Computation of Topological Entropy via φ-expansion, an Inverse Problem for the Dynamical Systems $\beta x + \alpha \bmod 1$

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Abstract. We give an algorithm, based on the φ-expansion of Parry, in order to compute the topological entropy of a class of shift spaces. The idea is to solve an inverse problem for the dynamical systems $\beta x + \alpha \bmod 1$. The first part is an exposition of the φ-expansion applied to piecewise monotone dynamical systems. We formulate for the validity of the φ-expansion, necessary and sufficient conditions, which are different from those in Parry's paper [P2].

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1. Introduction

In 1957 Rényi published his paper [R] about representations for real numbers by f-expansions, called hereafter φ-expansions, which had tremendous impact in Dynamical Systems Theory. The ideas of Rényi were further developed by Parry in [P1] and [P2]. See also the book of Schweiger [Sch]. The first part of the paper, section 2, is an exposition of the theory of φ-expansions in the setting of piecewise monotone dynamical systems. Although many of the results of section 2 are known, for example see [Bo] chapter 9 for Theorem 2.5, we state necessary and sufficient conditions for the validity of the φ-expansion, which are different from those in Parry's paper [P2], Theorem 2.2 and Theorem 2.3.

We then use φ-expansions to study two interesting and related problems in sections 3 and 4. When one applies the method of section 2 to the dynamical system $\beta x + \alpha \bmod 1$, one obtains a symbolic shift which is entirely described by two strings $\underline{u}^{\alpha,\beta}$ and $\underline{v}^{\alpha,\beta}$ of symbols in a finite alphabet $\mathtt{A} = \{0, \dots, k-1\}$. The shift space is given by

$$\mathbf{\Sigma}(\underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta}) = \{\underline{x} \in \mathtt{A}^{\mathbf{Z}_+} : \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{x} \preceq \underline{v}^{\alpha,\beta} \;\; \forall n \geq 0\}\,, \tag{1.1}$$

where $\preceq$ is the lexicographic order and σ the shift map. The particular case $\alpha = 0$ has been much studied from many different viewpoints (β-shifts). For $\alpha \neq 0$ the structure of the shift space is richer. A natural problem is to study all shift spaces $\Sigma(\underline{u}, \underline{v})$ of the form (1.1) when we replace $\underline{u}^{\alpha,\beta}$ and $\underline{v}^{\alpha,\beta}$ by a pair of strings $\underline{u}$ and $\underline{v}$. In section 3 we give an algorithm, Theorem 3.1, based on the φ-expansion, which allows to compute the topological entropy of shift spaces $\Sigma(\underline{u}, \underline{v})$. One of the essential tool is the follower-set graph associated to the shift space. This graph is presented in details in subsection 3.1. The algorithm is given in subsection 3.2 and the computations of the topological entropy in subsection 3.3. The basic idea of the algorithm is to compute two real numbers $\bar{\alpha}$ and $\bar{\beta}$, given the strings $\underline{u}$ and $\underline{v}$, and to show that the shift space $\mathbf{\Sigma}(\underline{u}, \underline{v})$ is a modification of the shift space $\Sigma(\underline{u}^{\bar{\alpha},\bar{\beta}}, \underline{v}^{\bar{\alpha},\bar{\beta}})$ obtained from the dynamical system $\bar{\beta}x + \bar{\alpha} \bmod 1$, and that the topological entropies of the two shift spaces are the same. In the last section we consider the following inverse problem for the dynamical systems $\beta x + \alpha \bmod 1$: given $\underline{u}$ and $\underline{v}$, find α and β so that

$$\underline{u} = \underline{u}^{\alpha,\beta} \quad \text{and} \quad \underline{v} = \underline{v}^{\alpha,\beta}\,.$$

The solution of this problem is given in Theorems 4.1 and 4.2 for all $\beta > 1$.

2. φ-expansion for piecewise monotone dynamical systems

2.1. Piecewise monotone dynamical systems

Let $X := [0, 1]$ (with the euclidean distance). We consider the case of piecewise monotone dynamical systems of the following type. Let $0 = a_0 < a_1 < \cdots < a_k = 1$ and

$I_j := (a_j, a_{j+1})$, $j \in \mathtt{A}$. We set $\mathtt{A} := \{0, \ldots, k-1\}$, $k \geq 2$, and

$$S_0 := X \backslash \bigcup_{j \in \mathtt{A}} I_j \,.$$

For each $j \in \mathtt{A}$ let

$$f_j : I_j \mapsto J_j := f_j(I_j) \subset [0,1]$$

be a strictly monotone continuous map. When necessary we also denote by f_j the continuous extension of the map on the closure $\overline{I}_j$ of I_j. We define a map T on $X \backslash S_0$ by setting

$$T(x) := f_j(x) \quad \text{if } x \in I_j \,.$$

The map T is left undefined on S_0. We also assume that

$$(\bigcup_{i \in \mathtt{A}} J_i) \cap I_j = I_j \quad \forall j \,. \tag{2.1}$$

We introduce sets X_j, S_j, and S by setting $X_0 := [0,1]$ and for $j \geq 1$

$$X_j := X_{j-1} \backslash S_{j-1} \,, \quad S_j := \{x \in X_j : T(x) \in S_{j-1}\} \,, \quad S := \bigcup_{j \geq 0} S_j \,.$$

Lemma 2.1 *Under the condition (2.1),* $T^n(X_{n+1}) = X_1$ *and* $T(X \backslash S) = X \backslash S$*.* $X \backslash S$ *is dense in* X*.*

Proof: Condition (2.1) is equivalent to $T(X_1) \supset X_1$. Since $X_2 = X_1 \backslash S_1$ and $S_1 = \{x \in X_1 \colon T(x) \notin X_1\}$, we have $T(X_2) = X_1$. Suppose that $T^n(X_{n+1}) = X_1$; we prove that $T^{n+1}(X_{n+2}) = X_1$. One has $X_{n+1} = X_{n+2} \cup S_{n+1}$ and

$$X_1 = T^n(X_{n+1}) = T^n(X_{n+2}) \cup T^n(S_{n+1}) \,.$$

Applying once more T,

$$X_1 \subset T(X_1) = T^{n+1}(X_{n+2}) \cup T^{n+1}(S_{n+1}) \,.$$

T^{n+1} is defined on X_{n+1} and $S_{n+1} \subset X_{n+1}$.

$$T^{n+1} S_{n+1} = \{x \in X_{n+1} : T^{n+1}(x) \in S_0\} = \{x \in X_{n+1} : T^{n+1}(x) \notin X_1\} \,.$$

Hence $T^{n+1}(X_{n+2}) = X_1$. Clearly $T(X \backslash S) \subset X \backslash S$ and $T(S \backslash S_0) \subset S$. Since X_1 is the disjoint union of $X \backslash S$ and $S \backslash S_0$, and $TX_1 \supset X_1$, we have $T(X \backslash S) = X \backslash S$. The sets $X \backslash S_k$ are open and dense in X. By Baire's Theorem $X \backslash S = \bigcap_k (X \backslash S_k)$ is dense. □

Let $\mathbf{Z}_+ := \{0, 1, 2, \ldots\}$ and $\mathtt{A}^{\mathbf{Z}_+}$ be equipped with the product topology. Elements of $\mathtt{A}^{\mathbf{Z}_+}$ are called strings and denoted by $\underline{x} = (x_0, x_1, \ldots)$. A finite string $\underline{w} = (w_0, \cdots, w_{n-1})$, $w_j \in \mathtt{A}$, is a word; we also use the notation $\underline{w} = w_0 \cdots w_{n-1}$. The length of $\underline{w}$ is $|\underline{w}| = n$. A n-word is a word of length n. There is a single word of length 0, the empty-word ϵ. The set of all words is $\mathtt{A}^*$. The shift-map $\sigma \colon \mathtt{A}^{\mathbf{Z}_+} \to \mathtt{A}^{\mathbf{Z}_+}$ is defined by

$$\sigma(\underline{x}) := (x_1, x_2, \ldots) \,.$$

We define two operations $\mathtt{p}$ and $\mathtt{s}$ on $\mathtt{A}^*\backslash\{\epsilon\}$,

$$\mathtt{p}\underline{w} := \begin{cases} w_0\cdots w_{n-2} & \text{if } \underline{w}=w_0\cdots w_{n-1} \text{ and } n\geq 2 \\ \epsilon & \text{if } \underline{w}=w_0 \end{cases} \tag{2.2}$$

$$\mathtt{s}\underline{w} := \begin{cases} w_1\cdots w_{n-1} & \text{if } \underline{w}=w_0\cdots w_{n-1} \text{ and } n\geq 2 \\ \epsilon & \text{if } \underline{w}=w_0. \end{cases} \tag{2.3}$$

On $\mathtt{A}^{\mathbf{Z}_+}$ we define a total order, denoted by $\prec$. We set

$$\delta(j) := \begin{cases} +1 & \text{if } f_j \text{ is increasing} \\ -1 & \text{if } f_j \text{ is decreasing,} \end{cases}$$

and for a word $\underline{w}$,

$$\delta(\underline{w}) := \begin{cases} \delta(w_0)\cdots\delta(w_{n-1}) & \text{if } \underline{w}=w_0\cdots w_{n-1} \\ 1 & \text{if } \underline{w}=\epsilon. \end{cases}$$

Let $\underline{x}'\neq\underline{x}''$ belong to $\mathtt{A}^{\mathbf{Z}_+}$; define j as the smallest integer with $x'_j\neq x''_j$. By definition

$$\underline{x}'\prec\underline{x}'' \iff \begin{cases} x'_j<x''_j & \text{if } \delta(x'_0\cdots x'_{j-1})=1 \\ x'_j>x''_j & \text{if } \delta(x'_0\cdots x'_{j-1})=-1 \end{cases}.$$

As usual $\underline{x}'\preceq\underline{x}''$ if and only if $\underline{x}'\prec\underline{x}''$ or $\underline{x}'=\underline{x}''$. When all maps f_j are increasing this order is the lexicographic order.

2.2. φ-expansion

We give an alternative description of a piecewise monotone dynamical system as in Parry's paper [P2]. In this description, when all maps f_j are increasing, one could use instead of the intervals I_j the intervals $I'_j := [a_j, a_{j+1})$, $j\in\mathtt{A}$. In that case $S_0=\{a_k\}$ and $S_j=\emptyset$ for all $j\geq 1$. This would correspond to the setting of Parry's paper [P2].

We define a map φ on the disjoint union

$$\mathrm{dom}\varphi := \bigcup_{j=0}^{k-1} j+J_j\subset\mathbf{R},$$

by setting

$$\varphi(x) := f_j^{-1}(t) \quad \text{if } x=j+t \text{ and } t\in J_j\,. \tag{2.4}$$

The map φ is continuous, injective with range X_1. On X_1 the inverse map is

$$\varphi^{-1}(x)=j+Tx \quad \text{if } x\in I_j\,.$$

For each j, such that f_j is increasing, we define $\overline{\varphi}^j$ on $j+[0,1]$ (using the extension of f_j to $[a_j,a_{j+1}]$) by

$$\overline{\varphi}^j(x) := \begin{cases} a_j & \text{if } x=j+t \text{ and } t\leq f_j(a_j) \\ f_j^{-1}(t) & \text{if } x=j+t \text{ and } t\in J_j \\ a_{j+1} & \text{if } x=j+t \text{ and } f_j(a_{j+1})\leq t. \end{cases} \tag{2.5}$$

For each j, such that f_j is decreasing, we define $\overline{\varphi}^j$ on $j+[0,1]$ by

$$\overline{\varphi}^j(x) := \begin{cases} a_{j+1} & \text{if } x = j+t \text{ and } t \leq f_j(a_{j+1}) \\ f_j^{-1}(t) & \text{if } x = j+t \text{ and } t \in J_j \\ a_j & \text{if } x = j+t \text{ and } f_j(a_j) \leq t. \end{cases} \tag{2.6}$$

It is convenient below to consider the family of maps $\overline{\varphi}^j$ as a single map defined on $[0,k]$, which is denoted by $\overline{\varphi}$. In order to avoid ambiguities at integers, where the map may be multi-valued, we always write a point of $[j,j+1]$ as $x = j+t$, $t \in [0,1]$, so that

$$\overline{\varphi}(j+t) \equiv \overline{\varphi}(x) := \overline{\varphi}^j(t)\,.$$

We define the coding map $\mathtt{i} : X \backslash S \to \mathtt{A}^{\mathbf{Z}_+}$ by

$$\mathtt{i}(x) := (\mathtt{i}_0(x), \mathtt{i}_1(x), \ldots) \quad \text{with } \mathtt{i}_n(x) := j \text{ if } T^n x \in I_j\,.$$

The φ-code of $x \in X\backslash S$ is the string $\mathtt{i}(x)$, and we set

$$\Sigma = \{\underline{x} \in \mathtt{A}^{\mathbf{Z}_+} : \underline{x} = \mathtt{i}(x) \text{ for some } x \in X\backslash S\}\,.$$

For $x \in X\backslash S$ and any $n \geq 0$,

$$\varphi^{-1}(T^n x) = \mathtt{i}_n(x) + T^{n+1}x \quad \text{and} \quad \mathtt{i}(T^n x) = \sigma^n \mathtt{i}(x)\,. \tag{2.7}$$

Let $z_j \in \mathtt{A}$, $1 \leq j \leq n$, and $t \in [0,1]$; we set

$$\overline{\varphi}_1(z_1 + t) := \overline{\varphi}(z_1 + t)$$

and

$$\overline{\varphi}_n(z_1, \ldots, z_n + t) := \overline{\varphi}_{n-1}(z_1, \ldots, z_{n-1} + \overline{\varphi}(z_n + t))\,. \tag{2.8}$$

For $n \geq 1$ and $m \geq 1$ we have

$$\overline{\varphi}_{n+m}(z_1, \ldots, z_{n+m} + t) = \overline{\varphi}_n(z_1, \ldots, z_n + \overline{\varphi}_m(z_{n+1}, \ldots, z_{n+m} + t))\,. \tag{2.9}$$

The map $t \mapsto \overline{\varphi}_n(x_0, \ldots, x_{n-1} + t)$ is increasing if $\delta(x_0 \cdots x_{n-1}) = 1$ and decreasing if $\delta(x_0 \cdots x_{n-1}) = -1$. We also write $\overline{\varphi}_n(\underline{x})$ for $\overline{\varphi}_n(x_0, \ldots, x_{n-1})$.

Definition 2.1 *The real number s has a φ-*expansion *$\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$ if the following limit exists,*

$$s = \lim_{n\to\infty} \overline{\varphi}_n(\underline{x}) \equiv \overline{\varphi}(x_0 + \overline{\varphi}(x_1 + \ldots)) \equiv \overline{\varphi}_\infty(\underline{x})\,.$$

*The φ-*expansion is well-defined *if for all $\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$, $\lim_{n\to\infty} \overline{\varphi}_n(\underline{x}) = \overline{\varphi}_\infty(\underline{x})$ exists.*
*The φ-*expansion is valid *if for all $x \in X\backslash S$ the φ-code $\mathtt{i}(x)$ of x is a φ-*expansion *of x.*

If the φ-expansion is valid, then for $x \in X\backslash S$, using (2.9), (2.7) and the continuity of the maps $\overline{\varphi}^j$,

$$\begin{aligned} x &= \lim_{n\to\infty} \overline{\varphi}_n(\mathtt{i}_0(x), \ldots, \mathtt{i}_{n-1}(x)) \\ &= \lim_{m\to\infty} \overline{\varphi}_n(\mathtt{i}_0(x), \ldots, \mathtt{i}_{n-1}(x) + \overline{\varphi}_m(\mathtt{i}_n(x), \ldots, \mathtt{i}_{n+m-1}(x))) \\ &= \overline{\varphi}_n(\mathtt{i}_0(x), \ldots, \mathtt{i}_{n-1}(x) + \overline{\varphi}_\infty(\mathtt{i}(T^n x))\,. \end{aligned} \tag{2.10}$$

The basic and elementary fact of the φ-expansion is

$$a, b \in [0,1] \text{ and } x_0 < x_0' \implies \overline{\varphi}(x_0 + a) \leq \overline{\varphi}(x_0' + b)\,. \tag{2.11}$$

We begin with two lemmas on the φ-code (for Lemma 2.2 see e.g. [CoE]).

Lemma 2.2 *The φ-code $\mathtt{i}$ is $\preceq$-order-preserving on $X\backslash S$: $x \leq y$ implies $\mathtt{i}(x) \preceq \mathtt{i}(y)$.*

Proof: Let $x < y$. Either $\mathtt{i}_0(x) < \mathtt{i}_0(y)$, or $\mathtt{i}_0(x) = \mathtt{i}_0(y)$; in the latter case, the strict monotonicity of $f_{\mathtt{i}_0(x)}$ implies

$$\begin{aligned}\varphi^{-1}(x) = \mathtt{i}_0(x) + T(x) < \varphi^{-1}(y) = \mathtt{i}_0(x) + T(y) \quad \text{if } \delta(\mathtt{i}_0(x)) = +1\\ \varphi^{-1}(x) = \mathtt{i}_0(x) + T(x) > \varphi^{-1}(y) = \mathtt{i}_0(x) + T(y) \quad \text{if } \delta(\mathtt{i}_0(x)) = -1\,.\end{aligned}$$

Repeating this argument we get $\mathtt{i}(x) \preceq \mathtt{i}(y)$. □

Lemma 2.3 *The φ-code $\mathtt{i}$ is continuous‡ on $X\backslash S$.*

Proof: Let $x \in X\backslash S$ and $\{x^n\} \subset X\backslash S$, $\lim_n x^n = x$. Let $x \in I_{j_0}$. For n large enough $x^n \in I_{j_0}$ and $\mathtt{i}_0(x^n) = \mathtt{i}_0(x) = j_0$. Let $j_1 := \mathtt{i}_1(x)$; we can choose n_1 so large that for $n \geq n_1$ $Tx_n \in I_{j_1}$. Hence $\mathtt{i}_0(x^n) = j_0$ and $\mathtt{i}_1(x^n) = j_1$ for all $n \geq n_1$. By induction we can find an increasing sequence $\{n_m\}$ such that $n \geq n_m$ implies $\mathtt{i}_j(x) = \mathtt{i}_j(x^n)$ for all $j = 0, \ldots, m$. □

The next lemmas give the essential properties of the map $\overline{\varphi}_\infty$.

Lemma 2.4 *Let $\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$. Then there exist $y_\uparrow(\underline{x})$ and $y_\downarrow(\underline{x})$ in $[0,1]$, such that $y_\uparrow(\underline{x}) \leq y_\downarrow(\underline{x})$; $y_\uparrow(\underline{x})$ and $y_\downarrow(\underline{x})$ are the only possible cluster points of the sequence $\{\overline{\varphi}_n(\underline{x})\}_n$.*
Let $x \in X\backslash S$ and set $\underline{x} := \mathtt{i}(x)$. Then

$$a_j \leq y_\uparrow(\underline{x}) \leq x \leq y_\downarrow(\underline{x}) \leq a_{j+1} \quad \text{if } x_0 = j\,.$$

If the φ-expansion is valid, then each $y \in X\backslash S$ has a unique φ-expansion§,

$$y = \overline{\varphi}_\infty(\underline{x}) \in X\backslash S \iff \underline{x} = \mathtt{i}(y)\,.$$

Proof: Consider the map

$$t \mapsto \overline{\varphi}_n(x_0, \ldots, x_{n-1} + t)\,.$$

Suppose that $\delta(x_0 \cdots x_{n-1}) = -1$. Then it is decreasing, and for any m

$$\begin{aligned}\overline{\varphi}_{n+m}(x_0, \ldots, x_{n+m-1}) &= \overline{\varphi}_n(x_0, \ldots, x_{n-1} + \overline{\varphi}_m(x_n, \ldots, x_{n+m-1}))\\ &\leq \overline{\varphi}_n(x_0, \ldots, x_{n-1})\,.\end{aligned}$$

In particular the subsequence $\{\overline{\varphi}_n(\underline{x})\}_n$ of all n such that $\delta(x_0 \cdots x_{n-1}) = -1$ is decreasing with limit‖ $y_\downarrow(\underline{x})$. When there is no n such that $\delta(x_0 \cdots x_{n-1}) = -1$, we set $y_\downarrow(\underline{x}) := a_{x_0+1}$. Similarly, the subsequence $\{\overline{\varphi}_n(\underline{x})\}_n$ of all n such that $\delta(x_0 \cdots x_{n-1}) = 1$ is increasing with limit $y_\uparrow(\underline{x}) \leq y_\downarrow(\underline{x})$. When there is no n such that $\delta(x_0 \cdots x_{n-1}) = 1$, we set $y_\uparrow(\underline{x}) := a_{x_0}$. Since any $\overline{\varphi}_n(\underline{x})$ appears in one of these sequences, there are at most two cluster points for $\{\overline{\varphi}_n(\underline{x})\}_n$.

‡ If we use the intervals $I'_j = [a_j, a_{j+1})$, then we have only right-continuity
§ If we use the intervals $I'_j = [a_j, a_{j+1})$, this statement is not correct.
‖ If the subsequence is finite, then $y_\downarrow(\underline{x})$ is the last point of the subsequence.

Let $x \in X\backslash S$; $x = \varphi(\varphi^{-1}(x))$ and by (2.7)

$$\begin{aligned} x &= \varphi(\mathtt{i}_0(x) + Tx) = \varphi(\mathtt{i}_0(x) + \varphi(\varphi^{-1}(Tx))) \\ &= \varphi(\mathtt{i}_0(x) + \varphi(\mathtt{i}_1(x) + T^2x)) = \cdots \\ &= \varphi(\mathtt{i}_0(x) + \varphi(\mathtt{i}_1(x) + \ldots + \varphi(\mathtt{i}_{n-1}(x) + T^n x)))\,. \end{aligned} \tag{2.12}$$

By monotonicity

$$\begin{aligned} &(x \in X\backslash S \text{ and } \delta(\mathtt{i}_0(x)\cdots\mathtt{i}_{n-1}(x)) = -1) \implies \\ &\overline{\varphi}_n(\mathtt{i}_0(x),\cdots,\mathtt{i}_{n-1}(x)) \geq x\,, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &(x \in X\backslash S \text{ and } \delta(\mathtt{i}_0(x)\cdots\mathtt{i}_{n-1}(x)) = 1) \implies \\ &\overline{\varphi}_n(\mathtt{i}_0(x),\cdots,\mathtt{i}_{n-1}(x)) \leq x\,. \end{aligned} \tag{2.14}$$

The inequalities of Lemma 2.4 follow from (2.13), (2.14) and $\overline{\varphi}(\mathtt{i}_0(x)+t) \in [a_{x_0}, a_{x_0+1}]$.

