Centres of centralizers of unipotent elements in simple algebraic groups

R. Lawther and D.M. Testerman

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Contents

1 Introduction

Let G be a simple algebraic group defined over an algebraically closed field k whose characteristic is either 0 or a good prime for G, and let $u \in G$ be unipotent. We study the centralizer $C_G(u)$, especially its centre $Z(C_G(u))$. We calculate the Lie algebra of $Z(C_G(u))$, in particular determining its dimension; in the case where G is of exceptional type we find the upper central series of the Lie algebra of $R_u(C_G(u))$, writing each term explicitly as a direct sum of indecomposable tilting modules for a reductive complement to $R_u(C_G(u))$ in $C_G(u)^\circ$.

Work on $C_G(u)$ dates back to 1966, when Springer showed in [44] that if u is regular then $C_G(u)^\circ$ is abelian. Subsequently, Kurtzke established the converse of this result in [17], in the case where $\mathrm{char}(k)$ is either 0 or a good prime for G. Further study of $C_G(u)$ in the case where G is of exceptional type was undertaken by Chang in [7] and Stuhler in [48] for G of type G_2, Shinoda in [40] and Shoji in [41] for G of type F_4, and Mizuno in [27, 28] for G of type E_6, E_7 and E_8. In more recent work, Sommers in [43], McNinch and Sommers in [25] and Premet in [31] have obtained results on the component group $C_G(u)/C_G(u)^\circ$, while Liebeck and Seitz in [21] have a new approach to the classification of unipotent and nilpotent orbits.

Our interest in $Z(C_G(u))$ is motivated by the desire to embed u in a connected abelian unipotent subgroup of G satisfying certain uniqueness properties. In [38], Seitz considered this question in the case where either $\mathrm{char}(k) = 0$, or $\mathrm{char}(k)$ is a good prime p for G and u has order p; he constructed a 1-dimensional connected subgroup U of G, intrinsically associated to u, such that $u \in U$ and the centralizers in G of u, the subgroup U and the Lie algebra of U all coincide. In [34], Proud showed that if $\mathrm{char}(k)$ is a good prime p for G, and u has order p^t, then there exists a closed connected t-dimensional abelian unipotent subgroup containing u; however the subgroup satisfies no uniqueness properties. A natural candidate for a canonically defined abelian overgroup of a unipotent element u is $Z(C_G(u))$. In [35], Proud turned to the study of $Z(C_G(u))$, and in particular proved that if $\mathrm{char}(k)$ is either 0 or a good prime for G then $Z(C_G(u))^\circ$ is unipotent. Seitz carried this further in [39], showing that $Z(C_G(u))^\circ$ has a decomposition into Witt vector groups such that u is contained in one (not uniquely determined) factor; he pointed out that while $Z(C_G(u))^\circ$ is 'of considerable interest ... even the dimension of this subgroup remains a mystery.' We attempt to shed some light upon this mystery here.

Using the existence of a G-equivariant homeomorphism between the varieties of unipotent elements of G and nilpotent elements of its Lie algebra (see Theorem 2.2), we replace u by a nilpotent element e and subsequently study $C_G(e)$ and $Z(C_G(e))$; our findings will then equally apply to $C_G(u)$ and $Z(C_G(u))$. We begin by proving some preliminary results valid for all G. After dealing fairly quickly with the classical groups, we move on to consider the exceptional groups. For each of these we obtain a set of representatives of the non-zero nilpotent orbits, and proceed with a case-by-case analysis of their centralizers: for each orbit representative e we find a basis for the Lie algebra of $R_u(C_G(e))$ and generators for a reductive complement to $R_u(C_G(e))$ in $C_G(e)$.

In order to state the theorems which are direct consequences of our findings, we need to introduce some notation and terminology. Throughout this work we will write $\mathfrak{L}(H)$ for the Lie algebra of an algebraic group H. To avoid repetition in the statements of the theorems which follow, we shall say that the pair (G, e) *satisfies Hypothesis (H)* if

> G is a simple algebraic group defined over an algebraically closed field k whose characteristic is either 0 or a good prime for G, and $e \in \mathfrak{L}(G)$ is nilpotent.

Given such a pair (G, e), we take a cocharacter $\tau : k^* \to G$ associated to e (see Definition 2.6); any two such cocharacters are conjugate by an element of $C_G(e)$. From τ one obtains a unique labelling of the Dynkin diagram of G, with labels taken from the set $\{0, 1, 2\}$, which determines the set of τ-weights on $\mathfrak{L}(G)$ with multiplicities (see §3); the corresponding labelled diagram Δ is that attached to the G-orbit of e in the Bala-Carter-Pommerening classification of nilpotent orbits in $\mathfrak{L}(G)$. We write $n_2(\Delta)$ for the number of labels in Δ which are equal to 2.

Note that in two of the results below we shall use dots in labelled diagrams to denote an arbitrary number of unspecified labels, where the underlying group is of type D_ℓ in both cases; thus $\cdots{}^{2}_{2}$ will mean any labelled diagram in which the labels of the last two nodes are 2, and $1\cdots{}^{1}_{1}$ will mean one in which the labels of the three endnodes are 1.

Our first result concerns the case where e is distinguished in $\mathfrak{L}(G)$ (see Definition 2.5); we determine $\dim Z(C_G(e))$ and the action of $\mathrm{im}(\tau)$ on $\mathfrak{L}(Z(C_G(e)))$. Note that we may regard τ-weights as integers.

Theorem 1 *Let (G, e) satisfy Hypothesis (H), with associated cocharacter τ and labelled diagram Δ. Let ℓ be the rank of G, and write $d_1, \ldots, d_\ell$ for the degrees of the invariant polynomials of the Weyl group of G, ordered such that d_ℓ is ℓ if G is of type D_ℓ and is $\max\{d_i\}$ otherwise, and $d_i < d_j$ if $i < j < \ell$. Assume e is distinguished in $\mathfrak{L}(G)$. Then*

(i) $\dim Z(C_G(e)) = n_2(\Delta) = \dim Z(C_G(\mathrm{im}(\tau)))$*; and*

(ii) the τ-weights on $\mathfrak{L}(Z(C_G(e)))$ are the $n_2(\Delta)$ integers $2d_i - 2$ for $i \in S_\Delta$, where

$$S_\Delta = \begin{cases} \{1, \ldots, n_2(\Delta) - 1, \ell\} & \text{if } G \text{ is of type } D_\ell \text{ and } \Delta = \cdots{}^{2}_{2}; \\ \{1, \ldots, n_2(\Delta) - 1, n_2(\Delta)\} & \text{otherwise.} \end{cases}$$

Leaving aside the observation about the action of $\mathrm{im}(\tau)$, we may generalize to the case of nilpotent elements whose cocharacters have labelled diagrams with only even labels.

Theorem 2 *Let (G, e) satisfy Hypothesis (H), with associated cocharacter τ and labelled diagram Δ. Assume Δ has no label equal to 1. Then*

$$\dim Z(C_G(e)) = n_2(\Delta) = \dim Z(C_G(\mathrm{im}(\tau))).$$

Theorem 2 is subsumed by a more general result, the statement of which requires the following. Given a labelled diagram Δ for the group G, we define the 2*-free core* of Δ to be the sub-labelled diagram Δ_0 obtained by removing from Δ all labels equal to 2, together with the corresponding nodes. We let G_0 be a semisimple algebraic group (of any isogeny type) defined over k whose Dynkin diagram is the underlying diagram of Δ_0; thus $\operatorname{rank} G_0 = \operatorname{rank} G - n_2(\Delta)$. Note that the assumption that the pair (G, e) satisfies Hypothesis (H) implies that if $\mathrm{char}(k)$ is positive, it is a good prime for G_0; thus the Bala-Carter-Pommerening classification of nilpotent orbits applies to $\mathfrak{L}(G_0)$, and there is a bijection between the set of nilpotent orbits in $\mathfrak{L}(G_0)$ and the set of labelled diagrams for G_0 which are unions of labelled diagrams for the simple factors.

Theorem 3 *Let (G, e) satisfy Hypothesis (H), with associated cocharacter τ and labelled diagram Δ. Let Δ_0 be the 2-free core of Δ, with corresponding algebraic group G_0. Then there exists a nilpotent G_0-orbit in $\mathfrak{L}(G_0)$ having labelled diagram Δ_0. Let $e_0 \in \mathfrak{L}(G_0)$ be a representative of this orbit. Then*

(i) $\dim C_G(e) - \dim C_{G_0}(e_0) = n_2(\Delta)$*; and*

(ii) $\dim Z(C_G(e)) - \dim Z(C_{G_0}(e_0)) = n_2(\Delta)$.

In fact Theorem 3(ii) follows immediately from the following more general result.

Theorem 4 *Let (G, e) satisfy Hypothesis (H), with associated cocharacter τ and labelled diagram Δ having labels $a_1, \ldots, a_\ell$. Then*

$$\dim Z(C_G(e)) = \lceil \tfrac{1}{2} \sum a_j \rceil + \epsilon, \qquad \textit{where } \epsilon \in \{0, \pm 1\}.$$

Moreover the value of ϵ may be explicitly described as follows. Let Δ_0 be the 2-free core of Δ, with corresponding algebraic group G_0. Then, provided Δ_0 is not the empty diagram, there exists a connected component Γ_0 of Δ_0 such that all labels in $\Delta_0 \setminus \Gamma_0$ are 0. Let H_0 be the corresponding simple factor of G_0. Then $\epsilon = 0$ with the following exceptions.

$\epsilon = 1$:

H_0	Γ_0
F_4	1010
E_7	$\substack{101000\\0}$, $\substack{001010\\0}$
E_8	$\substack{0000101\\0}$, $\substack{0100001\\0}$, $\substack{1000100\\0}$, $\substack{0010100\\0}$

$\epsilon = -1$:

H_0	Γ_0
D_ℓ	$1 \cdots {}^{1}_{1}$
E_6	$\substack{10101\\0}$, $\substack{11011\\1}$
E_7	$\substack{101010\\0}$
E_8	$\substack{1000101\\0}$, $\substack{1010100\\0}$

In fact the cases in Theorem 4 where $\epsilon = 1$ may be described combinatorially as follows. For a simple algebraic group H_0 with simple roots $\alpha_1, \ldots, \alpha_\ell$, write the highest root as $\sum n_i \alpha_i$; given a labelled diagram Γ_0 for H_0 with labels $a_1, \ldots, a_\ell \in \{0, 1\}$, we have $\epsilon = 1$ precisely if $\{i : a_i = 1\} = \{j_1, j_2\}$, where n_{j_1} and n_{j_2} are even and differ by 2. It seems however to be harder to describe similarly the cases where $\epsilon = -1$.

The information presented here in the case where G is of exceptional type includes an explicit decomposition of $\mathfrak{L}(R_u(C_G(e)))$ as a direct sum of indecomposable tilting modules for a reductive complement to $R_u(C_G(e))$ in $C_G(e)^\circ$. We mention here an additional potential application of these results. Consider the case where $k = \mathbb{C}$, and fix an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{L}(G)$. Let $(\ ,\ )$ be a G-invariant bilinear form on $\mathfrak{L}(G)$ with $(e, f) = 1$, and define $\chi \in \mathfrak{L}(G)^*$ by setting $\chi(x) = (e, x)$ for all $x \in \mathfrak{L}(G)$. Let Q_χ be the generalized Gelfand-Graev module for the universal enveloping algebra $U(\mathfrak{L}(G))$ associated with the triple (e, h, f) (see [32]); set $H_\chi = \mathrm{End}_{\mathfrak{L}(G)}(Q_\chi)^{\mathrm{op}}$. The interest in the algebra H_χ arises in part from its connection with quantizations of certain transverse slices of the nilpotent cone $\mathcal{N}$ of $\mathfrak{L}(G)$. Set $S = e + \ker(\mathrm{ad}\, f)$, a so-called Slodowy slice to the adjoint orbit $\mathrm{Ad}(G)e$, as in [42]. In [32] Premet shows that, for each algebra homomorphism $\eta : Z(H_\chi) \to \mathbb{C}$, the algebra $H_\chi \otimes_{Z(H_\chi)} \mathbb{C}_\eta$ (where $\mathbb{C}_\eta$ is the 1-dimensional $Z(H_\chi)$-module induced by η) is a quantization of $\mathbb{C}[\mathcal{N} \cap S]$. Moreover, in [33] he considers the case where e is a long root element and gives an explicit presentation of H_χ by generators and relations; he further applies this to study the representation theory of H_χ. His calculations use knowledge of the action on $\mathfrak{L}(R_u(C_G(e)))$ of a reductive complement to $R_u(C_G(e))$ in $C_G(e)^\circ$. One can hope to treat other nilpotent orbits in a similar fashion.

The remainder of the present work is organized as follows. In §2 we fix notation and recall results from the literature which will be necessary in what follows. In §3 we first prove those parts of the main theorems stated here which require no case analysis, and then establish a structural result which, for $e \in \mathfrak{L}(G)$ nilpotent, reduces the determination of $\mathfrak{L}(Z(C_G(e)))$ to that of the fixed points of a certain reductive group acting on a certain subalgebra of $\mathfrak{L}(G)$; the identification of this subalgebra is a lengthy but tractable calculation. In §4 we prove our main theorems for groups G of classical type. In the remainder of the work we therefore restrict our attention to groups G of exceptional type. In §5 we establish a list of non-zero nilpotent orbit representatives in $\mathfrak{L}(G)$, and thereafter work only with such elements e. In §6 we fix a cocharacter associated to e.

In §7 we treat the connected centralizer of e; we begin with its unipotent radical, whose Lie algebra we determine explicitly, and then exhibit a reductive complement. In §8 we describe the action of this reductive complement on $\mathfrak{L}(C_G(e))$. In §9 we apply the results of the preceding sections, and consider the action of the full centralizer in those cases where it is not connected, to obtain a basis of the Lie subalgebra $\mathfrak{L}(Z(C_G(e)))$. In §10 we provide tables summarizing the results obtained, and prove our main theorems for groups of exceptional type. Finally, in §11 we describe and present our detailed information for each non-zero nilpotent orbit representative e.

The authors are particularly grateful to Professor J.-P. Serre for his close reading of parts of this work and his pertinent comments thereon, which have been most helpful.

2 Notation and preliminary results

In this section, we fix notation which will be used throughout, and recall definitions and results from the literature which play an essential role in what follows.

Let G be a simple algebraic group defined over an algebraically closed field k. Fix a maximal torus T of G and write $\Phi = \Phi(G)$ for the root system of G with respect to T. Write $\dim T = \ell$, and fix a set of simple roots $\Pi = \Pi(G) = \{\alpha_1, \ldots, \alpha_\ell\}$, where we number the roots in the associated Dynkin diagram as in [4]; let Φ^+ and Φ^- be the sets of positive and negative roots determined by Π. We shall write roots as linear combinations of simple roots, and represent them as ℓ-tuples of coefficients arranged as in a Dynkin diagram; thus for example if G is of type E_8 the high root is denoted $\substack{2465432\\3}$.

Let $W = N_G(T)/T$ be the Weyl group of G with respect to T; for $\alpha \in \Phi$, write $w_\alpha \in W$ for the reflection associated to α, and let U_α be the T-root subgroup corresponding to α. A closed connected semisimple subgroup of G generated by T-root subgroups will be called a *subsystem subgroup*. Note that if S is a torus in G, then $C_G(S)$ is the Levi factor of a parabolic subgroup of G, and the derived group $[C_G(S), C_G(S)]$ is a subsystem subgroup. Throughout this work we shall call a Levi factor of a parabolic subgroup of G a *Levi subgroup of* G; as is well known, if L is a Levi subgroup of G then $C_G(Z(L)) = L$.

By the Chevalley construction ([8]), there exists a basis $\mathcal{B}$ of $\mathfrak{L}(G)$ of the form $\mathcal{B} = \{e_\alpha : \alpha \in \Phi\} \cup \{h_{\alpha_i} : 1 \leq i \leq \ell\}$, where $\mathfrak{L}(U_\alpha) = \langle e_\alpha \rangle$ for each $\alpha \in \Phi$, $\mathfrak{L}(T) = \langle h_{\alpha_i} : 1 \leq i \leq \ell \rangle$, and the structure constants of $\mathfrak{L}(G)$ with respect to the basis $\mathcal{B}$ lie in $\mathbb{Z} \cdot 1_k$. There is no canonical choice of structure constants. In the cases where we will need to perform calculations, $\mathfrak{L}(G)$ will be of exceptional type: for $\mathfrak{L}(G)$ of type G_2, we will use the structure constants of [5, p.211]; for $\mathfrak{L}(G)$ of type F_4, we will use those given in [41]; for $\mathfrak{L}(G)$ of type E_ℓ, we will use those given in the appendix of [20]. For $\alpha \in \Phi^+$ we may also write f_α for $e_{-\alpha}$.

For each $\alpha \in \Phi$, we fix an isomorphism $x_\alpha : \mathbf{G}_a \to U_\alpha$ such that $dx_\alpha(1) = e_\alpha$. The action of $x_\alpha(t)$ on $\mathfrak{L}(G)$ with respect to the basis $\mathcal{B}$ is then given as follows. Let $\mathfrak{L}_\mathbb{C}(G) = (\bigoplus_{v \in \mathcal{B}} \mathbb{Z}v) \otimes_\mathbb{Z} \mathbb{C}$. For $\gamma \in \mathbb{C}$, let $A_\alpha(\gamma)$ be the matrix representing the action of $\exp(\gamma \operatorname{ad} e_\alpha)$ on $\mathfrak{L}_\mathbb{C}(G)$ with respect to the basis $\{b \otimes 1 : b \in \mathcal{B}\}$. For X an indeterminate, there exist polynomials f_{ij} in $\mathbb{Z}[X]$ such that $A_\alpha(\gamma)_{ij} = f_{ij}(\gamma)$. Let $A_\alpha(X)$ be the $|\mathcal{B}| \times |\mathcal{B}|$ matrix whose i,j-entry is $f_{ij}(X)$, and $\bar{A}_\alpha(X)$ be the image of $A_\alpha(X)$ under the natural homomorphism $\mathbb{Z}[X] \to (\mathbb{Z} \cdot 1_k)[X]$. For $t \in k$, the matrix $\bar{A}_\alpha(t)$ represents the adjoint action of $x_\alpha(t)$ on $\mathfrak{L}(G)$ with respect to the basis $\mathcal{B}$.

