i) = B(f_1 + \sigma(a) f_2 + \tau(a') f_i) . \tag{6}$$

Since $ae_2 + e_i \in A(e_1 + ae_2 + e_i) + Ae_1$, using (6) and (5) we have $\alpha A(ae_2 + e_i) =$ $= B(b(f_1 + \sigma(a) f_2 + f_i) + cf_1)$. Since $\alpha A(ae_2 + e_i) \subset Bf_2 + Bf_i$, we get $b + c = 0$ and this proves

$$A(a e_2 + e_i) = B(\sigma(a) f_2 + f_i) . \tag{7}$$

Now using (6) and (5), we have for $a, a' \in A$, $\alpha A(e_1 + (a+a') e_2 + e_i) = B(f_1 + \sigma(a +$ $+ a') f_2 + f_i)$. But $\alpha A(e_1 + (a+a') e_2 + e_i) \subset \alpha A(e_1 + ae_2) + \alpha A(a' e_2 + e_i)$. Using (7), we therefore have $\alpha A(e_1 + (a+a') e_2 + e_i) \subset B(f_1 + \sigma(a) f_2) + B(\sigma(a') f_2 + f_i)$. Using the above, we see that for $a, a' \in A$, we have

$$\sigma(a + a') = \sigma(a) + \sigma(a') . \tag{8}$$

Now for $a, a' \in A$, we have, using (6), that $\alpha A(e_1 + aa' e_2 + ae_i) = B(f_1 + \sigma(aa') f_2 +$

$+\tau(a)f_i)$. On the other hand, $\alpha A(e_1+aa'e_2+ae_i)\subset\alpha Ae_1+\alpha A(a'e_2+e_i)$ which implies that $\alpha A(e_1+aa'e_2+ae_i)=B(bf_1+b'(\sigma(a')f_2+f_i))$. Comparing coefficients, we find that $\sigma(aa')=\tau(a)\sigma(a')$. Setting $a'=1$, we get

$$\sigma(a)=\tau(a) \quad \text{for all} \quad a\in A \tag{9}$$

and

$$\sigma(a\,a')=\sigma(a)\,\sigma(a') \quad \text{for } a, a'\in A. \tag{10}$$

Thus, the map $\sigma: A\to B$ defined by (2) is a homomorphism. Replacing α by α^{-1}, we can define a homomorphism $\sigma': B\to A$ satisfying

$$\alpha^{-1}B(f_1+b\,f_2)=A(e_1+\sigma'(b)\,e_2)$$

and clearly σ and σ' are inverses of each other. Thus $\sigma: A\to B$ is an isomorphism.

We now show that, for $a_2,\dots,a_n\in A$, we have

$$\alpha A(e_1+a_2e_2+\dots+a_ne_n)=B(f_1+\sigma(a_2)f_2+\dots+\sigma(a_n)f_n). \tag{11}$$

We can assume by induction that

$$\alpha A(e_1+a_2e_2+\dots+a_{n-1}e_{n-1})=B(f_1+\sigma(a_2)f_2+\dots+\sigma(a_{n-1})f_{n-1}).$$

Since

$$\alpha A(e_1+a_2e_2+\dots+a_ne_n)\subset\alpha A(e_1+a_2e_2+\dots+a_{n-1}e_{n-1})+\alpha Ae_n,$$

we have

$$\alpha A(e_1+a_2e_2+\dots+a_ne_n)=B(b(f_1+\sigma(a_2)f_2+\dots+\sigma(a_{n-1})f_{n-1})+b'f_n).$$

On the other hand, we also have

$$\alpha A(e_1+a_2e_2+\dots+a_ne_n)\subset\alpha A(e_1+a_ne_n)+\alpha Ae_2+\dots+\alpha Ae_{n-1}.$$

Comparing coefficients we find that $b'=b\sigma(a_n)$ and this proves (11).

If $a_2,\dots,a_n\in A$ are such that $a_2e_2+\dots+a_ne_n\in M$ is unimodular, we have

$$\alpha A(a_2e_2+\dots+a_ne_n)\subset A(e_1+a_2e_2+\dots+a_ne_n)+\alpha Ae_1.$$

Using (11) we have

$$\alpha A(a_2e_2+\dots+a_ne_n)=B(b(f_1+\sigma(a_2)f_2+\dots+\sigma(a_n)f_n)+b'f_1).$$

We also have

$$\alpha A(a_2e_2+\dots+a_ne_n)\subset Bf_2+\dots+Bf_n.$$

Combining these two facts, we get

$$\alpha A(a_2e_2+\dots+a_ne_n)=B(\sigma(a_2)f_2+\dots+\sigma(a_n)f_n). \tag{12}$$

We now assert that for any $a_1,\dots,a_{i-1},a_{i+1},\dots,a_n\in A$ and $i=2,\dots,n$,

$$\alpha A(e_i+a_1e_1+\dots+a_{i-1}e_{i-1}+a_{i+1}e_{i+1}+\dots+a_ne_n)$$
$$=B(f_i+\sigma(a_1)f_1+\dots+\sigma(a_n)f_n). \tag{13}$$

To prove (13), we first observe, using (1) and (12) that $\alpha A(e_i+e_j)=B(f_i+f_j)$ for any $j\neq i$. Fixing an i and replacing e_1 by e_i, we can repeat the previous arguments to get an isomorphism $\varrho: A\to B$ such that for $a_1,\ldots, a_{i-1}, a_{i+1},\ldots, a_n\in A$, we have the following equation:

$$\alpha A(e_i + a_1 e_1 + \cdots + a_{i-1} e_{i-1} + a_{i+1} e_{i+1} + \cdots + a_n e_n) = B(f_i + \varrho(a_1) f_1 + \cdots + \varrho(a_n) f_n). \quad (14)$$

instead of (11).