Suppose that the φ-expansion is valid and that $\overline{\varphi}_\infty(\underline{x}) = y \in X\backslash S$. We prove that $\underline{x} = \mathtt{i}(y)$. By hypothesis $y \in I_{x_0}$; using (2.10) and the fact that I_{x_0} is open, we can write

$$y = \overline{\varphi}(x_0 + \overline{\varphi}(x_1 + \overline{\varphi}(x_2 + \ldots))) = \varphi(x_0 + \overline{\varphi}(x_1 + \overline{\varphi}(x_2 + \ldots)))\,.$$

This implies that

$$\varphi^{-1}(y) = \mathtt{i}_0(y) + Ty = x_0 + \overline{\varphi}(x_1 + \overline{\varphi}(x_2 + \ldots))\,.$$

Since $Ty \in X\backslash S$, we can iterate this argument. □

Lemma 2.5 *Let $\underline{x}, \underline{x}' \in \mathtt{A}^{\mathbf{Z}_+}$ and $\underline{x} \preceq \underline{x}'$. Then any cluster point of $\{\overline{\varphi}_n(\underline{x})\}_n$ is smaller then any cluster point of $\{\overline{\varphi}_n(\underline{x}')\}_n$. In particular, if $\overline{\varphi}_\infty$ is well-defined on $\mathtt{A}^{\mathbf{Z}_+}$, then $\overline{\varphi}_\infty$ is order-preserving.*

Proof: Let $\underline{x} \prec \underline{x}'$ with $x_k = x'_k$, $k = 0,\ldots,m-1$ and $x_m \neq x'_m$. We have

$$\overline{\varphi}_{m+n}(\underline{x}) = \overline{\varphi}_m(x_0,\ldots,x_{m-1} + \overline{\varphi}_n(\sigma^m\underline{x}))\,.$$

By (2.11), if $\delta(x_0\cdots x_{m-1}) = 1$, then $x_m < x'_m$ and for any $n \geq 1$, $\ell \geq 1$,

$$\overline{\varphi}_n(\sigma^m\underline{x}) = \overline{\varphi}_1(x_m + \overline{\varphi}_{n-1}(\sigma^{m+1}\underline{x})) \leq \overline{\varphi}_\ell(\sigma^m\underline{x}') = \overline{\varphi}_1(x'_m + \overline{\varphi}_{\ell-1}(\sigma^{m+1}\underline{x}'))\,;$$

if $\delta(x_0\cdots x_{m-1}) = -1$, then $x_m > x'_m$ and

$$\overline{\varphi}_n(\sigma^m\underline{x}) = \overline{\varphi}_1(x_m + \overline{\varphi}_{n-1}(\sigma^{m+1}\underline{x})) \geq \overline{\varphi}_\ell(\sigma^m\underline{x}') = \overline{\varphi}_1(x'_m + \overline{\varphi}_{\ell-1}(\sigma^{m+1}\underline{x}'))\,.$$

Therefore, in both cases, for any $n \geq 1$, $\ell \geq 1$,

$$\overline{\varphi}_{m+n}(\underline{x}) \leq \overline{\varphi}_{m+\ell}(\underline{x}')\,.$$

□

Lemma 2.6 *Let $\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$ and $x_0 = j$.*
1) Let $\delta(j) = 1$ and $y_\uparrow(\underline{x}) \in \overline{I}_j$ be a cluster point of $\{\overline{\varphi}_n(\underline{x})\}$. Then $f_j(y_\uparrow(\underline{x})) \geq y_\uparrow(\sigma\underline{x})$ if $y_\uparrow(\underline{x}) = a_j$, $f_j(y_\uparrow(\underline{x})) \leq y_\uparrow(\sigma\underline{x})$ if $y_\uparrow(\underline{x}) = a_{j+1}$ and $f_j(y_\uparrow(\underline{x})) = y_\uparrow(\sigma\underline{x})$ otherwise. The same conclusions hold when $y_\downarrow(\underline{x})$ is a cluster point of $\{\overline{\varphi}_n(\underline{x})\}$.
2) Let $\delta(j) = -1$ and $y_\uparrow(\underline{x}) \in \overline{I}_j$ be a cluster point of $\{\overline{\varphi}_n(\underline{x})\}$. Then $f_j(y_\uparrow(\underline{x})) \leq y_\downarrow(\sigma\underline{x})$ if $y_\uparrow(\underline{x}) = a_j$, $f_j(y_\uparrow(\underline{x})) \geq y_\downarrow(\sigma\underline{x})$ if $y_\uparrow(\underline{x}) = a_{j+1}$ and $f_j(y_\uparrow(\underline{x})) = y_\downarrow(\sigma\underline{x})$ otherwise. The same conclusions hold when $y_\downarrow(\underline{x})$ is a cluster point of $\{\overline{\varphi}_n(\underline{x})\}$.

Proof: Set $f_j(\overline{I}_j) := [\alpha_j, \beta_j]$. Suppose for example that $\delta(j) = -1$ and that n_k is the subsequence of all m such that $\delta(x_0, \ldots, x_m) = 1$. Since $\delta(j) = -1$ the sequence $\{\overline{\varphi}_{n_k - 1}(\sigma \underline{x})\}_k$ is decreasing. Hence by continuity

$$y_\uparrow(\underline{x}) = \lim_k \overline{\varphi}_{n_k}(\underline{x}) = \overline{\varphi}(j + \lim_k \overline{\varphi}_{n_k - 1}(\sigma \underline{x})) = \overline{\varphi}(j + y_\downarrow(\sigma \underline{x}))\,. \tag{2.15}$$

If $y_\uparrow(\underline{x}) = a_j$, then $f_j(a_j) = \beta_j \leq y_\downarrow(\sigma \underline{x})$; if $y_\uparrow(\underline{x}) = a_{j+1}$, then $f_j(a_{j+1}) = \alpha_j \geq y_\downarrow(\sigma \underline{x})$; if $a_j < y_\uparrow(\underline{x}) < a_{j+1}$, then

$$j + f_j(y_\uparrow(\underline{x})) = \varphi^{-1}(\varphi(j + \lim_k \overline{\varphi}_{n_k - 1}(\sigma \underline{x}))) = j + y_\downarrow(\sigma \underline{x})\,.$$

Similar proofs for the other cases. □

Lemma 2.7 *Let* $\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$.
1) If $\{\overline{\varphi}_n(\underline{x})\}$ *has two cluster points, and if* $y \in (y_\uparrow(\underline{x}), y_\downarrow(\underline{x}))$, *then* $y \in X \backslash S$, $\mathtt{i}(y) = \underline{x}$ *and* y *has no* φ*-expansion.*
Let $x \in X \backslash S$ *and set* $\underline{x} := \mathtt{i}(x)$.
2) If $\lim_n \overline{\varphi}_n(\underline{x}) = y_\uparrow(\underline{x})$ *and if* $y \in (y_\uparrow(\underline{x}), x)$, *then* $y \in X \backslash S$, $\mathtt{i}(y) = \underline{x}$ *and* y *has no* φ*-expansion.*
3) If $\lim_n \overline{\varphi}_n(\underline{x}) = y_\downarrow(\underline{x})$ *and if* $y \in (x, y_\downarrow(\underline{x}))$, *then* $y \in X \backslash S$, $\mathtt{i}(y) = \underline{x}$ *and* y *has no* φ*-expansion.*

Proof: Suppose that $y_\uparrow(\underline{x}) < y < y_\downarrow(\underline{x})$. Then $y \in I_{x_0}$ and $\mathtt{i}_0(y) = x_0$. From Lemma 2.6

$$y_\uparrow(\sigma \underline{x}) < Ty < y_\downarrow(\sigma \underline{x}) \quad \text{if } \delta(x_0) = 1\,,$$

and

$$y_\downarrow(\sigma \underline{x}) > Ty > y_\uparrow(\sigma \underline{x}) \quad \text{if } \delta(x_0) = -1\,.$$

Iterating this argument we prove that $T^n y \in I_{x_n}$ and $\mathtt{i}_n(y) = x_n$ for all $n \geq 1$. Suppose that y has a φ-expansion, $y = \overline{\varphi}_\infty(\underline{x}')$. If $\underline{x}' \prec \underline{x}$, then by Lemma 2.5 $\overline{\varphi}_\infty(\underline{x}') \leq y_\uparrow(\underline{x})$ and if $\underline{x} \prec \underline{x}'$, then by Lemma 2.5 $y_\downarrow(\underline{x}) \leq \overline{\varphi}_\infty(\underline{x}')$, which leads to a contradiction. Similar proofs in cases 2 and 3. □

Lemma 2.8 *Let* $\underline{x}' \in \mathtt{A}^{\mathbf{Z}_+}$ *and* $x \in X \backslash S$. *Then*

$$y_\downarrow(\underline{x}') < x \implies \underline{x}' \preceq \mathtt{i}(x) \quad \text{and} \quad x < y_\uparrow(\underline{x}') \implies \mathtt{i}(x) \preceq \underline{x}'.$$

Proof: Suppose that $y_\downarrow(\underline{x}') < x$ and $y_\downarrow(\underline{x}')$ is a cluster point. Either $x'_0 < \mathtt{i}_0(x)$ or $x'_0 = \mathtt{i}_0(x)$ and by Lemma 2.6

$$y_\downarrow(\sigma \underline{x}') < Tx \quad \text{if } \delta(x'_0) = 1,$$

or

$$y_\uparrow(\sigma \underline{x}') > Tx \quad \text{if } \delta(x'_0) = -1\,.$$

Since $y_\downarrow(\sigma \underline{x}')$ or $y_\uparrow(\sigma \underline{x}')$ is a cluster point we can repeat the argument and conclude that $\underline{x}' \preceq \mathtt{i}(x)$. If $y_\downarrow(\underline{x}')$ is not a cluster point, then we use the cluster point $y_\uparrow(\underline{x}') < y_\downarrow(\underline{x}')$ for the argument. □

Theorem 2.1 [P2] *A φ-expansion is valid if and only if the φ-code* $\mathtt{i}$ *is injective on* $X \backslash S$.

Proof: Suppose that the φ-expansion is valid. If $x \neq z$, then

$$x = \overline{\varphi}(\mathtt{i}_0(x) + \overline{\varphi}(\mathtt{i}_1(x) + \ldots)) \neq \overline{\varphi}(\mathtt{i}_0(z) + \overline{\varphi}(\mathtt{i}_1(z) + \ldots)) = z\,,$$

and therefore $\mathtt{i}(x) \neq \mathtt{i}(z)$. Conversely, assume that $x \neq z$ implies $\mathtt{i}(x) \neq \mathtt{i}(z)$. Let $x \in X\backslash S$, $\underline{x} = \mathtt{i}(x)$, and suppose for example that $y_\uparrow(\underline{x}) < y_\downarrow(\underline{x})$ are two cluster points. Then by Lemma 2.7 any y such that $y_\uparrow(\underline{x}) < y < y_\downarrow(\underline{x})$ is in $X\backslash S$ and $\mathtt{i}(y) = \underline{x}$, contradicting the hypothesis. Therefore $z := \lim_n \overline{\varphi}_n(\underline{x})$ exists. If $z \neq x$, then we get again a contradiction using Lemma 2.7. □

Theorem 2.1 states that the validity of the φ-expansion is equivalent to the injectivity of the map $\mathtt{i}$ defined on $X\backslash S$. One can also state that the validity of the φ-expansion is equivalent to the surjectivity of the map $\overline{\varphi}_\infty$.

Theorem 2.2 *A φ-expansion is valid if and only if $\overline{\varphi}_\infty : \mathtt{A}^{\mathbf{Z}_+} \rightarrow [0,1]$ is well-defined on $\mathtt{A}^{\mathbf{Z}_+}$ and surjective.*

Proof: Suppose that the φ-expansion is valid. Let $\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$ and suppose that $\{\overline{\varphi}_n(\underline{x})\}_n$ has two different accumulation points $y_\uparrow < y_\downarrow$. By Lemma 2.7 we get a contradiction. Thus $\overline{\varphi}_\infty(\underline{x})$ is well-defined for any $\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$.

To prove the surjectivity of $\overline{\varphi}_\infty$ it is sufficient to consider $s \in S$. The argument is a variant of the proof of Lemma 2.7. Let $\underline{x}'$ be a string such that for any $n \geq 1$

$$f_{x'_{n-1}} \circ \cdots \circ f_{x'_0}(s) \in \overline{I}_{x'_n}\,.$$

We use here the extension of f_j to $\overline{I}_j$; we have a choice for x'_n whenever $f_{x'_{n-1}} \circ \cdots \circ f_{x'_0}(s) \in S_0$. Suppose that $\overline{\varphi}_\infty(\underline{x}') < s$ and that $\overline{\varphi}_\infty(\underline{x}') < z < s$. Since $s, \overline{\varphi}_\infty(\underline{x}') \in \overline{I}_{x'_0}$, we have $z \in I_{x'_0}$ and therefore $\mathtt{i}(z) = x'_0$. Moreover,

$$\overline{\varphi}_\infty(\sigma\underline{x}') < Tz < f_{x'_0}(s) \quad \text{if } \delta(x'_0) = 1$$

or

$$f_{x'_0}(s) < Tz < \overline{\varphi}_\infty(\sigma\underline{x}') \quad \text{if } \delta(\mathrm{x}'_0) = -1\,.$$

Iterating the argument we get $z \in X\backslash S$ and $\mathtt{i}(z) = \underline{x}'$, contradicting the validity of the φ-expansion. Similarly we exclude the possibility that $\overline{\varphi}_\infty(\underline{x}') > s$, thus proving the surjectivity of the map $\overline{\varphi}_\infty$.

Suppose that $\overline{\varphi}_\infty : \mathtt{A}^{\mathbf{Z}_+} \rightarrow [0,1]$ is well-defined and surjective. Let $x \in X\backslash S$ and $\underline{x} = \mathtt{i}(x)$. Suppose that $x < \overline{\varphi}_\infty(\underline{x})$. By Lemma 2.7 any z, such that $x < z < \overline{\varphi}_\infty(\underline{x})$, does not have a φ-expansion. This contradicts the hypothesis that $\overline{\varphi}_\infty$ is surjective. Similarly we exclude the possibility that $x > \overline{\varphi}_\infty(\underline{x})$. □

Theorem 2.3 *A φ-expansion is valid if and only if $\overline{\varphi}_\infty : \mathtt{A}^{\mathbf{Z}_+} \rightarrow [0,1]$ is well-defined, continuous and there exist $\underline{x}^+$ with $\overline{\varphi}_\infty(\underline{x}^+) = 1$ and $\underline{x}^-$ with $\overline{\varphi}_\infty(\underline{x}^-) = 0$.*

Proof: Suppose that the φ-expansion is valid. By Theorem 2.2 $\overline{\varphi}_\infty$ is well-defined and surjective so that there exist $\underline{x}^+$ and $\underline{x}^-$ with $\overline{\varphi}_\infty(\underline{x}^+) = 1$ and $\overline{\varphi}_\infty(\underline{x}^-) = 0$. Suppose that $\underline{x}^n \downarrow \underline{x}$ and set $y := \overline{\varphi}_\infty(\underline{x})$, $x_n := \overline{\varphi}_\infty(\underline{x}^n)$. By Lemma 2.5 the sequence $\{x_n\}$ is monotone decreasing; let $x := \lim_n x_n$. Suppose that $y < x$ and $y < z < x$. Since $y < z < x_n$ for any $n \geq 1$ and $\lim_n \underline{x}^n = \underline{x}$, we prove, as in the beginning of the proof of Lemma 2.7, that $z \in X\backslash S$. The validity of the φ-expansion implies that $z = \overline{\varphi}_\infty(\mathtt{i}(z))$. By Lemma 2.8

$$\underline{x} \preceq \mathtt{i}(z) \preceq \underline{x}^n\,.$$

Since these inequalities are valid for any z, with $y < z < x$, the validity of φ-expansion implies that we have strict inequalities, $\underline{x} \prec \mathtt{i}(z) \prec \underline{x}^n$. This contradicts the hypothesis that $\lim_{n\to\infty} \underline{x}^n = \underline{x}$. A similar argument holds in the case $\underline{x}^n \uparrow \underline{x}$. Hence

$$\lim_{n\to\infty} \underline{x}^n = \underline{x} \implies \lim_{n\to\infty} \overline{\varphi}_\infty(\underline{x}^n) = \overline{\varphi}_\infty(\underline{x})\,.$$

Conversely, suppose that $\overline{\varphi}_\infty \colon \mathtt{A}^{\mathbf{Z}_+} \to [0,1]$ is well-defined and continuous. Then, given $\delta > 0$ and $\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$, $\exists n$ so that

$$\begin{aligned} 0 \leq \sup\{\overline{\varphi}_\infty(\underline{x}') \colon x'_j = x_j\ j = 0,\ldots,n-1\} \\ - \inf\{\overline{\varphi}_\infty(\underline{x}') \colon x'_j = x_j\ j = 0,\ldots,n-1\} \leq \delta\,. \end{aligned}$$

We set

$$\underline{x}^{n,-} := x_0 \cdots x_{n-1}\underline{x}^- \quad \text{and} \quad \underline{\mathrm{x}}^{n,+} := \mathrm{x}_0 \cdots \mathrm{x}_{n-1}\underline{\mathrm{x}}^+\,.$$

For any $x \in X\backslash S$ we have the identity (2.12),

$$\begin{aligned} x &= \varphi(\mathtt{i}_0(x) + \varphi(\mathtt{i}_1(x) + \ldots + \varphi(\mathtt{i}_{n-1} + T^n x))) \\ &= \overline{\varphi}_n(\mathtt{i}_0(x), \ldots, \mathtt{i}_{n-1}(x) + T^n x)\,. \end{aligned}$$

If $\delta(\mathtt{i}_0(x) \cdots \mathtt{i}_{n-1}(x)) = 1$, then

$$\begin{aligned} \overline{\varphi}_\infty(\underline{x}^{n,-}) :&= \overline{\varphi}_n(\mathtt{i}_0(x), \ldots, \mathtt{i}_{n-1}(x) + \overline{\varphi}_\infty(\underline{x}^-)) \\ &= \overline{\varphi}_n(\mathtt{i}_0(x), \ldots, \mathtt{i}_{n-1}(x)) \\ &\leq \overline{\varphi}_n(\mathtt{i}_0(x), \ldots, \mathtt{i}_{n-1}(x) + T^n x) \\ &\leq \overline{\varphi}_n(\mathtt{i}_0(x), \ldots, \mathtt{i}_{n-1}(x) + 1) \\ &= \overline{\varphi}_n(\mathtt{i}_0(x), \ldots, \mathtt{i}_{n-1}(x) + \overline{\varphi}_\infty(\underline{x}^+)) =: \overline{\varphi}_\infty(\underline{x}^{n,+})\,. \end{aligned}$$

If $\delta(\mathtt{i}_0(x) \cdots \mathtt{i}_{n-1}(x)) = -1$, then the inequalities are reversed. Letting n going to infinity, we get $\overline{\varphi}_\infty(\mathtt{i}(x)) = x$. □

Remark 2.1 *When the maps f_0 and f_{k-1} are increasing, then we can take*

$$\underline{x}^+ = (k-1, k-1, \ldots) \quad \text{and} \quad \underline{\mathrm{x}}^- = (0, 0, \ldots)\,.$$

Theorem 2.4 [P2] *A necessary and sufficient condition for a φ-expansion to be valid is that S is dense in $[0,1]$. A sufficient condition is $\sup_t |\varphi'(t)| < 1$.*

For each $j \in \mathtt{A}$ we define (the limits are taken with $x \in X\backslash S$)

$$\underline{u}^j := \lim_{x \downarrow a_j} \mathtt{i}(x) \quad \text{and} \quad \underline{v}^j := \lim_{x \uparrow a_{j+1}} \mathtt{i}(x)\,. \tag{2.16}$$

The strings $\underline{u}^j$ and $\underline{v}^j$ are called virtual itineraries. Notice that $\underline{v}^j \prec \underline{u}^{j+1}$ since $v_0^j < u_0^{j+1}$.

$$\sigma^k \underline{u}^j = \sigma^k(\lim_{x\downarrow a_j} \mathtt{i}(x)) = \lim_{x\downarrow a_j} \sigma^k \mathtt{i}(x) = \lim_{x\downarrow a_j} \mathtt{i}(T^k x) \qquad (x \in X\backslash S)\,. \tag{2.17}$$

Proposition 2.1 *Suppose that $\underline{x}' \in \mathtt{A}^{\mathbf{Z}_+}$ verifies $\underline{u}^{x'_n} \prec \sigma^n \underline{x}' \prec \underline{v}^{x'_n}$ for all $n \geq 0$. Then there exists $x \in X\backslash S$ such that $\mathtt{i}(x) = \underline{x}'$.*

Notice that we do not assume that the φ-expansion is valid or that the map $\overline{\varphi}_\infty$ is well-defined. For unimodal maps see e.g. Theorem II.3.8 in [CoE]. Our proof is different.

Proof: If $y_\uparrow(\underline{x}') < y_\downarrow(\underline{x}')$ are two cluster points, then this follows from Lemma 2.7. Therefore, assume that $\lim_n \overline{\varphi}_n(\underline{x}')$ exists. Either there exists $m > 1$ so that $y_\uparrow(\sigma^m \underline{x}') < y_\downarrow(\sigma^m \underline{x}')$ are two cluster points, or $\lim_n \overline{\varphi}_n(\sigma^m \underline{x}')$ exists for all $m \geq 1$.