We define certain specific elements of $N_G(T)$. For $\alpha \in \Phi$ and $c \in k^*$, set $n_\alpha(c) = x_\alpha(c)x_{-\alpha}(-c^{-1})x_\alpha(c)$, $h_\alpha(c) = n_\alpha(t)n_\alpha(1)^{-1}$ and $n_\alpha = n_\alpha(1)$. Then we have $T = \langle h_{\alpha_i}(c) : c \in k^*, 1 \leq i \leq \ell \rangle$, and $n_\alpha \in N_G(T)$ induces on T the reflection associated to α; that is, $n_\alpha h_\beta(c) {n_\alpha}^{-1} = h_{w_\alpha(\beta)}(c)$ for all $\beta \in \Phi(G)$ and $c \in k^*$.

Recall that a prime p is said to be *bad* for G if it divides the coefficient of some α_i in the high root of Φ, and to be *good* for G otherwise. Thus if G is of type A_ℓ then all primes are good; if G is of type B_ℓ, C_ℓ or D_ℓ then 2 is bad; if G is of type G_2, F_4, E_6 or E_7 then 2 and 3 are bad; and if G is of type E_8 then 2, 3 and 5 are bad. A prime p is said to be *very good* for G if it is good for G, and additionally does not divide $\ell + 1$ if G is of type A_ℓ.

We now turn to some structural results concerning unipotent elements of G and nilpotent elements of $\mathfrak{L}(G)$. For H a closed nilpotent subgroup of G, let H_u denote the closed subgroup of unipotent elements in H. In [35] Proud studied uniform properties of $Z(C_G(u))$, for u a unipotent element in a simple algebraic group. He established the following result (subsequently proven differently by Seitz in [39]).

Theorem 2.1 ([35, 39]) *Let u be a unipotent element of G.*

(a) We have $Z(C_G(u)) = (Z(C_G(u)))_u \times Z(G)$.

(b) Assume $\mathrm{char}(k)$ *is either* 0 *or a good prime for G. Then $Z(C_G(u))^\circ = (Z(C_G(u)))_u = ((Z(C_G(u)))_u)^\circ$.*

In §3 we shall show that a certain Lie algebra calculation leads to the determination of $\dim Z(C_G(u))$. In order to establish this, we require a theorem of Springer which describes the connection between the varieties $\mathcal{U}$ of unipotent elements of G and $\mathcal{N}$ of nilpotent elements of $\mathfrak{L}(G)$. A number of variants of this result are known, with slightly different hypotheses and conclusions (see the discussion in [13, 6.20], for example); the version we shall use is the following.

Theorem 2.2 ([23, Proposition 29]) *Assume* $\mathrm{char}(k)$ *is either* 0 *or a good prime for G. Then there exists a G-equivariant homeomorphism $f : \mathcal{N} \to \mathcal{U}$.*

A map as in Theorem 2.2 will be called a *Springer map*; its G-equivariance means that results proved about centralizers of nilpotent elements apply equally to centralizers of unipotent elements. In his appendix to [24], Serre shows that any two Springer maps give the same bijection between G-orbits on $\mathcal{N}$ and on $\mathcal{U}$.

We shall also need the following result of Slodowy on the smoothness of centralizers of unipotent or nilpotent elements.

Theorem 2.3 ([42, p.38]) *Assume* $\mathrm{char}(k)$ *is either* 0 *or a very good prime for G. Given $u \in \mathcal{U}$ and $e \in \mathcal{N}$, we have*

$$\mathfrak{L}(C_G(u)) = C_{\mathfrak{L}(G)}(u) \qquad \textit{and} \qquad \mathfrak{L}(C_G(e)) = C_{\mathfrak{L}(G)}(e).$$

We now recall some results on cocharacters and nilpotent elements. Given a cocharacter $\lambda : k^* \to G$, we define an associated grading on $\mathfrak{L}(G)$. Indeed, if $\mathfrak{A} \subseteq \mathfrak{L}(G)$ is any $\mathrm{im}(\lambda)$-invariant subalgebra, for $m \in \mathbb{Z}$ set $\mathfrak{A}(m;\lambda) = \{x \in \mathfrak{A} : \mathrm{Ad}\,\lambda(c)x = c^m x\}$; then $\mathfrak{A} = \bigoplus_{m\in\mathbb{Z}} \mathfrak{A}(m;\lambda)$, and for all $m, n \in \mathbb{Z}$ we have $[\mathfrak{A}(m;\lambda), \mathfrak{A}(n;\lambda)] \subseteq \mathfrak{A}(m+n;\lambda)$. We will denote by $\mathfrak{A}_{+,\lambda}$ the sum of the graded parts $\mathfrak{A}(m;\lambda)$ with $m > 0$.

In addition, we have the following result.

Proposition 2.4 ([46, Proposition 8.4.5]) *Given a cocharacter $\lambda : k^* \to G$, set*

$$P_\lambda = \{x \in G : \lim_{a\to 0} \lambda(a)x\lambda(a)^{-1} \text{ exists}\};$$

then P_λ is a parabolic subgroup of G.

In fact the proof of [46, Proposition 8.4.5] shows that $C_G(\mathrm{im}(\lambda))$ is a Levi factor L_λ of P_λ, and $\mathfrak{L}(P_\lambda) = \mathfrak{L}(L_\lambda) \oplus \mathfrak{L}(R_u(P_\lambda))$, with $\mathfrak{L}(L_\lambda) = \mathfrak{L}(G)(0;\lambda)$ and $\mathfrak{L}(R_u(P_\lambda)) = \bigoplus_{m>0} \mathfrak{L}(G)(m;\lambda)$. In particular, the subspace $\mathfrak{L}(G)_{+,\lambda}$ consists of nilpotent elements.

Given a nilpotent element, we will choose a certain cocharacter related to it; we require the following definition.

Definition 2.5 *Let L be a Levi subgroup of G. A unipotent element $u \in L$ is said to be* distinguished in L *if each torus of $C_L(u)$ is contained in $Z(L)$; similarly a nilpotent element $e \in \mathfrak{L}(L)$ is said to be* distinguished in $\mathfrak{L}(L)$ *if each torus of $C_L(e)$ is contained in $Z(L)$.*

In particular, a unipotent or nilpotent element is distinguished in G or $\mathfrak{L}(G)$ precisely if its connected centralizer in G is a unipotent group. It is easy to see that every unipotent element is distinguished in some Levi subgroup of G. Indeed, given $u \in \mathcal{U}$ let S be a maximal torus of $C_G(u)$; then u is distinguished in $C_G(S)$. Similarly we find that every nilpotent element of $\mathfrak{L}(G)$ is distinguished in the Lie algebra of some Levi subgroup of G.

The cocharacters with which we will be concerned are as follows.

Definition 2.6 *Let e be a nilpotent element in $\mathfrak{L}(G)$. A cocharacter $\tau : k^* \to G$ is called* associated to e *if*

(a) $e \in \mathfrak{L}(G)(2;\tau)$*; and*

(b) there exists a Levi subgroup L of G with $\operatorname{im}(\tau) \subseteq [L, L]$ *and e distinguished in* $\mathfrak{L}(L)$.

If τ is a cocharacter associated to a nilpotent element $e \in \mathfrak{L}(G)$, then $\operatorname{im}(\tau)$ normalizes $C_G(e)$ and therefore stabilizes $\mathfrak{L}(C_G(e))$. In [15, §5] there is a lengthy discussion of the properties of associated cocharacters and their connection to the structure of the group centralizer of a nilpotent element. We recall here the results which will be used in what follows. The first concerns the existence and conjugacy of cocharacters associated to e.

Proposition 2.7 ([15, Lemma 5.3]) *Let e be a nilpotent element in $\mathfrak{L}(G)$.*

(a) Assume $\operatorname{char}(k)$ *is either* 0 *or a good prime for G. Then cocharacters associated to e exist.*

(b) Two cocharacters associated to e are conjugate under $C_G(e)^\circ$.

Before stating the other results, we simplify the notation we have introduced for the grading on $\mathfrak{L}(G)$. Usually we have a fixed cocharacter τ in mind; for an $\operatorname{im}(\tau)$-invariant subalgebra $\mathfrak{A} \subseteq \mathfrak{L}(G)$, we may then write $\mathfrak{A}_m$ and $\mathfrak{A}_+$ for $\mathfrak{A}(m;\tau)$ and $\mathfrak{A}_{+,\tau}$ respectively.

Proposition 2.8 ([15, Proposition 5.9]) *Assume* $\operatorname{char}(k)$ *is either* 0 *or a good prime for G. Let $e \in \mathfrak{L}(G)$ be nilpotent and τ be a cocharacter associated to e. Let $P = P_\tau$ be the parabolic subgroup associated to τ as in Proposition 2.4. Then*

(a) the group P depends only on e and not on τ;

(b) $C_G(e) = C_P(e)$*; and*

(c) $\overline{\operatorname{Ad}(P)e} = \bigoplus_{m \geq 2} \mathfrak{L}(G)_m$.

Assume $\operatorname{char}(k)$ is either 0 or a good prime for G, and let e, τ and P be as in Proposition 2.8. Let L_τ denote the Levi factor $C_G(\operatorname{im}(\tau))$ of P, and $R_u(P)$ the unipotent radical of P; then P is the semidirect product (as an algebraic group) of L_τ and $R_u(P)$, and we have $\mathfrak{L}(L_\tau) = \mathfrak{L}(G)_0$ and $\mathfrak{L}(R_u(P)) = \mathfrak{L}(G)_+$. Set $C = C_G(e) \cap L_\tau$ and $R = C_G(e) \cap R_u(P)$.

Proposition 2.9 ([15, Proposition 5.10])

(a) Assume $\operatorname{char}(k)$ *is either* 0 *or a good prime for G. Then $C_G(e)$ is the semidirect product (as an algebraic group) of the subgroup C and the normal subgroup R.*

(b) Assume $\operatorname{char}(k)$ *is either* 0 *or a very good prime for G. Then* $\mathfrak{L}(C) = C_{\mathfrak{L}(G)}(e)_0$ *and* $\mathfrak{L}(R) = C_{\mathfrak{L}(G)}(e)_+$.

(c) The group R is connected and unipotent.

(In fact, the statement of (b) in [15] is slightly different from that given here: there it is assumed that G is simply connected and the characteristic is either 0 or a good prime for G. However, the condition on G is only required to ensure $\mathfrak{L}(C_G(e)) = C_{\mathfrak{L}(G)}(e)$, which here follows from Theorem 2.3, given the extra requirement on the field characteristic.)

We note that the proofs of the above results, as given in [15], do not depend upon case-by-case consideration of the list of nilpotent orbits. However, they do rely upon the Bala-Carter-Pommerening classification of G-orbits of nilpotent elements in $\mathfrak{L}(G)$, valid in good characteristic, which we now go on to describe.

First we must recall a result of Richardson; the original of this was proved in [36], but the version most useful for the present purposes is due to Steinberg.

Theorem 2.10 ([6, Theorem 5.2.3]) *Let H be a connected reductive group defined over an algebraically closed field. Let P be a parabolic subgroup of H with unipotent radical U. Let $\mathcal{C}$ be the unique nilpotent orbit in $\mathfrak{L}(H)$ under the adjoint action of H such that $\mathcal{C} \cap \mathfrak{L}(U)$ is an open dense subset of $\mathfrak{L}(U)$. Then $\mathcal{C} \cap \mathfrak{L}(U)$ is a single P-orbit under the adjoint action of P on $\mathfrak{L}(U)$.*

In order to describe the Bala-Carter-Pommerening classification, we require the following definition.

Definition 2.11 *A parabolic subgroup P of G with unipotent radical U is said to be* distinguished *if* $\dim U/[U,U] = \dim P/U$.

We can now state the theorem.

Theorem 2.12 ([1, 2, 29, 30]) *Assume* $\mathrm{char}(k)$ *is either* 0 *or a good prime for G.*

(a) There is a bijective map between G-orbits of distinguished nilpotent elements of $\mathfrak{L}(G)$ and conjugacy classes of distinguished parabolic subgroups of G. The G-orbit corresponding to a given parabolic subgroup P contains the dense orbit of P acting on the Lie algebra of its unipotent radical.

(b) There is a bijective map between G-orbits of nilpotent elements of $\mathfrak{L}(G)$ and G-classes of pairs (L, P_L), where L is a Levi subgroup of G and P_L is a distinguished parabolic subgroup of $[L, L]$. The G-orbit corresponding to a pair (L, P_L) contains the dense orbit of P_L acting on the Lie algebra of its unipotent radical.

We note in passing that a recent alternative proof of this theorem, avoiding case analysis, is given by Premet in [31].

This concludes our survey of known results. We end this section with a result which will enable us to work with groups of arbitrary isogeny type. We shall employ the set-up of [15, 2.7]: we write G_{sc} for a simply connected group over k of the same type as G, and let $\pi : G_{sc} \to G$ be a surjective morphism of algebraic groups with $\ker \pi \subseteq Z(G_{sc})$; then certainly $\pi(Z(G_{sc})) = Z(G)$. Also $\ker d\pi$ is an ideal of $\mathfrak{L}(G_{sc})$; thus, provided that $\mathrm{char}(k)$ is either 0 or a good prime for G, by [37, 1.9] we see that $\ker d\pi \subseteq Z(\mathfrak{L}(G_{sc}))$. We write $\mathcal{U}_{sc}$ and $\mathcal{N}_{sc}$ for the varieties of unipotent elements of G_{sc} and nilpotent elements of $\mathfrak{L}(G_{sc})$. By [15, Proposition 2.7(a)] the restriction of $d\pi$ to $\mathcal{N}_{sc}$ is an isomorphism $d\pi : \mathcal{N}_{sc} \to \mathcal{N}$.

The result we shall prove relates centres of group centralizers of elements of $\mathcal{N}_{sc}$ and $\mathcal{N}$; note that Theorem 2.2 implies that Theorem 2.1 applies to these centres.

Proposition 2.13 *Assume* $\mathrm{char}(k)$ *is either* 0 *or a good prime for* G*. Let* e *be an element of* $\mathcal{N}$*, and* e_{sc} *be the unique element of* $\mathcal{N}_{sc}$ *with* $d\pi(e_{sc}) = e$*.*

(a) We have $\pi(Z(C_{G_{sc}}(e_{sc}))) = Z(C_G(e))$*.*

(b) The map $d\pi : \mathfrak{L}(Z(C_{G_{sc}}(e_{sc}))^\circ) \to \mathfrak{L}(Z(C_G(e))^\circ)$ *is a Lie algebra isomorphism.*

(c) Let $\tau : k^* \to G_{sc}$ *be a cocharacter associated to* e_{sc}*. Then* $\pi \circ \tau : k^* \to G$ *is a cocharacter associated to* e*; and the set of* τ*-weights, with multiplicities, on* $\mathfrak{L}(Z(C_{G_{sc}}(e_{sc})))$ *is the same as the set of* $\pi \circ \tau$*-weights, with multiplicities, on* $\mathfrak{L}(Z(C_G(e)))$*.*

Proof. By [15, Proposition 2.7(a)] we have $\pi(C_{G_{sc}}(e_{sc})) = C_G(e)$; it is then immediate that $\pi(Z(C_{G_{sc}}(e_{sc}))) \subseteq Z(C_G(e))$. Now take $\pi(x) \in Z(C_G(e))$. By Theorem 2.1 we have $\pi(x) = \pi(y)\pi(s)$, where $\pi(y) \in \mathcal{U}$ and $\pi(s) \in Z(G) = \pi(Z(G_{sc}))$. Hence $x = vz$, where $v \in \mathcal{U}_{sc}$ and $z \in Z(G_{sc})$. Now for all $\pi(g) \in C_G(e)$ we have $\pi(x)\pi(g) = \pi(g)\pi(x)$. Hence given $g \in C_{G_{sc}}(e_{sc})$ there exists $n_g \in \ker \pi$ such that $xg = gxn_g$. Thus $vzg = gvzn_g$, and so $vg = gvn_g$, whence $g^{-1}vg = vn_g$; since v is unipotent and $n_g \in Z(G_{sc})$, the uniqueness of Jordan decomposition forces $n_g = 1$, so that $xg = gx$. As this is true for all $g \in C_{G_{sc}}(e_{sc})$, we have $x \in Z(C_{G_{sc}}(e_{sc}))$, proving (a).

By Theorem 2.1 both $Z(C_{G_{sc}}(e_{sc}))^\circ$ and $Z(C_G(e))^\circ$ are unipotent groups, so their Lie algebras consist of nilpotent elements. By (a) the dimensions of these Lie algebras are equal; as $d\pi : \mathcal{N}_{sc} \to \mathcal{N}$ is bijective, it follows that $d\pi : \mathfrak{L}(Z(C_{G_{sc}}(e_{sc}))^\circ) \to \mathfrak{L}(Z(C_G(e))^\circ)$ is a Lie algebra isomorphism, proving (b).

