Nilpotent Subalgebras of Semisimple Lie Algebras

Sous-algèbres Nilpotentes d’Algèbres de Lie Semi-simples

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Abstract

Let $g$ be the Lie algebra of a semisimple linear algebraic group. Under mild conditions on the characteristic of the underlying field, one can show that any subalgebra of $g$ consisting of nilpotent elements is contained in some Borel subalgebra. In this note, we provide examples for each semisimple group $G$ and for each of the torsion primes for $G$ of nil subalgebras not lying in any Borel subalgebra of $g$. To cite this article: P. Levy, G. McNinch, D. Testerman C. R. Acad. Sci. Paris, Ser. I 336 (2003).

Résumé


Version française abrégée

Soit $k$ un corps algébriquement clos de caractéristique $p > 0$. Par ‘groupe algébrique sur $k$’ nous entendons un schéma en groupes affine de type fini sur $k$. Soit $G$ un groupe algébrique semi-simple défini sur $k$ ($G$ est lisse et connexe) et soit $U$ un sous-groupe (algébrique) unipotent de $G$. Si $U$ est réduit, on

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saït que \( U \) est contenu dans un sous-groupe de Borel de \( G \) (cf. [6, 30.4]). Nous nous intéressons au cas où \( U \) n’est pas réduit, plus précisément au cas des \( p \)-sous-algèbres de \( \text{Lie}(G) \).

**Theorem 0.1** Supposons que \( p \) ne soit pas un nombre premier de torsion de \( G \). Alors tout sous-groupe unipotent (non nécessairement réduit) de \( G \) est contenu dans un sous-groupe de Borel de \( G \).

La démonstration repose essentiellement sur Theorem A de [9].

**Theorem 0.2** Supposons que \( p \) soit un nombre premier de torsion pour \( G \). Il existe un sous-groupe unipotent de \( G \), de dimension 0, qui n’est contenu dans aucun sous-groupe de Borel de \( G \).

On démontre ce théorème en construisant des \( p \)-sous-algèbres de \( \text{Lie}(G) \), formées d’éléments nilpotents, et qui ne sont contenues dans aucune sous-algèbre de Borel. Il y a deux types de constructions :

a) Si \( \tilde{G} \to G \) est le revêtement universel de \( G \) et \( p \) divise l’ordre du noyau (schématique) de \( \tilde{G} \to G \), on peut construire une \( p \)-sous-algèbre commutative de \( \text{Lie}(G) \), formée d’éléments nilpotents, dont l’image réciproque dans \( \text{Lie}(\tilde{G}) \) n’est pas commutative ; une telle sous-algèbre n’est pas contenue dans une sous-algèbre de Borel de \( G \). Lorsque \( G \) est simple, l’algèbre ainsi construite est de dimension 2, et elle est annulée par la puissance \( p \)-ième.

b) Si \( p \) est de torsion pour le système de racines de \( G \) (par exemple \( p = 2, 3, \) ou 5 si \( G \) est de type \( E_8 \)), il existe une \( p \)-sous-algèbre commutative de \( \text{Lie}(G) \), de dimension 3, annulée par la puissance \( p \)-ième, et non contenue dans une sous-algèbre de Borel.

1. Introduction

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and let \( G \) be a semisimple linear algebraic group over \( k \). Let \( g \) be the Lie algebra of \( G \). Under mild conditions on \( G \) and \( p \) it is straightforward to show that any nil subalgebra of \( g \), that is, a subalgebra consisting of nilpotent elements, is contained in a Borel subalgebra (see §2 below). J.-P. Serre has asked the following question: is it true that if \( p \) is a torsion prime for \( G \) then there exists a nil subalgebra of \( g \) which is contained in no Borel subalgebra? In this note, we establish a positive answer to this question. Moreover, if \( p \) is not a torsion prime for \( G \), every nil subalgebra of \( g \) lies in a Borel subalgebra. Our argument in fact applies to the more general setting of unipotent subgroup schemes of a semisimple group scheme over \( k \).