In the first case, there exists $z_m \in X\backslash S$,

$$y_\uparrow(\sigma^m \underline{x}') < z_m < y_\downarrow(\sigma^m \underline{x}') \quad \text{and} \quad \mathtt{i}(z_m) = \sigma^m \underline{x}'\,.$$

Let

$$z_{m-1} := \overline{\varphi}(x'_{m-1} + z_m)\,.$$

We show that $a_{x'_{m-1}} < z_{m-1} < a_{x'_{m-1}+1}$. This implies that $z_m \in \mathrm{int}(\mathrm{dom}\varphi)$ so that

$$\varphi^{-1}(z_{m-1}) = x'_{m-1} + T z_{m-1} = x'_{m-1} + z_m\,.$$

Suppose that $\delta(x'_{m-1}) = 1$ and $a_{x'_{m-1}} = z_{m-1}$. Then for any $y \in X\backslash S$, $y > a_{x'_{m-1}}$, we have $Ty > z_m$. Therefore, by Lemma 2.2, $\mathtt{i}(Ty) \succeq \mathtt{i}(z_m) = \sigma^m \underline{x}'$; $\mathtt{i}_0(y) = x'_{m-1}$ when y is close to $a_{x'_{m-1}}$, so that

$$\lim_{y \downarrow a_{x'_{m-1}}} \mathtt{i}(y) = \underline{u}^{x'_{m-1}} \succeq \sigma^{m-1}\underline{x}'\,,$$

which is a contradiction. Similarly we exclude the cases $\delta(x'_{m-1}) = 1$ and $a_{x'_{m-1}+1} = z_{m-1}$, $\delta(x'_{m-1}) = -1$ and $a_{x'_{m-1}} = z_{m-1}$, $\delta(x'_{m-1}) = -1$ and $a_{x'_{m-1}+1} = z_{m-1}$. Iterating this argument we get the existence of $z_0 \in X\backslash S$ with $\mathtt{i}(z_0) = \underline{x}'$.

In the second case, $\lim_n \overline{\varphi}_n(\sigma^m \underline{x}')$ exists for all $m \geq 1$. Let $x := \lim_n \overline{\varphi}_n(\underline{x}')$. Suppose that $x'_0 = j$, so that $\underline{u}^j \prec \underline{x}' \prec \underline{v}^j$. By Lemma 2.2 and definition of $\underline{u}^j$ and $\underline{v}^j$ there exist $z_1, z_2 \in I_j$ such that

$$z_1 < x < z_2 \quad \text{and} \quad \underline{u}^j \preceq \mathtt{i}(z_1) \prec \underline{x}' \prec \mathtt{i}(z_2) \preceq \underline{v}^j\,.$$

Therefore $a_j < x < a_{j+1}$, $\mathtt{i}_0(x) = x'_0$ and $Tx = \overline{\varphi}_\infty(\sigma \underline{x}')$ (Lemma 2.6). Iterating this argument we get $\underline{x}' = \mathtt{i}(x)$. □

Theorem 2.5 *Suppose that the φ-expansion is valid. Then*

(i) $\Sigma := \{\mathtt{i}(x) \in \mathtt{A}^{\mathbf{Z}_+} : x \in X\backslash S\} = \{\underline{x} \in \mathtt{A}^{\mathbf{Z}_+} : \underline{u}^{x_n} \prec \sigma^n \underline{x} \prec \underline{v}^{x_n} \quad \forall\, n \geq 0\}$.

(ii) The map $\mathtt{i}: X\backslash S \to \Sigma$ is bijective, $\overline{\varphi}_\infty \circ \mathtt{i} = \mathrm{id}$ and $\mathtt{i} \circ \overline{\varphi}_\infty = \mathrm{id}$.
Both maps $\mathtt{i}$ and $\overline{\varphi}_\infty$ are order-preserving.

(iii) $\sigma(\Sigma)=\Sigma$ *and* $\overline{\varphi}_\infty(\sigma\underline{x})=T\overline{\varphi}_\infty(\underline{x})$ *if* $\underline{x}\in\Sigma$.

(iv) *If* $\underline{x}\in \mathtt{A}^{\mathbf{Z}_+}\backslash\Sigma$, *then there exist* $m\in\mathbf{Z}_+$ *and* $j\in\mathtt{A}$ *such that* $\overline{\varphi}_\infty(\sigma^m\underline{x})=a_j$.

(v) $\forall n\geq 0\,,\forall j\in\mathtt{A}:\quad \underline{u}^{u^j_n}\preceq\sigma^n\underline{u}^j\prec\underline{v}^{u^j_n}$ *if* $\delta(\underline{u}^j_0\cdots\underline{u}^j_{n-1})=1$ *and* $\underline{u}^{u^j_n}\prec\sigma^n\underline{u}^j\preceq\underline{v}^{v^j_n}$ *if* $\delta(\underline{u}^j_0\cdots\underline{u}^j_{n-1})=-1$.

(vi) $\forall n\geq 0\,,\forall j\in\mathtt{A}:\quad \underline{u}^{u^j_n}\preceq\sigma^n\underline{v}^j\prec\underline{v}^{u^j_n}$ *if* $\delta(\underline{v}^j_0\cdots\underline{v}^j_{n-1})=-1$ *and* $\underline{u}^{u^j_n}\prec\sigma^n\underline{v}^j\preceq\underline{v}^{v^j_n}$ *if* $\delta(\underline{u}^j_0\cdots\underline{u}^j_{n-1})=1$.

Proof: Let $x\in X\backslash S$. Clearly, by monotonicity,

$$\underline{u}^{\mathtt{i}_k(x)}\preceq\sigma^k\mathtt{i}(x)\preceq\underline{v}^{\mathtt{i}_k(x)}\quad\forall\,k\in\mathbf{Z}_+\,.$$

Suppose that there exist $x\in X\backslash S$ and k such that $\sigma^k\mathtt{i}(x)=\underline{v}^{\mathtt{i}_k(x)}$. Since $(\sigma^k\mathtt{i}(x))_0=\mathtt{i}_0(T^kx)$, we can assume, without restricting the generality, that $k=0$ and $\mathtt{i}_0(x)=j$. Therefore $x\in(a_j,a_{j+1})$, and for all $y\in X\backslash S$, such that $x\leq y<a_{j+1}$, we have by Lemma 2.2 that $\mathtt{i}(y)=\mathtt{i}(x)=\underline{v}^j$. By Theorem 2.1 this contradicts the hypothesis that the φ-expansion is valid. The other case, $\sigma^k\mathtt{i}(x)=\underline{u}^{\mathtt{i}_k(x)}$, is treated similarly. This proves half of the first statement. The second half is a consequence of Proposition 2.1. The second statement also follows, as well as the third, since $T(X\backslash S)=X\backslash S$ (we assume that (2.1) holds).

Let $\underline{x}\in\mathtt{A}^{\mathbf{Z}_+}\backslash\Sigma$ and $m\in\mathbf{Z}_+$ be the smallest integer such that one of the conditions defining Σ is not verified. Then either $\sigma^m\underline{x}\preceq\underline{u}^{x_m}$, or $\sigma^m\underline{x}\succeq\underline{v}^{x_m}$. The map $\overline{\varphi}_\infty$ is continuous (Theorem 2.3). Hence, for any $j\in\mathtt{A}$,

$$\overline{\varphi}_\infty(\underline{u}^j)=a_j\quad\text{and}\quad\overline{\varphi}_\infty(\underline{\mathrm{v}}^{\mathrm{j}})=\mathrm{a}_{\mathrm{j}+1}\,.$$

Let $\sigma^m\underline{x}\preceq\underline{u}^{x_m}$. Since $\underline{v}^{x_m-1}\prec\sigma^m\underline{x}$,

$$a_{x_m}=\overline{\varphi}_\infty(\underline{v}^{x_m-1})\leq\overline{\varphi}_\infty(\sigma^m\underline{x})\leq\overline{\varphi}_\infty(\underline{u}^{x_m})=a_{x_m}\,.$$

The other case is treated in the same way. From definition (2.16) $\underline{u}^{u^j_n}\preceq\sigma^n\underline{u}^j\preceq\underline{v}^{u^j_n}$. Suppose that $\delta(\underline{u}^j_0\cdots\underline{u}^j_{n-1})=1$ and $\sigma^n\underline{u}^j=\underline{v}^{u^j_n}$. By continuity of the φ-code there exists $x\in X\backslash S$ such that $x>a_j$ and $\mathtt{i}_k(x)=\underline{u}^j_k$, $k=0,\ldots,n$. Let $a_j<y<x$. Since $\delta(\underline{u}^j_0\cdots\underline{u}^j_{n-1})=1$, $T^ny<T^nx$ and consequently

$$\lim_{y\downarrow a_j}\mathtt{i}(T^ny)=\sigma^n\underline{u}^j\preceq\mathtt{i}(T^nx)\preceq\underline{v}^{u^j_n}\,.$$

Hence $\sigma^n\mathtt{i}(x)=\underline{v}^{x_n}$, which is a contradiction. The other cases are treated similarly. □

2.3. Dynamical system $\beta x+\alpha \bmod 1$

We consider the family of dynamical systems $\beta x+\alpha\bmod 1$ with $\beta>1$ and $0\leq\alpha<1$. For given α and β, the dynamical system is described by $k=\lceil\alpha+\beta\rceil$ intervals I_j and maps f_j, $I_0=(0,\beta^{-1}(1-\alpha))$, $I_{k-1}=(\beta^{-1}(k-1-\alpha),1)$ and

$$I_j=(\beta^{-1}(j-\alpha),\beta^{-1}(j+1-\alpha))\,,\;j=1,\ldots,k-2$$

and

$$f_j(x)=\beta x+\alpha-j\,,\;j=0,\ldots k-1\,.$$

The maps $T_{\alpha,\beta}$, $\varphi^{\alpha,\beta}$ and $\overline{\varphi}^{\alpha,\beta}$ are defined as in subsection 2.1.

$$\overline{\varphi}^{\alpha,\beta}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \alpha \\ \beta^{-1}(t-\alpha) & \text{if } \alpha \leq t \leq \alpha+\beta \\ 1 & \text{if } \alpha+\beta \leq t \leq \lceil \alpha+\beta \rceil \end{cases} \tag{2.18}$$

and

$$S_0 = \{a_j : j = 1, \ldots, k-1\} \cup \{0,1\} \text{with} \quad \mathrm{a_j} := \beta^{-1}(\mathrm{j}-\alpha)\,. \tag{2.19}$$

Since all maps are increasing the total order on $\mathtt{A}^{\mathbf{Z}_+}$ is the lexicographic order. We have $2k$ virtual orbits, but only two of them are important. Indeed, if we set

$$\underline{u}^{\alpha,\beta} := \underline{u}^0 \quad \text{and} \quad \underline{\mathrm{v}}^{\alpha,\beta} := \underline{\mathrm{v}}^{\mathrm{k}-1}\,,$$

then

$$\underline{u}^j = j\underline{u}^{\alpha,\beta}\,,\ j = 1, \ldots k-1$$

and

$$\underline{v}^j = j\underline{v}^{\alpha,\beta}\,,\ j = 0, \ldots, k-2\,.$$

Proposition 2.2 *Let $\beta > 1$ and $0 \leq \alpha < 1$. The φ-expansion for the dynamical system $\beta x + \alpha \bmod 1$ is valid.*

$$\Sigma^{\alpha,\beta} := \{\mathtt{i}(x) \in \mathtt{A}^{\mathbf{Z}_+} : x \in X \backslash S\} = \{\underline{x} \in \mathtt{A}^{\mathbf{Z}_+} : \underline{u}^{\alpha,\beta} \prec \sigma^n \underline{x} \prec \underline{v}^{\alpha,\beta}\ \forall n \geq 0\}\,.$$

Moreover

$$\underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{u}^{\alpha,\beta} \prec \underline{v}^{\alpha,\beta} \quad \text{and} \quad \underline{\mathrm{u}}^{\alpha,\beta} \prec \sigma^{\mathrm{n}} \underline{\mathrm{v}}^{\alpha,\beta} \preceq \underline{\mathrm{v}}^{\alpha,\beta} \quad \forall \mathrm{n} \geq 0\,.$$

The closure of $\Sigma^{\alpha,\beta}$ is the shift space

$$\mathbf{\Sigma}(\underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta}) := \{\underline{x} \in \mathtt{A}^{\mathbf{Z}_+} : \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{x} \preceq \underline{v}^{\alpha,\beta} \quad \forall n \geq 0\}\,. \tag{2.20}$$

We define the orbits of 0, resp. 1 as, (the limits are taken with $x \in X \backslash S$)

$$T^k_{\alpha,\beta}(0) := \lim_{x \downarrow 0} T^k_{\alpha,\beta}(x)\,,\ k \geq 0 \quad \text{resp.} \quad \mathrm{T}^{\mathrm{k}}_{\alpha,\beta}(1) := \lim_{\mathrm{x} \uparrow 1} \mathrm{T}^{\mathrm{k}}_{\alpha,\beta}(\mathrm{x})\,,\ \mathrm{k} \geq 0\,.$$

From (2.16) and (2.17) the coding of these orbits is $\underline{u}^{\alpha,\beta}$, resp. $\underline{v}^{\alpha,\beta}$,

$$\sigma^k \underline{u}^{\alpha,\beta} = \lim_{x \downarrow 0} \mathtt{i}(T^k_{\alpha,\beta}(x)) \quad \text{and} \quad \sigma^{\mathrm{k}} \underline{\mathrm{v}}^{\alpha,\beta} = \lim_{\mathrm{x} \uparrow 1} \mathtt{i}(\mathrm{T}^{\mathrm{k}}_{\alpha,\beta}(\mathrm{x}))\,. \tag{2.21}$$

Notice that $T^k_{\alpha,\beta}(0) < 1$ and $T^k_{\alpha,\beta}(1) > 0$ for all $k \geq 0$.

The virtual itineraries $\underline{u} \equiv \underline{u}^{\alpha,\beta}$ and $\underline{v} \equiv \underline{v}^{\alpha,\beta}$ of the dynamical system $\beta x + \alpha \bmod 1$ verify the conditions

$$\underline{u} \preceq \sigma^n \underline{u} \preceq \underline{v} \quad \forall\, n \geq 0 \quad \text{and} \quad \underline{\mathrm{u}} \preceq \sigma^{\mathrm{n}} \underline{\mathrm{v}} \preceq \underline{\mathrm{v}} \quad \forall\, \mathrm{n} \geq 0\,. \tag{2.22}$$

By Theorem 2.3, (2.21) and Theorem 2.5 we have ($x \in X \backslash S$)

$$\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^k \underline{u}) = \lim_{x \downarrow 0} \overline{\varphi}^{\alpha,\beta}_\infty(\mathtt{i}(T^k_{\alpha,\beta}(x)) = \lim_{x \downarrow 0} T^k_{\alpha,\beta}(x) \equiv T^k_{\alpha,\beta}(0) \tag{2.23}$$

$$\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^k \underline{v}) = \lim_{x \uparrow 1} \overline{\varphi}^{\alpha,\beta}_\infty(\mathtt{i}(T^k_{\alpha,\beta}(x)) = \lim_{x \uparrow 1} T^k_{\alpha,\beta}(x) \equiv T^k_{\alpha,\beta}(1)\,. \tag{2.24}$$

Hence $\underline{u}$ and $\underline{v}$ verify the equations¶

$$\overline{\varphi}_\infty^{\alpha,\beta}(\underline{u}) = 0\,,\ \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{u}) = \alpha \quad \text{and} \quad \overline{\varphi}_\infty^{\alpha,\beta}(\underline{\mathrm{v}}) = 1\,,\ \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{\mathrm{v}}) = \gamma\,, \tag{2.25}$$

with

$$\gamma := \alpha + \beta - k + 1 \in (0,1]\,. \tag{2.26}$$

The strings $\underline{u}^{\alpha,\beta}$ and $\underline{v}^{\alpha,\beta}$ are φ-expansions of 0 and 1. Because of the presence of discontinuities for the transformation $T_{\alpha,\beta}$ at $a_1,\ldots a_{k-1}$, there are other strings $\underline{u}$, $\underline{v}$ which verify (2.22) and (2.25), and which are also φ-expansions of 0 and 1. For latter purposes we need to decribe these strings; this is the content of Proposition 2.3, Proposition 2.4 and Proposition 2.5. We also take into consideration the borderline cases $\alpha = 1$ and $\gamma = 0$. When $\alpha = 1$ or $\gamma = 0$ the dynamical system $T_{\alpha,\beta}$ is defined using formula (2.7). The orbits of 0 and 1 are defined as before. For example, if $\alpha = 1$ it is the same dynamical system as $T_{0,\beta}$, but with different symbols for the coding of the orbits. The orbit of 0 is coded by $\underline{u}^{1,\beta} = (1)^\infty$, that is $\underline{u}_j^{1,\beta} = 1$ for all $j \geq 0$. Similarly, if $\gamma = 0$ the orbit of 1 is coded by $\underline{v}^{\alpha,\beta} = (k-2)^\infty$. *We always assume that* $\alpha \in [0,1]$, $\gamma \in [0,1]$ *and* $\beta \geq 1$.

Lemma 2.9 *The equation*

$$y = \overline{\varphi}^{\alpha,\beta}(x_k + t)\,, \quad y \in [0,1]$$

can be solved uniquely if $y \notin S_0$, *and its solution is* $x_k = \mathtt{i}_0(y)$ *and* $t = T_{\alpha,\beta}(y) \in (0,1)$. *If* $y < y'$, *then the solutions of the equations*

$$y = \overline{\varphi}^{\alpha,\beta}(x_k + t) \quad \text{and} \quad \mathrm{y}' = \overline{\varphi}^{\alpha,\beta}(\mathrm{x}_\mathrm{k}' + \mathrm{t}')$$

are such that either $x_k = x_k'$ *and* $T_{\alpha,\beta}(y') - T_{\alpha,\beta}(y) = \beta(y'-y)$, *or* $x_k < x_k'$.

Proof: The proof is elementary. It suffices to notice that

$$y \notin S_0 \implies y = \varphi^{\alpha,\beta}(x_k + t)\,.$$

The second statement follows by monotonicity. □

Proposition 2.3 *Let* $0 \leq \alpha < 1$ *and assume that the* φ*-expansion is valid. The following assertions are equivalent.*
1) There is a unique solution ($\underline{u} = \underline{u}^{\alpha,\beta}$) *of the equations*

$$\overline{\varphi}_\infty^{\alpha,\beta}(\underline{u}) = 0 \quad \text{and} \quad \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{\mathrm{u}}) = \alpha\,. \tag{2.27}$$

2) The orbit of 0 *is not periodic or* $x = 0$ *is a fixed point of* $T_{\alpha,\beta}$.
3) $\underline{u}^{\alpha,\beta}$ *is not periodic or* $\underline{u}^{\alpha,\beta} = \underline{0}$, *where* $\underline{0}$ *is the string* $\underline{x}$ *with* $x_j = 0\ \forall j \geq 0$.

Proposition 2.4 *Let* $0 < \gamma \leq 1$ *and assume that the* φ*-expansion is valid. The following assertions are equivalent.*
1) There is a unique solution ($\underline{v} = \underline{v}^{\alpha,\beta}$) *of the equations*

$$\overline{\varphi}_\infty^{\alpha,\beta}(\underline{v}) = 1 \quad \text{and} \quad \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{\mathrm{v}}) = \gamma\,. \tag{2.28}$$

¶ If the φ-expansion is not valid, which happens when $\beta = 1$ and $\alpha \in \mathbf{Q}$, then (2.23) and (2.24) are not necessarily true, as simple examples show. Hence $\underline{u}^{\alpha,\beta}$ and $\underline{v}^{\alpha,\beta}$ do not necessarily verify (2.25).

2) The orbit of 1 *is not periodic or* $x=1$ *is a fixed point of* $T_{\alpha,\beta}$.
3) $\underline{v}^{\alpha,\beta}$ *is not periodic or* $\underline{v}^{\alpha,\beta}=(k-1)^\infty$.

Proof: We prove Proposition 2.3. Assume 1. The validity of the φ-expansion implies that $\underline{u}^{\alpha,\beta}$ is a solution of (2.27). If $\alpha=0$, then $\underline{u}^{0,\beta}=\underline{0}$ is the only solution of (2.27) since $\underline{x}\neq\underline{0}$ implies $\overline{\varphi}_\infty^{0,\beta}(\underline{x})>0$ and $x=0$ is a fixed point of $T_{0,\beta}$. Let $0<\alpha<1$. Using Lemma 2.9 we deduce that $u_0=0$ and

$$\alpha=T_{\alpha,\beta}(0)=\overline{\varphi}^{\alpha,\beta}(u_1+\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^2\underline{u}))\,.$$

If $\alpha=a_j$, $j=1,\ldots,k-1$ (see (2.19)), then (2.27) has at least two solutions, which are $0j(\sigma^2\underline{u}^{\alpha,\beta})$ with $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^2\underline{u}^{\alpha,\beta})=T^2(0)=0$ (see (2.23)), and $0(j-1)\underline{v}^{\alpha,\beta}$ with $\overline{\varphi}_\infty^{\alpha,\beta}(\underline{v}^{\alpha,\beta})=1$. Therefore, by our hypothesis we have $\alpha\not\in\{a_1,\ldots,a_{k-1}\}$, $u_1=u_1^{\alpha,\beta}$ and $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^2\underline{u}^{\alpha,\beta})=T^2(0)\in(0,1)$. Iterating this argument we conclude that $1\implies 2$. Assume 2. If $x=0$ is a fixed point, then $\alpha=0$ and $\underline{u}^{0,\beta}=\underline{0}$. If the orbit of 0 is not periodic, (2.21) and the validity of the φ-expansion imply

$$\sigma^k\underline{u}^{\alpha,\beta}=\lim_{x\downarrow 0}\mathtt{i}(T_{\alpha,\beta}^k(x))\succ\lim_{x\downarrow 0}\mathtt{i}(x)=\underline{u}^{\alpha,\beta}\,.$$

Assume 3. From (2.23) and the validity of the φ-expansion we get

$$\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^k\underline{u}^{\alpha,\beta})=T_{\alpha,\beta}^k(0)>\overline{\varphi}_\infty^{\alpha,\beta}(\underline{u}^{\alpha,\beta})=0\,,$$

so that the orbit of 0 is not periodic. The orbit of 0 is not periodic if and only if $T_{\alpha,\beta}^k(0)\not\in\{a_1,\ldots,a_{k-1}\}$ for all $k\geq 1$. Using Lemma 2.9 we conclude that (2.27) has a unique solution. □

Propositions 2.3 and 2.4 give necessary and sufficient conditions for the existence and uniqueness of the solution of equations (2.25). In the following discussion we consider the case when there are several solutions. The main results are summarize in Proposition 2.5. We assume the validity of the φ-expansion.