Finally, we observe that π maps Levi subgroups of G_{sc} to those of G; and if L_{sc} is a Levi subgroup of G_{sc} such that e_{sc} is distinguished in $\mathfrak{L}(L_{sc})$ and $\mathrm{im}(\tau) \subseteq [L_{sc}, L_{sc}]$, then e is distinguished in $\mathfrak{L}(\pi(L_{sc}))$ and $\mathrm{im}(\pi \circ \tau) \subseteq [\pi(L_{sc}), \pi(L_{sc})]$. Moreover, given $g_{sc} \in G_{sc}$ we have $\pi \circ \mathrm{Int}(g_{sc}) = \mathrm{Int}(\pi(g_{sc})) \circ \pi$, where $\mathrm{Int} : G_{sc} \to \mathrm{Aut}(G_{sc})$ is the morphism sending g_{sc} to the inner automorphism given by g_{sc}; taking differentials gives $d\pi \circ \mathrm{Ad}(g_{sc}) = \mathrm{Ad}(\pi(g_{sc})) \circ d\pi$. Both assertions of (c) follow. □

For the most part, in the sections which follow we shall make no assumption on the isogeny type of G. However, when treating the classical groups in §4 it will be convenient to assume that G is $\mathrm{SL}(V)$, $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$ as appropriate; on the other hand, when concluding the treatment of the exceptional groups in §9 we shall take G to be of adjoint type. In each case Proposition 2.13 will then imply that our results on $\mathfrak{L}(Z(C_G(e)))$ apply to all other isogenous groups.

3 Reduction of the problem

Throughout this section, we take G to be a simple algebraic group defined over an algebraically closed field k, and T to be a fixed maximal torus of G; to begin with we assume $\mathrm{char}(k)$ is either 0 or a good prime for G. Fix a nilpotent element $e \in \mathfrak{L}(G)$ and an associated cocharacter $\tau : k^* \to T$. As in §2, τ induces a grading on $\mathfrak{L}(G)$; we write $\mathfrak{L}(G)_+$ for the sum of the strictly positive τ-weight spaces.

Let P_τ be the parabolic subgroup associated to τ as in Proposition 2.4. In particular $C_G(\mathrm{im}(\tau))$ is a Levi factor of P_τ. As in the paragraph before Proposition 2.9, we have $C_G(e) = R.C$ where $R = C_G(e) \cap R_u(P_\tau)$ and $C = C_G(e) \cap C_G(\mathrm{im}(\tau))$.

We begin by proving certain parts of the main theorems of §1; to do this we must first explain how the cocharacter τ gives rise to the labelled diagram Δ. For each simple root α_i, let a_i be the τ-weight of the root vector e_{α_i}, regarded as an integer; thus for all $c \in k^*$ we have $\tau(c)e_{\alpha_i} = c^{a_i}e_{\alpha_i}$. The labelled diagram of τ is the Dynkin diagram of G, in which the node corresponding to α_i is labelled with a_i. This is not in general uniquely determined by the nilpotent orbit containing e, since conjugation by elements of W need not preserve τ. There is however a unique W-conjugate τ^+ of τ with the property that all labels in its labelled diagram are non-negative; it is this labelled diagram which is called Δ. By [30, 31], Δ is the diagram which corresponds to the orbit of e, and appears for example in the tables of [6, 13.1]; each of its labels is 0, 1 or 2.

Now the centralizer $C_G(\mathrm{im}(\tau^+))$ is generated by T and the T-root subgroups U_α such that $\alpha|_{\mathrm{im}(\tau^+)} = 0$. Hence $C_G(\mathrm{im}(\tau^+))$ is a Levi subgroup whose semisimple rank is equal to the number of zeros in Δ; as τ is a conjugate of τ^+ the same is true of $C_G(\mathrm{im}(\tau))$. Thus $\dim Z(C_G(\mathrm{im}(\tau)))$ is the number of nonzero entries in Δ; so if Δ has no label equal to 1, then $\dim Z(C_G(\mathrm{im}(\tau))) = n_2(\Delta)$. As each orbit of distinguished elements of $\mathfrak{L}(G)$ has an even diagram, this proves the second equality in each of Theorems 1(i) and 2.

Next recall from §1 the definitions of the 2-free core Δ_0 of Δ and the semisimple group G_0; note that we may take G_0 to be the subgroup $\langle U_{\pm\alpha} : \alpha \text{ a node of } \Delta_0\rangle$ of G. Modulo the existence statement of Theorem 3, we may now prove Theorem 3(i).

Proposition 3.1 *Let e have labelled diagram Δ; let Δ_0 be the 2-free core of Δ and G_0 be a corresponding semisimple group. Assume that there exists $e_0 \in \mathfrak{L}(G_0)$ with labelled diagram Δ_0. Then $\dim C_G(e) - \dim C_{G_0}(e_0) = n_2(\Delta)$.*

Proof. Let τ^+ be the conjugate of τ giving rise to Δ. Proposition 2.8 gives

$$\begin{aligned}\dim C_G(e) = \dim C_{P_\tau}(e) &= \dim P_\tau - \dim \bigoplus_{m\geq 2} \mathfrak{L}(G)(m;\tau)\\ &= \dim \mathfrak{L}(P_\tau) - \dim \bigoplus_{m\geq 2} \mathfrak{L}(G)(m;\tau)\\ &= \dim \mathfrak{L}(G)(0;\tau) + \dim \mathfrak{L}(G)(1;\tau)\\ &= \dim \mathfrak{L}(G)(0;\tau^+) + \dim \mathfrak{L}(G)(1;\tau^+).\end{aligned}$$

Similarly we have $\dim C_{G_0}(e_0) = \dim \mathfrak{L}(G_0)(0;{\tau_0}^+) + \dim \mathfrak{L}(G_0)(1;{\tau_0}^+)$, where τ_0 is a cocharacter associated to e_0 and ${\tau_0}^+$ is the conjugate of τ_0 giving rise to Δ_0. Taking $G_0 = \langle U_{\pm\alpha} : \alpha \text{ a node of } \Delta_0\rangle \subseteq G$, we see that τ^+ and ${\tau_0}^+$ have the same weights on root vectors e_α for $\alpha \in \Phi(G_0)$, and the weight of τ^+ on e_α is at least 2 if $\alpha \in \Phi^+ \setminus \Phi(G_0)$ and at most -2 if $\alpha \in \Phi^- \setminus \Phi(G_0)$. Thus $\mathfrak{L}(G)(1;\tau^+) = \mathfrak{L}(G_0)(1;{\tau_0}^+)$ and $\mathfrak{L}(G)(0;\tau^+) = \mathfrak{L}(G_0)(0;{\tau_0}^+) + \mathfrak{L}(T)$; thus $\mathfrak{L}(G)(0;\tau^+)/\mathfrak{L}(T) \cong \mathfrak{L}(G_0)(0;{\tau_0}^+)/\mathfrak{L}(T \cap G_0)$, and hence $\dim C_G(e) - \dim C_{G_0}(e_0) = \dim \mathfrak{L}(T) - \dim \mathfrak{L}(T \cap G_0) = \mathrm{rank}\, G - \mathrm{rank}\, G_0 = n_2(\Delta)$. □

We now turn to the consideration of $\mathfrak{L}(Z(C_G(e)))$. Set

$$\mathcal{Z} = (Z(C_{\mathfrak{L}(G)}(e)_+))^C,$$

where the superscript denotes the collection of fixed points under C. In the remainder of this section we shall prove that, under a slight strengthening of the assumption on $\mathrm{char}(k)$, we have equality between $\mathfrak{L}(Z(C_G(e)))$ and $\mathcal{Z}$. We shall obtain this equality by showing both inclusions; for the first we shall require the following result, which holds under the characteristic assumption stated at the beginning of this section.

Proposition 3.2 *We have* $\mathfrak{L}(Z(C_G(e))) \subseteq \mathfrak{L}(G)_+$.

PROOF. First let H be a semisimple group defined over k, such that $\mathrm{char}(k)$ is either 0 or a good prime for each simple factor of H. Let e_H be a distinguished nilpotent element in $\mathfrak{L}(H)$, with associated cocharacter τ_H. We shall prove that $\mathfrak{L}(Z(C_H(e_H))) \subseteq \mathfrak{L}(H)_+$; the proof will use various results which are stated for simple groups but hold for semisimple groups.

By Theorem 2.12 there exists a distinguished parabolic subgroup $P \subseteq H$ such that e lies in the dense orbit of P on $\mathfrak{L}(R_u(P))$. Let T' be a maximal torus of P. By [15, Lemma 5.2] there exist a cocharacter $\tau' : k^* \to T'$ and a base of the root system with respect to T', such that $C_H(\mathrm{im}(\tau'))$ is a Levi factor of P containing T' and simple roots outside $\Phi(C_H(\mathrm{im}(\tau')))$ afford τ'-weight 2; thus $P = P_{\tau'}$ by the proof given in [46] of Proposition 2.4. Now by [6, Proposition 5.8.5] we may assume e_H lies in the 2-weight space for τ'; as e_H is distinguished in $\mathfrak{L}(H)$, the cocharacter τ' is associated to e_H. By Proposition 2.8 we have $P_{\tau_H} = P_{\tau'} = P$, and then using [6, Corollary 5.8.6] as well gives $C_H(e_H)^\circ = C_P(e_H)^\circ \subseteq R_u(P)$ (note that although Proposition 5.8.5 and Corollary 5.8.6 of [6] are stated under the assumption that the characteristic is either 0 or quite large, their proofs are valid provided the characteristic is either 0 or any good prime). Thus certainly $Z(C_H(e_H))^\circ \subseteq R_u(P)$, and so $\mathfrak{L}(Z(C_H(e_H))) = \mathfrak{L}(Z(C_H(e_H))^\circ) \subseteq \mathfrak{L}(R_u(P)) = \mathfrak{L}(H)_+$ as claimed.

Now let L be a Levi subgroup of G such that e is distinguished in $\mathfrak{L}(L)$ and $\mathrm{im}(\tau) \subseteq [L, L]$. Clearly $Z(L) \subseteq C_G(e)$, so $Z(C_G(e)) \subseteq C_G(Z(L)) = L$. Moreover, by Theorem 2.1 $Z(C_G(e))^\circ$ is a connected unipotent group and hence lies in $[L, L]$; so $Z(C_G(e))^\circ \subseteq Z(C_{[L,L]}(e))$. Setting $H = [L, L]$, $e_H = e$ and $\tau_H = \tau$ in the previous paragraph gives $\mathfrak{L}(Z(C_{[L,L]}(e))) \subseteq \mathfrak{L}([L, L])_+$, whence we have $\mathfrak{L}(Z(C_G(e))) = \mathfrak{L}(Z(C_G(e))^\circ) \subseteq \mathfrak{L}(Z(C_{[L,L]}(e))) \subseteq \mathfrak{L}(G)_+$. □

For the remainder of this section we make the slightly stronger assumption that

$$\mathrm{char}(k) \text{ is either 0 or a very good prime for } G.$$

Proposition 2.9 then gives $\mathfrak{L}(R) = C_{\mathfrak{L}(G)}(e)_+$ and $\mathfrak{L}(C) = C_{\mathfrak{L}(G)}(e)_0$.

Proposition 3.3 *We have* $\mathfrak{L}(Z(C_G(e))) \subseteq (Z(C_{\mathfrak{L}(G)}(e)))^C$.

PROOF. Recall that $\mathfrak{L}(Z(H)) \subseteq Z(\mathfrak{L}(H))$ for an arbitrary closed subgroup H of G. Combining this with the fact that $\mathfrak{L}(C_G(e)) = C_{\mathfrak{L}(G)}(e)$ by Theorem 2.3, we have

$$\mathfrak{L}(Z(C_G(e))) \subseteq Z(C_{\mathfrak{L}(G)}(e)).$$

As C acts trivially on $Z(C_G(e))$ by conjugation, it also acts trivially on $\mathfrak{L}(Z(C_G(e)))$; so

$$\mathfrak{L}(Z(C_G(e))) \subseteq (C_{\mathfrak{L}(G)}(e))^C.$$

The result follows. □

The first of our two inclusions is now immediate.

Corollary 3.4 *We have* $\mathfrak{L}(Z(C_G(e))) \subseteq \mathcal{Z}$.

PROOF. Apply Propositions 3.2 and 3.3. □

In Proposition 3.7 below, we shall show that $C_G(e)$ is generated by a certain family of closed commutative subgroups, together with the group C. We begin with a lemma.

Lemma 3.5 *Let* $X, Y \in \mathfrak{L}(G)$ *with* $[X, Y] = 0$. *Then* $Z(C_G(Y)) \subset C_G(X)$, *and we have* $[Z(C_G(X)), Z(C_G(Y))] = 1$.

PROOF. The group $Z(C_G(Y))$ acts trivially on $C_G(Y)$ and therefore on $\mathfrak{L}(C_G(Y))$, which by Theorem 2.3 equals $C_{\mathfrak{L}(G)}(Y)$. The latter subalgebra contains X and hence $Z(C_G(Y)) \subseteq C_G(X)$, proving the results. □

We will also need the following result of [26].

Proposition 3.6 ([26, Theorem A]) *If* $X \in \mathcal{N}$, *then* $X \in \mathfrak{L}(Z(C_G(X)))$.

Our result on $C_G(e)$ is now as follows.

Proposition 3.7 *We have* $C_G(e) = \langle C, Z(C_G(X)) : X \in C_{\mathfrak{L}(G)}(e)_+ \rangle$.

PROOF. Write $H = \langle C, Z(C_G(X)) : X \in C_{\mathfrak{L}(G)}(e)_+ \rangle$. First note that $\mathfrak{L}(G)_+$ consists of nilpotent elements and so $C_{\mathfrak{L}(G)}(e)_+ \subseteq \mathcal{N}$. By Lemma 3.5, for all $X \in C_{\mathfrak{L}(G)}(e)_+$ we have $Z(C_G(X)) \subseteq C_G(e)$; so $H \subseteq C_G(e)$. On the other hand Proposition 3.6 shows that $\mathfrak{L}(H) \supseteq C_{\mathfrak{L}(G)}(e)_+ = \mathfrak{L}(R)$, and by the definition of H we have $\mathfrak{L}(H) \supseteq \mathfrak{L}(C)$. As $\mathfrak{L}(R) \cap \mathfrak{L}(C) = C_{\mathfrak{L}(G)}(e)_+ \cap C_{\mathfrak{L}(G)}(e)_0 = \{0\}$, it follows that

$$\dim H = \dim \mathfrak{L}(H) \geq \dim \mathfrak{L}(R) + \dim \mathfrak{L}(C) = \dim R + \dim C = \dim C_G(e).$$

Thus $H^\circ = C_G(e)^\circ$. Since R is connected and lies in $C_G(e)$, it lies in H; thus we have $H \supseteq \langle C, R \rangle = R.C = C_G(e)$, whence $H = C_G(e)$ as required. □

We may now show the second of our two inclusions.

Proposition 3.8 *We have* $\mathcal{Z} \subseteq \mathfrak{L}(Z(C_G(e)))$.

PROOF. Take $v \in \mathcal{Z}$. Lemma 3.5 shows that $Z(C_G(v))$ commutes with $Z(C_G(X))$ for all $X \in C_{\mathfrak{L}(G)}(e)_+$; as $v \in C_{\mathfrak{L}(G)}(C)$, we have $C \subseteq C_G(v)$, and so $Z(C_G(v))$ also commutes with C. By Proposition 3.7 we therefore have $Z(C_G(v)) \subseteq Z(C_G(e))$. Thus $\mathfrak{L}(Z(C_G(v))) \subseteq \mathfrak{L}(Z(C_G(e)))$, so by Proposition 3.6 we have $v \in \mathfrak{L}(Z(C_G(e)))$. □

As a consequence we have our main result of this section.

Theorem 3.9 *We have* $\mathfrak{L}(Z(C_G(e))) = \mathcal{Z} = (Z(C_{\mathfrak{L}(G)}(e)_+))^C$, *so that* $\dim Z(C_G(e)) = \dim \mathcal{Z}$. *Moreover, if* $Z(C_{\mathfrak{L}(G)}(e)) \subseteq \mathfrak{L}(G)_+$ *then in fact* $\mathfrak{L}(Z(C_G(e))) = (Z(C_{\mathfrak{L}(G)}(e)))^C$.

PROOF. The first statement is immediate from Corollary 3.4 and Proposition 3.8. If $Z(C_{\mathfrak{L}(G)}(e))$ is contained in $\mathfrak{L}(G)_+$, then it lies in $Z(C_{\mathfrak{L}(G)}(e)_+)$; taking fixed points under C and applying Propositions 3.3 and 3.8 gives the second statement. □

We conclude this section by noting that, for groups of all types other than A_ℓ, the concepts of 'good prime' and 'very good prime' coincide.

4 Classical groups

Let G be a simple algebraic group of classical type defined over an algebraically closed field k whose characteristic is either 0 or a good prime for G. In this section we will prove that the conclusions of the main theorems of §1 hold for G. The arguments employed will be rather different from those employed in the sections which follow, where groups of exceptional type are considered. We begin with the case where G is of type A_ℓ. We then treat the case where G is of type B_ℓ, C_ℓ or D_ℓ; here we make use of a result of Yakimova which shortens the argument considerably.

4.1 G of type A_ℓ

Let G be of type A_ℓ. By Proposition 2.13 we may assume $G = \mathrm{SL}(V)$, where V is a vector space over k of dimension $\ell+1$; then $\mathfrak{L}(G) = \mathfrak{sl}(V)$, and the action of G on $\mathfrak{L}(G)$ is given by conjugation. Here the distinction between good and very good primes is an obstacle preventing us from using the results of §3. However, in this case a more direct approach is available.

Take a nilpotent element $e \in \mathfrak{L}(G)$, with associated cocharacter τ and labelled diagram Δ having labels $a_1, \dots, a_\ell$; let Δ_0 be the 2-free core of Δ, with corresponding algebraic group G_0. From [15, 3.5 and 5.4] we see that the collection of τ-weights on V is the union (with multiplicities) of the sets $\{d-1, d-3, \dots, 3-d, 1-d\}$, where the union runs over the Jordan blocks of e on V and d is the size of the block. Write this collection of τ-weights as $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_\ell \geq \xi_{\ell+1}$; observe that for each i we have $\xi_i + \xi_{\ell+2-i} = 0$. From [6, p.393] we see that the ξ_i determine the labels of Δ as follows: for $1 \leq i \leq \ell$ we have $a_i = \xi_i - \xi_{i+1}$.