We outline two separate cases. In the first case, assume that \( G \) is simply connected. The scheme-theoretic centre \( Z \) of \( G \) is a finite group scheme. Now by a Heisenberg-type subalgebra of \( g \), we mean a \( p \)-subalgebra which is a central extension of an abelian nil algebra by a 1-dimensional algebra. If \( p \) divides the order of \( Z \), we exhibit a Heisenberg-type restricted subalgebra of \( g \) whose centre is central in \( g \). This gives a construction of a suitable nil algebra in \( \text{Lie}(G_{ad}) \), where \( G_{ad} \) is the corresponding adjoint group. In [3], Borel, Friedman and Morgan study a similar situation. More precisely, for \( K \) a compact, connected and semisimple Lie group with simply connected cover \( \hat{K} \), they study pairs and triples of elements in \( \hat{K} \) whose images commute in \( K \). Secondly, assume \( p \) is a torsion prime for the root system of \( G \). Then we will exhibit a commutative 3-dimensional restricted nil subalgebra of \( g \) which is not contained in any Borel subalgebra.

In [5], Draisma, Kraft and Kuttler study subspaces of \( g \), rather than subalgebras, consisting of nilpotent elements. Under certain restrictions on \( p \), they show that the dimension of such a subspace is bounded above by the dimension of the nil-radical of a Borel subalgebra. Moreover, they show that when the restrictions on the prime are relaxed there exist subspaces of this maximal possible dimension which do not lie in a Borel subalgebra. We refer the reader as well to the article of Vasiu ([11]) in which he studies normal unipotent subgroup schemes of reductive groups.
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2. Good characteristics

Throughout this note, $k$ is an algebraically closed field of characteristic $p > 0$. By ‘linear algebraic group defined over $k$’ we mean an affine group scheme of finite type over $k$. Let $G$ be a semisimple linear algebraic group over $k$; in particular, $G$ is a smooth group scheme with restricted Lie algebra $\mathfrak{g}$, the $p$-operation being denoted by $X \mapsto X^p$. Let $T$ be a fixed maximal torus of $G$, $W = W(G,T)$ the Weyl group of $G$, $\Phi = \Phi(G,T)$ the root system, $\Phi^+$ a positive system in $\Phi$, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ the corresponding basis and $B \subset G$ the associated Borel subgroup containing $T$. For $\alpha \in \Phi$, let $\alpha^\vee$ denote the corresponding coroot. If $\Phi$ is an irreducible root system then there is a unique root of maximal height with respect to $\Delta$, $B$ basis and $G$ group defined over $k$.

Throughout this note, $k$ is an algebraically closed field of characteristic $p > 0$. By ‘linear algebraic group defined over $k$’ we mean an affine group scheme of finite type over $k$. Let $G$ be a semisimple linear algebraic group over $k$; in particular, $G$ is a smooth group scheme with restricted Lie algebra $\mathfrak{g}$, the $p$-operation being denoted by $X \mapsto X^p$. Let $T$ be a fixed maximal torus of $G$, $W = W(G,T)$ the Weyl group of $G$, $\Phi = \Phi(G,T)$ the root system, $\Phi^+$ a positive system in $\Phi$, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ the corresponding basis and $B \subset G$ the associated Borel subgroup containing $T$. For $\alpha \in \Phi$, let $\alpha^\vee$ denote the corresponding coroot. If $\Phi$ is an irreducible root system then there is a unique root of maximal height with respect to $\Delta$, noted here by $\beta$. Write $\beta = \sum_{i=1}^\ell m_i \alpha_i$ and $\beta^\vee = \sum_{i=1}^\ell m'_i \alpha_i^\vee$. Recall that $p$ is bad for $\Phi$ if $m_i = p$ for some $i$, $1 \leq i \leq \ell$, and $p$ is torsion for $\Phi$ if $m_i' = p$ for some $i$, $1 \leq i \leq \ell$. (If the Dynkin diagram is simply-laced then $m_i = m'_i$ for all $i$.) We say that $p$ is good for $\Phi$ if $p$ is not bad for $\Phi$ and that $p$ is very good for $\Phi$ if $p$ is good for $\Phi$ and $p \nmid (\ell + 1)$ when $\Phi$ is of type $A_\ell$. Finally, we will say $p$ is good, (respectively, very good) for $G$ if $p$ is good (resp. very good) for every irreducible component of $\Phi = \Phi(G,T)$. We will say that $p$ is bad for $G$ if $p$ is bad for some irreducible component of $\Phi$ and that $p$ is torsion for $G$ if $p$ is torsion for some irreducible component of $\Phi$ or $p$ divides the order of the fundamental group of $G$.