Suppose first that the orbit of 1 is not periodic and that the orbit of 0 is periodic, with minimal period $p:=\min\{k:T^k(0)=0\}>1$. Hence $0<\gamma<1$ and $0<\alpha<1$. Let $\underline{u}$ be a solution of equations (2.27) and suppose furthermore that $\underline{w}$ is a φ-expansion of 1 such that

$$\forall n:\ \underline{u}\preceq\sigma^n\underline{u}\preceq\underline{w}\quad\text{with}\quad\overline{\varphi}_\infty^{\alpha,\beta}(\underline{\mathrm{w}})=1\,,\ \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{\mathrm{w}})\leq\gamma\,.$$

By Lemma 2.9 we conclude that

$$u_j=u_j^{\alpha,\beta}\quad\text{and}\quad\mathrm{T}_{\alpha,\beta}^{\mathrm{j+1}}(0)=\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^{\mathrm{j+1}}\underline{\mathrm{u}})\,,\quad\mathrm{j}=1,\ldots,\mathrm{p}-2\,.$$

Since $T^p(0)=0$, $T^{p-1}(0)\in\{a_1,\ldots,a_{k-1}\}$ and the equation

$$T_{\alpha,\beta}^{p-1}(0)=\overline{\varphi}_\infty^{\alpha,\beta}(u_{p-1}+\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^p\underline{u}))$$

has two solutions. Either $u_{p-1}=u_{p-1}^{\alpha,\beta}$ and $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^p\underline{u})=0$ or $u_{p-1}=u_{p-1}^{\alpha,\beta}-1$ and $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^p\underline{u})=1$. Let $\underline{a}$ be the prefix of $\underline{u}^{\alpha,\beta}$ of length p and $\underline{a}'$ the word of length p obtained by changing the last letter of $\underline{a}$ into[+] $u_{p-1}^{\alpha,\beta}-1$. We have $\underline{a}'<\underline{a}$. If $u_{p-1}=u_{p-1}^{\alpha,\beta}$,

[+] $u_{p-1}^{\alpha,\beta}\geq 1$. $u_{p-1}^{\alpha,\beta}=0$ if and only if $p=1$ and $\alpha=0$.

then we can again determine uniquely the next $p-1$ letters u_i. The condition $\underline{u} \leq \sigma^k \underline{u}$ for $k = p$ implies that we have $u_{2p-1} = u_{p-1}^{\alpha,\beta}$ so that, by iteration, we get the solution $\underline{u} = \underline{u}^{\alpha,\beta}$ for the equations (2.27). If $u_{p-1} = u_{p-1}^{\alpha,\beta} - 1$, then

$$1 = \overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma^p \underline{u}) = \overline{\varphi}_{\infty}^{\alpha,\beta}(u_p + \overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma^{p+1}\underline{u}))\,.$$

When $\overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma^p \underline{u}) = 1$, by our hypothesis on $\underline{u}$ we also have $\overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma^{p+1}\underline{u}) = \gamma$. By Proposition 2.4 the equations

$$\overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma^p \underline{u}) = 1 \quad \text{and} \quad \overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma^{p+1}\underline{u}) = \gamma$$

have a unique solution, since we assume that the orbit of 1 is not periodic. The solution is $\sigma^p \underline{u} = \underline{v}^{\alpha,\beta}$, so that $\underline{u} = \underline{a}'\underline{v}^{\alpha,\beta} \prec \underline{u}^{\alpha,\beta}$ is also a solution of (2.27). In that case there is no other solution for (2.27). The borderline case $\alpha = 1$ corresponds to the periodic orbit of the fixed point 0, $\underline{u}^{1,\beta} = (1)^{\infty}$. Notice that $\overline{\varphi}_{\infty}^{1,\beta}(\sigma \underline{u}^{1,\beta}) \neq 1$. We can also consider $\overline{\varphi}_{\infty}^{1,\beta}$-expansions of 0 with $u_0 = 0$ and $\overline{\varphi}_{\infty}^{1,\beta}(\sigma \underline{u}) = 1$. Our hypothesis on $\underline{u}$ imply that $\overline{\varphi}_{\infty}^{1,\beta}(\sigma^2 \underline{u}) = \gamma$. Hence, $\underline{u} = 0\underline{v}^{1,\beta} = \underline{a}'\underline{v}^{\alpha,\beta} \prec \underline{u}^{\alpha,\beta}$ is a solution of (2.27) and a $\overline{\varphi}_{\infty}^{1,\beta}$-expansion of 0.

We can treat similarly the case when $\underline{u}^{\alpha,\beta}$ is not periodic, but $\underline{v}^{\alpha,\beta}$ is periodic. When both $\underline{u}^{\alpha,\beta}$ and $\underline{v}^{\alpha,\beta}$ are periodic we have more solutions, but the discussion is similar. Assume that $\underline{u}^{\alpha,\beta}$ has (minimal) period $p > 1$ and $\underline{v}^{\alpha,\beta}$ has (minimal) period $q > 1$. Define $\underline{a}$, $\underline{a}'$ as before, $\underline{b}$ as the prefix of length q of $\underline{v}^{\alpha,\beta}$, and $\underline{b}'$ as the word of length q obtained by changing the last letter of $\underline{b}$ into $v_{q-1}^{\alpha,\beta} + 1$. When $0 < \alpha < 1$ and $0 < \gamma < 1$, one shows as above that the elements $\underline{u} \neq \underline{u}^{\alpha,\beta}$ and $\underline{v} \neq \underline{v}^{\alpha,\beta}$ which are $\overline{\varphi}^{\alpha,\beta}$-expansions of 0 and 1 are of the form

$$\underline{u} = \underline{a}'\underline{b}^{n_1}\underline{b}'\underline{a}^{n_2}\cdots\,,\ n_i \geq 0 \quad \text{and} \quad \underline{v} = \underline{b}'\underline{a}^{m_1}\underline{a}'\underline{b}^{m_2}\cdots\,,\ m_i \geq 0\,.$$

The integers n_i and m_i must be such that (2.22) is verified. The largest solution of (2.27) is $\underline{u}^{\alpha,\beta}$ and the smallest one is $\underline{a}'\underline{v}^{\alpha,\beta}$.

Proposition 2.5 *Assume that the φ-expansion is valid.*
1) Let $\underline{u}$ be a solution of (2.27), such that $\underline{u} \preceq \sigma^n \underline{u}$ for all $n \geq 1$, and let $\underline{v}$ be a solution of (2.28), such that $\sigma^n \underline{v} \preceq \underline{v}$ for all $n \geq 1$. Then

$$\underline{u} \preceq \underline{u}^{\alpha,\beta} \quad \text{and} \quad \underline{v}^{\alpha,\beta} \preceq \underline{v}\,.$$

2) Let $\underline{u}$ be a solution of (2.27), and let $\underline{u}^{\alpha,\beta} = (\underline{a})^{\infty}$ be periodic with minimal period $p > 1$, and suppose that there exists $\underline{w}$ such that

$$\forall n:\ \underline{u} \preceq \sigma^n \underline{u} \preceq \underline{w} \quad \text{with} \quad \overline{\varphi}_{\infty}^{\alpha,\beta}(\underline{w}) = 1\,,\ \overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma \underline{w}) \leq \gamma\,.$$

Then

$$\underline{u}_*^{\alpha,\beta} \preceq \underline{u} \preceq \underline{u}^{\alpha,\beta} \quad \text{where} \quad \underline{u}_*^{\alpha,\beta} := \underline{a}'\underline{v}^{\alpha,\beta} \text{ and } \underline{a}' := (p\underline{a})(a_{p-1} - 1)\,. \qquad (2.29)$$

Moreover, $\underline{u} = \underline{u}^{\alpha,\beta} \iff \underline{a}$ is a prefix of $\underline{u} \iff \overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma^p \underline{u}) < 1$.
3) Let $\underline{v}$ be a solution of (2.28), and let $\underline{v}^{\alpha,\beta} = (\underline{b})^{\infty}$ be periodic with minimal period $q > 1$, and suppose that there exists $\underline{w}$ such that

$$\forall n:\ \underline{w} \preceq \sigma^n \underline{v} \preceq \underline{v} \quad \text{with} \quad \overline{\varphi}_{\infty}^{\alpha,\beta}(\underline{w}) = 0\,,\ \overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma \underline{w}) \geq \alpha\,.$$

Then

$$\underline{v}^{\alpha,\beta} \preceq \underline{v} \preceq \underline{v}_*^{\alpha,\beta} \quad \text{where} \quad \underline{v}_*^{\alpha,\beta} := \underline{b}'\underline{u}^{\alpha,\beta} \text{ and } \underline{b}' := (\mathrm{p}\underline{b})(\mathrm{b}_{q-1}+1)\,. \tag{2.30}$$

Moreover, $\underline{v} = \underline{v}^{\alpha,\beta} \iff \underline{b}$ *is a prefix of* $\underline{v} \iff \overline{\varphi}_\infty^{\alpha,\beta}(\sigma^q\underline{v}) > 0$.

3. Shift space $\mathbf{\Sigma}(\underline{u},\underline{v})$

Let $\underline{u} \in \mathtt{A}^{\mathbf{Z}_+}$ and $\underline{v} \in \mathtt{A}^{\mathbf{Z}_+}$, such that $u_0 = 0$, $v_0 = k-1$ ($k \geq 2$) and (2.22) holds. These assumptions are valid for the whole section, except subsection 3.2. We study the shift-space

$$\mathbf{\Sigma}(\underline{u},\underline{v}) := \{\underline{x} \in \mathtt{A}^{\mathbf{Z}_+} : \underline{u} \preceq \sigma^n\underline{x} \preceq \underline{v} \quad \forall\, n \geq 0\}\,. \tag{3.1}$$

It is useful to extend the relation $\prec$ to words or to words and strings. We do it only in the following case. Let $\underline{a}$ and $\underline{b}$ be words (or strings). Then

$$\underline{a} \prec \underline{b} \quad \text{iff } \exists \underline{c} \in \mathtt{A}^*\,,\ \exists \mathrm{k} \geq 0 \text{ such that } \underline{a} = \underline{c}\mathrm{a}_\mathrm{k}\cdots\,,\ \underline{b} = \underline{c}\mathrm{b}_\mathrm{k}\cdots \text{ and } \mathrm{a}_\mathrm{k} < \mathrm{b}_\mathrm{k}\,.$$

If $\underline{a} \prec \underline{b}$ then neither $\underline{a}$ is a prefix of $\underline{b}$, nor $\underline{b}$ is a prefix of $\underline{a}$.

In subsection 3.1 we introduce one of the main tool for studying the shift-space $\mathbf{\Sigma}(\underline{u},\underline{v})$, the follower-set graph. In subsection 3.2 we give an algorithm which assigns to a pair of strings $(\underline{u},\underline{v})$ a pair of real numbers $(\bar{\alpha},\bar{\beta}) \in [0,1]\times[1,\infty)$. Finally in subsection 3.3 we compute the topological entropy of the shift space $(\underline{u},\underline{v})$.

3.1. Follower-set graph $\mathcal{G}(\underline{u},\underline{v})$

We associate to $\mathbf{\Sigma}(\underline{u},\underline{v})$ a graph $\mathcal{G}(\underline{u},\underline{v})$, called the follower-set graph (see [LiM]). The graph $\mathcal{G}(\underline{u},\underline{v})$, or its variants, have been systematically studied by Hofbauer in his works about piecewise monotone one-dimensional dynamical systems; see [Ho1], [Ho2] and [Ho3] in the context of this paper, as well as [Ke] and [BrBr]. Our presentation differs from that of Hofbauer, but several proofs are directly inspired by [Ho2] and [Ho3].

We denote by $\mathcal{L}(\underline{u},\underline{v})$ the language of $\mathbf{\Sigma}(\underline{u},\underline{v})$, that is the set of words, which are factors of $\underline{x} \in \mathbf{\Sigma}(\underline{u},\underline{v})$ (including the empty word ϵ). Since $\sigma\mathbf{\Sigma}(\underline{u},\underline{v}) \subset \mathbf{\Sigma}(\underline{u},\underline{v})$, the language is also the set of prefixes of the strings $\underline{x} \in \mathbf{\Sigma}(\underline{u},\underline{v})$. To simplify the notations we set in this subsection $\mathbf{\Sigma} := \mathbf{\Sigma}(\underline{u},\underline{v})$, $\mathcal{L} := \mathcal{L}(\underline{u},\underline{v})$, $\mathcal{G} := \mathcal{G}(\underline{u},\underline{v})$.

Let $\mathcal{C}_u$ be the set of words $\underline{w} \in \mathcal{L}$ such that

$$\underline{w} = \begin{cases} \underline{w}' : \underline{w}' \neq \epsilon\,,\ \underline{w}' \text{ is a prefix of } \underline{\mathrm{u}} \\ w_0\underline{w}' : w_0 \neq u_0\,,\ \underline{w}' \text{ is a prefix of } \underline{\mathrm{u}}\,,\ \text{possibly } \epsilon\,. \end{cases}$$

Similarly we introduce $\mathcal{C}_v$ as the set of words $\underline{w} \in \mathcal{L}$ such that

$$\underline{w} = \begin{cases} \underline{w}' : \underline{w}' \neq \epsilon\,,\ \underline{w}' \text{ is a prefix of } \underline{\mathrm{v}} \\ w_0\underline{w}' : w_0 \neq v_0\,,\ \underline{w}' \text{ is a prefix of } \underline{v}\,,\ \text{possibly } \epsilon <,. \end{cases}$$

Definition 3.1 *Let* $\underline{w} \in \mathcal{L}$. *The longest suffix of* $\underline{w}$*, which is a prefix of* $\underline{v}$*, is denoted by* $v(\underline{w})$. *The longest suffix of* $\underline{w}$*, which is a prefix of* $\underline{u}$*, is denoted by* $u(\underline{w})$. *The* u-parsing of $\underline{w}$ *is the following decomposition of* $\underline{w}$ *into* $\underline{w} = \underline{a}^1\cdots\underline{a}^k$ *with* $\underline{a}^j \in \mathcal{C}_u$. *The first word*

$\underline{a}^1$ *is the longest prefix of* $\underline{w}$ *belonging to* $\mathcal{C}_u$. *If* $\underline{w} = \underline{a}^1\underline{w}'$ *and* $\underline{w}' \neq \epsilon$, *then the next word* $\underline{a}^2$ *is the longest prefix of* $\underline{w}'$ *belonging to* $\mathcal{C}_u$ *and so on.*

The v-parsing of $\underline{w}$ *is the analogous decomposition of* $\underline{w}$ *into* $\underline{w} = \underline{b}^1 \cdots \underline{b}^\ell$ *with* $\underline{b}^j \in \mathcal{C}_v$.

Lemma 3.1 *Let* $\underline{wc}$ *and* $\underline{cw}'$ *be prefixes of* $\underline{u}$ *(respectively of* $\underline{v}$*). If* $\underline{wcw}' \in \mathcal{L}$, *then* $\underline{wcw}'$ *is a prefix of* $\underline{u}$ *(respectively of* $\underline{v}$*). Let* $\underline{w} \in \mathcal{L}$. *If* $\underline{a}^1 \cdots \underline{a}^k$ *is the* u*-parsing of* $\underline{w}$, *then only the first word* $\underline{a}^1$ *can be a prefix of* $\underline{u}$, *otherwise* $u(\underline{a}^j) = \mathtt{s}\underline{a}^j$. *Moreover* $u(\underline{a}^k) = u(\underline{w})$. *Analogous properties hold for the* v*-parsing of* $\underline{w}$.

Proof: Suppose that $\underline{wc}$ and $\underline{cw}'$ are prefixes of $\underline{u}$. Then $\underline{w}$ is a prefix of $\underline{u}$. Assume that $\underline{wcw}' \in \mathcal{L}$ is not a prefix of $\underline{u}$. Then $\underline{u} \prec \underline{wcw}'$. Since $\underline{w}$ is a prefix of $\underline{u}$, $\sigma^{|\underline{w}|}\underline{u} \prec \underline{cw}'$. This contradicts the fact that $\underline{cw}'$ is a prefix of $\underline{u}$. By applying this result with $\underline{c} = \epsilon$ we get the result that only the first word in the u-parsing of $\underline{w}$ can be a prefix of $\underline{u}$. Suppose that the u-parsing of $\underline{w}$ is $\underline{a}^1 \cdots \underline{a}^k$. Let $k \geq 2$ and assume that $u(\underline{w})$ is not a suffix of $\underline{a}^k$ (the case $k = 1$ is obvious). Since $\underline{a}^k$ is not a prefix of $\underline{u}$, $u(\underline{w})$ has $\underline{a}^k$ as a proper suffix. By the first part of the lemma this contradicts the maximality property of the words in the u-parsing. □

Lemma 3.2 *Let* $\underline{w} \in \mathcal{L}$. *Let* $p = |u(\underline{w})|$ *and* $q = |v(\underline{w})|$. *Then*

$$\{\underline{x} \in \mathbf{\Sigma} \colon \underline{w} \text{ is a prefix of } \underline{\mathrm{x}}\} = \{\underline{\mathrm{x}} \in \mathtt{A}^{\mathbf{Z}_+} : \underline{\mathrm{x}} = \underline{\mathrm{wy}},\ \underline{\mathrm{y}} \in \mathbf{\Sigma},\ \sigma^{\mathrm{p}}\underline{\mathrm{u}} \preceq \underline{\mathrm{y}} \preceq \sigma^{\mathrm{q}}\underline{\mathrm{v}}\}.$$

Moreover,

$$\{\underline{y} \in \mathbf{\Sigma} \colon \underline{wy} \in \mathbf{\Sigma}\} = \{\underline{y} \in \mathbf{\Sigma} \colon u(\underline{w})\underline{y} \in \mathbf{\Sigma}\} \quad \text{if p} > \text{q}$$

$$\{\underline{y} \in \mathbf{\Sigma} \colon \underline{wy} \in \mathbf{\Sigma}\} = \{\underline{y} \in \mathbf{\Sigma} \colon v(\underline{w})\underline{y} \in \mathbf{\Sigma}\} \quad \text{if q} > \text{p}\,.$$

Proof: Suppose that $\underline{x} \in \mathbf{\Sigma}$ and $\underline{w}$, $|\underline{w}| = n$, is a prefix of $\underline{x}$. Let $n \geq 1$ (the case $n = 0$ is trivial). We can write $\underline{x} = \underline{wy}$. Since $\underline{x} \in \mathbf{\Sigma}$,

$$\underline{u} \preceq \sigma^{\ell+n}\underline{x} \preceq \underline{v} \quad \forall\, \ell \geq 0\,,$$

so that $\underline{y} \in \mathbf{\Sigma}$. We have

$$\underline{u} \preceq \sigma^{n-p}\underline{x} = u(\underline{w})\underline{y}\,.$$

Since $u(\underline{w})$ is a prefix of $\underline{u}$ of length p, we get $\sigma^p\underline{u} \preceq \underline{y}$. Similarly we prove that $\underline{y} \preceq \sigma^q\underline{v}$.

Suppose that $\underline{x} = \underline{wy}$, $\underline{y} \in \mathbf{\Sigma}$ and $\sigma^p\underline{u} \preceq \underline{y} \preceq \sigma^q\underline{v}$. To prove that $\underline{x} \in \mathbf{\Sigma}$, it is sufficient to prove that $\underline{u} \preceq \sigma^m\underline{x} \preceq \underline{v}$ for $m = 0, \ldots, n-1$. We prove $\underline{u} \preceq \sigma^m\underline{x}$ for $m = 0, \ldots, n-1$. The other case is similar. Let $\underline{w} = \underline{a}^1 \cdots \underline{a}^\ell$ be the u-parsing of $\underline{w}$, $|\underline{w}| = n$ and $p = |u(\underline{w})|$. We have

$$\sigma^p\underline{u} \preceq \underline{y} \implies \underline{u} \preceq \sigma^j\underline{u} \preceq \sigma^j u(\underline{w})\underline{y} \quad \forall\, j = 0, \ldots, p\,.$$

If $\underline{a}^\ell$ is not a prefix of $\underline{u}$, then $p = n - 1$ and we also have $\underline{u} \preceq \underline{a}^k\underline{y}$. If $\underline{a}^\ell$ is a prefix of $\underline{u}$, then $p = n$ (and $\ell = 1$). This proves the result for $\ell = 1$. Let $\ell \geq 2$. Then $\underline{a}^\ell$ is not a prefix of $\underline{u}$ and $\underline{a}^{\ell-1}\underline{a}^\ell \in \mathcal{L}$. Suppose that $\underline{a}^{\ell-1}$ is not a prefix of $\underline{u}$. In that case $\underline{u} \preceq \underline{a}^{\ell-1}\underline{a}^\ell\underline{y}$ and we want to prove that $\underline{u} \preceq \sigma^j\underline{a}^{\ell-1}\underline{a}^\ell\underline{y}$ for $j = 1, \ldots, |\underline{a}^{\ell-1}|$. We know that $\sigma\underline{a}^{\ell-1}$ is a prefix of $\underline{u}$, and by maximality of the words in the u-parsing and Lemma 3.1 $\underline{u} \prec \sigma\underline{a}^{\ell-1}\underline{a}^{\ell}$; hence $\underline{u} \prec \sigma\underline{a}^{\ell-1}\underline{a}^{\ell}\underline{y}$. Therefore

$$\underline{u} \preceq \sigma^j\underline{u} \preceq \sigma^j\underline{a}^{\ell-1}\underline{a}^{\ell}\underline{y} \quad \forall j = 0, \ldots, |\underline{a}^{\ell-1}|\,.$$

Similar proof if $\ell = 2$ and $\underline{a}^{\ell-1}$ is a prefix of $\underline{u}$. Iterating this argument we prove that $\underline{u} \preceq \sigma^m\underline{x}$ for $m = 0, \ldots, n-1$.