We begin by establishing the existence statements of Theorems 3 and 4. Let $S_2(\Delta) = \{i : a_i = 2\}$. If $S_2(\Delta)$ is empty, then $\Delta_0 = \Delta$ and we may take $e_0 = e$ and $\Gamma_0 = \Delta$; so assume $S_2(\Delta)$ is non-empty. The formula above for a_i shows that if $i \in S_2(\Delta)$ then $\ell+1-i \in S_2(\Delta)$; let $j \leq \lfloor \frac{\ell+1}{2} \rfloor$ be maximal with $j \in S_2(\Delta)$.

We have $2 = a_j = \xi_j - \xi_{j+1}$, and hence V has no τ-weight $\xi_{j+1}+1$; so V also has no τ-weight $\xi_{j+1} + 2n + 1$ for $n \in \mathbb{N}$. Therefore each τ-weight ξ_i for $i < j$ must have the same parity as ξ_j; so if $i < j$ then $a_i = a_{\ell+1-i} = \xi_i - \xi_{i+1} \in \{0, 2\}$. Thus if $j < \lfloor \frac{\ell+1}{2} \rfloor$ we may let Γ_0 be the connected component of Δ_0 with nodes $\alpha_{j+1}, \dots, \alpha_{\ell-j}$, and then all labels in $\Delta_0 \setminus \Gamma_0$ are 0. Moreover if each Jordan block size greater than $\xi_{j+1}+1$ is replaced by $\xi_{j+1}+1$, then the set of numbers obtained (with multiplicities) is the list of Jordan block sizes of a nilpotent element e_0 of $\mathfrak{L}(H_0)$, where H_0 is the simple factor of G_0 corresponding to Γ_0; and the labelled diagram of e_0, regarded as an element of $\mathfrak{L}(G_0)$, is Δ_0. If on the other hand $j = \lfloor \frac{\ell+1}{2} \rfloor$, then for all i we have $a_i \in \{0, 2\}$; we may therefore take $e_0 = 0$, and Γ_0 to be any component of Δ_0 (provided Δ_0 is not the empty diagram). This proves the existence statements of Theorems 3 and 4.

We now turn to the proof of the dimension formula in Theorem 4, which in this case states that

$$\dim Z(C_G(e)) = \lceil \tfrac{1}{2} \textstyle\sum a_j \rceil;$$

to show this we shall identify $Z(C_G(e))^\circ$ explicitly. Note that we have

$$\sum a_j = \xi_1 - \xi_{\ell+1} = 2\xi_1.$$

Let the Jordan blocks of e on V have sizes $\ell_1+1, \dots, \ell_n+1$, where $\ell_1 \geq \cdots \geq \ell_n$ and $\sum_{i=1}^n (\ell_i + 1) = \ell + 1$. There are then vectors $w_1, \dots, w_n \in V$, with $e^{\ell_i+1} w_i = 0$ for

$1 \le i \le n$, such that

$$\{e^s w_i : 1 \le i \le n,\ 0 \le s \le \ell_i\}$$

is a basis for V. Write $V_i = \langle w_i, ew_i, \dots, e^{\ell_i} w_i \rangle$, so that $V = V_1 \oplus \cdots \oplus V_n$. In what follows, we treat elements of both G and $\mathfrak{L}(G)$ as matrices with respect to the given basis, whose elements we order as

$$e^{\ell_1} w_1, \dots, ew_1, w_1, \quad e^{\ell_2} w_2, \dots, ew_2, w_2, \quad \dots, \quad e^{\ell_n} w_n, \dots, ew_n, w_n.$$

We write $E_{i,j}$ for the elementary matrix with 1 in (i,j)-position. For $0 \le i \le n$ set $s_i = \sum_{j=1}^{i} (\ell_j + 1)$, so that $s_0 = 0$ and $s_n = \ell + 1$; then we have $e = \sum_{i=1}^{n} e_i$, where for $1 \le i \le n$ we set $e_i = \sum_{h=1}^{\ell_i} E_{s_i+h, s_i+h+1}$. For each i, the nilpotent element e_i is regular in $\mathfrak{L}(L_i)$, where $L_i = \mathrm{SL}(V_i)$; thus if we let L be the Levi subgroup whose derived group is $L_1 \dots L_n$ then e is distinguished in $\mathfrak{L}(L)$. Note that we have $\xi_1 = \ell_1$.

We first observe that if $n = 1$, so that e is a regular nilpotent element, a straightforward calculation shows that for any $(\ell+1) \times (\ell+1)$ matrix $A = (a_{ij})$ we have

$$Ae = eA \iff a_{ij} = \begin{cases} 0 & \text{if } i \ge j, \\ a_{1,j-i+1} & \text{if } i < j \end{cases} \iff A \in \langle e^0 = I, e, e^2, \dots, e^{\ell} \rangle.$$

Now return to the general case; choose $\lambda_1, \dots, \lambda_n \in k^*$ such that $\lambda_i \ne \lambda_j$ if $i \ne j$, and $\prod_{i=1}^{n} {\lambda_i}^{\ell_i+1} = 1$. Let $A = (a_{ij})$ be the diagonal matrix with $a_{jj} = \lambda_i$ if $s_i < j \le s_{i+1}$; then A lies in $C_G(e)$. Hence

$$Z(C_G(e)) \subseteq C_G(A) = (\mathrm{GL}(V_1) \oplus \cdots \oplus \mathrm{GL}(V_n)) \cap G.$$

From the regular case, we see that if $z \in Z(C_G(e))$ then we have $z = \sum_{i=1}^{n} \sum_{j=0}^{\ell_i} \mu_{ij} e_i^j$ for some $\mu_{ij} \in k$. By Theorem 2.1, the elements of $Z(C_G(e))^\circ$ are unipotent, so we have $\mu_{i0} = 1$ for all i.

Now given $1 \le i < n$, set $C_i = I + \sum_{h=1}^{\ell_{i+1}+1} E_{s_{i-1}+h, s_i+h}$; then we find that $C_i \in C_G(e)$. Thus if $z \in Z(C_G(e))^\circ$ we must have $zC_i = C_i z$; equating entries gives $\mu_{ij} = \mu_{i+1,j}$ for all $1 \le j \le \ell_{i+1}$. Letting i vary, we see that z is of the form $I + v$ for some $v \in \langle e, e^2, \dots, e^{\ell_1} \rangle$. Conversely, any such matrix clearly lies in $Z(C_G(e))^\circ$; so we have

$$Z(C_G(e))^\circ = \left\{ I + \mu_1 e + \mu_2 e^2 + \cdots + \mu_{\ell_1} e^{\ell_1} : \mu_1, \mu_2, \dots, \mu_{\ell_1} \in k \right\}.$$

Hence we have

$$\dim Z(C_G(e)) = \dim Z(C_G(e))^\circ = \ell_1 = \xi_1 = \lceil \tfrac{1}{2} \sum a_j \rceil$$

as required.

It remains to consider Theorem 1(ii). The only distinguished orbit in $\mathfrak{L}(G)$ is that containing regular nilpotent elements; a result of Kostant in [16] says that if e is regular nilpotent then the τ-weights in $C_{\mathfrak{L}(G)}(e)$ are $2d_1 - 2, \dots, 2d_\ell - 2$, where $d_1 < \cdots < d_\ell$ are the degrees of the invariant polynomials of the Weyl group of G. This completes the treatment of groups G of type A_ℓ.

4.2 G of type B_ℓ, C_ℓ or D_ℓ

Let G be of type B_ℓ, C_ℓ or D_ℓ, so that the characteristic of k is not 2. By Proposition 2.13 we may assume $G = \mathrm{SO}(V)$, $\mathrm{Sp}(V)$ or $\mathrm{SO}(V)$, where V is a vector space over k of dimension $2\ell+1$, 2ℓ or 2ℓ respectively; then $\mathfrak{L}(G) = \mathfrak{so}(V)$, $\mathfrak{sp}(V)$ or $\mathfrak{so}(V)$, and the action of G on $\mathfrak{L}(G)$ is given by conjugation. Take a nilpotent element $e \in \mathfrak{L}(G)$, with associated cocharacter τ and labelled diagram Δ having labels $a_1, \ldots, a_\ell$; let Δ_0 be the 2-free core of Δ, with corresponding algebraic group G_0. From [15, 3.5 and 5.4] we see that the collection of τ-weights on V is the union (with multiplicities) of the sets $\{d-1, d-3, \ldots, 3-d, 1-d\}$, where the union runs over the Jordan blocks of e on V and d is the size of the block. Write this collection of τ-weights as $\xi_1 \geq \xi_2 \geq \cdots$. From [6, pp.394–396] we see that, with one exception, the ξ_i determine the labels of Δ as follows: for $1 \leq i \leq \ell - 1$ we have $a_i = \xi_i - \xi_{i+1}$, while $a_\ell = \xi_\ell$, $2\xi_\ell$ or $\xi_{\ell-1} + \xi_\ell$ according as G is of type B_ℓ, C_ℓ or D_ℓ; the single exception is that in type D_ℓ the labels $a_{\ell-1}$ and a_ℓ, corresponding to the last two nodes of the diagram, may be interchanged if $\xi_\ell > 0$.

We begin by establishing the existence statements of Theorems 3 and 4. Let $S_2(\Delta) = \{i : a_i = 2\}$. If $S_2(\Delta)$ is empty, then $\Delta_0 = \Delta$ and we may take $e_0 = e$ and $\Gamma_0 = \Delta$; so assume $S_2(\Delta)$ is non-empty and let j be its maximal element.

First suppose $j < \ell$; then we have $2 = a_j = \xi_j - \xi_{j+1}$, and hence e has no τ-weight $\xi_{j+1} + 1$. Just as in §4.1, we see that if $i < j$ then $a_i = \xi_i - \xi_{i+1} \in \{0, 2\}$. Thus we may let Γ_0 be the connected component of Δ_0 with nodes $\alpha_{j+1}, \ldots, \alpha_\ell$, and then all labels in $\Delta_0 \setminus \Gamma_0$ are 0. Moreover if each Jordan block size greater than $\xi_{j+1} + 1$ is replaced by $\xi_{j+1} + 1$, then the parity of the number of blocks of a given size is unchanged, and so the set of numbers obtained (with multiplicities) is the list of Jordan block sizes of a nilpotent element e_0 of $\mathfrak{L}(H_0)$, where H_0 is the simple factor of G_0 corresponding to Γ_0; and the labelled diagram of e_0, regarded as an element of $\mathfrak{L}(G_0)$, is Δ_0.

Now suppose $j = \ell$. If G is of type B_ℓ then $\xi_\ell = 2$, while $\xi_{\ell+1} = 0$; if G is of type C_ℓ then $\xi_\ell = 1$, and so $\xi_{\ell+1} = -\xi_\ell = -1$; if G is of type D_ℓ then $\xi_{\ell-1} + \xi_\ell = 2$, so $(\xi_{\ell-1}, \xi_\ell) = (1, 1)$ or $(2, 0)$. In all cases it follows that all τ-weights have the same parity; so again for all i we have $a_i \in \{0, 2\}$. Hence we may take $e_0 = 0$, and Γ_0 to be any component of Δ_0 (provided Δ_0 is not the empty diagram). This proves the existence statements of Theorems 3 and 4.

We now turn to the proof of the dimension formula in Theorem 4. This states that

$$\dim Z(C_G(e)) = \lceil \tfrac{1}{2} \textstyle\sum a_i \rceil + \epsilon, \qquad \text{where } \epsilon = \begin{cases} -1 & \text{if } \Gamma_0 = {\scriptstyle 1 \cdots {}^{1}_{1}}, \\ 0 & \text{otherwise;} \end{cases}$$

recall that Γ_0 is a connected component of Δ_0 such that all labels in $\Delta_0 \setminus \Gamma_0$ are 0, and dots in labelled diagrams are used to denote an arbitrary number of unspecified labels, so that ${\scriptstyle 1 \cdots {}^{1}_{1}}$ represents any labelled diagram of type D_ℓ in which the labels of the three endnodes are 1. Note that we have

$$\sum a_i = \begin{cases} \xi_1 & \text{if } G \text{ is of type } B_\ell, \\ \xi_1 + \xi_\ell & \text{if } G \text{ is of type } C_\ell, \\ \xi_1 + \xi_{\ell-1} & \text{if } G \text{ is of type } D_\ell. \end{cases}$$

Let the Jordan blocks of e on V have sizes $\ell_1 + 1, \ldots, \ell_n + 1$, where $\ell_1 \geq \cdots \geq \ell_n$ and $\sum_{i=1}^{n} (\ell_i + 1) = \ell + 1$. There are then vectors $w_1, \ldots, w_n \in V$, with $e^{\ell_i + 1} w_i = 0$ for $1 \leq i \leq n$, such that

$$\{e^s w_i : 1 \leq i \leq n,\ 0 \leq s \leq \ell_i\}$$

is a basis for V. Write $V_i = \langle w_i, e w_i, \ldots, e^{\ell_i} w_i \rangle$, so that $V = V_1 \oplus \cdots \oplus V_n$. In what follows, we treat elements of both G and $\mathfrak{L}(G)$ as matrices with respect to the given basis.

We shall make use of work of Yakimova in [51], which identifies the centre of $C_{\mathfrak{L}(G)}(e)$, correcting Proposition 3.5 of [17]. In order to state her result, we must introduce some notation, which in places will differ slightly from hers to avoid clashing with what has already been established.

Let $(\ , \ )$ be the non-degenerate symmetric or skew-symmetric bilinear form on V stabilized by G; then $\mathfrak{L}(G)$ is the set of $\zeta \in \mathfrak{L}(\mathrm{GL}(V))$ satisfying $(\zeta v, w) = -(v, \zeta w)$ for all $v, w \in V$. As is well known (see for example [15, 1.11]), the vectors w_i may be chosen such that there is an involution $i \mapsto i'$ on the set $\{1, \ldots, m\}$ satisfying the following conditions:

- $\ell_i = \ell_{i'}$;
- $(e^s w_i, e^t w_j) = 0$ unless $j = i'$ and $s + t = n_i$;
- $i = i'$ if and only if ℓ_i is even if G is of type B_ℓ or D_ℓ, or odd if G is of type C_ℓ.

The restriction of the form $(\ , \ )$ to the subspace $V_i + V_{i'}$ is then non-degenerate. We scale the vectors to ensure that $(w_i, e^{\ell_i} w_{i'}) = 1$ if $i \leq i'$.

Given $1 \leq i, j \leq m$ and $s \in \mathbb{Z}$ satisfying $\max\{\ell_j - \ell_i, 0\} \leq s \leq \ell_j$, we define $\zeta_i^{j,s} \in \mathfrak{L}(\mathrm{GL}(V))$ to be the unique map commuting with e which sends w_i to $e^s w_j$ and all other w_h to 0. The maps $\zeta_i^{j,s}$ then form a basis of $C_{\mathfrak{L}(\mathrm{GL}(V))}(e)$; with appropriate choices of signs, the maps $\zeta_i^{j,\ell_j - s} \pm \zeta_{j'}^{i',\ell_i - s}$ which are non-zero, where $0 \leq s \leq \min\{\ell_i, \ell_j\}$, form a basis of $C_{\mathfrak{L}(G)}(e)$.

From [15, 3.5 and 5.4] we see that for all $1 \leq i \leq n$ and $0 \leq s \leq \ell_i$, and all $c \in k^*$, we have $\tau(c).e^s w_i = c^{2s - n_i} e^s w_i$. Thus for all relevant i, j, s, t, and all $c \in k^*$, we have

$$\begin{aligned} ((\mathrm{Ad}\, \tau(c))\zeta_i^{j,s}).e^t w_i &= \tau(c)\zeta_i^{j,s}\tau(c)^{-1}.e^t w_i = \tau(c)\zeta_i^{j,s}.c^{n_i - 2t} e^t w_i \\ &= c^{n_i - 2t}\tau(c)\zeta_i^{j,s}.e^t w_i = c^{n_i - 2t}\tau(c).e^{s+t} w_j \\ &= c^{n_i - 2t} c^{2s + 2t - n_j} e^{s+t} w_j = c^{2s + n_i - n_j}\zeta_i^{j,s}.e^t w_i; \end{aligned}$$

as any basis vector $e^t w_h$ with $h \neq i$ is sent to 0 by both $(\mathrm{Ad}\, \tau(c))\zeta_i^{j,s}$ and $\zeta_i^{j,s}$, we have $(\mathrm{Ad}\, \tau(c))\zeta_i^{j,s} = c^{2s + n_i - n_j}\zeta_i^{j,s}$. Therefore the τ-weight of $\zeta_i^{j,s}$ is $2s + \ell_i - \ell_j$, and hence that of $\zeta_i^{j,\ell_j - s} \pm \zeta_{j'}^{i',\ell_i - s}$ is $(\ell_i - s) + (\ell_j - s)$.

Let $Y = \langle e, e^3, e^5, \ldots \rangle$; observe that $\dim Y = \lceil \frac{1}{2}\xi_1 \rceil$, and that $Y \subset \mathfrak{L}(G)$. The result of Yakimova may then be stated as follows.

Theorem 4.1 ([51, Theorem 2]) *With the notation established, if G is of type B_ℓ or D_ℓ, and ℓ_1 and ℓ_2 are both even with $\ell_2 > \ell_3$, then we have $Z(C_{\mathfrak{L}(G)}(e)) = Y \oplus \langle x \rangle$, where $x = \zeta_1^{2,\ell_2} - \zeta_2^{1,\ell_1}$; in all other cases we have $Z(C_{\mathfrak{L}(G)}(e)) = Y$.*

For convenience we refer to blocks of odd size as 'odd blocks' and blocks of even size as 'even blocks'; thus from Theorem 4.1 we see that $Z(C_{\mathfrak{L}(G)}(e))$ properly contains Y precisely when G is of type B_ℓ or D_ℓ, and the two largest blocks of e are odd and are strictly larger than all others.