Before considering the case of non-torsion primes, we introduce one further definition:

**Definition 2.1** ([1, Exposé XVII, 1.1]) An algebraic group $U$ over $k$ is said to be unipotent if $U$ admits a composition series whose successive quotients are isomorphic to some subgroup scheme of the algebraic group $G_u$.

We include the proof of the following theorem which follows directly from the literature in the case of very good primes.

**Theorem 2.2** Let $G$ be a semisimple group and $p$ a non-torsion prime for $G$. Let $U$ be a unipotent subgroup scheme of $G$. Then $U$ is contained in a Borel subgroup of $G$.

**Proof.** Consider first the case where $G$ is of type $A_\ell$. The result follows from [1, 3.2, Exposé XVII] and induction if $G = SL_{\ell+1}$. For the other cases, as $p$ does not divide the order of the fundamental group of $G$, we have a separable isogeny $\pi : SL_{\ell+1} \rightarrow G$ which induces a bijection on the set of Borel subgroups, whence the result follows.

In case $G = Sp_{2\ell}$, we argue similarly: a unipotent subgroup of $G$ fixes a nonzero, isotropic vector in the natural representation of $G$ and again by induction lies in a Borel subgroup of $G$. Indeed, this argument works as well for the orthogonal groups when $p \neq 2$.

Consider now the case where $G = G_2$ and $p = 3$. By the result for $SO_7$, we know that $U$ fixes a nontrivial singular vector in the action of $G$ on its $7$-dimensional orthogonal representation. One checks that the stabilizer of such a vector is a parabolic subgroup of $G_2$. Indeed this is clear for the group of $k$-points as the long root parabolic lies in the stabilizer and is a maximal subgroup. One checks directly that the stabilizer in $\mathfrak{g}$ of a maximal vector with respect to the fixed Borel subgroup is indeed a parabolic subalgebra with Levi factor a long root $\mathfrak{sl}_2$.

Now consider the case where $p$ is a very good prime for $G$. As $G$ is separably isogenous to a simply connected group, we may take $G$ to be simply connected. Then $G$ satisfies the following so-called standard hypotheses for a reductive group $G$ (cf. [7, 5.8]):
- $p$ is good for each irreducible component of the root system of $G$,
- the derived subgroup $(G,G)$ is simply connected, and
- there exists a non-degenerate $G$-equivariant symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$.

We proceed by induction on $\dim G$, the case where $\dim G = 3$ and $G = \text{SL}_2$ having been handled above. By [1, Exposé XVII, 3.5], $U$ has a nontrivial center $Z(U)$ and either there exists $X \in \text{Lie}(Z(U))$ with $X^p = 0$ and so $U \subset C_G(X)$ or there exists $u \in Z(U)$ with $u^p = 1$ and $U \subset C_G(u)$. Then applying Theorem A of [9], together with a Springer isomorphism between the variety of nilpotent elements and the variety of unipotent elements, we have that $U$ lies in a proper parabolic subgroup $P$ of $G$. Let $L$ be a Levi subgroup of $P$; then $L$ satisfies the standard hypotheses as well. Taking the image of $U$ in $P/R_u(P)$, we obtain a unipotent subgroup scheme of $(L,L)$ which is, by induction on the dimension of $G$, contained in a Borel subgroup $B_L$ of $L$. We then have that $B_L \cdot R_u(P)$ is a Borel subgroup of $G$ containing $U$.

It remains to consider the case where the root system of $G$ is not irreducible and $p$ is not a very good prime for $G$. In this case, $G$ is separably isogenous to a direct product of simply connected almost simple groups, and the result follows as in the case of type $A_r$ above. \square

We note that the conclusion of the proposition holds for reduced unipotent subgroup schemes even if the characteristic is a torsion prime for $G$. (See [6, 30.4].)