Suppose that $|u(\underline{w})| > |v(\underline{w})|$ and set $\underline{a} = u(\underline{w})$. We prove that $v(\underline{a}) = v(\underline{w})$. By definition $v(\underline{w})$ is the longest suffix of $\underline{w}$ which is a prefix of $\underline{v}$; it is also a suffix of $\underline{a}$, whence it is also the longest suffix of $\underline{a}$ which is a prefix of $\underline{v}$. Therefore, from the first part of the lemma we get

$$\{\underline{y} \in \mathbf{\Sigma} \colon \underline{w}\underline{y} \in \mathbf{\Sigma}\} = \{\underline{y} \in \mathbf{\Sigma} \colon u(\underline{w})\underline{y} \in \mathbf{\Sigma}\}\,.$$

□

Definition 3.2 *Let* $\underline{w} \in \mathcal{L}$. *The* follower-set* *of* $\underline{w}$ *is the set*

$$\mathcal{F}_{\underline{w}} := \{\underline{y} \in \mathbf{\Sigma} \colon \underline{w}\underline{y} \in \mathbf{\Sigma}\}\,.$$

Lemma 3.2 gives the important results that $\mathcal{F}_{\underline{w}} = \mathcal{F}_{u(\underline{w})}$ if $|u(\underline{w})| > |v(\underline{w})|$, and $\mathcal{F}_{\underline{w}} = \mathcal{F}_{v(\underline{w})}$ if $|v(\underline{w})| > |u(\underline{w})|$. Moreover,

$$\mathcal{F}_{\underline{w}} = \{\underline{y} \in \mathbf{\Sigma} \colon \sigma^p\underline{u} \preceq \underline{y} \preceq \sigma^q\underline{v}\} \text{where p} = |\text{u}(\underline{\text{w}})| \text{ and q} = |\text{v}(\underline{\text{w}})|\,. \tag{3.2}$$

We can define an equivalence relation between words of $\mathcal{L}$,

$$\underline{w} \sim \underline{w}' \iff \mathcal{F}_{\underline{w}} = \mathcal{F}_{\underline{w}'}\,.$$

The collection of follower-sets is entirely determined by the strings $\underline{u}$ and $\underline{v}$. Moreover, the strings $\underline{u}$ and $\underline{v}$ are eventually periodic if and only if this collection is finite. Notice that $\mathbf{\Sigma} = \mathcal{F}_{\epsilon} = \mathcal{F}_{\underline{w}}$ when $p = q = 0$.

Definition 3.3 *The* follower-set graph $\mathcal{G}$ *is the labeled graph whose set of vertices is the collection of all follower-sets. Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be two vertices. There is an edge, labeled by* $a \in \mathtt{A}$*, from* $\mathcal{C}$ *to* $\mathcal{C}'$ *if and only if there exists* $\underline{w} \in \mathcal{L}$ *so that* $\underline{w}a \in \mathcal{L}$, $\mathcal{C} = \mathcal{F}_{\underline{w}}$ *and* $\mathcal{C}' = \mathcal{F}_{\underline{w}a}$. $\mathcal{F}_{\epsilon}$ *is called the* root *of* $\mathcal{G}$.

The following properties of $\mathcal{G}$ are immediate. From any vertex there is at least one out-going edge and at most $|\mathtt{A}|$. If $\mathtt{A} = \{0, 1, \ldots, k-1\}$ and $k \geq 3$, then for each $j \in \{1, \ldots, k-2\}$ there is an edge labeled by j from $\mathcal{F}_{\epsilon}$ to $\mathcal{F}_{\epsilon}$. The out-going edges from $\mathcal{F}_{\underline{w}}$ are labeled by the first letters of the strings $\underline{y} \in \mathcal{F}_{\underline{w}}$. The follower-set graph $\mathcal{G}$ is right-resolving. Given $\underline{w} \in \mathcal{L}$, there is a unique path labeled by $\underline{w}$ from $\mathcal{F}_{\epsilon}$ to $\mathcal{F}_{\underline{w}}$.

Lemma 3.3 *Let* $\underline{a}$ *be a u-prefix and suppose that* $\underline{b} = v(\underline{a})$. *Let* $p = |\underline{a}|$ *and* $q = |\underline{b}|$ *so that* $\mathcal{F}_{\underline{a}} = \{\underline{y} \in \mathbf{\Sigma} \colon \sigma^p\underline{u} \preceq \underline{y} \preceq \sigma^q\underline{v}\}$. *Then there are more than one out-going edges from* $\mathcal{F}_{\underline{a}}$ *if and only if* $u_p < v_q$.

* Usually the follower-set is defined as $\mathcal{F}_{\underline{w}} = \big\{\underline{y} \in \mathcal{L} \colon \underline{w}\underline{y} \in \mathcal{L}\big\}$. Since $\mathcal{L}$ is a dynamical language, i.e. for each $\underline{w} \in \mathcal{L}$ there exists a letter $e \in \mathtt{A}$ such that $\underline{w}e \in \mathcal{L}$, the two definitions agree.

Assume that $u_p < v_q$. Then there is an edge labeled by v_q from $\mathcal{F}_{\underline{a}}$ to $\mathcal{F}_{\underline{b}v_q}$, an edge labeled by u_p from $\mathcal{F}_{\underline{a}}$ to $\mathcal{F}_{\underline{a}u_p}$ and $v(\underline{a}u_p\underline{c}) = v(\underline{c})$. If there exists $u_p < \ell < v_q$, there is an edge labeled by ℓ from $\mathcal{F}_{\underline{a}}$ to $\mathcal{F}_\epsilon$. Moreover, there are at least two out-going edges from $\mathcal{F}_{\underline{b}}$, one labeled by v_q to $\mathcal{F}_{\underline{b}v_q}$ and one labeled by $\ell' = u_{|u(\underline{b})|+1} < v_q$ to $\mathcal{F}_{u(\underline{b})\ell'}$. Furthermore $u(\underline{b}v_q\underline{c}) = u(\underline{c})$.

Proof: The first part of the lemma is immediate. Suppose that there is only one out-going edge from $\mathcal{F}_{\underline{b}}$, that is from $\mathcal{F}_{\underline{b}}$ to $\mathcal{F}_{\underline{b}v_q}$. This happens if and only if $u(\underline{b})v_q$ is a prefix of $\underline{u}$. By Lemma 3.1 we conclude that $\underline{a}v_q$ is a prefix of $\underline{u}$, which is a contradiction. Therefore $u(\underline{b}v_q) = \epsilon$; hence $u(\underline{b}v_q\underline{c}) = u(\underline{c})$. □

Lemma 3.4 *Let $\underline{b}$ be a v-prefix and suppose that $\underline{a} = u(\underline{b})$. Let $p = |\underline{a}|$ and $q = |\underline{b}|$ so that $\mathcal{F}_{\underline{b}} = \{\underline{y} \in \mathbf{\Sigma} \colon \sigma^p\underline{u} \preceq \underline{y} \preceq \sigma^q\underline{v}\}$. Then there are more than one out-going edges from $\mathcal{F}_{\underline{b}}$ if and only if $u_p < v_q$.*

Assume that $u_p < v_q$. Then there is an edge labeled by u_p from $\mathcal{F}_{\underline{b}}$ to $\mathcal{F}_{\underline{a}u_p}$, an edge labeled by v_q from $\mathcal{F}_{\underline{b}}$ to $\mathcal{F}_{\underline{b}v_q}$ and $u(\underline{b}v_q\underline{c}) = u(\underline{c})$. If there exists $u_p < \ell < v_q$, there is an edge labeled by ℓ from $\mathcal{F}_{\underline{b}}$ to $\mathcal{F}_\epsilon$. Moreover, there are at least two out-going edges from $\mathcal{F}_{\underline{a}}$, one labeled by u_p to $\mathcal{F}_{\underline{a}u_p}$ and one labeled by $\ell' = v_{|v(\underline{a})|+1} > u_p$ to $\mathcal{F}_{v(\underline{a})\ell'}$. Furthermore $v(\underline{a}u_p\underline{c}) = v(\underline{c})$.

Scholium 3.1 *The main property of the graph is the following one. Let $\underline{w}$ be a prefix of $\underline{u}$ for example. Suppose that $v(\underline{w}) = \underline{b}$, that the letter following $\underline{w}$ in $\underline{u}$ is e'', and that the letter following $\underline{b}$ in $\underline{v}$ is e'. We have an out-going edge labeled by e', from $\mathcal{F}_{\underline{w}}$ to $\mathcal{F}_{\underline{b}e'}$ if and only if $e'' \prec e'$. Moreover we* necessarily *have an an edge labeled by e from $\mathcal{F}_{\underline{b}}$ to $\mathcal{F}_{\underline{a}e}$ with $e \prec e'$, where $\underline{a} = u(\underline{b})$. We also have $e \preceq e''$.*

Definition 3.4 *Let $\mathbf{\Sigma}$ be a shift-space and $\mathcal{L}$ its language. We denote by $\mathcal{L}_n$ the set of all words of $\mathcal{L}$ of length n. The* entropy *of $\mathbf{\Sigma}$ is*

$$h(\mathbf{\Sigma}) := \lim_{n\to\infty} \frac{1}{n} \log_2 \mathrm{card}(\mathcal{L}_n)\,.$$

The number $h(\mathbf{\Sigma})$ is also equal to the topological entropy of the dynamical system $(\mathbf{\Sigma}, \sigma)$ [LiM]. In our case we can give an equivalent definition using the graph $\mathcal{G}$. We set

$$\ell(n) := \mathrm{card}\{n-\text{paths in } \mathcal{G} \text{ starting at the root } \mathcal{F}_\epsilon\}\,.$$

Since the graph is right-resolving and for any $\underline{w} \in \mathcal{L}_n$ there is a unique path labeled by $\underline{w}$, starting at the root $\mathcal{F}_\epsilon$, so that $h(\mathbf{\Sigma}) = h(\mathcal{G})$ where

$$h(\mathcal{G}) = \lim_{n\to\infty} \frac{1}{n} \log_2 \ell(n)\,.$$

3.2. The algorithm for finding $(\bar{\alpha}, \bar{\beta})$

We describe an algorithm, which assigns to a pair of strings $(\underline{u}, \underline{v})$, such that $u_0 = 0$ and $v_0 = k-1$, a pair of real numbers $(\bar{\alpha}, \bar{\beta}) \in [0,1] \times [1,\infty)$. We assume tacitly that for the pair (α, β) one has $\alpha \in [0,1]$, $\beta \le k$, and that the map $\overline{\varphi}^{\alpha,\beta}$ verifies

$$0 < \overline{\varphi}^{\alpha,\beta}(t) < 1 \quad \forall t \in (1, k-1)\,.$$

In particular $\beta \geq k-2$. When $k=2$ we assume that $\beta \geq 1$. Recall that $\overline{\varphi}^{\alpha,\beta}(t)$ is given by (2.18),

$$\gamma = \alpha + \beta - k + 1\,,$$

and notice that our assumptions imply that $0 \leq \gamma \leq 1$.

Definition 3.5 *The map $\overline{\varphi}^{\alpha,\beta}$* dominates *the map $\overline{\varphi}^{\alpha',\beta'}$ if and only if $\overline{\varphi}^{\alpha,\beta}(t) \geq \overline{\varphi}^{\alpha',\beta'}(t)$ for all $t \in [0,k]$ and there exists $s \in [0,k]$ such that $\overline{\varphi}^{\alpha,\beta}(s) > \overline{\varphi}^{\alpha',\beta'}(s)$.*

Lemma 3.5 *If $\overline{\varphi}^{\alpha,\beta}$ dominates $\overline{\varphi}^{\alpha',\beta'}$, then, for all $\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$, $\overline{\varphi}^{\alpha,\beta}_\infty(\underline{x}) \geq \overline{\varphi}^{\alpha',\beta'}_\infty(\underline{x})$. If*

$$0 < \overline{\varphi}^{\alpha,\beta}_\infty(\underline{x}) < 1 \quad \text{or} \quad 0 < \overline{\varphi}^{\alpha',\beta'}_\infty(\underline{\mathrm{x}}) < 1\,,$$

then the inequality is strict.

Proof: If $\overline{\varphi}^{\alpha,\beta}$ dominates $\overline{\varphi}^{\alpha',\beta'}$, then by our implicit assumptions we get by inspection of the graphs that

$$\forall t \geq t' : \ \overline{\varphi}^{\alpha,\beta}(t) > \overline{\varphi}^{\alpha',\beta'}(t') \quad \text{if} \quad \mathrm{t},\mathrm{t}' \in (\alpha, \alpha' + \beta')\,,$$

otherwise $\overline{\varphi}^{\alpha,\beta}(t) \geq \overline{\varphi}^{\alpha',\beta'}(t')$. Therefore, for all $n \geq 1$,

$$\overline{\varphi}^{\alpha,\beta}_n(\underline{x}) \geq \overline{\varphi}^{\alpha',\beta'}_n(\underline{x})\,.$$

Suppose that $0 < \overline{\varphi}^{\alpha,\beta}_\infty(\underline{x}) < 1$. Then $x_0 + \overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{x}) \in (\alpha, \alpha+\beta)$ and

$$\overline{\varphi}^{\alpha,\beta}_\infty(\underline{x}) = \overline{\varphi}^{\alpha,\beta}(x_0 + \overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{x})) > \overline{\varphi}^{\alpha',\beta'}(x_0 + \overline{\varphi}^{\alpha',\beta'}_\infty(\sigma\underline{x})) = \overline{\varphi}^{\alpha',\beta'}_\infty(\underline{x})\,.$$

Similar proof for $0 < \overline{\varphi}^{\alpha',\beta'}_\infty(\underline{x}) < 1$. □

Lemma 3.6 *Let $\alpha = \alpha' \in [0,1]$ and $1 \leq \beta < \beta'$. Then, for $\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$,*

$$0 \leq \overline{\varphi}^{\alpha,\beta}_\infty(\underline{x}) - \overline{\varphi}^{\alpha,\beta'}_\infty(\underline{x}) \leq \frac{|\beta-\beta'|}{\beta'-1}\,.$$

Let $\gamma = \gamma' \in [0,1]$, $0 \leq \alpha' < \alpha \leq 1$ and $\beta' > 1$. Then, for $\underline{x} \in \mathtt{A}^{\mathbf{Z}_+}$,

$$0 \leq \overline{\varphi}^{\alpha',\beta'}_\infty(\underline{x}) - \overline{\varphi}^{\alpha,\beta}_\infty(\underline{x}) \leq \frac{|\alpha-\alpha'|}{\beta'-1}\,.$$

The map $\beta \mapsto \overline{\varphi}^{\alpha,\beta}_\infty(\underline{x})$ is continuous at $\beta = 1$.

Proof: Let $\alpha = \alpha' \in [0,1]$ and $1 \leq \beta < \beta'$. For $t,t' \in [0,k]$,

$$\begin{aligned}|\overline{\varphi}^{\alpha,\beta'}(t') - \overline{\varphi}^{\alpha,\beta}(t)| &\leq |\overline{\varphi}^{\alpha,\beta'}(t') - \overline{\varphi}^{\alpha,\beta'}(t)| + |\overline{\varphi}^{\alpha,\beta'}(t) - \overline{\varphi}^{\alpha,\beta}(t)| \\ &\leq \frac{|t-t'|}{\beta'} + \frac{|\beta-\beta'|}{\beta'}\,.\end{aligned}$$

(The maximum of $|\overline{\varphi}^{\alpha,\beta'}(t) - \overline{\varphi}^{\alpha,\beta}(t)|$ is taken at $\alpha+\beta$). By induction

$$|\overline{\varphi}^{\alpha,\beta'}_n(x_0,\ldots,x_{n-1}) - \overline{\varphi}^{\alpha,\beta}_n(x_0,\ldots,x_{n-1})| \leq |\beta-\beta'| \sum_{j=1}^{n} (\beta')^{-j}\,.$$

Since $\beta' > 1$ the sum is convergent. This proves the first statement. The second statement is proved similarly using

$$\begin{aligned}|\overline{\varphi}^{\alpha',\beta'}(t') - \overline{\varphi}^{\alpha,\beta}(t)| &\leq |\overline{\varphi}^{\alpha',\beta'}(t') - \overline{\varphi}^{\alpha',\beta'}(t)| + |\overline{\varphi}^{\alpha',\beta'}(t) - \overline{\varphi}^{\alpha,\beta}(t)| \\ &\leq \frac{|t-t'|}{\beta'} + \frac{|\alpha-\alpha'|}{\beta'}\end{aligned}$$

which is valid for $\gamma = \gamma' \in [0,1]$ and $0 \leq \alpha' < \alpha \leq 1$. We prove the last statement. Given $\varepsilon > 0$ there exists n^*

$$\overline{\varphi}_{n^*}^{\alpha,1}(\underline{x}) \geq \overline{\varphi}_{\infty}^{\alpha,1}(\underline{x}) - \varepsilon\,.$$

Since $\beta \mapsto \overline{\varphi}_{n^*}^{\alpha,\beta}(\underline{x})$ is continuous, there exists β' so that for $1 \leq \beta \leq \beta'$,

$$\overline{\varphi}_{n}^{\alpha,\beta}(\underline{x}) \geq \overline{\varphi}_{n^*}^{\alpha,\beta'}(\underline{x}) \geq \overline{\varphi}_{n^*}^{\alpha,1}(\underline{x}) - \varepsilon \quad \forall n \geq n^*\,.$$

Hence

$$\overline{\varphi}_{\infty}^{\alpha,1}(\underline{x}) - 2\varepsilon \leq \overline{\varphi}_{\infty}^{\alpha,\beta}(\underline{x}) \leq \overline{\varphi}_{\infty}^{\alpha,1}(\underline{x})\,.$$

□

Corollary 3.1 *Given $\underline{x}$ and $0 \leq \alpha^* \leq 1$, let*

$$g_{\alpha^*}(\gamma) := \overline{\varphi}_{\infty}^{\alpha^*,\beta(\gamma)}(\underline{x}) \quad \text{with} \quad \beta(\gamma) := \gamma - \alpha^* + \mathrm{k} - 1\,.$$

For $k \geq 3$ the map g_{α^*} is continuous and non-increasing on $[0,1]$. If $0 < g_{\alpha^*}(\gamma_0) < 1$, then the map is strictly decreasing in a neighborhood of γ_0. If $k = 2$ then the same statements hold on $[\alpha^*, 1]$.

Corollary 3.2 *Given $\underline{x}$ and $0 < \gamma^* \leq 1$, let*

$$h_{\gamma^*}(\alpha) := \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\underline{x})\text{with} \quad \beta(\alpha) := \gamma^* - \alpha + \mathrm{k} - 1\,.$$

For $k \geq 3$ the map h_{γ^*} is continuous and non-increasing on $[0,1]$. If $0 < h_{\gamma^*}(\alpha_0) < 1$, then the map is strictly decreasing in a neighborhood of α_0. If $k = 2$ then the same statements hold on $[0, \gamma^*)$.

Proposition 3.1 *Let $k \geq 2$, $\underline{u}, \underline{v} \in \mathtt{A}^{\mathbf{Z}_+}$ verifying $u_0 = 0$ and $v_0 = k - 1$ and*

$$\sigma\underline{u} \preceq \underline{v} \quad \text{and} \quad \underline{\mathrm{u}} \preceq \sigma\underline{\mathrm{v}}\,.$$

If $k = 2$ we also assume that $\sigma\underline{u} \preceq \sigma\underline{v}$. Then there exist $\bar{\alpha} \in [0,1]$ and $\bar{\beta} \in [1,\infty)$ so that $\bar{\gamma} \in [0,1]$. If $\bar{\beta} > 1$, then

$$\overline{\varphi}_{\infty}^{\bar{\alpha},\bar{\beta}}(\sigma\underline{u}) = \bar{\alpha} \quad \text{and} \quad \overline{\varphi}_{\infty}^{\bar{\alpha},\bar{\beta}}(\sigma\underline{\mathrm{v}}) = \bar{\gamma}\,.$$

Proof: We consider separately the cases $\sigma\underline{v} = \underline{0}$ and $\sigma\underline{u} = (k-1)^\infty$ (i.e. $u_j = k - 1$ for all $j \geq 1$). If $\sigma\underline{v} = \underline{0}$, then $\underline{u} = \underline{0}$ and $\underline{v} = (k-1)\underline{0}$; we set $\bar{\alpha} := 0$ and $\bar{\beta} := k-1$ ($\bar{\gamma} = 0$). If $\sigma\underline{u} = (k-1)^\infty$, then $\underline{v} = (k-1)^\infty$ and $\underline{u} = 0(k-1)^\infty$; we set $\bar{\alpha} := 1$ and $\bar{\beta} := k$.