As e^j has τ-weight $2j$, and $\zeta_1^{2,\ell_2} - \zeta_2^{1,\ell_1}$ has τ-weight $\ell_1 + \ell_2$, Theorem 4.1 also gives $Z(C_{\mathfrak{L}(G)}(e)) \subseteq \mathfrak{L}(G)_+$; thus by Theorem 3.9 we have $\mathfrak{L}(Z(C_G(e))) = (Z(C_{\mathfrak{L}(G)}(e)))^C$, where $C = C_G(\mathrm{im}(\tau)) \cap C_G(e)$. So in order to prove that the conclusion of Theorem 4 holds for the groups treated here, we must show that in each case we have

$$\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \lceil \tfrac{1}{2} \textstyle\sum a_i \rceil + \epsilon.$$

We must thus consider the action of C on $Z(C_{\mathfrak{L}(G)}(e))$; as the action of G on its Lie algebra is by conjugation, any element of G which fixes e will clearly fix any power e^j, whence certainly $Y \subseteq Z(C_{\mathfrak{L}(G)}(e))^C$. Therefore to determine $Z(C_{\mathfrak{L}(G)}(e))^C$ it only remains to consider those cases in which $Z(C_{\mathfrak{L}(G)}(e)) = Y \oplus \langle x \rangle$; so G will be of type B_ℓ or D_ℓ.

Proposition 4.2 *Assume $Z(C_{\mathfrak{L}(G)}(e)) = Y \oplus \langle x \rangle$, with x as given in Theorem 4.1; then $x \in (Z(C_{\mathfrak{L}(G)}(e)))^C$ if and only if e has exactly two odd blocks.*

Proof. First assume there exists some $i > 2$ with ℓ_i even, so that e has a third odd block; let g be the element of $\mathrm{GL}(V)$ which acts as -1 on $V_1 \oplus V_i$ and as 1 on $\bigoplus_{j \neq 1,i} V_j$. As each of V_1 and V_i is orthogonal to all other blocks, it is clear that g stabilizes the bilinear form $(\ ,\ )$; as g has determinant 1 we have $g \in G$. Since g acts as a scalar on each block, it lies in $C_G(e)$; as g commutes with $\mathrm{im}(\tau)$ it lies in C. However, as gxg^{-1} maps w_1 and w_2 to $-e^{\ell_2}w_2$ and $e^{\ell_1}w_1$ respectively, and sends all other w_j to 0, we have $gxg^{-1} = -x$; so $x \notin (Z(C_{\mathfrak{L}(G)}(e)))^C$.

Now assume ℓ_i is odd for all $i > 2$, so that e has only the two odd blocks; write $d = \frac{1}{2}(\ell_1 + \ell_2)$. Set $V' = V_1 \oplus V_2$ and $V'' = V_3 \oplus \cdots \oplus V_n$, so that $V = V' \oplus V''$. The choice of the vectors w_i means that both V' and V'' are non-degenerate subspaces. Let J be the subgroup of G preserving V' and fixing V'' pointwise, and H be the subgroup of G fixing V' pointwise and preserving V''; then J is of type D_{d+1} and H is of type $D_{\ell-d-1}$. We have $e \in \mathfrak{L}(JH) = \mathfrak{L}(J) \oplus \mathfrak{L}(H)$; let e_J and e_H be the projections of e in $\mathfrak{L}(J)$ and $\mathfrak{L}(H)$, and note that $x \in \mathfrak{L}(J)$. We have $\mathrm{im}(\tau) \subseteq JH$; define $\tau_J : k^* \to J$ and $\tau_H : k^* \to H$ by letting $\tau_J(c)$ and $\tau_H(c)$ be the projections of $\tau(c)$ in J and H for all $c \in k^*$.

Recall that $\mathfrak{L}(C) = C_{\mathfrak{L}(G)}(e)_0$. As mentioned above, with appropriate choices of signs a basis of $C_{\mathfrak{L}(G)}(e)$ is given by the vectors $\zeta_i^{j,\ell_j-s} \pm \zeta_{j'}^{i',\ell_i-s}$, of τ-weight $(\ell_i - s) + (\ell_j - s)$; since this value is only 0 if $\ell_i = s = \ell_j$, which as $\ell_2 > \ell_3$ means that either $i, j \leq 2$ or $i, j \geq 3$, it follows that $C_{\mathfrak{L}(G)}(e)_0 = C_{\mathfrak{L}(J)}(e_J)_0 \oplus C_{\mathfrak{L}(H)}(e_H)_0$. We shall identify both summands as the Lie algebras of certain subgroups of J and H respectively.

We begin with $C_{\mathfrak{L}(H)}(e_H)_0$. Set $K = C_H(e_H) = C_H(e)$; then we have $C_K(\mathrm{im}(\tau_H)) = C_K(\mathrm{im}(\tau)) = K \cap L_\tau$. As $\mathrm{im}(\tau_H)$ is a torus of H which normalizes K, by [3, p.229] we have $C_{\mathfrak{L}(K)}(\mathrm{im}(\tau_H)) = \mathfrak{L}(C_K(\mathrm{im}(\tau_H)))$; moreover Theorem 2.3 gives $C_{\mathfrak{L}(H)}(e_H) = \mathfrak{L}(K)$. Thus

$$C_{\mathfrak{L}(H)}(e_H)_0 = C_{C_{\mathfrak{L}(H)}(e_H)}(\mathrm{im}(\tau_H)) = C_{\mathfrak{L}(K)}(\mathrm{im}(\tau_H)) = \mathfrak{L}(C_K(\mathrm{im}(\tau_H))) = \mathfrak{L}(K \cap L_\tau).$$

We now turn to $C_{\mathfrak{L}(J)}(e_J)_0$. For $i \in \{1, 2\}$ we have $i' = i$ since ℓ_i is even, and it follows that the sign in the vector $\zeta_i^{j,\ell_j-s} \pm \zeta_j^{i,\ell_i-s}$ is $(-1)^{s+1}$. Hence if $\ell_1 > \ell_2$ then $C_{\mathfrak{L}(J)}(e_J)_0 = 0$, while if $\ell_1 = \ell_2$ then $C_{\mathfrak{L}(J)}(e_J)_0 = \langle y \rangle$ where $y = \zeta_1^{2,0} - \zeta_2^{1,0}$.

Thus suppose for the moment that $\ell_1 = \ell_2 = d$. Again, from [15, 3.5 and 5.4] we see that for $j \in \{1, 2\}$ and $0 \leq s \leq d$, and for all $c \in k^*$, we have $\tau_J(c).e^s w_j = c^{2s-d} e^s w_j$. Take $i \in k$ with $i^2 = -1$, and define $\lambda : k^* \to J$ as follows: for $c \in k^*$ let $\lambda(c)$ be the linear map satisfying

$$\begin{aligned} e^s w_1 &\mapsto \tfrac{1}{2}(c + c^{-1}) e^s w_1 - \tfrac{i}{2}(c - c^{-1}) e^s w_2, \\ e^s w_2 &\mapsto \tfrac{i}{2}(c - c^{-1}) e^s w_1 + \tfrac{1}{2}(c + c^{-1}) e^s w_2, \end{aligned}$$

for $0 \leq s \leq d$. Write $T_1 = \mathrm{im}(\lambda)$. A straightforward check reveals that each $\lambda(c)$ preserves the bilinear form $(\ ,\ )$ and has determinant 1, so $T_1 \subseteq J$; it is immediate that each $\lambda(c)$ commutes with e_J, so $T_1 \subseteq C_J(e_J)$ and thus $\mathfrak{L}(T_1) \subseteq \mathfrak{L}(C_J(e_J)) = C_{\mathfrak{L}(J)}(e_J)$. For each $c \in k^*$, the map $\lambda(c)$ preserves the subspaces $\langle e^s w_1, e^s w_2 \rangle$ for $0 \leq s \leq d$, upon each of which $\mathrm{im}(\tau_J)$ acts as a scalar, so for all $c, c' \in k^*$ we have $[\lambda(c), \tau_J(c')] = 1$; thus we have $T_1 \subseteq C_J(\mathrm{im}(\tau_J))$, whence $\mathfrak{L}(T_1) \subseteq \mathfrak{L}(C_J(\mathrm{im}(\tau_J))) = C_{\mathfrak{L}(J)}(\mathrm{im}(\tau_J))$, and so $\mathfrak{L}(T_1) \subseteq C_{\mathfrak{L}(J)}(e_J)_0$. Since $\dim C_{\mathfrak{L}(J)}(e_J)_0 = 1$ we must have $\mathfrak{L}(T_1) = C_{\mathfrak{L}(J)}(e_J)_0$.

Now return to the general case of $\ell_1 \geq \ell_2$. Set $T_J = 1$ or T_1 according as $\ell_1 > \ell_2$ or $\ell_1 = \ell_2$; then by the above we have

$$\begin{aligned}\mathfrak{L}(C) = C_{\mathfrak{L}(G)}(e)_0 &= C_{\mathfrak{L}(J)}(e_J)_0 \oplus C_{\mathfrak{L}(H)}(e_H)_0 \\ &= \mathfrak{L}(T_J) \oplus \mathfrak{L}(K \cap L_\tau) = \mathfrak{L}(T_J.(K \cap L_\tau)).\end{aligned}$$

Now $K \cap L_\tau = C_H(e) \cap L_\tau \subseteq C_G(e) \cap L_\tau = C$, and if $\ell_1 = \ell_2$ then $T_1 \subseteq C_J(e_J) \cap L_\tau \subseteq C_G(e) \cap L_\tau = C$; thus $T_J.(K \cap L_\tau) \subseteq C$. Hence $C^\circ = (T_J.(K \cap L_\tau))^\circ$; by [6, p.399] we see that in the adjoint group the centralizer of e is connected, so that here we have $C = C^\circ.\{\pm I\} = (T_J.(K \cap L_\tau))^\circ.\{\pm I\}$. Since x lies in $\mathfrak{L}(J)$ it is fixed by $K \cap L_\tau$; if $\ell_1 = \ell_2$, an easy check shows that each $\lambda(c)$ commutes with x, so that x is also fixed by T_J; and certainly x is fixed by $-I$. Thus x is fixed by C, and we have $x \in (Z(C_{\mathfrak{L}(G)}(e)))^C$ as required. □

In order to establish the equality $\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \lceil \frac{1}{2} \sum a_i \rceil + \epsilon$, we may now work through the possibilities in turn.

First assume G is of type B_ℓ; then evidently $\epsilon = 0$. Since the number of odd blocks is odd, by Proposition 4.2 we have

$$\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \dim Y = \lceil \tfrac{1}{2}\xi_1 \rceil = \lceil \tfrac{1}{2} \textstyle\sum a_i \rceil + \epsilon.$$

Next assume G is of type C_ℓ; again $\epsilon = 0$. As $0 \leq \xi_\ell - \xi_{\ell+1} \leq 2$ and $\xi_{\ell+1} = -\xi_\ell$, we have $\xi_\ell \in \{0, 1\}$. If $\xi_\ell = 0$ then certainly we have

$$\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \dim Y = \lceil \tfrac{1}{2}\xi_1 \rceil = \lceil \tfrac{1}{2} \textstyle\sum a_i \rceil + \epsilon.$$

If instead $\xi_\ell = 1$ then all weights are odd, and in particular ξ_1 is odd; so we have

$$\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \dim Y = \lceil \tfrac{1}{2}\xi_1 \rceil = \lceil \tfrac{1}{2}(\xi_1 + 1) \rceil = \lceil \tfrac{1}{2} \textstyle\sum a_i \rceil + \epsilon.$$

Finally assume G is of type D_ℓ. As above we have $\xi_\ell \in \{0, 1\}$; since $0 \leq \xi_{\ell-1} - \xi_\ell \leq 2$, and odd weights must occur with even multiplicity, we have four possibilities for the ordered pair $(\xi_{\ell-1}, \xi_\ell)$: (a) $(0, 0)$; (b) $(1, 0)$; (c) $(2, 0)$; (d) $(1, 1)$.

(a) If $(\xi_{\ell-1}, \xi_\ell) = (0, 0)$ then there are at least four 0 weights, so at least four odd blocks; since $(a_{\ell-1}, a_\ell) = (0, 0)$, we have $\epsilon = 0$. Thus by Proposition 4.2 we have

$$\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \dim Y = \lceil \tfrac{1}{2}\xi_1 \rceil = \lceil \tfrac{1}{2} \textstyle\sum a_i \rceil + \epsilon.$$

(b) If $(\xi_{\ell-1}, \xi_\ell) = (1, 0)$ then there are exactly two 0 weights and some odd weights, so there are exactly two odd blocks and some even blocks; let the high weights of the odd blocks be $2s$ and $2t$ with $s \geq t$, and the high weight of the largest even block be $2r + 1$. We have $(a_{\ell-1}, a_\ell) = (1, 1)$, so we must consider further to decide the value of ϵ: we see that $\Gamma_0 = {}_1\cdots{}^1_1$ if and only if there exists $i \leq \ell - 3$ satisfying $\xi_i - \xi_{i+1} = 1$, where if $i > 1$ we also require $\xi_{i-1} - \xi_i = 2$. For this to be true, the multiplicity of the weight ξ_i must be one, forcing ξ_i to be even; so ξ_{i+1} must be the highest odd weight $2r + 1$. It follows that

$$\epsilon = -1 \iff 2s > 2r + 1 > 2t.$$

We therefore have three subcases.

(i) If $2r+1 > 2s \geq 2t$ then $\epsilon = 0$ and $\xi_1 = 2r+1$ is odd; as $Z(C_{\mathfrak{L}(G)}(e)) = Y$, we have

$$\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \dim Y = \lceil \tfrac{1}{2}\xi_1 \rceil = \lceil \tfrac{1}{2}(\xi_1 + 1) \rceil = \lceil \tfrac{1}{2} \sum a_i \rceil + \epsilon.$$

(ii) If $2s > 2r+1 > 2t$ then $\epsilon = -1$ and $\xi_1 = 2s$ is even; as $Z(C_{\mathfrak{L}(G)}(e)) = Y$, we have

$$\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \dim Y = \lceil \tfrac{1}{2}\xi_1 \rceil = \lceil \tfrac{1}{2}(\xi_1 + 1) \rceil - 1 = \lceil \tfrac{1}{2} \sum a_i \rceil + \epsilon.$$

(iii) If $2s \geq 2t > 2r+1$ then $\epsilon = 0$ and $\xi_1 = 2s$ is even; as $Z(C_{\mathfrak{L}(G)}(e)) = Y \oplus \langle x \rangle$, by Proposition 4.2 we have

$$\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \dim Y + 1 = \lceil \tfrac{1}{2}\xi_1 \rceil + 1 = \lceil \tfrac{1}{2}(\xi_1 + 1) \rceil = \lceil \tfrac{1}{2} \sum a_i \rceil + \epsilon.$$

(c) If $(\xi_{\ell-1}, \xi_\ell) = (2, 0)$ then there are exactly two 0 weights and no odd weights, so there are exactly two blocks, both of which are odd; since $(a_{\ell-1}, a_\ell) = (2, 2)$, we have $\epsilon = 0$. As $Z(C_{\mathfrak{L}(G)}(e)) = Y \oplus \langle x \rangle$, by Proposition 4.2 we have

$$\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \dim Y + 1 = \lceil \tfrac{1}{2}\xi_1 \rceil + 1 = \lceil \tfrac{1}{2}(\xi_1 + 2) \rceil = \lceil \tfrac{1}{2} \sum a_i \rceil + \epsilon.$$

(d) If $(\xi_{\ell-1}, \xi_\ell) = (1, 1)$ then there are no 0 weights, so no odd blocks; since $(a_{\ell-1}, a_\ell) = (2, 0)$ or $(0, 2)$, we have $\epsilon = 0$. As $Z(C_{\mathfrak{L}(G)}(e)) = Y$, and ξ_1 is odd, we have

$$\dim(Z(C_{\mathfrak{L}(G)}(e)))^C = \dim Y = \lceil \tfrac{1}{2}\xi_1 \rceil = \lceil \tfrac{1}{2}(\xi_1 + 1) \rceil = \lceil \tfrac{1}{2} \sum a_i \rceil + \epsilon.$$

We have therefore shown that in all cases the dimension formula in Theorem 4 holds.

It remains to consider Theorem 1(ii); thus assume now that e is distinguished in $\mathfrak{L}(G)$. We have observed that the basis vectors $e, e^3, e^5, \dots$ of Y have τ-weights $2, 6, 10, \dots$, and $x = \zeta_1^{2,\ell_2} - \zeta_2^{1,\ell_1}$ has τ-weight $\ell_1 + \ell_2$. The degrees d_i of the invariant polynomials of the Weyl group of G, ordered as in the statement of Theorem 1, are

$$\begin{cases} 2, 4, 6, \dots, 2\ell-2, 2\ell & \text{if } G \text{ is of type } B_\ell \text{ or } C_\ell, \\ 2, 4, 6, \dots, 2\ell-2, \ell & \text{if } G \text{ is of type } D_\ell. \end{cases}$$

Thus if G is of type B_ℓ or C_ℓ, the first $n_2(\Delta)$ of the integers $2d_i - 2$ are indeed the τ-weights on $\mathfrak{L}(Z(C_G(e))) = Y$. We may therefore assume G is of type D_ℓ. Since e is distinguished in $\mathfrak{L}(G)$, all its blocks are odd and of distinct sizes (see for example [50, Proposition 3.2]). If e has more than two blocks, we are in case (a) above, where $(a_{\ell-1}, a_\ell) = (0, 0)$ so that $\Delta = \cdots {}^0_0$; here we again have $\mathfrak{L}(Z(C_G(e))) = Y$, and the τ-weights are simply the first $n_2(\Delta)$ of the integers $2d_i - 2$. If however e has just two blocks, we are in case (c) above, where $(a_{\ell-1}, a_\ell) = (2, 2)$ so that $\Delta = \cdots {}^2_2$; this time we have $x \in \mathfrak{L}(Z(C_G(e)))$ of τ-weight $\ell_1 + \ell_2 = 2\ell - 2 = 2d_\ell - 2$, and the remaining τ-weights are the first $n_2(\Delta) - 1$ of the integers $2d_i - 2$. This completes the proof of Theorem 1(ii) for G of type B_ℓ, C_ℓ or D_ℓ.