Taking in (14) $a_1=0$ and comparing this equation with (12), we find that $\sigma=\varrho$. Now (14) gives (13).

Let $e=\sum_{1\leqslant i\leqslant n} a_i e_i\in M$ be a unimodular element. We now show that

$$\alpha A(a_1 e_1 + \cdots + a_n e_n) = B(\sigma(a_1) f_1 + \cdots + \sigma(a_n) f_n). \quad (15)$$

Since for $i=1, 2, 3$, we have $\alpha Ae\subset\alpha Ae_i+\alpha A(e_i+\cdots+\widehat{a_i e_i}+\cdots)$ (where $\wedge$ indicates that the corresponding term is omitted), we can write $\alpha Ae=Bf$ where

$$\begin{aligned} f &= b_1 \sigma(a_1) f_1 + c_1 \sigma(a_2) f_2 + c_1 \sigma(a_3) f_3 + \cdots \\ &= c_2 \sigma(a_1) f_1 + b_2 \sigma(a_2) f_2 + c_2 \sigma(a_3) f_3 + \cdots \\ &= c_3 \sigma(a_1) f_1 + c_3 \sigma(a_2) f_2 + b_3 \sigma(a_3) f_3 + \cdots. \end{aligned}$$

Comparing coefficients, we find

$$\left.\begin{array}{l} b_1 \sigma(a_1) \sigma(a_2) = c_3 \sigma(a_1) \sigma(a_2) = c_1 \sigma(a_1) \sigma(a_2) \\ \text{and for every } i \geqslant 3, \text{ we have} \\ b_1 \sigma(a_1) \sigma(a_i) = c_2 \sigma(a_1) \sigma(a_i) = c_1 \sigma(a_1) \sigma(a_i). \end{array}\right\} \quad (16)$$

Since $e=\sum a_i e_i$ is unimodular, it follows that $\sum \sigma(a_i) f_i$ is unimodular and hence there exist $k_1,\ldots, k_n\in B$ such that $\sum \sigma(a_i)\, k_i=1$. Set

$$d = b_1 \sigma(a_1) k_1 + c_1 \sigma(a_2) k_2 + \cdots + c_1 \sigma(a_n) k_n.$$

Using the equations (16), we easily verify that $d\sigma(a_1)=b_1\sigma(a_1)$ and $d\sigma(a_i)=c_1\sigma(a_i)$ for $i\geqslant 2$. Then d is a unit and (15) is proved.

Let $\Phi: M\to N$ be the σ-semilinear isomorphism $M\to N$ defined by $\Phi(e_i)=f_i$. The equation (15) shows that $\alpha=P(\Phi)$. The proof of the second statement of the theorem is the same as in the classical case which can be found for instance in E. Artin [1, chap. II].

§ 3 A Counter-Example

If M, N are finite dimensional vector spaces of the same rank over fields A and B respectively and if $\alpha: P(M)\to P(N)$ is a bijection which is such that for any $p_1, p_2, p_3,\in P(M)$ with $p_1\subset p_2+p_3$, we have $\alpha p_1\subset\alpha p_2+\alpha p_3$, it can be proved (see for instance Artin [1, chap. II]) that α is a projectivity. We now give an example to show that this need not be the case if A and B are arbitrary rings.

Let K be a field; let $A=K\langle x\rangle$ be the ring of formal power series in x and B the quotient field of A. The canonical inclusion $\sigma: A \to B$ induces a σ-semilinear map $A^3 \to B^3$ which in turn gives rise to a map $P(\sigma): P(A^3) \to P(B^3)$.

PROPOSITION 2.*) *The map $P(\sigma)$ is a bijection such that for any $p_1, p_2, p_3 \in P(A^3)$ with $p_1 \subset p_2 + p_3$, we have $P(\sigma)\, p_1 \subset P(\sigma)\, p_2 + P(\sigma)\, p_3$. However $P(\sigma)$ is not a projectivity.*

Proof. Let (a_1, a_2, a_3), (a'_1, a'_2, a'_3) be unimodular elements of A^3 which represent the same element of $P(B^3)$. We then have $a, a' \in A$, $a \neq 0$, $a' \neq 0$ such that $a'(a'_1, a'_2, a'_3) = a(a_1, a_2, a_3)$, i.e. $a' a'_i = a a_i\, 1 \leqslant i \leqslant 3$. If $\sum_{1 \leqslant i \leqslant 3} a_i k_i = 1$, we have $a' \lambda = a$ with $\lambda = \sum a_i k_i A$. Similarly, $a \mu = a$ for some $\mu \in A$. This implies that a and a' differ by a unit of A and hence $A(a_1, a_2, a_3) = A(a'_1, a'_2, a'_3)$. This proves that $P(\sigma)$ is injective. Given any element of $P(B^3)$, we can write it in the form Be where $e \in A^3$. Dividing if necessary by a suitable power of x, we may assume that at least one coordinate of e has a nonzero constant term and hence is a unit in A. Therefore we may assume that e is a unimodular element of A^3 and this proves that $P(\sigma)$ is surjective. If $p_1, p_2, p_3 \in P(A^3)$ are such that $p_1 \subset p_2 + p_3$, it is trivial to check that $P(\sigma)\, p_1 \subset P(\sigma)\, p_2 + P(\sigma)\, p_3$. Now, $P(\sigma)\ A(1, 0, 0) = B(1, 0, 0) = B(x, 0, 0) \subset P(\sigma)\ A(x, 1, 0) + P(\sigma)\ A(0, 1, 0)$. However, $(1, 0, 0) \notin A(x, 1, 0) + A(0, 1, 0)$. This shows that $P(\sigma)$ is not a projectivity.

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*) (Added in proof.) This proposition and its proof are valid equally for any unique factorisation domain.