From now on we assume that $\underline{0} \prec \sigma\underline{v}$ and $\sigma\underline{u} \prec (k-1)^\infty$. Set $\alpha_0 := 0$ and $\beta_0 := k$. We consider in details the case $k = 2$, so that we also assume that $\sigma\underline{u} \preceq \sigma\underline{v}$.
Step 1. Set $\alpha_1 := \alpha_0$ and solve the equation

$$\overline{\varphi}_{\infty}^{\alpha_1,\beta}(\sigma\underline{v}) = \beta + \alpha_1 - k + 1\,.$$

There exists a unique solution, β_1, such that $k - 1 < \beta_1 \leq k$. Indeed, the map

$$G_{\alpha_1}(\gamma) := g_{\alpha_1}(\gamma) - \gamma$$

with

$$g_{\alpha_1}(\gamma) := \overline{\varphi}_{\infty}^{\alpha_1,\beta(\gamma)}(\sigma\underline{v}) \text{ and } \beta(\gamma) := \gamma - \alpha_1 + \mathrm{k} - 1$$

is continuous and strictly decreasing on $[\alpha_1, 1]$ (see Corollary 3.1). If $\sigma\underline{v} = (k-1)^\infty$, then $G_{\alpha_1}(1) = 0$ and we set $\beta_1 := k$ and we have $\gamma_1 = 1$. If $\sigma\underline{v} \neq (k-1)^\infty$, then there exists a smallest $j \geq 1$ so that $v_j \leq (k-2)$. Therefore $\overline{\varphi}_\infty^{\alpha_1,k}(\sigma^j\underline{v}) < 1$ and

$$\overline{\varphi}_\infty^{\alpha_1,k}(\sigma\underline{v}) = \overline{\varphi}_{j-1}^{\alpha_1,k}(v_1, \ldots, v_{j-1} + \overline{\varphi}_\infty^{\alpha_1,k}(\sigma^j\underline{v})) < 1\,,$$

so that $G_{\alpha_1}(1) < 0$. On the other hand, since $\sigma\underline{v} \neq \underline{0}$, $\overline{\varphi}_\infty^{\alpha_1,k-1}(\sigma\underline{v}) > 0$, so that $G_{\alpha_1}(0) > 0$. There exists a unique $\gamma_1 \in (0,1)$ with $G_{\alpha_1}(\gamma_1) = 0$. Define $\beta_1 := \beta(\gamma_1) = \gamma_1 - \alpha_1 + k - 1$.

Step 2. Solve in $[0, \gamma_1)$ the equation

$$\overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{u}) = \alpha \quad \text{with} \quad \beta(\alpha) := \gamma_1 - \alpha + \mathrm{k} - 1 = \beta_1 + \alpha_1 - \alpha\,.$$

If $\sigma\underline{u} = 0$, then set $\bar{\alpha} := 0$ and $\bar{\beta} := \beta_1$. Let $\sigma\underline{u} \neq 0$. There exists a smallest $j \geq 1$ such that $u_j \geq 1$. This implies that $\overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma^j\underline{u}) > 0$ and consequently

$$\overline{\varphi}_\infty^{\alpha_1,\beta(\alpha_1)}(\sigma\underline{u}) = \overline{\varphi}_{j-1}^{\alpha_1,\beta_1}(u_1, \ldots, u_{j-1} + \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma^j\underline{u})) > 0\,.$$

Since $\sigma\underline{u} \preceq \sigma\underline{v}$,

$$0 < \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{u}) \leq \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{v}) = \gamma_1\,.$$

We have $\gamma_1 = 1$ only in the case $\sigma\underline{v} = (k-1)^\infty$; in that case we also have $\overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{u}) < 1$. By Corollary 3.2, for any $\alpha > \alpha_1$ we have $\overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{u}) > \overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{u})$. Therefore, the map

$$H_{\gamma_1}(\alpha) := h_{\gamma_1}(\alpha) - \alpha \text{with} \quad \mathrm{h}_{\gamma_1}(\alpha) := \overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{\mathrm{u}})$$

is continuous and strictly decreasing on $[0, \gamma_1)$, $H_{\gamma_1}(\alpha_1) > 0$ and $\lim_{\alpha\uparrow\gamma_1} H_{\gamma_1}(\alpha) < 0$. There exists a unique $\alpha_2 \in (\alpha_1, \gamma_1)$ such that $H_{\gamma_1}(\alpha_2) = 0$. Set $\beta_2 := \gamma_1 - \alpha_2 + k - 1 = \alpha_1 + \beta_1 - \alpha_2$ and $\gamma_2 := \alpha_2 + \beta_2 - k + 1 = \gamma_1$. Since $\alpha_2 \in [0, \gamma_1)$, we have $\beta_2 > 1$. Hence

$$\alpha_1 < \alpha_2 < \gamma_1 \quad \text{and} \quad 1 < \beta_2 < \beta_1 \quad \text{and} \quad \gamma_2 = \gamma_1\,. \tag{3.3}$$

If $\sigma\underline{v} = (k-1)^\infty$, $\gamma_2 = 1$ and we set $\bar{\alpha} := \alpha_2$ and $\bar{\beta} := \beta_2$.

Step 3. From now on $\sigma\underline{u} \neq \underline{0}$ and $\sigma\underline{v} \neq (k-1)^\infty$. Set $\alpha_3 := \alpha_2$ and solve in $[\alpha_3, 1]$ the equation

$$\overline{\varphi}_\infty^{\alpha_3,\beta(\gamma)}(\sigma\underline{v}) = \gamma \text{with} \quad \beta(\gamma) := \gamma - \alpha_3 + \mathrm{k} - 1\,.$$

By Lemma 3.5 ($k = 2$),

$$\overline{\varphi}_\infty^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{v}) = \overline{\varphi}_\infty^{\alpha_2,1}(\sigma\underline{v}) \geq \overline{\varphi}_\infty^{\alpha_2,1}(\sigma\underline{u}) > \overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{u}) = \alpha_2\,,$$

since $0 < \alpha_2 < 1$. On the other hand by Corollary 3.2,

$$\overline{\varphi}_\infty^{\alpha_3,\beta(\gamma_1)}(\sigma\underline{v}) = \overline{\varphi}_\infty^{\alpha_3,1+\gamma_1-\alpha_3}(\sigma\underline{v}) < \overline{\varphi}_\infty^{\alpha_1,1+\gamma_1-\alpha_1}(\sigma\underline{v}) = \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{v}) = \gamma_1 \tag{3.4}$$

since $0 < \gamma_1 < 1$. Therefore, the map G_{α_3} is continuous and strictly decreasing on $[\alpha_3, 1]$, $G_{\alpha_3}(\alpha_3) > 0$ and $G_{\alpha_3}(\gamma_1) < 0$. There exists a unique $\gamma_3 \in (\alpha_3, \gamma_1)$ such that $G_{\alpha_3}(\gamma_3) = 0$. Set $\beta_3 := \gamma_3 - \alpha_3 + k - 1$, so that $\beta_3 < \gamma_1 - \alpha_2 + k - 1 = \beta_2$. Hence

$$\alpha_3 = \alpha_2 \quad \text{and} \quad 1 < \beta_3 < \beta_2 \quad \text{and} \quad 0 < \gamma_3 < \gamma_2 < 1\,. \tag{3.5}$$

Step 4. Solve in $[0, \gamma_3)$ the equation

$$\overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{u}) = \alpha \quad \text{with} \quad \beta(\alpha) := \gamma_3 - \alpha + \mathrm{k} - 1 = \beta_3 + \alpha_3 - \alpha\,.$$

By Lemma 3.5

$$\overline{\varphi}_{\infty}^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{u}) = \overline{\varphi}_{\infty}^{\alpha_3,\beta_3}(\sigma\underline{u}) > \overline{\varphi}_{\infty}^{\alpha_3,\beta_2}(\sigma\underline{u}) = \overline{\varphi}_{\infty}^{\alpha_2,\beta_2}(\sigma\underline{u}) = \alpha_2\,, \tag{3.6}$$

since $0 < \alpha_2 < 1$. On the other hand,

$$0 < \overline{\varphi}_{\infty}^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{u}) = \overline{\varphi}_{\infty}^{\alpha_3,\beta_3}(\sigma\underline{u}) \leq \overline{\varphi}_{\infty}^{\alpha_3,\beta_3}(\sigma\underline{v}) = \gamma_3 < 1\,.$$

By Corollary 3.2

$$\overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{u}) < \overline{\varphi}_{\infty}^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{u}) \quad \forall\alpha \in (\alpha_3,\gamma_3)\,.$$

Therefore, the map

$$H_{\gamma_3}(\alpha) := h_{\gamma_3}(\alpha) - \alpha \quad \text{with} \quad \mathrm{h}_{\gamma_3}(\alpha) := \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\mathrm{\underline{u}})$$

is continuous and strictly decreasing on $[\alpha_3,\gamma_3)$, $H_{\gamma_3}(\alpha_3) > 0$ and $\lim_{\alpha\uparrow\gamma_3} H_{\gamma_3}(\alpha) < 0$. There exists a unique $\alpha_4 \in (\alpha_3,\gamma_3)$. Set $\beta_4 := \gamma_3 - \alpha_4 + k - 1 = \alpha_3 + \beta_3 - \alpha_4$ and $\gamma_4 := \alpha_4 + \beta_4 - k + 1 = \gamma_3$. Hence

$$\alpha_3 < \alpha_4 < \gamma_3 \quad \text{and} \quad 1 < \beta_4 < \beta_3 \quad \text{and} \quad \gamma_4 = \gamma_3\,. \tag{3.7}$$

Repeating steps 3 and 4 we get two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$. We set $\bar{\alpha} := \lim_{n\to\infty}\alpha_n$ and $\bar{\beta} := \lim_{n\to\infty}\beta_n$.

We consider briefly the changes which occur when $k \geq 3$. Step 1 remains the same. In step 2 we solve the equation $H_{\gamma_1}(\alpha) = 0$ on $[0,1)$ instead of $[0,\gamma_1)$. The proof that $H_{\gamma_1}(\alpha_1) > 0$ remains the same. We prove that $\lim_{\alpha\uparrow 1} H_{\gamma_1}(\alpha) < 0$. Corollary 3.2 implies that

$$\gamma_1 = \overline{\varphi}_{\infty}^{\alpha_1,\beta_1}(\sigma\underline{v}) = \overline{\varphi}_{\infty}^{\alpha_1,\beta(\alpha_1)}(\sigma\underline{v}) > \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{v}) \quad \forall\alpha > \alpha_1\,.$$

Since $\sigma\underline{u} \preceq \underline{v}$ and $\beta(\alpha_1) = \beta_1$,

$$\overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{u}) \leq \overline{\varphi}^{\alpha,\beta(\alpha)}(v_0 + \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{v})) \leq \overline{\varphi}^{\alpha_1,\beta(\alpha_1)}(v_0 + \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{v})) < 1\,.$$

Instead of (3.3) we have

$$\alpha_1 < \alpha_2 < 1 \quad \text{and} \quad 1 < \beta_2 < \beta_1 \quad \text{and} \quad \gamma_2 = \gamma_1\,.$$

Estimate (3.4) is still valid in step 3 with $k \geq 3$. Hence $G_{\alpha_3}(\gamma_1) < 0$. We solve the equation $G_{\alpha_3}(\gamma) = 0$ on $[0,\gamma_1]$. We have

$$\overline{\varphi}_{\infty}^{\alpha_3,\beta(\gamma_1)}(\sigma\underline{u}) = \overline{\varphi}_{\infty}^{\alpha_2,\beta_2}(\sigma\underline{u}) = \alpha_2\,.$$

By Corollary 3.1 we get

$$\overline{\varphi}_{\infty}^{\alpha_3,\beta(\gamma)}(\sigma\underline{u}) > \overline{\varphi}_{\infty}^{\alpha_2,\beta_2}(\sigma\underline{u}) = \alpha_2 \quad \forall\gamma < \gamma_1\,.$$

Since $\underline{u} \preceq \sigma\underline{v}$,

$$\begin{aligned}\overline{\varphi}_{\infty}^{\alpha_3,\beta(0)}(\sigma\underline{v}) &\geq \overline{\varphi}^{\alpha_3,\beta(0)}(u_0 + \overline{\varphi}_{\infty}^{\alpha_3,\beta(0)}(\sigma\underline{u})) \\ &\geq \overline{\varphi}^{\alpha_2,\beta(\gamma_1)}(u_0 + \overline{\varphi}_{\infty}^{\alpha_3,\beta(0)}(\sigma\underline{u})) > 0\,.\end{aligned}$$

Estimate (3.6) is still valid in step 4 so that $H_{\gamma_3}(\alpha_3) > 0$. Corollary 3.2 implies that

$$\gamma_3 = \overline{\varphi}_{\infty}^{\alpha_3,\beta_3}(\sigma\underline{v}) = \overline{\varphi}_{\infty}^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{v}) > \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{v}) \quad \forall\alpha > \alpha_3\,.$$

Therefore

$$\overline{\varphi}_\infty^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{u}) \leq \overline{\varphi}^{\alpha,\beta(\alpha)}(v_0 + \overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{v})) \leq \overline{\varphi}^{\alpha_3,\beta(\alpha_3)}(v_0 + \overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{v})) < 1\,.$$

Instead of (3.7) we have

$$\alpha_3 < \alpha_4 < 1 \quad \text{and} \quad 1 < \beta_4 < \beta_3 \quad \text{and} \quad \gamma_4 = \gamma_3\,.$$

Assume that $\bar{\beta} > 1$. Then $1 < \bar{\beta} \leq \beta_n$ for all n. We have

$$\overline{\varphi}_\infty^{\alpha_n,\beta_n}(\sigma\underline{v}) = \gamma_n\,, \quad n \text{ odd}$$

and

$$\overline{\varphi}_\infty^{\alpha_n,\beta_n}(\sigma\underline{u}) = \alpha_n\,, \quad n \text{ even}\,.$$

Let $\bar{\gamma} = \bar{\alpha} + \bar{\beta} - k + 1$. For n odd, let $\beta_n^* := \bar{\gamma} - \alpha_n + k - 1$; using Lemma 3.6 we get

$$\begin{aligned}|\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma\underline{v}) - \bar{\gamma}| &\leq |\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma\underline{v}) - \overline{\varphi}_\infty^{\alpha_n,\beta_n^*}(\sigma\underline{v})| + |\overline{\varphi}_\infty^{\alpha_n,\beta_n^*}(\sigma\underline{v}) - \overline{\varphi}_\infty^{\alpha_n,\beta_n}(\sigma\underline{v})| \\ &\quad + |\gamma_n - \gamma| \\ &\leq \frac{1}{\bar{\beta} - 1}(2|\bar{\alpha} - \alpha_n| + |\bar{\beta} - \beta_n|) + |\gamma_n - \gamma|\,,\end{aligned}$$

since $\beta_n^* = \bar{\beta} + \bar{\alpha} - \alpha_n$. Letting n going to infinity we get $\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma\underline{v}) = \bar{\gamma}$. Similarly we prove $\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma\underline{v}) = \bar{\alpha}$. □

Corollary 3.3 *Suppose that $(\underline{u}, \underline{v})$, respectively $(\underline{u}', \underline{v}')$, verify the hypothesis of Proposition 3.1 with $k \geq 2$, respectively with $k' \geq 2$. If $k \geq k'$, $\underline{u} \preceq \underline{u}'$ and $\underline{v}' \preceq \underline{v}$, then $\bar{\beta}' \leq \bar{\beta}$ and $\bar{\alpha}' \geq \bar{\alpha}$.*

Proof: We consider the case $k = k'$, whence $\sigma\underline{v}' \preceq \sigma\underline{v}$. From the proof of Proposition 3.1 we get $\gamma_1' \leq \gamma_1$ and $\alpha_1' \geq \alpha_1$. Suppose that $\gamma_j' \leq \gamma_j$ and $\alpha_j' \geq \alpha_j$ for $j = 1, \ldots, n$. If n is even, then $\alpha_{n+1}' = \alpha_n'$ and $\alpha_{n+1} = \alpha_n$. We prove that $\gamma_{n+1}' \leq \gamma_{n+1}$. We have

$$\begin{aligned}\gamma_{n+1}' = \overline{\varphi}_\infty^{\alpha_{n+1}',\beta(\gamma_{n+1}')}(\sigma\underline{v}') \leq \overline{\varphi}_\infty^{\alpha_{n+1}',\beta(\gamma_{n+1}')}(\sigma\underline{v}) \leq \overline{\varphi}_\infty^{\alpha_{n+1},\beta(\gamma_{n+1}')}(\sigma\underline{v}) \\ \implies \gamma_{n+1} \geq \gamma_{n+1}'\,.\end{aligned}$$

If n is odd, then $\gamma_{n+1}' = \gamma_n'$ and $\gamma_{n+1} = \gamma_n$. We prove that $\alpha_{n+1}' \geq \alpha_{n+1}$. We have

$$\begin{aligned}\alpha_{n+1} = \overline{\varphi}_\infty^{\alpha_{n+1},\beta(\alpha_{n+1})}(\sigma\underline{u}) \leq \overline{\varphi}_\infty^{\alpha_{n+1},\beta(\alpha_{n+1})}(\sigma\underline{u}') = \overline{\varphi}_\infty^{\alpha_{n+1},\gamma_{n+1}-\alpha_{n+1}+k-1)}(\sigma\underline{u}') \\ \leq \overline{\varphi}_\infty^{\alpha_{n+1},\gamma_{n+1}'-\alpha_{n+1}+k-1)}(\sigma\underline{u}') \implies \alpha_{n+1}' \geq \alpha_{n+1}\,.\end{aligned}$$

□

We state a uniqueness result. The proof uses Theorem 3.1.

Proposition 3.2 *Let $k \geq 2$, $\underline{u}, \underline{v} \in \mathtt{A}^{\mathbf{Z}_+}$, $u_0 = 0$ and $v_0 = k - 1$, and assume that (2.22) holds. Then there is at most one solution $(\alpha, \beta) \in [0, 1] \times [1, \infty)$ for the equations*

$$\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{u}) = \alpha \quad \text{and} \quad \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{v}) = \gamma\,.$$

Proof: Assume that there are two solutions (α_1, β_1) and (α_2, β_2) with $\beta_1 \leq \beta_2$. If $\alpha_2 > \alpha_1$, then

$$\alpha_2 - \alpha_1 = \overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{u}) - \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{u}) \leq 0\,,$$

which is impossible. Therefore $\alpha_2 \leq \alpha_1$. If $\beta_1 = \beta_2$, then

$$0 \geq \alpha_2 - \alpha_1 = \overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{u}) - \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{u}) \geq 0\,,$$

which implies $\alpha_2 = \alpha_1$. Therefore we assume that $\alpha_2 \leq \alpha_1$ and $\beta_1 < \beta_2$. However, Theorem 3.1 implies that

$$\log_2 \beta_1 = h(\mathbf{\Sigma}(\underline{u}, \underline{v})) = \log_2 \beta_2\,,$$

which is impossible. □

3.3. Computation of the topological entropy of $\mathbf{\Sigma}(\underline{u}, \underline{v})$

We compute the entropy of the shift space $\mathbf{\Sigma}(\underline{u}, \underline{v})$ where $\underline{u}$ and $\underline{v}$ is a pair of strings verifying $u_0 = 0$, $v_0 = k-1$ and (2.22). The main result is Theorem 3.1. The idea for computing the topological entropy is to compute $\bar{\alpha}$ and $\bar{\beta}$ by the algorithm of section 3.2 and to use the fact that $h(\mathbf{\Sigma}(\underline{u}^{\bar{\alpha},\bar{\beta}}, \underline{v}^{\bar{\alpha},\bar{\beta}})) = \log_2 \bar{\beta}$ (see e.g. [Ho1]). The most difficult case is when $\underline{u}$ and $\underline{v}$ are both periodic. Assume that the string $\underline{u} := \underline{a}^\infty$ has minimal period p, $|\underline{a}| = p$, and that the string $\underline{v} := \underline{b}^\infty$ has minimal period q, $|\underline{b}| = q$. If $a_0 = a_{p-1} = 0$, then $\underline{u} = \underline{0}$ and $p = 1$. Indeed, if $a_0 = a_{p-1} = 0$, then $\underline{a}\underline{a} = (\mathtt{p}\underline{a})00(\mathtt{s}\underline{a})$; the result follows from (2.22). Similarly, if $b_0 = b_{q-1} = k-1$, then $\underline{v} = (k-1)^\infty$ and $q = 1$. These cases are similar to the case when only one of the strings $\underline{u}$ and $\underline{v}$ is periodic and are simpler than the generic case of two periodic strings, which we treat in details.

The setting for subsection 3.3 is the following one. The string $\underline{u} := \underline{a}^\infty$ has minimal period $p \geq 2$ with $u_0 = 0$, or $\underline{u} = (1)^\infty$. The string $\underline{v} := \underline{b}^\infty$ has minimal period $q \geq 2$ with $v_0 = k-1$, or $\underline{v} = (k-2)^\infty$. We also consider the strings $\underline{u}^* = \underline{a}'\underline{b}^\infty$ and $\underline{v}^* = \underline{b}'\underline{a}^\infty$ with $\underline{a}' = \mathtt{p}\underline{a}(u_{p-1} - 1)$ and $\underline{b}' = \mathtt{p}\underline{b}(v_{q-1} + 1)$. We write $\mathbf{\Sigma} \equiv \mathbf{\Sigma}(\underline{u}, \underline{v})$, $\mathbf{\Sigma}^* \equiv \mathbf{\Sigma}(\underline{u}^*, \underline{v}^*)$, $\mathcal{G} \equiv \mathcal{G}(\underline{u}, \underline{v})$ and $\mathcal{G}^* \equiv \mathcal{G}(\underline{u}^*, \underline{v}^*)$. The main point is to prove that $h(\mathcal{G}) = h(\mathcal{G}^*)$ by comparing the follower-set graphs $\mathcal{G}$ and $\mathcal{G}^*$.