5 Exceptional groups: nilpotent orbit representatives

From now on, unless otherwise stated G will be a simple algebraic group of exceptional type defined over an algebraically closed field k whose characteristic is either 0 or a good prime for G; again, T will be a fixed maximal torus of G. In this section, we produce lists of elements, given in terms of T-root vectors, which will be shown to represent the non-zero nilpotent G-orbits. In view of the Bala-Carter-Pommerening classification (see Theorem 2.12), it is necessary to determine the G-classes of Levi subgroups of G.

In most instances, isomorphic Levi subgroups are conjugate, as may be seen from [2]. For exceptional groups, there are two types of cases involving pairs of isomorphic but non-conjugate Levi subgroups; we describe them in terms of the underlying root systems. In the groups G_2 and F_4 there are pairs where the root systems are of type A_1, A_2 or $A_2 + A_1$ (the last two only occurring in F_4); in these the two may be distinguished by root length, and we use a tilde to denote a root system consisting of short roots. On the other hand, in E_7 there are pairs of non-conjugate Levi subgroups where the root systems are of type $3A_1$, $A_3 + A_1$ or A_5; in each instance exactly one of the two root systems has a conjugate contained in $\langle \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle$, and following Dynkin in [10] we denote that particular Levi subgroup with a double prime superscript and the other with a single prime superscript.

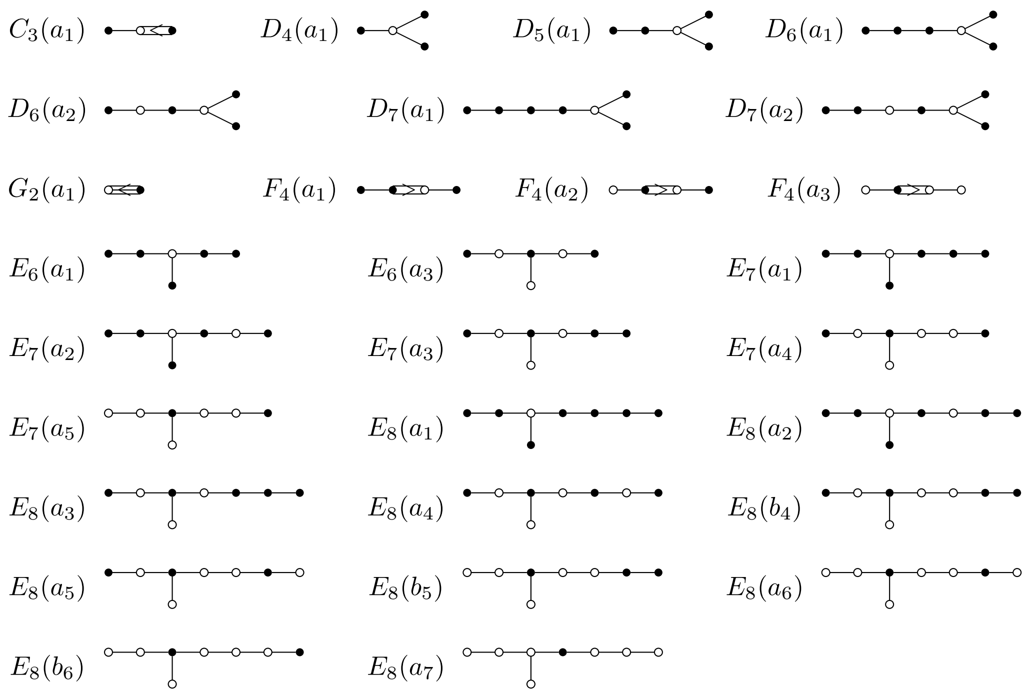
We can now explain the notation used in the Bala-Carter-Pommerening classification of nilpotent orbits (and unipotent classes). Let L be a Levi subgroup of G, with $[L, L] = L_1 \dots L_t$ where each L_i is simple. A distinguished orbit in $\mathfrak{L}(L)$ is represented by $e_1 + \cdots + e_t$, where each e_i is a distinguished element in $\mathfrak{L}(L_i)$. In $\mathfrak{L}(L_i)$ the regular nilpotent orbit corresponds to the Borel subgroup, and is denoted by the type of L_i (with a tilde if the root system of L_i consists of short roots). For non-regular distinguished nilpotent orbits the notation used is given in Table 1; here each distinguished parabolic subgroup other than the Borel subgroup is indicated by giving the Dynkin diagram with black and white nodes, where the simple roots corresponding to the white nodes form a basis of the root system of the Levi factor. As is customary, the nilpotent orbit concerned is usually denoted $L_i(a_j)$ where j is the number of white nodes; if however $L_i = E_8$ and $j \in \{4, 5, 6\}$ there are two such distinguished parabolic subgroups, and the corresponding nilpotent orbits are denoted $E_8(a_j)$ and $E_8(b_j)$. Finally the name of the orbit containing e is obtained by combining those of the elements e_i; for example, if $t = 2$ and the names of the orbits containing e_1 and e_2 are A_4 and A_2 respectively, then that of the orbit containing $e_1 + e_2$ is $A_4 A_2$. Note that we write ${A_1}^2$ for $A_1 A_1$ etc.

We now proceed to obtain the desired lists. For each of the exceptional Lie algebras $\mathfrak{L}(G)$, the second column of Table 2 contains a collection of non-zero nilpotent elements, numbered in the first column for ease of reference. By calculating the Jordan block structure in the adjoint representation one can show that, in each of the exceptional Lie algebras, the nilpotent elements e listed both represent distinct G-orbits and have $\dim C_{\mathfrak{L}(G)}(e)$ as given in the third column. Moreover the number of elements equals the number of non-zero nilpotent orbits in $\mathfrak{L}(G)$, the latter being given in good characteristic by the Bala-Carter-Pommerening classification. Therefore for each of the exceptional Lie algebras we have a set of non-zero nilpotent orbit representatives.

The fourth column of Table 2 contains the name of each orbit given by the Bala-Carter-Pommerening classification; we must now show that these have been correctly assigned. Let L be a Levi subgroup of G, and as before write $[L, L] = L_1 \dots L_t$ where each L_i is simple. It suffices to treat the distinguished orbits in each of the $\mathfrak{L}(L_i)$.

If L_i is of type A_1, with root system $\{\pm\alpha\}$, clearly a nilpotent orbit representative in $\mathfrak{L}(L_i)$ may be taken to be e_α. Now assume by induction on the rank of L_i that all

Table 1: Diagrams of distinguished parabolic subgroups



orbits lying in proper Levi subalgebras of $\mathfrak{L}(L_i)$ have been identified; this leaves just the distinguished nilpotent elements in $\mathfrak{L}(L_i)$ to consider. In each case we have a set of non-conjugate distinguished elements in bijection with the set of distinguished parabolic subgroups of L_i, in such a way that each given element lies in the Lie algebra of the unipotent radical of the corresponding parabolic subgroup. We may now appeal to the following result to deduce that our representatives of non-zero nilpotent orbits of L_i are named correctly.

Lemma 5.1 ([50, Lemma 3.3]) *Let H be a simple algebraic group and $P_1, \ldots, P_s$ be a complete list of non-conjugate distinguished parabolic subgroups of H. Suppose $x_1, \ldots, x_s$ are non-conjugate distinguished nilpotent elements of $\mathfrak{L}(H)$ such that $x_i \in \mathfrak{L}(R_u(P_i))$ for $i = 1, \ldots, s$. Then x_i lies in the dense orbit of P_i on $\mathfrak{L}(R_u(P_i))$ for $i = 1, \ldots, s$.*

Proof. For $i = 1, \ldots, s$ let $\mathcal{O}_i$ be the distinguished nilpotent orbit in $\mathfrak{L}(H)$ such that $x_i \in \mathcal{O}_i$, and $\mathrm{Rich}(P_i)$ denote the nilpotent orbit containing the dense orbit of P_i on $\mathfrak{L}(R_u(P_i))$. As the P_i and the x_i form complete sets of distinguished parabolic subgroups and representatives of distinguished nilpotent orbits in $\mathfrak{L}(H)$ respectively, either we have $\mathcal{O}_i = \mathrm{Rich}(P_i)$ for all i, or there exists j such that (after renumbering) we have $\mathcal{O}_i = \mathrm{Rich}(P_{i+1})$ for $i < j$ and $\mathcal{O}_j = \mathrm{Rich}(P_1)$. Now if $n \in \mathrm{Rich}(P_i)$ we have $\dim C_H(n) = \dim C_{P_i}(n) = \dim P_i - \dim R_u(P_i)$, while if $n' \in \mathfrak{L}(R_u(P_i))$ with $n' \notin \mathrm{Rich}(P_i)$ we have $\dim C_H(n') \geq \dim C_{P_i}(n') > \dim P_i - \dim R_u(P_i)$. Thus if $\mathcal{O}_i = \mathrm{Rich}(P_{i+1})$ for $i < j$ and $\mathcal{O}_j = \mathrm{Rich}(P_1)$, we have $\dim C_H(x_1) < \dim C_H(x_2) < \cdots < \dim C_H(x_j) < \dim C_H(x_1)$, a contradiction; so we must have $\mathcal{O}_i = \mathrm{Rich}(P_i)$ for $i = 1, \ldots, s$. □

Remark In fact, now that we have shown that the names in the fourth column of Table 2 are correct, it may be seen that in good characteristic the Jordan block structure of each nilpotent element e is the same as that of the unipotent element having the same name; lists of these Jordan block sizes can be found in [18].

We conclude with some additional remarks about our choice of orbit representatives. Firstly, for the regular orbit in $\mathfrak{L}(L_i)$ we have simply taken the sum of the simple root vectors. In the majority of the remaining cases a non-regular distinguished nilpotent element of $\mathfrak{L}(L_i)$ is regular in a naturally defined subalgebra $\mathfrak{L}(H^\sigma)$, where H is a subsystem subgroup of L_i and H^σ is its fixed point subgroup under a (possibly trivial) graph automorphism σ. If L_i is of classical type, the orbits concerned are $C_3(a_1)$ and various $D_r(a_j)$ for $4 \leq r \leq 7$. An argument analogous to that of [50, Proposition 3.1] shows that the $C_3(a_1)$ orbit is represented by a regular element in a subalgebra of type $C_2 + C_1$. Similarly, by [50, Proposition 3.2] the $D_r(a_j)$ orbit is represented by a regular element in a subalgebra of type $B_{r-j-1} + B_j$; such subalgebras are explicitly given in [50, p.67]. If L_i is of exceptional type the orbits concerned are the following.

Orbit	Subalgebra
$G_2(a_1)$	A_2
$F_4(a_3)$	$A_2\tilde{A}_2$
$F_4(a_2)$	C_3A_1
$F_4(a_1)$	B_4
$E_6(a_3)$	A_5A_1
$E_6(a_1)$	C_4
$E_7(a_5)$	A_5A_2
$E_7(a_4)$	$B_4B_1A_1$
$E_7(a_3)$	D_6A_1
$E_8(a_7)$	${A_4}^2$
$E_8(b_6)$	C_4A_2
$E_8(a_6)$	A_8
$E_8(b_5)$	E_6A_2
$E_8(a_5)$	B_6B_1
$E_8(a_4)$	D_8
$E_8(a_3)$	E_7A_1

This leaves five cases where a distinguished nilpotent element of $\mathfrak{L}(L_i)$ is not regular in any such naturally defined subalgebra, namely $E_7(a_2)$, $E_7(a_1)$, $E_8(b_4)$, $E_8(a_2)$ and $E_8(a_1)$. Of these, the orbit $E_8(b_4)$ is represented by an element of the subalgebra E_7A_1 whose projections in the simple algebras lie in the orbits $E_7(a_1)$ and A_1 respectively.

Table 2: Nilpotent orbit representatives

$G = G_2$			
Orbit	e	$\dim C_{\mathfrak{L}(G)}(e)$	Name
1	e_{01}	8	A_1
2	e_{10}	6	$\tilde{A}_1$
3	$e_{01} + e_{31}$	4	$G_2(a_1)$
4	$e_{10} + e_{01}$	2	G_2

$G = F_4$			
Orbit	e	$\dim C_{\mathfrak{L}(G)}(e)$	Name
1	e_{1000}	36	A_1
2	e_{0001}	30	$\tilde{A}_1$
3	$e_{1000} + e_{0001}$	24	$A_1\tilde{A}_1$
4	$e_{1000} + e_{0100}$	22	A_2
5	$e_{0010} + e_{0001}$	22	$\tilde{A}_2$
6	$e_{1000} + e_{0100} + e_{0001}$	18	$A_2\tilde{A}_1$
7	$e_{0100} + e_{0010}$	16	B_2
8	$e_{0010} + e_{0001} + e_{1000}$	16	$\tilde{A}_2A_1$
9	$e_{0001} + e_{0120} + e_{0100}$	14	$C_3(a_1)$
10	$e_{0100} + e_{1120} + e_{1110} + e_{0121}$	12	$F_4(a_3)$
11	$e_{1000} + e_{0100} + e_{0010}$	10	B_3
12	$e_{0001} + e_{0010} + e_{0100}$	10	C_3
13	$e_{1110} + e_{0001} + e_{0120} + e_{0100}$	8	$F_4(a_2)$
14	$e_{0100} + e_{1000} + e_{0120} + e_{0001}$	6	$F_4(a_1)$
15	$e_{1000} + e_{0100} + e_{0010} + e_{0001}$	4	F_4

Table 2: Nilpotent orbit representatives (continued)

$G = E_6$			
Orbit	e	$\dim C_{\mathfrak{L}(G)}(e)$	Name
1	$e_{\substack{10000\\0}}$	56	A_1
2	$e_{\substack{10000\\0}} + e_{\substack{00001\\0}}$	46	${A_1}^2$
3	$e_{\substack{10000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00001\\0}}$	38	${A_1}^3$
4	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}}$	36	A_2
5	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00000\\1}}$	32	A_2A_1
6	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$	30	${A_2}^2$
7	$e_{\substack{00000\\1}} + e_{\substack{00100\\0}} + e_{\substack{10000\\0}} + e_{\substack{00001\\0}}$	28	$A_2{A_1}^2$
8	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}}$	26	A_3
9	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}} + e_{\substack{00000\\1}}$	24	${A_2}^2A_1$
10	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00001\\0}}$	22	A_3A_1
11	$e_{\substack{01000\\0}} + e_{\substack{00100\\1}} + e_{\substack{00110\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$	20	$D_4(a_1)$
12	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}}$	18	A_4
13	$e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$	18	D_4
14	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}} + e_{\substack{00001\\0}}$	16	A_4A_1
15	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$	14	A_5
16	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\1}} + e_{\substack{00110\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$	14	$D_5(a_1)$
17	$e_{\substack{01100\\1}} + e_{\substack{10000\\0}} + e_{\substack{01110\\0}} + e_{\substack{00001\\0}} + e_{\substack{00110\\1}} + e_{\substack{00100\\0}}$	12	$E_6(a_3)$
18	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$	10	D_5
19	$e_{\substack{10000\\0}} + e_{\substack{00001\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00110\\0}} + e_{\substack{01100\\0}} + e_{\substack{00000\\1}}$	8	$E_6(a_1)$
20	$e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$	6	E_6

Table 2: Nilpotent orbit representatives (continued)

$G = E_7$			
Orbit	e	$\dim C_{\mathfrak{L}(G)}(e)$	Name
1	$e_{\substack{100000\\0}}$	99	A_1
2	$e_{\substack{100000\\0}} + e_{\substack{000010\\0}}$	81	${A_1}^2$
3	$e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$	79	$({A_1}^3)''$
4	$e_{\substack{100000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000010\\0}}$	69	$({A_1}^3)'$
5	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}}$	67	A_2
6	$e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$	63	${A_1}^4$
7	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000000\\1}}$	57	A_2A_1
8	$e_{\substack{000000\\1}} + e_{\substack{001000\\0}} + e_{\substack{100000\\0}} + e_{\substack{000010\\0}}$	51	$A_2{A_1}^2$
9	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}}$	49	A_3
10	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}}$	49	${A_2}^2$
11	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$	49	$A_2{A_1}^3$
12	$e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{000000\\1}}$	47	$(A_3A_1)''$
13	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000000\\1}}$	43	${A_2}^2A_1$
14	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000010\\0}}$	41	$(A_3A_1)'$
15	$e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$	39	$D_4(a_1)$
16	$e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}}$	39	$A_3{A_1}^2$
17	$e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$	37	D_4
18	$e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$	37	$D_4(a_1)A_1$
19	$e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$	35	A_3A_2
20	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}}$	33	A_4
21	$e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000000\\1}}$	33	$A_3A_2A_1$
22	$e_{\substack{000000\\1}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$	31	$(A_5)''$
23	$e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$	31	D_4A_1
24	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000010\\0}}$	29	A_4A_1
25	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$	27	$D_5(a_1)$
26	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$	27	A_4A_2
27	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}}$	25	$(A_5)'$
28	$e_{\substack{000000\\1}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{100000\\0}}$	25	A_5A_1

Table 2: Nilpotent orbit representatives (continued)