Before presenting our examples, we fix some additional notation. If $G$ is separably isogenous to a simply connected group then we can and will choose a Chevalley basis $\{h_i, e_\alpha, f_\alpha : 1 \leq i \leq \ell, \alpha \in \Phi^+\}$ for $\mathfrak{g}$, satisfying the usual relations. If $G$ is not separably isogenous to a simply connected group, then we can choose $\{h_i, e_\alpha, f_\alpha : 1 \leq i \leq \ell, \alpha \in \Phi^+\}$ satisfying the usual Chevalley relations; however, the $h_i$ will not be linearly independent and a basis of $\mathfrak{g}$ can be obtained by extending $\{h_i : 1 \leq i \leq \ell\}$ to a basis of $\text{Lie}(T)$. We use the structure constants given in [10] for $\mathfrak{g}$ of type $F_4$; for $\mathfrak{g}$ of type $E_i$, we use those given in [8]. Our labelling of Dynkin diagrams is taken as in [4]. It will sometimes be convenient to represent roots as the $\ell$-tuple of integers giving the coefficients of the simple roots, arranged as in a Dynkin diagram.

3. Heisenberg-type subalgebras

Here we take $G$ to be simply connected. For $G = \text{SL}_{mp}$, let $E_{ij}$ denote the elementary $mp \times mp$ matrix with $(r,s)$ entry $\delta_{ir}\delta_{js}$. Set $X = \sum_{j=0}^{m-1} \sum_{p=1}^{n-1} E_{jp+p,jp+p+1}\mathbb{Z}$ and $Y = \sum_{j=0}^{m-1} \sum_{p=1}^{n-1} iE_{jp+p+1,jp+p+1}\mathbb{Z}$. Then $X^p = 0 = Y^p$, $[X,Y] = I$ and hence the Lie algebra generated by $X$ and $Y$ is nilpotent.

Similar examples exist for other types with a non-trivial centre:
- if $p = 2$ and $G = \text{Spin}(2\ell + 1,k)$ then let $X = e_{\alpha\ell}$ and $Y = f_{\alpha\ell}$.
- if $p = 2$ and $G = \text{Sp}(2\ell,k)$ then let $X = \sum_{j=0}^{\ell/2} e_{2j-1\ell}$ and $Y = \sum_{j=0}^{\ell/2} i f_{2j-1\ell}$.
- if $p = 2$ and $G = \text{Spin}(2\ell,k)$ then let $X = e_{\alpha\ell-1} + e_{\alpha\ell}$ and $Y = f_{\alpha\ell-1} + f_{\alpha\ell}$.
- if $p = 3$ and $G$ is of type $E_6$ then let $X = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_6}$ and $Y = f_{\alpha_1} - f_{\alpha_2} + f_{\alpha_3} - f_{\alpha_6}$.
- if $p = 2$ and $G$ is of type $E_7$ then let $X = e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_7}$ and $Y = f_{\alpha_2} + f_{\alpha_3} + f_{\alpha_7}$.

In each of the above cases $X^p = 0 = Y^p$ and $[X,Y]$ is a nontrivial element of $\mathfrak{g}(\mathfrak{g})$, the center of $\mathfrak{g}$; in particular $[X,Y]$ is a nontrivial semisimple element. Hence there does not exist a Borel subalgebra of $\mathfrak{g}$ which contains both $X$ and $Y$.

Now let $G_{ad}$ denote an adjoint type group with root system $\Phi$ and $\pi : G \to G_{ad}$ the corresponding central isogeny (cf. §22 of [2]); then $\ker(d\pi)$ is central in $\mathfrak{g}$. Applying 22.6 of [2], we see that $\pi$ induces a bijection between Borel subgroups of $G$ and Borel subgroups of $G_{ad}$. Moreover, by ([2, 22.4]), $d\pi$ is bijective on nilpotent elements in the unipotent radical of a Borel subgroup. We deduce that there is no Borel subalgebra of $\text{Lie}(G_{ad})$ which contains both $d\pi(X)$ and $d\pi(Y)$. Setting $\mathfrak{h} = kd\pi(X) + kd\pi(Y)$, we have our desired example.
Suppose now that the root system of $G$ is not irreducible. Set $X = \sum_{i=1}^{r} e_{\alpha_i} \in \mathfrak{g}$, so $X \in \text{Lie}(B)$. Then there exists a cocharacter $\tau : G_m \to T$ with $X$ in $\mathfrak{g}(\tau;2)$, the 2-weight space with respect to $\tau$ and $\text{Lie}(B) = \oplus_{i \geq 0} \mathfrak{g}(\tau;i)$. In particular, $\text{ad}(X) : \mathfrak{g}(\tau;i) \to \mathfrak{g}(\tau;i+2)$ for all $i \in \mathbb{Z}$. It is clear that $\text{ad}(X) : \mathfrak{g}(\tau;-2) \to \mathfrak{g}(\tau;0) = \text{Lie}(T)$ is surjective.