Lemma 3.7 *1) In the above setting the vertices of the graph* $\mathcal{G}$ *are* $\mathcal{F}_\epsilon$, $\mathcal{F}_{\underline{w}}$ *with* $\underline{w}$ *a prefix of* $\mathtt{p}\underline{a}$ *or of* $\mathtt{p}\underline{b}$, $\mathtt{p}\underline{a}$ *and* $\mathtt{p}\underline{b}$ *included.*
2) Let $r := |v(\mathtt{p}\underline{a})|$. *If* $u_{p-1} \neq v_r$, *then* $\mathcal{F}_{\underline{a}} = \mathcal{F}_\epsilon$ *and there is an edge labeled by* u_{p-1} *from* $\mathcal{F}_{\mathtt{p}\underline{a}}$ *to* $\mathcal{F}_\epsilon$. *If* $u_{p-1} = v_r$, *then* $\mathcal{F}_{\underline{a}} = \mathcal{F}_{v(\mathtt{p}\underline{a})v_r}$ *and there is a single edge, labeled by* $u_{p-1} = v_r$, *from* $\mathcal{F}_{\mathtt{p}\underline{a}}$ *to* $\mathcal{F}_{v(\mathtt{p}\underline{a})v_r}$. *If* $k = 2$ *the first possibility is excluded.*
3) Let $s := |u(\mathtt{p}\underline{b})|$. *If* $v_{q-1} \neq u_s$, *then* $\mathcal{F}_{\underline{b}} = \mathcal{F}_\epsilon$ *and there is an edge labeled by* v_{q-1} *from* $\mathcal{F}_{\mathtt{p}\underline{b}}$ *to* $\mathcal{F}_\epsilon$. *If* $v_{q-1} = u_s$, *then* $\mathcal{F}_{\underline{b}} = \mathcal{F}_{u(\mathtt{p}\underline{b})u_s}$ *and there is a single edge, labeled by* $v_{q-1} = u_s$, *from* $\mathcal{F}_{\mathtt{p}\underline{b}}$ *to* $\mathcal{F}_{u(\mathtt{p}\underline{b})u_s}$. *If* $k = 2$ *the first possibility is excluded.*

Proof: Suppose that $\underline{w}$ and $\underline{ww'}$ are two prefixes of $\mathtt{p}\underline{a}$. We show that $\mathcal{F}_{\underline{w}} \neq \mathcal{F}_{\underline{ww'}}$. Write $\underline{u} = \underline{wx}$ and $\underline{u} = \underline{ww'y}$ and suppose that $\mathcal{F}_{\underline{w}} = \mathcal{F}_{\underline{ww'}}$. Then (see (3.2)) $\underline{x} = \underline{y} = \sigma^p\underline{u}$, so that $\underline{u} = (\underline{w}')^\infty$, contradicting the minimality of the period p. Consider the vertex $\mathcal{F}_{\mathtt{p}\underline{a}}$ of $\mathcal{G}$. We have

$$\mathcal{F}_{\mathtt{p}\underline{a}} = \{\underline{x} \in \mathbf{\Sigma} \colon \sigma^{p-1}\underline{u} \preceq \underline{x} \preceq \sigma^r \underline{v}\} \quad \text{where r} = |\text{v}(\text{p}\underline{\text{a}})|\,.$$

Let $\underline{d}$ be the prefix of $\underline{v}$ of length $r+1$, so that $\mathtt{p}\underline{d} = v(\mathtt{p}\underline{a})$. If $u_{p-1} \neq v_r$, then there are an edge labeled by u_{p-1} from $\mathcal{F}_{\mathtt{p}\underline{a}}$ to $\mathcal{F}_{\underline{a}} = \mathcal{F}_\epsilon$ (since $\sigma^p\underline{u} = \underline{u}$) and an edge labeled by v_r from $\mathcal{F}_{\mathtt{p}\underline{a}}$ to $\mathcal{F}_{\underline{d}}$. There may be other labeled edges from $\mathcal{F}_{\mathtt{p}\underline{a}}$ to $\mathcal{F}_\epsilon$ (see Lemma 3.3). If $u_{p-1} = v_r$, then there is a single out-going edge labeled by u_{p-1} from $\mathcal{F}_{\mathtt{p}\underline{a}}$ to $\mathcal{F}_{\underline{a}}$ and $v(\underline{a}) = \underline{d}$. We prove that $\mathcal{F}_{\underline{a}} = \mathcal{F}_{\underline{d}}$. If $u(\underline{d}) = \epsilon$, the result is true, since in that case

$$\mathcal{F}_{\underline{d}} = \{\underline{x} \in \mathbf{\Sigma} \colon \underline{u} \preceq \underline{x} \preceq \sigma^{r+1}\underline{v}\} = \{\underline{x} \in \mathbf{\Sigma} \colon \sigma^p\underline{u} \preceq \underline{x} \preceq \sigma^{r+1}\underline{v}\} = \mathcal{F}_{\underline{a}}\,.$$

We exclude the possibility $u(\underline{d}) \neq \epsilon$. Suppose that $\underline{w} := u(\underline{d})$ is non-trivial ($|u(\underline{d})| < p$). We can write $\underline{a} = \underline{a}''\underline{w}$ and $\underline{a} = \underline{w}\widehat{\underline{a}}$ since $\underline{w}$ is a prefix of $\underline{u}$, and consequently $\underline{aa} = \underline{a}''\underline{ww}\widehat{\underline{a}}$. From Lemma 3.1 we conclude that $\underline{ww}$ is a prefix of $\underline{u}$, so that $\underline{au} = \underline{a}''\underline{www}\cdots$, proving that $\underline{u}$ has period $|\underline{w}|$, contradicting the hypothesis that p is the minimal period of $\underline{u}$. If $k = 2$ the first possibility is excluded because $u_{p-1} \neq 0$ and we have $u_{p-1} \preceq v_r$ by $\sigma^{p-r}\underline{u} \preceq \underline{v}$. The discussion concerning the vertex $\mathcal{F}_{\mathtt{p}\underline{b}}$ is similar. □

Proposition 3.3 *Consider the above setting. If $h(\mathbf{\Sigma}) > 0$, then $h(\mathbf{\Sigma}) = h(\mathbf{\Sigma}^*)$.*

Proof: Consider the vertex $\mathcal{F}_{\mathtt{p}\underline{a}}$ of $\mathcal{G}^*$. In that case $(u_{p-1} - 1) \neq v_r$ so that we have an additional edge labeled by $u_{p-1} - 1$ from $\mathcal{F}_{\mathtt{p}\underline{a}}$ to $\mathcal{F}_{\underline{a}'}$ (see proof of Lemma 3.7), otherwise all out-going edges from $\mathcal{F}_{\mathtt{p}\underline{a}}$, which are present in the graph $\mathcal{G}$, are also present in $\mathcal{G}^*$. Let $v^*(\underline{w})$ be the longest suffix of $\underline{w}$, which is a prefix of $\underline{v}^*$. Then

$$\mathcal{F}_{\underline{a}'} = \{\underline{x} \in \mathbf{\Sigma}^* \colon \sigma^p\underline{u}^* \preceq \underline{x} \preceq \underline{v}^*\} = \{\underline{x} \in \mathbf{\Sigma}^* \colon \underline{v} \preceq \underline{x} \preceq \underline{v}^*\}\,.$$

Similarly, there is an additional edge labeled by $v_{q-1} + 1$ from $\mathcal{F}_{\mathtt{p}\underline{b}}$ to $\mathcal{F}_{\underline{b}'}$. Let $u^*(\underline{w})$ be the longest suffix of $\underline{w}$, which is a prefix of $\underline{u}^*$. Then

$$\mathcal{F}_{\underline{b}'} = \{\underline{x} \in \mathbf{\Sigma}^* \colon \underline{u}^* \preceq \underline{x} \preceq \sigma^q\underline{v}^*\} = \{\underline{x} \in \mathbf{\Sigma}^* \colon \underline{u}^* \preceq \underline{x} \preceq \underline{u}\}\,.$$

The structure of the graph $\mathcal{G}^*$ is very simple from the vertices $\mathcal{F}_{\underline{a}'}$ and $\mathcal{F}_{\underline{b}'}$. There is a single out-going edge from $\mathcal{F}_{\underline{a}'}$ to $\mathcal{F}_{\underline{a}'v_0}$, from $\mathcal{F}_{\underline{a}'v_0}$ to $\mathcal{F}_{\underline{a}'v_0v_1}$ and so on, until we reach the vertex $\mathcal{F}_{\underline{a}'\mathtt{p}\underline{b}}$. From that vertex there are an out-going edge labeled by v_{q-1} to $\mathcal{F}_{\underline{a}'}$ and an out-going edge labeled by $v_{q-1} + 1$ to $\mathcal{F}_{\underline{b}'}$. Similarly, there is a single out-going edge from $\mathcal{F}_{\underline{b}'}$ to $\mathcal{F}_{\underline{b}'u_0}$, from $\mathcal{F}_{\underline{b}'u_0}$ to $\mathcal{F}_{\underline{b}'u_0u_1}$ and so on, until we reach the vertex $\mathcal{F}_{\underline{b}'\mathtt{p}\underline{a}}$. From that vertex there are an out-going edge labeled by u_{p-1} to $\mathcal{F}_{\underline{b}'}$ and an out-going edge labeled by $u_{p-1} - 1$ to $\mathcal{F}_{\underline{a}'}$. Let us denote that part of $\mathcal{G}^*$ by $\mathcal{G}^*\backslash\mathcal{G}$. This subgraph is strongly connected. The graph $\mathcal{G}^*$ consists of the union of $\mathcal{G}$ and $\mathcal{G}^*\backslash\mathcal{G}$ with the addition of the two edges from $\mathcal{F}_{\mathtt{p}\underline{a}}$ to $\mathcal{F}_{\underline{a}'}$ and $\mathcal{F}_{\mathtt{p}\underline{b}}$ to $\mathcal{F}_{\underline{b}'}$. Using Theorem 1.7 of [BGMY] it easy to compute the entropy of the subgraph $\mathcal{G}^*\backslash\mathcal{G}$ (use as rome $\{\mathcal{F}_{\underline{a}'}, \mathcal{F}_{\underline{b}'}\}$). It is the largest root, say λ^*, of the equation

$$\lambda^{-q} + \lambda^{-p} - 1 = 0\,.$$

Hence λ^* is equal to the entropy of a graph with two cycles of periods p and q, rooted at a common point. To prove Proposition 3.3 it sufficient to exhibit a subgraph of $\mathcal{G}$ which has an entropy larger or equal to that of $\mathcal{G}^*\backslash\mathcal{G}$.

If $k \geq 4$, then there is a subgraph with two cycles of length 1 rooted at $\mathcal{F}_\epsilon$. Hence $h(\mathcal{G}) \geq \log_2 2 > \lambda^*$. If $\mathcal{F}_{\underline{a}} = \mathcal{F}_\epsilon$ or $\mathcal{F}_{\underline{b}} = \mathcal{F}_\epsilon$, which could happen only for $k \geq 3$ (see Lemma 3.7), then there is a subgraph of $\mathcal{G}$ consisting of two cycles rooted at $\mathcal{F}_\epsilon$, one of length p or of length q and another one of length 1. This also implies that $h(\mathcal{G}) \geq \lambda^*$. Since the minimal periods of $\underline{u}$ and $\underline{v}$ are p and q, it is impossible that $\mathcal{F}_{\underline{w}} = \mathcal{F}_\epsilon$ for $\underline{w}$ a non trivial prefix of $\mathtt{p}\underline{a}$ or $\mathtt{p}\underline{b}$. Therefore we assume that $k \leq 3$, $\mathcal{F}_{\underline{a}} \neq \mathcal{F}_\epsilon$ and $\mathcal{F}_{\underline{b}} \neq \mathcal{F}_\epsilon$.

Let $\mathcal{H}$ be a strongly connected component of $\mathcal{G}$ which has strictly positive entropy. If $\mathcal{F}_\epsilon$ is a vertex of $\mathcal{H}$, which happens only if $k = 3$, then we conclude as above that $h(\mathcal{G}) \geq \lambda^*$. Hence, we assume that $\mathcal{F}_\epsilon$ is not a vertex of $\mathcal{H}$. The vertices of $\mathcal{H}$ are indexed by prefixes of $\mathtt{p}\underline{a}$ and $\mathtt{p}\underline{b}$. Let $\mathcal{F}_{\underline{c}}$ be the vertex of $\mathcal{H}$ with $\underline{c}$ a prefix of $\underline{u}$ and $|\underline{c}|$ minimal; similarly, let $\mathcal{F}_{\underline{d}}$ be the vertex of $\mathcal{H}$ with $\underline{d}$ a prefix of $\underline{v}$ and $|\underline{d}|$ minimal. By our assumptions $r := |\underline{c}| \geq 1$ and $s := |\underline{d}| \geq 1$. The following argument is a simplified adaptation of the proof of Lemma 3 in [Ho3]. The core of the argument is the content of the Scholium 3.1. Consider the v-parsing of $\underline{a}$ from the prefix $\mathtt{p}\underline{c}$, and the u-parsing of $\underline{b}$ from the prefix $\mathtt{p}\underline{d}$,

$$\underline{a} = (\mathtt{p}\underline{c})\underline{a}^1 \cdots \underline{a}^k \quad \text{and} \quad \underline{\mathrm{b}} = (\mathtt{p}\underline{\mathrm{d}})\underline{\mathrm{b}}^1 \cdots \underline{\mathrm{b}}^\ell \,.$$

(From $\mathtt{p}\underline{c}$ the v-parsing of $\underline{a}$ does not depend on $\mathtt{p}\underline{c}$ since there is an in-going edge at $\mathcal{F}_{\underline{c}}$.)

We claim that there are an edge from $\mathcal{F}_{(\mathtt{p}\underline{c})\underline{a}^1}$ to $\mathcal{F}_{\underline{d}}$ and an edge from $\mathcal{F}_{(\mathtt{p}\underline{d})\underline{b}^1}$ to $\mathcal{F}_{\underline{c}}$. Suppose that this is not the case, for example, there is an edge from $\mathcal{F}_{(\mathtt{p}\underline{c})\underline{a}^1\cdots\underline{a}^j}$ to $\mathcal{F}_{\underline{d}}$, but no edge from $\mathcal{F}_{(\mathtt{p}\underline{c})\underline{a}^1\cdots\underline{a}^i}$ to $\mathcal{F}_{\underline{d}}$, $1 \leq i < j$. This implies that $v(\underline{a}^j) = \mathtt{s}\underline{a}^j = \mathtt{p}(\underline{d})$ and $(\mathtt{p}\underline{c})\underline{a}^1 \cdots \underline{a}^j f'$ is a prefix of $\underline{u}$ with $f' \prec f$ and f defined by $\underline{d} = (\mathtt{p}\underline{d})f$. On the other hand there exists an edge from $\mathcal{F}_{(\mathtt{p}\underline{c})\underline{a}^1\cdots\underline{a}^{j-1}}$ to $\mathcal{F}_{v((\mathtt{p}\underline{c})\underline{a}^1\cdots\underline{a}^{j-1})*} = \mathcal{F}_{v(\underline{a}^{j-1})*}$ with $*$ some letter of $\mathtt{A}$ and $v(\underline{a}^{j-1})* \neq \underline{d}$ by hypothesis. Let e be the first letter of $\underline{a}^j$. Then $* = (e+1)$ since we assume that $\mathcal{F}_\epsilon$ is not a vertex of $\mathcal{H}$ and consequently there are only two out-going edges from $\mathcal{F}_{(\mathtt{p}\underline{c})\underline{a}^1\cdots\underline{a}^{j-1}}$. There exists an edge from $\mathcal{F}_{v(\underline{a}^{j-1})}$ to $\mathcal{F}_{u(v(\underline{a}^{j-1}))*}$, where $*$ is some letter of $\mathtt{A}$ (see Scholium 3.1). Again, since $\mathcal{F}_\epsilon$ is not a vertex of $\mathcal{H}$ we must have $* = e$. Either $u(v(\underline{a}^{j-1}))e = \underline{c}$ or $u(v(\underline{a}^{j-1}))e \neq \underline{c}$. In the latter case, by the same reasoning, there exists an edge from $\mathcal{F}_{u(v(\underline{a}^{j-1}))}$ to $\mathcal{F}_{v(u(v(\underline{a}^{j-1})))(e+1)}$ and $v(u(v(\underline{a}^{j-1})))(e+1) \neq \underline{d}$ by hypothesis; there exists also an edge from $\mathcal{F}_{v(u(v(\underline{a}^{j-1})))}$ to $\mathcal{F}_{u(v(u(v(\underline{a}^{j-1}))))e}$. After a finite number of steps we get

$$u(\cdots v(u(v(\underline{a}^{j-1}))))e = \underline{c}\,.$$

This implies that $\mathtt{p}\underline{c}$ is a suffix of $\underline{a}^{j-1}$, and the last letter of $\underline{c}$ (or the first letter of $\underline{a}^1$) is e. Hence $\underline{a}^1 = e\underline{d}\cdots$. If we write $\underline{a}^{j-1} = \underline{g}(\mathtt{p}\underline{c})$ we have

$$(\mathtt{p}\underline{c})\underline{a}^1 = \underline{c}\underline{d}\cdots = \underline{c}(\mathtt{p}\underline{d})f\cdots \quad \text{and} \quad \underline{\mathrm{a}}^{j-1}\underline{\mathrm{a}}^{j}\mathrm{f} = \underline{\mathrm{g}}(\mathtt{p}\underline{\mathrm{c}})\mathrm{e}(\mathtt{p}\underline{\mathrm{d}})\mathrm{f}' = \underline{\mathrm{g}}\underline{\mathrm{c}}(\mathtt{p}\underline{\mathrm{d}})\mathrm{f}'\,.$$

We get a contradiction with (2.22) since $\underline{c}(\mathtt{p}\underline{d})f' \prec \underline{c}(\mathtt{p}\underline{d})f$.

Consider the smallest strongly connected subgraph $\mathcal{H}'$ of $\mathcal{H}$ which contains the vertices $\mathcal{F}_{\underline{c}}$, $\mathcal{F}_{(\mathtt{p}\underline{c})\underline{a}^1}$, $\mathcal{F}_{\underline{d}}$ and $\mathcal{F}_{(\mathtt{p}\underline{d})\underline{b}^1}$. Since $\mathcal{H}$ has strictly positive entropy, there exists at least one edge from some other vertex A of $\mathcal{H}$ to $\mathcal{F}_{\underline{c}}$ or $\mathcal{F}_{\underline{d}}$, say $\mathcal{F}_{\underline{c}}$. Define $\mathcal{G}'$ as the smallest strongly connected subgraph of $\mathcal{H}$, which contains $\mathcal{H}'$ and A. This graph has two cycles: one passing through the vertices $\mathcal{F}_{\underline{c}}$, $\mathcal{F}_{(\mathtt{p}\underline{c})\underline{a}^1}$, $\mathcal{F}_{\underline{d}}$, $\mathcal{F}_{(\mathtt{p}\underline{d})\underline{b}^1}$ and $\mathcal{F}_{\underline{c}}$, the other one passing through the vertices $\mathcal{F}_{\underline{c}}$, $\mathcal{F}_{(\mathtt{p}\underline{c})\underline{a}^1}$, $\mathcal{F}_{\underline{d}}$, $\mathcal{F}_{(\mathtt{p}\underline{d})\underline{b}^1}$, A and $\mathcal{F}_{\underline{c}}$. The first cycle has length $|\underline{a}^1|+|\underline{b}^1|$, and the second cycle has length $|\underline{a}^1|+|\underline{b}^1|+\cdots+|\underline{b}^j|$ if $A=\mathcal{F}_{\mathtt{p}(\underline{d})\underline{b}^1\cdots\underline{b}^j}$. We also have

$$|\underline{c}|=|\underline{b}^1|=|\underline{b}^j| \quad \text{and} \quad |\underline{a}^1|=|\underline{d}|\,.$$

Therefore one cycle has period

$$|\underline{a}^1|+|\underline{b}^1|\leq|\underline{a}^1|+|\underline{c}|\leq p\,,$$

and the other one has period

$$|\underline{d}|+|\underline{b}^1|+\cdots+|\underline{b}^j|\leq q\,.$$

□

Theorem 3.1 *Let $k\geq 2$ and let $\underline{u}\in\mathtt{A}^{\mathbf{Z}_+}$ and $\underline{v}\in\mathtt{A}^{\mathbf{Z}_+}$, such that $u_0=0$, $v_0=k-1$ and*

$$\underline{u}\preceq\sigma^n\underline{u}\preceq\underline{v}\quad\forall n\geq 0\quad\text{and}\quad\underline{u}\preceq\sigma^n\underline{v}\preceq\underline{v}\quad\forall n\geq 0\,.$$

If $k=2$ we also assume that $\sigma\underline{u}\preceq\sigma\underline{v}$. Let $\bar{\alpha}$ and $\bar{\beta}$ be the two real numbers defined by the algorithm of Proposition 3.1. Then

$$h(\mathbf{\Sigma}(\underline{u},\underline{v}))=\log_2\bar{\beta}\,.$$

If $k=2$ and $\sigma\underline{v}\prec\sigma\underline{u}$, then $h(\mathbf{\Sigma}(\underline{u},\underline{v}))=0$.