$G = E_7$			
Orbit	e	$\dim C_{\mathfrak{L}(G)}(e)$	Name
29	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$	25	$D_5(a_1)A_1$
30	$e_{\substack{000001\\0}} + e_{\substack{000110\\0}} + e_{\substack{001000\\1}} - e_{\substack{011000\\0}} + e_{\substack{001100\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}}$	23	$D_6(a_2)$
31	$e_{\substack{011000\\1}} + e_{\substack{100000\\0}} + e_{\substack{011100\\0}} + e_{\substack{000010\\0}} + e_{\substack{001100\\1}} + e_{\substack{001000\\0}}$	23	$E_6(a_3)$
32	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$	21	D_5
33	$e_{\substack{111100\\0}} + e_{\substack{001110\\1}} + e_{\substack{000001\\0}} + e_{\substack{011110\\0}} + e_{\substack{111000\\1}} + e_{\substack{001000\\0}} + e_{\substack{011100\\1}}$	21	$E_7(a_5)$
34	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$	19	A_6
35	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$	19	D_5A_1
36	$e_{\substack{000001\\0}} + e_{\substack{000010\\0}} + e_{\substack{000100\\0}} + e_{\substack{001000\\1}} - e_{\substack{011000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}}$	19	$D_6(a_1)$
37	$e_{\substack{011000\\1}} + e_{\substack{100000\\0}} + e_{\substack{011100\\0}} + e_{\substack{000011\\0}} + e_{\substack{001110\\1}} + e_{\substack{000001\\0}} + e_{\substack{001100\\1}} + e_{\substack{001000\\0}}$	17	$E_7(a_4)$
38	$e_{\substack{000001\\0}} + e_{\substack{000010\\0}} + e_{\substack{000100\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}}$	15	D_6
39	$e_{\substack{100000\\0}} + e_{\substack{000010\\0}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{001100\\0}} + e_{\substack{011000\\0}} + e_{\substack{000000\\1}}$	15	$E_6(a_1)$
40	$e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}}$	13	E_6
41	$e_{\substack{011000\\1}} + e_{\substack{100000\\0}} + e_{\substack{011100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{001100\\1}} + e_{\substack{001000\\0}}$	13	$E_7(a_3)$
42	$e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000110\\0}} + e_{\substack{000011\\0}}$	11	$E_7(a_2)$
43	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{011000\\0}} + e_{\substack{001000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$	9	$E_7(a_1)$
44	$e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$	7	E_7

Table 2: Nilpotent orbit representatives (continued)

$G = E_8$			
Orbit	e	$\dim C_{\mathfrak{L}(G)}(e)$	Name
1	$e_{\substack{1000000\\0}}$	190	A_1
2	$e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}}$	156	${A_1}^2$
3	$e_{\substack{1000000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000100\\0}}$	136	${A_1}^3$
4	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}}$	134	A_2
5	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$	120	${A_1}^4$
6	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0000000\\1}}$	112	A_2A_1
7	$e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}}$	102	$A_2{A_1}^2$
8	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}}$	100	A_3
9	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$	94	$A_2{A_1}^3$
10	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}}$	92	${A_2}^2$
11	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000000\\1}}$	86	${A_2}^2A_1$
12	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000100\\0}}$	84	A_3A_1
13	$e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$	82	$D_4(a_1)$
14	$e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$	80	D_4
15	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000000\\1}}$ $+ e_{\substack{0000001\\0}}$	80	${A_2}^2{A_1}^2$
16	$e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$	76	$A_3{A_1}^2$
17	$e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$ $+ e_{\substack{0000010\\0}}$	72	$D_4(a_1)A_1$
18	$e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$	70	A_3A_2
19	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}}$	68	A_4
20	$e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}}$ $+ e_{\substack{0000000\\1}}$	66	$A_3A_2A_1$
21	$e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$	64	D_4A_1
22	$e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$ $+ e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$	64	$D_4(a_1)A_2$
23	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}}$	60	A_4A_1
24	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$ $+ e_{\substack{0000001\\0}}$	60	${A_3}^2$

Table 2: Nilpotent orbit representatives (continued)

$G = E_8$			
Orbit	e	$\dim C_{\mathfrak{L}(G)}(e)$	Name
25	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$	58	$D_5(a_1)$
26	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$	56	$A_4{A_1}^2$
27	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$	54	A_4A_2
28	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}}$	52	A_5
29	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$	52	$D_5(a_1)A_1$
30	$e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0000000\\1}}$	52	$A_4A_2A_1$
31	$e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$	50	D_4A_2
32	$e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}}$	50	$E_6(a_3)$
33	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$	48	D_5
34	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$	48	A_4A_3
35	$e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{1000000\\0}}$	46	A_5A_1
36	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$	46	$D_5(a_1)A_2$
37	$e_{\substack{0000010\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$	44	$D_6(a_2)$
38	$e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0000001\\0}}$	44	$E_6(a_3)A_1$
39	$e_{\substack{1111000\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0000010\\0}} + e_{\substack{0111100\\0}} + e_{\substack{1110000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0111000\\1}}$	42	$E_7(a_5)$
40	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$	40	D_5A_1

Table 2: Nilpotent orbit representatives (continued)

$G = E_8$			
Orbit	e	$\dim C_{\mathfrak{L}(G)}(e)$	Name
41	$e_{\substack{1111111\\0}} + e_{\substack{0121110\\1}} + e_{\substack{0001000\\0}} + e_{\substack{1121100\\1}} + e_{\substack{1221000\\1}} + e_{\substack{0011111\\1}} + e_{\substack{0111100\\0}} + e_{\substack{1111110\\1}}$	40	$E_8(a_7)$
42	$e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$	38	A_6
43	$e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$	38	$D_6(a_1)$
44	$e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}} + e_{\substack{1000000\\0}}$	36	A_6A_1
45	$e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0000010\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}}$	36	$E_7(a_4)$
46	$e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}}$	34	$E_6(a_1)$
47	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$	34	D_5A_2
48	$e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$	32	D_6
49	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}}$	32	E_6
50	$e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$	32	$D_7(a_2)$
51	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$	30	A_7
52	$e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000001\\0}}$	30	$E_6(a_1)A_1$
53	$e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}}$	28	$E_7(a_3)$
54	$e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{1111100\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0111110\\1}} - e_{\substack{1111110\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1110000\\1}}$	28	$E_8(b_6)$
55	$e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$	26	$D_7(a_1)$

Table 2: Nilpotent orbit representatives (continued)

$G = E_8$			
Orbit	e	$\dim C_{\mathfrak{L}(G)}(e)$	Name
56	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$	26	E_6A_1
57	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0000110\\0}}$	24	$E_7(a_2)$
58	$e_{\substack{0011000\\1}} + e_{\substack{1111100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001111\\0}} + e_{\substack{1110000\\1}} + e_{\substack{0111000\\0}}$	24	$E_8(a_6)$
59	$e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$	22	D_7
60	$e_{\substack{0011000\\1}} + e_{\substack{0000001\\0}} + e_{\substack{1111100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1110000\\1}}$	22	$E_8(b_5)$
61	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$	20	$E_7(a_1)$
62	$e_{\substack{0011000\\1}} + e_{\substack{0000111\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000010\\0}}$	20	$E_8(a_5)$
63	$e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0010000\\0}}$	18	$E_8(b_4)$
64	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$	16	E_7
65	$e_{\substack{0011000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0001110\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000001\\0}}$	16	$E_8(a_4)$
66	$e_{\substack{0000001\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0010000\\0}}$	14	$E_8(a_3)$
67	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0000001\\0}}$	12	$E_8(a_2)$
68	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$	10	$E_8(a_1)$
69	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$	8	E_8

6 Associated cocharacters

As before, G will be a simple algebraic group of exceptional type defined over an algebraically closed field k whose characteristic is either 0 or a good prime for G, and T will be a fixed maximal torus of G. In this section, for each of the non-zero nilpotent orbit representatives in $\mathfrak{L}(G)$ listed in Table 2, we will describe our choice of an associated cocharacter $\tau : k^* \to T$, and obtain its labelled diagram Δ.

Let e be a non-zero nilpotent orbit representative, distinguished in $\mathfrak{L}(L)$, where L is a Levi subgroup of G. Write $[L, L] = L_1 \dots L_t$, a product of simple factors. To indicate our choice of Levi subgroup, we shall give a Dynkin diagram with black and white nodes, such that the simple roots corresponding to the black nodes form a basis of the root system of $[L, L]$; this information is provided at the top of the page for e in §11.

For each simple group H occurring as a simple factor in some Levi subgroup of an exceptional group, Table 3 gives a list of cocharacters, one for each orbit of distinguished elements in $\mathfrak{L}(H)$; in this table the cocharacter $\tau : k^* \to T$ is defined by giving the expression $\tau(c) = \prod_{i=1}^{\ell} h_{\alpha_i}(c^{k_i})$.

Note that the given cocharacters are invariant under all graph automorphisms of H. We thus obtain cocharacters $\tau_1, \dots, \tau_t$, one for each simple factor, and the cocharacter which we will associate to e is the product of the τ_i; that is, $\tau(c) = \prod_{i=1}^{t} \tau_i(c)$ for all $c \in k^*$. In order to show that τ is an associated cocharacter for the element e, by Definition 2.6 we must verify that $\mathrm{im}(\tau) \subseteq [L, L] \cap T$ and $\tau(c)e = c^2 e$ for all $c \in k^*$; the first of these is true by construction, while the second is a simple calculation.

At the top of the page for e in §11 we shall represent τ by its labelled diagram, as explained at the beginning of §3; we shall also give the labelled diagram Δ of the nilpotent orbit containing e.

Table 3: Cocharacters in simple factors

Orbit	Cocharacter
A_1	$h_{\alpha_1}(c)$
A_2	$h_{\alpha_1}(c^2)h_{\alpha_2}(c^2)$
A_3	$h_{\alpha_1}(c^3)h_{\alpha_2}(c^4)h_{\alpha_3}(c^3)$
A_4	$h_{\alpha_1}(c^4)h_{\alpha_2}(c^6)h_{\alpha_3}(c^6)h_{\alpha_4}(c^4)$
A_5	$h_{\alpha_1}(c^5)h_{\alpha_2}(c^8)h_{\alpha_3}(c^9)h_{\alpha_4}(c^8)h_{\alpha_5}(c^5)$
A_6	$h_{\alpha_1}(c^6)h_{\alpha_2}(c^{10})h_{\alpha_3}(c^{12})h_{\alpha_4}(c^{12})h_{\alpha_5}(c^{10})h_{\alpha_6}(c^6)$
A_7	$h_{\alpha_1}(c^7)h_{\alpha_2}(c^{12})h_{\alpha_3}(c^{15})h_{\alpha_4}(c^{16})h_{\alpha_5}(c^{15})h_{\alpha_6}(c^{12})h_{\alpha_7}(c^7)$
B_2	$h_{\alpha_1}(c^4)h_{\alpha_2}(c^3)$
B_3	$h_{\alpha_1}(c^6)h_{\alpha_2}(c^{10})h_{\alpha_3}(c^6)$
C_3	$h_{\alpha_1}(c^5)h_{\alpha_2}(c^8)h_{\alpha_3}(c^9)$
$C_3(a_1)$	$h_{\alpha_1}(c^3)h_{\alpha_2}(c^4)h_{\alpha_3}(c^5)$

Table 3: Cocharacters in simple factors (continued)

Orbit	Cocharacter
D_4	$h_{\alpha_1}(c^6)h_{\alpha_2}(c^{10})h_{\alpha_3}(c^6)h_{\alpha_4}(c^6)$
$D_4(a_1)$	$h_{\alpha_1}(c^4)h_{\alpha_2}(c^6)h_{\alpha_3}(c^4)h_{\alpha_4}(c^4)$
D_5	$h_{\alpha_1}(c^8)h_{\alpha_2}(c^{14})h_{\alpha_3}(c^{18})h_{\alpha_4}(c^{10})h_{\alpha_5}(c^{10})$
$D_5(a_1)$	$h_{\alpha_1}(c^6)h_{\alpha_2}(c^{10})h_{\alpha_3}(c^{12})h_{\alpha_4}(c^7)h_{\alpha_5}(c^7)$
D_6	$h_{\alpha_1}(c^{10})h_{\alpha_2}(c^{18})h_{\alpha_3}(c^{24})h_{\alpha_4}(c^{28})h_{\alpha_5}(c^{15})h_{\alpha_6}(c^{15})$
$D_6(a_1)$	$h_{\alpha_1}(c^8)h_{\alpha_2}(c^{14})h_{\alpha_3}(c^{18})h_{\alpha_4}(c^{20})h_{\alpha_5}(c^{11})h_{\alpha_6}(c^{11})$
$D_6(a_2)$	$h_{\alpha_1}(c^6)h_{\alpha_2}(c^{10})h_{\alpha_3}(c^{14})h_{\alpha_4}(c^{16})h_{\alpha_5}(c^9)h_{\alpha_6}(c^9)$
D_7	$h_{\alpha_1}(c^{12})h_{\alpha_2}(c^{22})h_{\alpha_3}(c^{30})h_{\alpha_4}(c^{36})h_{\alpha_5}(c^{40})h_{\alpha_6}(c^{21})h_{\alpha_7}(c^{21})$
$D_7(a_1)$	$h_{\alpha_1}(c^{10})h_{\alpha_2}(c^{18})h_{\alpha_3}(c^{24})h_{\alpha_4}(c^{28})h_{\alpha_5}(c^{30})h_{\alpha_6}(c^{16})h_{\alpha_7}(c^{16})$
$D_7(a_2)$	$h_{\alpha_1}(c^8)h_{\alpha_2}(c^{14})h_{\alpha_3}(c^{18})h_{\alpha_4}(c^{22})h_{\alpha_5}(c^{24})h_{\alpha_6}(c^{13})h_{\alpha_7}(c^{13})$
E_6	$h_{\alpha_1}(c^{16})h_{\alpha_2}(c^{22})h_{\alpha_3}(c^{30})h_{\alpha_4}(c^{42})h_{\alpha_5}(c^{30})h_{\alpha_6}(c^{16})$
$E_6(a_1)$	$h_{\alpha_1}(c^{12})h_{\alpha_2}(c^{16})h_{\alpha_3}(c^{22})h_{\alpha_4}(c^{30})h_{\alpha_5}(c^{22})h_{\alpha_6}(c^{12})$
$E_6(a_3)$	$h_{\alpha_1}(c^8)h_{\alpha_2}(c^{10})h_{\alpha_3}(c^{14})h_{\alpha_4}(c^{20})h_{\alpha_5}(c^{14})h_{\alpha_6}(c^8)$
E_7	$h_{\alpha_1}(c^{34})h_{\alpha_2}(c^{49})h_{\alpha_3}(c^{66})h_{\alpha_4}(c^{96})h_{\alpha_5}(c^{75})h_{\alpha_6}(c^{52})h_{\alpha_7}(c^{27})$
$E_7(a_1)$	$h_{\alpha_1}(c^{26})h_{\alpha_2}(c^{37})h_{\alpha_3}(c^{50})h_{\alpha_4}(c^{72})h_{\alpha_5}(c^{57})h_{\alpha_6}(c^{40})h_{\alpha_7}(c^{21})$
$E_7(a_2)$	$h_{\alpha_1}(c^{22})h_{\alpha_2}(c^{31})h_{\alpha_3}(c^{42})h_{\alpha_4}(c^{60})h_{\alpha_5}(c^{47})h_{\alpha_6}(c^{32})h_{\alpha_7}(c^{17})$
$E_7(a_3)$	$h_{\alpha_1}(c^{18})h_{\alpha_2}(c^{25})h_{\alpha_3}(c^{34})h_{\alpha_4}(c^{50})h_{\alpha_5}(c^{39})h_{\alpha_6}(c^{28})h_{\alpha_7}(c^{15})$
$E_7(a_4)$	$h_{\alpha_1}(c^{14})h_{\alpha_2}(c^{19})h_{\alpha_3}(c^{26})h_{\alpha_4}(c^{38})h_{\alpha_5}(c^{29})h_{\alpha_6}(c^{20})h_{\alpha_7}(c^{11})$
$E_7(a_5)$	$h_{\alpha_1}(c^{10})h_{\alpha_2}(c^{15})h_{\alpha_3}(c^{20})h_{\alpha_4}(c^{30})h_{\alpha_5}(c^{23})h_{\alpha_6}(c^{16})h_{\alpha_7}(c^9)$
E_8	$h_{\alpha_1}(c^{92})h_{\alpha_2}(c^{136})h_{\alpha_3}(c^{182})h_{\alpha_4}(c^{270})h_{\alpha_5}(c^{220})h_{\alpha_6}(c^{168})h_{\alpha_7}(c^{114})h_{\alpha_8}(c^{58})$
$E_8(a_1)$	$h_{\alpha_1}(c^{72})h_{\alpha_2}(c^{106})h_{\alpha_3}(c^{142})h_{\alpha_4}(c^{210})h_{\alpha_5}(c^{172})h_{\alpha_6}(c^{132})h_{\alpha_7}(c^{90})h_{\alpha_8}(c^{46})$
$E_8(a_2)$	$h_{\alpha_1}(c^{60})h_{\alpha_2}(c^{88})h_{\alpha_3}(c^{118})h_{\alpha_4}(c^{174})h_{\alpha_5}(c^{142})h_{\alpha_6}(c^{108})h_{\alpha_7}(c^{74})h_{\alpha_8}(c^{38})$
$E_8(a_3)$	$h_{\alpha_1}(c^{52})h_{\alpha_2}(c^{76})h_{\alpha_3}(c^{102})h_{\alpha_4}(c^{152})h_{\alpha_5}(c^{124})h_{\alpha_6}(c^{96})h_{\alpha_7}(c^{66})h_{\alpha_8}(c^{34})$
$E_8(a_4)$	$h_{\alpha_1}(c^{44})h_{\alpha_2}(c^{64})h_{\alpha_3}(c^{86})h_{\alpha_4}(c^{128})h_{\alpha_5}(c^{104})h_{\alpha_6}(c^{80})h_{\alpha_7}(c^{54})h_{\alpha_8}(c^{28})$
$E_8(b_4)$	$h_{\alpha_1}(c^{40})h_{\alpha_2}(c^{58})h_{\alpha_3}(c^{78})h_{\alpha_4}(c^{116})h_{\alpha_5}(c^{94})h_{\alpha_6}(c^{72})h_{\alpha_7}(c^{50})h_{\alpha_8}(c^{26})$
$E_8(a_5)$	$h_{\alpha_1}(c^{36})h_{\alpha_2}(c^{52})h_{\alpha_3}(c^{70})h_{\alpha_4}(c^{104})h_{\alpha_5}(c^{84})h_{\alpha_6}(c^{64})h_{\alpha_7}(c^{44})h_{\alpha_8}(c^{22})$
$E_8(b_5)$	$h_{\alpha_1}(c^{32})h_{\alpha_2}(c^{48})h_{\alpha_3}(c^{64})h_{\alpha_4}(c^{96})h_{\alpha_5}(c^{78})h_{\alpha_6}(c^{60})h_{\alpha_7}(c^{42})h_{\alpha_8}(c^{22})$
$E_8(a_6)$	$h_{\alpha_1}(c^{28})h_{\alpha_2}(c^{42})h_{\alpha_3}(c^{56})h_{\alpha_4}(c^{84})h_{\alpha_5}(c^{68})h_{\alpha_6}(c^{52})h_{\alpha_7}(c^{36})h_{\alpha_8}(c^{18})$
$E_8(b_6)$	$h_{\alpha_1}(c^{24})h_{\alpha_2}(c^{36})h_{\alpha_3}(c^{48})h_{\alpha_4}(c^{72})h_{\alpha_5}(c^{58})h_{\alpha_6}(c^{44})h_{\alpha_7}(c^{30})h_{\alpha_8}(c^{16})$
$E_8(a_7)$	$h_{\alpha_1}(c^{16})h_{\alpha_2}(c^{24})h_{\alpha_3}(c^{32})h_{\alpha_4}(c^{48})h_{\alpha_5}(c^{40})h_{\alpha_6}(c^{30})h_{\alpha_7}(c^{20})h_{\alpha_8}(c^{10})$
F_4	$h_{\alpha_1}(c^{22})h_{\alpha_2}(c^{42})h_{\alpha_3}(c^{30})h_{\alpha_4}(c^{16})$
$F_4(a_1)$	$h_{\alpha_1}(c^{14})h_{\alpha_2}(c^{26})h_{\alpha_3}(c^{18})h_{\alpha_4}(c^{10})$
$F_4(a_2)$	$h_{\alpha_1}(c^{10})h_{\alpha_2}(c^{20})h_{\alpha_3}(c^{14})h_{\alpha_4}(c^8)$
$F_4(a_3)$	$h_{\alpha_1}(c^6)h_{\alpha_2}(c^{12})h_{\alpha_3}(c^8)h_{\alpha_4}(c^4)$
G_2	$h_{\alpha_1}(c^6)h_{\alpha_2}(c^{10})$
$G_2(a_1)$	$h_{\alpha_1}(c^2)h_{\alpha_2}(c^4)$