Suppose now that $G_0$ is isogenous to $G$ and $p$ divides the order of the fundamental group of $G_0$. Let $\pi : G \to G_0$ be a central isogeny; our assumption on $p$ implies that there exists $0 \neq W \in \ker(d\pi)$. Then $W \in \text{Lie}(T)$; hence there exists a unique $Y \in \mathfrak{g}(\tau;-2)$ for which $[X,Y] = W$. Set $\mathfrak{h} \subset \text{Lie}(G_0)$ to be the restricted subalgebra generated by $d\pi(X)$ and $d\pi(Y)$. The proof that $\mathfrak{h}$ does not lie in any Borel subalgebra of $\text{Lie}(G_0)$ goes through as above. Note that in most cases, $X^p \neq 0$.

4. Commutative subalgebras

In this section we study the case where $p$ is a torsion prime for an irreducible component of the root system of $G$. In each case we construct a 3-dimensional commutative restricted subalgebra of $\mathfrak{g}$ spanned by nilpotent elements $e, X, Y$, with $e^p = X^p = Y^p = 0$, which lies in no Borel subalgebra of $G$. It suffices to consider the case where $G$ is simple. In what follows we will use the Bala-Carter-Pommerening notation for nilpotent orbits in $\mathfrak{g}$.

The case $p = 2$.

Here we take $e$ to be an element of type $A_3^3$ if $G$ is of type $D_4$ or $E_6$, of type $A_1 \times \tilde{A}_1$ if $G$ is of type $B_3$ or $F_4$, and of type $A_1$ if $G$ is of type $G_2$.

If the Dynkin diagram of $G$ is simply-laced then it has a (unique) subdiagram of type $D_4$. We will work within this subsystem subalgebra. Set $e = e_{100} + e_{001} + e_{000}$, $X = e_{110} + e_{011} + e_{010}$, $Y = f_{111} + f_{110} + f_{011}$.

If $G$ is of type $B_3$ or $F_4$ then the Dynkin diagram of $G$ has a (unique) subdiagram of type $B_3$, which we label with roots $\beta_1, \beta_2, \beta_3$, where $\beta_3$ is short. Here we let $e = e_{\beta_1} + e_{\beta_3}$, $X = e_{110} + e_{011}$, $Y = f_{111} + f_{012}$.

Finally, if $G$ is of type $G_2$ then let $e = e_{\alpha_1}$, $X = e_{11}$, $Y = f_{21}$.

The case $p = 3$.

Here either $G$ is of type $E_6, E_7, E_8$ or $G$ is of type $F_4$. We take $e$ to be an element of type $A_3^3 \times A_1$ if $G$ is of type $E_8$ and of type $A_1 \times A_2$ if $G$ is of type $F_4$. If $G$ is of type $E_6, E_7$ or $E_8$ then we can restrict to the (standard) subsystem of type $E_6$: let $e = e_{10000} + e_{01000} + e_{00100} + e_{00010} + e_{00001} + e_{00000}$, $X = e_{11100} + e_{00110} + e_{00011} + e_{01011} + e_{01110}$, $Y = f_{11110} + f_{00111} + f_{11100} + f_{01110}$.

If $G$ is of type $F_4$ then let $e = e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_4}$, $X = e_{0111} + e_{1110} - e_{0120}$ and $Y = 2f_{1111} - 2f_{1110} + f_{012}$.

The case $p = 5$.

Here $G$ is of type $E_6$. We choose $e$ to be an element of type $A_4 \times A_3$. Let $e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6}$, $X = e_{1111000} + 2e_{0111110} + 2e_{1111100} + 2e_{0111111} + 2e_{0111110} - e_{0121000} - e_{0111100} + e_{0111000}$, $Y = f_{1111110} + f_{1110000} + f_{0111100} + 2f_{0011111} + 2f_{0011110} + f_{0012100} - 2f_{0011111}$.

Note that in each of the above cases, there exists $e_\alpha$ (resp. $e_\beta, f_\gamma$) in the expression for $e$ (resp. $X, Y$) such that $\alpha + \beta - \gamma = 0$.