Proof: Let $\bar{\beta}>1$. By Propositions 3.1 and 2.5 we have

$$\mathbf{\Sigma}(\underline{u}^{\bar{\alpha},\bar{\beta}},\underline{v}^{\bar{\alpha},\bar{\beta}})\subset\mathbf{\Sigma}(\underline{u},\underline{v})\subset\mathbf{\Sigma}(\underline{u}_*^{\bar{\alpha},\bar{\beta}},\underline{v}_*^{\bar{\alpha},\bar{\beta}})\,.$$

From Proposition 3.3 we get

$$h(\mathbf{\Sigma}(\underline{u}^{\bar{\alpha},\bar{\beta}},\underline{v}^{\bar{\alpha},\bar{\beta}}))=h(\mathbf{\Sigma}(\underline{u}_*^{\bar{\alpha},\bar{\beta}},\underline{v}_*^{\bar{\alpha},\bar{\beta}}))=\log_2\bar{\beta}\,.$$

Let $\lim_n\alpha_n=\bar{\alpha}$ and $\lim_n\beta_n=\bar{\beta}=1$. We have $\alpha_n<1$ and $\beta_n>1$ (see proof of Proposition 3.1). Let

$$\underline{u}^n:=\underline{u}_*^{\alpha_n,\beta_n}\quad\text{and}\quad\underline{\mathrm{v}}^{\mathrm{n}}:=\underline{\mathrm{v}}_*^{\alpha_{\mathrm{n}},\beta_{\mathrm{n}}}\,.$$

By Proposition 2.5 point 3,

$$\underline{v}^{\alpha_1,\beta_1}\preceq\underline{v}\preceq\underline{v}^1\,.$$

By monotonicity,

$$\overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{v}^1)\leq\overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{v}^1)=\gamma_1=\gamma_2\overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{v}^2)\,.$$

Therefore $\underline{v}^1\preceq\underline{v}^2$ ($v_0^1=v_0^2$) and by Proposition 2.5 point 2,

$$\underline{u}^2\preceq\underline{u}\preceq\underline{u}^{\alpha_2,\beta_2}\quad\text{and}\quad\underline{\mathrm{v}}\preceq\underline{\mathrm{v}}^2\,.$$

By monotonicity,

$$\overline{\varphi}_\infty^{\alpha_3,\beta_3}(\sigma\underline{u}^3)=\alpha_3=\alpha_2=\overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{u}^2)\leq\overline{\varphi}_\infty^{\alpha_3,\beta_3}(\sigma\underline{u}^2)\,.$$

Therefore $\underline{u}^3 \preceq \underline{u}^2$ and

$$\underline{u}^3 \preceq \underline{u} \quad \text{and} \quad \underline{\mathrm{v}}^{\alpha_3,\beta_3} \preceq \underline{\mathrm{v}} \preceq \underline{\mathrm{v}}^3\,.$$

Iterating this argument we conclude that

$$\underline{u}^n \preceq \underline{u} \quad \text{and} \quad \underline{\mathrm{v}} \preceq \underline{\mathrm{v}}^{\mathrm{n}}\,.$$

These inequalities imply

$$h(\mathbf{\Sigma}(\underline{u},\underline{v})) \leq h(\mathbf{\Sigma}(\underline{u}^n,\underline{v}^n)) = \log_2 \beta_n \to 0 \quad \text{for n} \to \infty\,.$$

Finally let $k=2$ and $\sigma\underline{v} \prec \sigma\underline{u}$. If $\sigma\underline{u} = (1)^\infty$, then $\underline{v}_j = 0$ for a single value of j, so that $h(\mathbf{\Sigma}(\underline{u},\underline{v})) = 0$. Suppose that $\sigma\underline{u} \neq (1)^\infty$ and fix any $\beta > 1$. The function $\alpha \mapsto \overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{u})$ is continuous and decreasing since $\overline{\varphi}^{\alpha,\beta}$ dominates $\overline{\varphi}^{\alpha',\beta}$ if $\alpha < \alpha'$. There exists $\alpha \in (0,1)$ such that $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{u}) = \alpha$. If $\underline{v}_0 < \underline{v}^{\alpha,\beta}_0$, then $\underline{v} \prec \underline{v}^{\alpha,\beta}$ and $\mathbf{\Sigma}(\underline{u},\underline{v}) \subset \mathbf{\Sigma}(\underline{u},\underline{v}^{\alpha,\beta})$, whence $h(\mathbf{\Sigma}(\underline{u},\underline{v})) \leq \log_2 \beta$. If $\underline{v}_0 = \underline{v}^{\alpha,\beta}_0 = 1$, then

$$\overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{v}) \leq \overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{u}) = \alpha < \gamma = \overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{v}^{\alpha,\beta})\,.$$

The map $\overline{\varphi}^{\alpha,\beta}_\infty$ is continuous and non-decreasing on $\mathtt{A}^{\mathbf{Z}_+}$ so that $\sigma\underline{v} \prec \sigma\underline{v}^{\alpha,\beta}$, whence $\underline{v} \prec \underline{v}^{\alpha,\beta}$ and $h(\mathbf{\Sigma}(\underline{u},\underline{v})) \leq \log_2 \beta$. Since $\beta > 1$ is arbitrary, $h(\mathbf{\Sigma}(\underline{u},\underline{v})) = 0$. □

4. Inverse problem for $\beta x + \alpha \bmod 1$

In this section we solve the inverse problem for $\beta x + \alpha \bmod 1$, namely the question: *given two strings $\underline{u}$ and $\underline{v}$ verifying*

$$\underline{u} \preceq \sigma^n\underline{u} \prec \underline{v} \quad \text{and} \quad \underline{\mathrm{u}} \prec \sigma^{\mathrm{n}}\underline{\mathrm{v}} \preceq \underline{\mathrm{v}} \quad \forall \mathrm{n} \geq 0\,, \tag{4.1}$$

can we find $\alpha \in [0,1)$ and $\beta \in (1,\infty)$ so that $\underline{u} = \underline{u}^{\alpha,\beta}$ and $\underline{v} = \underline{v}^{\alpha,\beta}$?

Proposition 4.1 *Let the φ-expansion be valid. Let $\underline{u}$ be a solution of (2.27) and $\underline{v}$ a solution of (2.28). If (4.1) holds, then*

$$\underline{u}^{\alpha,\beta} = \underline{u} \iff \forall n \geq 0\colon\ \overline{\varphi}^{\alpha,\beta}_\infty(\sigma^n\underline{u}) < 1$$
$$\iff \forall n \geq 0\colon\ \overline{\varphi}^{\alpha,\beta}_\infty(\sigma^n\underline{v}) > 0 \iff \underline{v}^{\alpha,\beta} = \underline{v}\,.$$

Proof: The φ-expansion is valid, so that (2.23) is true,

$$\forall n \geq 0\colon\ \overline{\varphi}^{\alpha,\beta}_\infty(\sigma^n\underline{u}^{\alpha,\beta}) = T^n_{\alpha,\beta}(0) < 1\,.$$

Proposition 2.3 and Proposition 2.5 point 2 imply

$$\underline{u} = \underline{u}^{\alpha,\beta} \iff \forall n \geq 0\colon\ \overline{\varphi}^{\alpha,\beta}_\infty(\sigma^n\underline{u}) < 1\,.$$

Similarly

$$\underline{v} = \underline{v}^{\alpha,\beta} \iff \forall n \geq 0\colon\ \overline{\varphi}^{\alpha,\beta}_\infty(\sigma^n\underline{v}) > 0\,.$$

Let $\underline{x} \prec \underline{x}'$, $\underline{x},\underline{x}' \in \mathbf{\Sigma}(\underline{u},\underline{v})$. Let $\ell := \min\{m \geq 0\colon x_m \neq x'_m\}$. Then

$$\overline{\varphi}^{\alpha,\beta}_\infty(\underline{x}) = \overline{\varphi}^{\alpha,\beta}_\infty(\underline{x}') \implies \overline{\varphi}^{\alpha,\beta}_\infty(\sigma^{\ell+1}\underline{x}) = 1 \quad \text{and} \quad \overline{\varphi}^{\alpha,\beta}_\infty(\sigma^{\ell+1}\underline{\mathrm{x}}) = 0\,.$$

Indeed,

$$\overline{\varphi}^{\alpha,\beta}_{\ell+1}(x_0,\ldots,x_{\ell-1},x_\ell+\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^{\ell+1}\underline{x}))=\overline{\varphi}^{\alpha,\beta}_{\ell+1}(x_0,\ldots,x_{\ell-1},x'_\ell+\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^{\ell+1}\underline{x}'))$$

Therefore $x'_\ell = x_\ell + 1$, $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^{\ell+1}\underline{x}) = 1$ and $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^{\ell+1}\underline{x}) = 0$. Suppose that $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^k\underline{u}) = 1$, and apply the above result to $\sigma^k\underline{u}$ and $\underline{v}$ to get the existence of m with $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^m\underline{v}) = 0$. □

Let $\underline{u}\in \mathtt{A}^{\mathbf{Z}_+}$ with $u_0 = 0$ and $\underline{u}\preceq\sigma^n\underline{u}$ for all $n\geq 0$. We introduce the quantity

$$\widehat{\underline{u}} := \sup\{\sigma^n\underline{u} : n\geq 0\}\,.$$

We have

$$\sigma^n\widehat{\underline{u}}\leq\widehat{\underline{u}}\quad\forall n\geq 0\,.$$

Indeed, if $\widehat{u}$ is periodic, then this is immediate. Otherwise there exists n_j, with $n_j\uparrow\infty$ as $j\to\infty$, so that $\widehat{\underline{u}} = \lim_j\sigma^{n_j}\underline{u}$. By continuity

$$\sigma^n\widehat{\underline{u}} = \lim_{j\to\infty}\sigma^{n+n_j}\underline{u}\leq\widehat{\underline{u}}\,.$$

Example. We consider the strings $\underline{u}' = (01)^\infty$ and $\underline{v}' = (110)^\infty$. One can prove that $\underline{u}' = \underline{u}^{\alpha,\beta}$ and $\underline{v}' = \underline{v}^{\alpha,\beta}$ where β is the largest solution of

$$\beta^6-\beta^5-\beta = \beta(\beta^2-\beta+1)(\beta^3-\beta-1) = 0$$

and $\alpha = (1+\beta)^{-1}$. With the notations of Proposition 2.5 we have

$$\underline{a} = 01\quad \underline{a}' = 00\quad \underline{b} = 110\quad \underline{b}' = 111\,.$$

Let

$$\underline{u} := (00110111)^\infty = (\underline{a}'\underline{b}\underline{b}')^\infty\,.$$

We have

$$\widehat{\underline{u}} = (11100110)^\infty = (\underline{b}'\underline{a}'\underline{b})^\infty\,.$$

By definition $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{u}) = \alpha$. We have

$$(\underline{b})^\infty\preceq\widehat{\underline{u}}\preceq\underline{b}'(\underline{a})^\infty\,.$$

From Proposition 2.5 point 3 and Proposition 3.3 we conclude that $\log_2\beta = h(\mathbf{\Sigma}(\underline{u},\widehat{\underline{u}}))$.

Theorem 4.1 *Let $k\geq 2$ and let $\underline{u}\in\mathtt{A}^{\mathbf{Z}_+}$ and $\underline{v}\in\mathtt{A}^{\mathbf{Z}_+}$, such that $u_0 = 0$, $v_0 = k-1$ and (4.1) holds. If $k = 2$ we also assume that $\sigma\underline{u}\preceq\sigma\underline{v}$. Set $\log_2\widehat{\beta} := h(\mathbf{\Sigma}(\underline{u},\widehat{\underline{u}}))$. Let $\bar{\alpha}$ and $\bar{\beta}$ be defined by the algorithm of Proposition 3.1.*
1) If $\widehat{\beta}<\bar{\beta}$, then $\underline{u} = \underline{u}^{\bar{\alpha},\bar{\beta}}$ and $\underline{v} = \underline{v}^{\bar{\alpha},\bar{\beta}}$.
2) If $\widehat{\beta} = \bar{\beta} > 1$ and $\underline{u}^{\bar{\alpha},\bar{\beta}}$ and $\underline{v}^{\bar{\alpha},\bar{\beta}}$ are not both periodic, then $\underline{u} = \underline{u}^{\bar{\alpha},\bar{\beta}}$ and $\underline{v} = \underline{v}^{\bar{\alpha},\bar{\beta}}$.
3) If $\widehat{\beta} = \bar{\beta} > 1$ and $\underline{u}^{\bar{\alpha},\bar{\beta}}$ and $\underline{v}^{\bar{\alpha},\bar{\beta}}$ are both periodic, then $\underline{u}\neq\underline{u}^{\bar{\alpha},\bar{\beta}}$ and $\underline{v}\neq\underline{v}^{\bar{\alpha},\bar{\beta}}$.

Proof: Let $\widehat{\beta}<\bar{\beta}$. Suppose that $\underline{u}\neq\underline{u}^{\bar{\alpha},\bar{\beta}}$ or $\underline{v}\neq\underline{v}^{\bar{\alpha},\bar{\beta}}$. By Proposition 4.1 $\underline{u}\neq\underline{u}^{\bar{\alpha},\bar{\beta}}$ and $\underline{v}\neq\underline{v}^{\bar{\alpha},\bar{\beta}}$, and there exists n such that $\overline{\varphi}^{\bar{\alpha},\bar{\beta}}_\infty(\sigma^n\underline{u}) = 1$. Hence $\overline{\varphi}^{\bar{\alpha},\bar{\beta}}_\infty(\widehat{\underline{u}}) = 1$. If $\bar{\gamma}>0$, then $\widehat{\underline{u}}_0 = v_0 = k-1$ whence $\sigma\widehat{\underline{u}}\preceq\sigma\underline{v}$, so that $\overline{\varphi}^{\bar{\alpha},\bar{\beta}}_\infty(\sigma\widehat{\underline{u}}) = \bar{\gamma}$. By Propositions 2.5 and 3.3 we deduce that

$$\log_2\widehat{\beta} = h(\mathbf{\Sigma}(\underline{u},\widehat{\underline{u}})) = h(\mathbf{\Sigma}(\underline{u},\underline{v})) = \log_2\bar{\beta}\,,$$

a contradiction. If $\bar{\gamma} = 0$, either $\widehat{\underline{u}}_0 = k-1$ and $\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma\widehat{\underline{u}}) = \bar{\gamma}$, and we get a contradiction as above, or $\widehat{\underline{u}}_0 = k-2$ and $\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma\widehat{\underline{u}}) = 1$. In the latter case, since $\sigma\widehat{\underline{u}} \preceq \widehat{\underline{u}}$, we conclude that $\widehat{\underline{u}}_1 = k-2$ and $\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma^2\widehat{\underline{u}}) = 1$. Using $\sigma^n\widehat{\underline{u}} \preceq \widehat{\underline{u}}$ we get $\widehat{\underline{u}} = (k-2)^\infty = \underline{v}^{\bar{\alpha},\bar{\beta}}$, so that $h(\mathbf{\Sigma}(\underline{u},\widehat{\underline{u}})) = h(\mathbf{\Sigma}(\underline{u},\underline{v}))$, a contradiction.
We prove 2. Suppose for example that $\underline{u}^{\bar{\alpha},\bar{\beta}}$ is not periodic. This implies that $\bar{\alpha} < 1$, so that Proposition 2.3 implies that $\underline{u} = \underline{u}^{\bar{\alpha},\bar{\beta}}$. We conclude using Proposition 4.1. Similar proof if $\underline{v}^{\bar{\alpha},\bar{\beta}}$ is not periodic.
We prove 3. By Proposition 4.1, $\underline{u} = \underline{u}^{\bar{\alpha},\bar{\beta}}$ or $\underline{v} = \underline{v}^{\bar{\alpha},\bar{\beta}}$ if and only if $\underline{u} = \underline{u}^{\bar{\alpha},\bar{\beta}}$ and $\underline{v} = \underline{v}^{\bar{\alpha},\bar{\beta}}$. Suppose $\underline{u} = \underline{u}^{\bar{\alpha},\bar{\beta}}$, then $\underline{u}$ is periodic so that $\widehat{u} = \sigma^p\underline{u}$ for some p. This implies that

$$\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma\widehat{\underline{u}}) \leq \overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\widehat{\underline{u}}) = \overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma^p\underline{u}) < 1\,.$$

by Proposition 4.1. Let $\widehat{\underline{u}}_0 \equiv \widehat{k}-1$. We can apply the algorithm of Proposition 3.1 to the pair $(\underline{u},\widehat{\underline{u}})$ and get two real numbers $\widetilde{\alpha}$ and $\widetilde{\beta}$ (if $\widehat{k} = 2$, using $\widehat{\beta} > 1$ and Theorem 3.1, we have $\sigma\underline{u} \preceq \sigma\widehat{\underline{u}}$). Theorem 3.1 implies $\widehat{\beta} = \widetilde{\beta}$, whence $\widetilde{\beta} = \bar{\beta}$. The map $\alpha \mapsto \overline{\varphi}_\infty^{\alpha,\bar{\beta}}(\sigma\underline{u})$ is continuous and decreasing, so that $\alpha \mapsto \overline{\varphi}_\infty^{\alpha,\bar{\beta}}(\sigma\underline{u}) - \alpha$ is strictly decreasing, whence there exists a unique solution to the equation $\overline{\varphi}_\infty^{\alpha,\bar{\beta}}(\sigma\underline{u}) - \alpha = 0$, which is $\bar{\alpha} = \widetilde{\alpha}$. Therefore $\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma\widehat{\underline{u}}) < 1$ and we must have $\widehat{k} = k$, whence

$$\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma\widehat{\underline{u}}) = \bar{\alpha} + \bar{\beta} - k + 1 = \overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\sigma\underline{v})\,.$$

But this implies $\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\widehat{\underline{u}}) = 1$, a contradiction □

Theorem 4.2 *Let $k \geq 2$ and let $\underline{u} \in \mathtt{A}^{\mathbf{Z}_+}$ and $\underline{v} \in \mathtt{A}^{\mathbf{Z}_+}$, such that $u_0 = 0$, $v_0 = k-1$ and (4.1 holds. If $k = 2$ we also assume that $\sigma\underline{u} \preceq \sigma\underline{v}$. Let $\bar{\alpha}$ and $\bar{\beta}$ be defined by the algorithm of Proposition 3.1. If $h(\mathbf{\Sigma}(\underline{u},\widehat{\underline{u}})) > 1$, then there exists $\underline{u}_* \succeq \widehat{\underline{u}}$ such that*

$$\underline{u}_* \prec \underline{v} \Longrightarrow \underline{u} = \underline{u}^{\bar{\alpha},\bar{\beta}} \text{ and } \mathrm{\underline{v}} = \mathrm{\underline{v}}^{\bar{\alpha},\bar{\beta}}$$
$$\underline{u}_* \succ \underline{v} \Longrightarrow \underline{u} \neq \underline{u}^{\bar{\alpha},\bar{\beta}} \text{ and } \mathrm{\underline{v}} \neq \mathrm{\underline{v}}^{\bar{\alpha},\bar{\beta}}\,.$$

Proof: As in the proof of Theorem 4.1 we define $\widetilde{k}$ and, by the algorithm of Proposition 3.1 applied to the pair $(\underline{u},\widehat{\underline{u}})$, two real numbers $\widetilde{\alpha}$ and $\widetilde{\beta}$. By Theorem 3.1, $\log_2\widetilde{\beta} = h(\mathbf{\Sigma}(\underline{u},\widehat{\underline{u}}))$. We set

$$\underline{u}_* := \begin{cases} \underline{v}_*^{\widetilde{\alpha},\widetilde{\beta}} & \text{if } \underline{v}^{\widetilde{\alpha},\widetilde{\beta}} \text{ is periodic} \\ \underline{v}^{\widetilde{\alpha},\widetilde{\beta}} & \text{if } \underline{v}^{\widetilde{\alpha},\widetilde{\beta}} \text{ is not periodic.} \end{cases}$$

It is sufficient to show that $\underline{u}_* \prec \underline{v}$ implies $\bar{\beta} > \widetilde{\beta}$ (see Theorem 4.1 point 1). Suppose the contrary, $\bar{\beta} = \widetilde{\beta}$. Then

$$1 = \overline{\varphi}_\infty^{\widetilde{\alpha},\bar{\beta}}(\widehat{\underline{u}}) \leq \overline{\varphi}_\infty^{\widetilde{\alpha},\bar{\beta}}(\underline{v})\,.$$

We have $\overline{\varphi}_\infty^{\bar{\alpha},\bar{\beta}}(\underline{v}) = 1$ and for $\alpha > \bar{\alpha}$, $\overline{\varphi}_\infty^{\alpha,\bar{\beta}}(\underline{v}) < 1$ (see Lemma 3.5). Therefore $\widetilde{\alpha} \leq \bar{\alpha}$. On the other hand, applying Corollary 3.3 we get $\widetilde{\alpha} \geq \bar{\alpha}$ so that $\widetilde{\alpha} = \bar{\alpha}$ and $\widetilde{k} = k$. From Propositions 2.4 or 2.5 we get $\underline{v} \preceq \underline{u}_*$, a contradiction.

Suppose that $\underline{u}_* \succ \underline{v}$. We have $\widehat{u} \preceq \underline{v} \prec \underline{u}_*$, whence $h(\mathbf{\Sigma}(\underline{u},\widehat{\underline{u}})) = h(\mathbf{\Sigma}(\underline{u},\underline{u}_*))$ and therefore $\bar{\beta} = \widetilde{\beta}$. As above we show that $\bar{\alpha} = \widetilde{\alpha}$. Notice that if $\underline{u}^{\widetilde{\alpha},\widetilde{\beta}}$ is not periodic, then by Proposition 2.3 $\underline{u}^{\widetilde{\alpha},\widetilde{\beta}} = \underline{u}$. If $\underline{v}^{\widetilde{\alpha},\widetilde{\beta}}$ is not periodic, then by Proposition 2.4 $\underline{v}^{\widetilde{\alpha},\widetilde{\beta}} = \underline{v}$. If $\underline{v}^{\widetilde{\alpha},\widetilde{\beta}}$ is periodic, then inequalities (4.1) imply that we must have $\underline{v}_*^{\widetilde{\alpha},\widetilde{\beta}} \prec \underline{v}$. Therefore we may have $\underline{u}_* \succ \underline{v}$ and inequalities (4.1) only if $\underline{u}^{\widetilde{\alpha},\widetilde{\beta}}$ and $\underline{v}^{\widetilde{\alpha},\widetilde{\beta}}$ are periodic. Suppose that it is the case. If $\underline{u}$ is not periodic, then using Proposition 4.1 the second statement is true. If $\underline{u}$ is periodic, then $\widehat{\underline{u}} = \sigma^p \underline{u}$ for some p, whence $\overline{\varphi}_\infty^{\widetilde{\alpha},\widetilde{\beta}}(\sigma^p \underline{u}) = 1$; by Proposition 4.1 $\underline{u} \neq \underline{u}^{\widetilde{\alpha},\widetilde{\beta}}$. □

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