**Proposition 4.1** Let $\mathfrak{h} = ke + kX + kY$, with $e, X, Y$ as above. Then $\mathfrak{h}$ is not contained in any Borel subalgebra of $\mathfrak{g}$.

**Proof.** Suppose $\mathfrak{h}$ is contained in a Borel subalgebra. Then for some $g \in G$, $\text{Ad}(g(\mathfrak{h})) \subset \mathfrak{h}$, where $\mathfrak{b}$ is the Borel subalgebra corresponding to the positive Weyl chamber. By the Bruhat decomposition, we have $g = u^\prime nu$, where $u, u'^\prime \in U^+$ and $n \in N_G(T)$. But now $\text{Ad}(\mathfrak{h}) \subset \mathfrak{b}$ if and only if $\text{Ad}(nu)(\mathfrak{h}) \subset \mathfrak{b}$, thus we may assume that $u' = 1$. Let $w = nT \in W$. We will explain our argument for the case where $G$
is of type $D_4$ and $p = 2$. Note that $\text{Ad}(u(e)) = e + x$, where $x$ is in the span of all positive root subspaces for roots of length greater than 1. Thus $\text{Ad}(u(e)) \in \mathfrak{b}$ implies, in particular, that $w(\alpha_1) \in \Phi^+$. Applying a similar argument to $X$ and $Y$, we see that $w(\alpha_2 + \alpha_3) \in \Phi^+$ and $w(-\alpha_1 + \alpha_2 + \alpha_3) \in \Phi^+$. Taking the sum $w(\alpha_1) + w(\alpha_2 + \alpha_3) + w(-\alpha_1 + \alpha_2 + \alpha_3) = 0$, we have a contradiction. This argument works for all the examples given above, using the observation that if $e_1$ and $e_2$ have non-zero coefficients in the expression for $e$ then $\alpha$ and $\beta$ are not congruent modulo the subgroup $\mathbb{Z}\Phi$ (and similarly for $X$, $Y$).

Finally, the examples of §3 and Proposition 4.1 give the following result:

**Theorem 4.2** Let $G$ be a semisimple algebraic group over $k$ and $p$ a torsion prime for $G$. Then there exists a non-reduced unipotent subgroup scheme of $G$ which does not lie in any Borel subgroup of $G$.

We conclude with one further proposition which describes to some extent the nature of the 3-dimensional subalgebras defined above.

**Proposition 4.3** Let $e$, $X$ and $Y$ be as in Proposition 4.1. Any non-zero element of $\mathfrak{h} = ke \oplus kX \oplus kY$ is conjugate to $e$ and $N_G(\mathfrak{h})/C_G(\mathfrak{h}) \cong \text{SL}(3, k)$.

**Proof.** In each case, $e$ is a regular nilpotent element in $\text{Lie}(L, L)$, for some Levi factor $L$ of $G$ normalized by $T$. Note that $(L, L)$ is a commuting product of type $A_m$ subgroups and hence $p$ is good for $(L, L)$. We choose $\tau$ to be a cocharacter of $(L, L)$ (and hence a cocharacter of $G$), associated to $e$ (see [7, 5.3]). In particular $e \in g(2; \tau)$. Then one checks that $g(\tau; -1) \cap C_G(e) = kX \oplus kY$. This then implies that the group $C = C_G(e) \cap C_G(\tau(k^\times))$ normalizes $\mathfrak{h}$. It can be checked that the adjoint representation induces a surjective morphism $C \twoheadrightarrow \text{SL}(kX \oplus kY)$. But we can apply a similar argument to an analogous subgroup of $C_G(Y)$. Thus $N_G(\mathfrak{h})$ contains the subgroups $\text{SL}(ke \oplus kX)$ and $\text{SL}(kX \oplus kY)$, and hence contains $\text{SL}(\mathfrak{h})$. In particular, all non-zero elements of $\mathfrak{h}$ are conjugate by an element of $N_G(\mathfrak{h})$. It follows from our remark on root elements in the expressions for $e$, $X$ and $Y$ that there can be no cocharacter in $G$ for which $e$, $X$ and $Y$ are all in the sum of positive weight spaces. This then implies that $N_G(\mathfrak{h})/C_G(\mathfrak{h})$ is isomorphic to $\text{SL}(\mathfrak{h})$.

**References**