7 The connected centralizer

Again, G will be a simple algebraic group of exceptional type defined over an algebraically closed field k whose characteristic is either 0 or a good prime for G, and T will be a fixed maximal torus of G. Let $e \in \mathfrak{L}(G)$ be a non-zero nilpotent orbit representative listed in Table 2, with associated cocharacter $\tau : k^* \to T$ and Levi subgroup L of G as given at the top of the page for e in §11, so that $T \subset L$ and e is distinguished in $\mathfrak{L}(L)$. Let P be the parabolic subgroup corresponding to τ as defined in §2. Recall that we have $C_G(e) = R.C$, where $R = C_G(e) \cap R_u(P)$ and $C = C_G(e) \cap C_G(\mathrm{im}(\tau))$, with $\mathfrak{L}(R) = C_{\mathfrak{L}(G)}(e)_+$ and $\mathfrak{L}(C) = C_{\mathfrak{L}(G)}(e)_0$. In this section we will obtain detailed information about both $\mathfrak{L}(R)$ and C°.

7.1 A basis of $C_{\mathfrak{L}(G)}(e)_+$ and its upper central series

We begin with $\mathfrak{L}(R) = C_{\mathfrak{L}(G)}(e)_+$; we shall first obtain a basis of this subalgebra. In order to do this we introduce a second grading on $\mathfrak{L}(G)$, which will lead to a refinement of that defined by the cocharacter τ.

We have the torus $Z(L)^\circ$; write $X(Z(L)^\circ)$ for its character group. For $\chi \in X(Z(L)^\circ)$ write

$$\mathfrak{L}(G)^\chi = \{v \in \mathfrak{L}(G) : (\mathrm{Ad}\, t)v = \chi(t)v \text{ for all } t \in Z(L)^\circ\}$$

for the corresponding $Z(L)^\circ$-weight space of $\mathfrak{L}(G)$; we then have the grading

$$\mathfrak{L}(G) = \bigoplus_{\chi \in X(Z(L)^\circ)} \mathfrak{L}(G)^\chi.$$

Since $Z(L)^\circ \subseteq T$, each $\mathfrak{L}(G)^\chi$ for $\chi \neq 0$ has a basis of root vectors, while $\mathfrak{L}(G)^0 = \mathfrak{L}(L)$; indeed, given $\beta, \gamma \in \Phi$ the root vectors e_β and e_γ lie in the same $Z(L)^\circ$-weight space if and only if $\beta - \gamma$ is a linear combination of roots in $\Pi([L, L])$.

As both $\mathrm{im}(\tau)$ and $Z(L)^\circ$ are subtori of T, each preserves the weight spaces of the other; accordingly we may decompose $\mathfrak{L}(G)$ into $\mathrm{im}(\tau)Z(L)^\circ$-weight spaces. Given a pair $(m, \chi) \in \mathbb{Z} \times X(Z(L)^\circ)$, we write

$$\mathfrak{L}(G)^\chi_m = \mathfrak{L}(G)_m \cap \mathfrak{L}(G)^\chi;$$

we then have the grading

$$\mathfrak{L}(G) = \bigoplus_{(m,\chi) \in \mathbb{Z} \times X(Z(L)^\circ)} \mathfrak{L}(G)^\chi_m.$$

This is a refinement of the grading defined by τ; for each pair $(m, \chi) \neq (0, 0)$ the $\mathrm{im}(\tau)Z(L)^\circ$-weight space $\mathfrak{L}(G)^\chi_m$ has a basis consisting of root vectors.

Proposition 7.1 *For $v \in \mathfrak{L}(G)$, write $v = \sum v^\chi_m$, where the sum runs over all pairs $(m, \chi) \in \mathbb{Z} \times X(Z(L)^\circ)$ and $v^\chi_m \in \mathfrak{L}(G)^\chi_m$. Then $v \in C_{\mathfrak{L}(G)}(e)$ if and only if $v^\chi_m \in C_{\mathfrak{L}(G)}(e)$ for all $(m, \chi) \in \mathbb{Z} \times X(Z(L)^\circ)$.*

Proof. Observe that $Z(L)^\circ$ lies in $C_G(e)$ and hence stabilizes $C_{\mathfrak{L}(G)}(e)$; recall that $\mathrm{im}(\tau)$ also stabilizes $C_{\mathfrak{L}(G)}(e)$. Thus $\mathrm{im}(\tau)Z(L)^\circ$ acts on $C_{\mathfrak{L}(G)}(e)$; the result follows. □

We note that for $(m, \chi) \in \mathbb{Z} \times X(Z(L)^\circ)$, we have $\mathrm{ad}(e) : \mathfrak{L}(G)^\chi_m \to \mathfrak{L}(G)^\chi_{m+2}$. If we take bases of the kernels of these maps for all pairs (m, χ) with $m > 0$, by Proposition 2.9(b) we see that their union is the desired basis of $C_{\mathfrak{L}(G)}(e)_+ = \bigoplus_{m>0} C_{\mathfrak{L}(G)}(e)_m$. In particular, $\dim C_{\mathfrak{L}(G)}(e)_+$ is the cardinality of this basis.

We now consider the upper central series

$$0 \subset Z_1(C_{\mathfrak{L}(G)}(e)_+) \subset Z_2(C_{\mathfrak{L}(G)}(e)_+) \subset \cdots,$$

where we recall that for a Lie algebra $\mathfrak{A}$ we have $Z_1(\mathfrak{A}) = Z(\mathfrak{A})$, and for $n \geq 2$ the term $Z_n(\mathfrak{A})$ contains $Z_{n-1}(\mathfrak{A})$ and satisfies $Z(\mathfrak{A}/Z_{n-1}(\mathfrak{A})) = Z_n(\mathfrak{A})/Z_{n-1}(\mathfrak{A})$. We note that both $\mathrm{im}(\tau)$ and $Z(L)^\circ$ act on $C_{\mathfrak{L}(G)}(e)_+$ as Lie algebra automorphisms, and hence stabilize each subspace $Z_n(C_{\mathfrak{L}(G)}(e)_+)$. So a vector v in $C_{\mathfrak{L}(G)}(e)_+$ lies in $Z_n(C_{\mathfrak{L}(G)}(e)_+)$ if and only if, for each pair $(m,\chi) \in \mathbb{Z} \times X(Z(L)^\circ)$, the projection v_m^χ of v into the $\mathrm{im}(\tau)Z(L)^\circ$-weight space $\mathfrak{L}(G)_m^\chi$ lies in $Z_n(C_{\mathfrak{L}(G)}(e)_+)$; in particular we have

$$Z_n(C_{\mathfrak{L}(G)}(e)_+) = \bigoplus_{m>0} Z_{n,m},$$

where for convenience we write $Z_{n,m} = (Z_n(C_{\mathfrak{L}(G)}(e)_+))_m$. We may thus successively identify the terms $Z_n(C_{\mathfrak{L}(G)}(e)_+)$ for $n = 1, 2, \ldots$ by calculating commutators of vectors in the basis of $C_{\mathfrak{L}(G)}(e)_+$ just obtained. Arguing by induction on n, one sees that if $n+m$ is greater than the largest τ-weight on $\mathfrak{L}(G)$, or more generally if there are fewer than n distinct τ-weights on $\mathfrak{L}(G)$ which are larger than m, then $C_{\mathfrak{L}(G)}(e)_m$ lies in $Z_n(C_{\mathfrak{L}(G)}(e)_+)$ and so $Z_{n,m} = C_{\mathfrak{L}(G)}(e)_m$.

Example To illustrate the above, we take an example in a classical group. Let $G = A_5$ and set $e = e_{10000} + e_{01000} + e_{00001}$; then e is distinguished in the Lie algebra of the Levi subgroup L of type A_2A_1 having simple system $\{\alpha_1, \alpha_2, \alpha_5\}$. We may take the associated cocharacter τ to be given by $\tau(c) = h_{\alpha_1}(c^2)h_{\alpha_2}(c^2)h_{\alpha_5}(c)$ for $c \in k^*$, since we then clearly have $\mathrm{im}(\tau) \subseteq [L,L] \cap T$ and $\tau(c)e = c^2 e$ for all $c \in k^*$. We shall give bases of the various spaces $\mathfrak{L}(G)_m^\chi$ in a table, in which the rows are labelled by the values of m and the columns correspond to the $Z(L)^\circ$-weights χ: for convenience we may represent each such χ as a pair (n_3, n_4), where the root vectors e_β lying in $\mathfrak{L}(G)^\chi$ are those for which the coefficients in β of α_3 and α_4 are n_3 and n_4 respectively.

	$(-1,-1)$	$(-1,0)$	$(0,-1)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
-4				f_{11000}			
-3	f_{11111}						e_{00110}
-2		f_{11100}		$f_{10000}, f_{01000}, f_{00001}$		e_{00100}	
-1	f_{11110}, f_{01111}		f_{00011}		e_{00010}		e_{01110}, e_{00111}
0		f_{01100}		$h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}, h_{\alpha_4}, h_{\alpha_5}$		e_{01100}	
1	f_{01110}, f_{00111}		f_{00010}		e_{00011}		e_{11110}, e_{01111}
2		f_{00100}		$e_{10000}, e_{01000}, e_{00001}$		e_{11100}	
3	f_{00110}						e_{11111}
4				e_{11000}			

By taking the kernels of the maps $\mathrm{ad}(e) : \mathfrak{L}(G)_m^\chi \to \mathfrak{L}(G)_{m+2}^\chi$ for $m > 0$, we obtain the following table giving a basis of $C_{\mathfrak{L}(G)}(e)_+$.

	$(-1,-1)$	$(-1,0)$	$(0,-1)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
1	$f_{01110} + f_{00111}$		f_{00010}		e_{00011}		$e_{11110} + e_{01111}$
2		f_{00100}		$e_{10000} + e_{01000}, e_{00001}$		e_{11100}	
3	f_{00110}						e_{11111}
4				e_{11000}			

Thus $\dim C_{\mathfrak{L}(G)}(e)_+ = 11$. Taking commutators then shows that $Z_{n,2} = \langle e \rangle$ or $C_{\mathfrak{L}(G)}(e)_2$ according as $n \leq 2$ or $n > 2$, while if $m \neq 2$ then $Z_{n,m} = 0$ or $C_{\mathfrak{L}(G)}(e)_m$ according as $n + m \leq 4$ or $n + m > 4$.

7.2 Identifying C°

We now turn to the subgroup C°, which was shown to be reductive by Premet in [31, Theorem 2.3]. In fact, we shall not need to appeal to Premet's result: we shall obtain explicit generators for C°, as these will be needed for the calculations in the following sections; consequently we shall show independently that C° is reductive, and give its rank and the type of its root system.

We have $\dim C = \dim C_G(e) - \dim R$, which is now known since the first term on the right is given in Table 2 while the second is equal to the dimension of $C_{\mathfrak{L}(G)}(e)_+$. The following result shows that the reductive rank of C is equal to $\ell - \operatorname{rank}[L, L]$.

Lemma 7.2 *The torus $Z(L)^\circ$ is a maximal torus of $C_G(e)$, and hence of C.*

Proof. Let S be a maximal torus of $C_G(e)$ with $Z(L)^\circ \subseteq S$; then $S \subset C_G(Z(L)^\circ) = L$. But then if $Z(L)^\circ \neq S$, we must have $S \cap [L, L]$ of positive dimension, contradicting the fact that e is distinguished in $\mathfrak{L}(L)$. □

A 1-dimensional $Z(L)^\circ$-invariant connected unipotent subgroup of C will be called a *$Z(L)^\circ$-root subgroup*.

In what follows, we will define a set of closed connected simple subgroups $C_1, \dots, C_r$, each generated by $Z(L)^\circ$-root subgroups and lying in $C_G(e) \cap C_G(\operatorname{im}(\tau))$. Note that to see that $C_i \subseteq C_G(\operatorname{im}(\tau))$ it suffices to show that the generating subgroups lie in products of T-root subgroups U_α with $\operatorname{im}(\tau) \subseteq \ker \alpha$. Consideration of dimensions will show that $C^\circ = C_1 \dots C_r Z(L)^\circ$; it will then follow that C° is reductive and $[C^\circ, C^\circ] = C_1 \dots C_r$.

To begin with we exclude the orbits in E_7 or E_8 labelled $A_2{A_1}^3$, ${A_2}^2{A_1}^2$ and $D_4(a_1)A_2$; these will be treated separately later.

We define a certain number s of subgroups C_i as follows. For each we give a subsystem subgroup G_i of G possessing a (possibly trivial) graph automorphism σ with the property that

$$\text{for all } \alpha \in \Phi(G_i), \text{ we have } \alpha \perp \{\sigma^j \alpha : \sigma^j \alpha \neq \alpha\},$$

and let C_i be the fixed point subgroup $(G_i)^\sigma$. We observe that since C_i is the fixed point subgroup of a graph automorphism of G_i, it is a semisimple algebraic group and the type of $\Phi(C_i)$ is easily deduced; indeed, G_i is chosen such that the fixed point subgroup $(G_i)^\sigma$ is simple. By construction, root elements of C_i are products of commuting root elements of G. The action of such an element on $\mathfrak{L}(G)$ is hence given by $\exp(\operatorname{ad}(b_1 e_{\gamma_1} + \cdots + b_n e_{\gamma_n}))$, for some $\gamma_1, \dots, \gamma_n \in \Phi(G)$ with $[e_{\gamma_j}, e_{\gamma_{j'}}] = 0$ for $j \neq j'$ and some $b_1, \dots, b_n \in k$; a check reveals that $[b_1 e_{\gamma_1} + \cdots b_n e_{\gamma_n}, e] = 0$. It follows that $C_i \subseteq C_G(e)$. On the page for e in §11 we give explicit expressions for a set of root subgroups of C_i corresponding to simple roots and their negatives.

At this point we observe that the subgroups $C_1, \dots, C_s$ thus defined all commute, and therefore that $\dim C_1 \dots C_s = \sum_{i=1}^{s} \dim C_i$. Moreover we find that if we set $d = \dim C - \dim Z(L)^\circ - \sum_{i=1}^{s} |\Phi(C_i)|$, then $d = 0$ or 2. If $d = 0$, we set $r = s$. If on the other hand $d = 2$, we set $r = s + 1$; this occurs in the following cases.

(i) G of type F_4: orbits labelled $A_1\tilde{A}_1$, $A_2\tilde{A}_1$ and B_3 (numbers 3, 6 and 11 respectively);

(ii) G of type E_6: orbit labelled $A_2{A_1}^2$ (number 7);

(iii) G of type E_7: orbits labelled $A_2{A_1}^2$, $A_3A_2A_1$, A_4A_2, $D_5(a_1)A_1$ and A_6 (numbers 8, 21, 26, 29 and 34 respectively);

(iv) G of type E_8: orbits labelled $A_2{A_1}^2$, $A_3A_2A_1$, A_4A_2, $D_5(a_1)A_1$, $A_4A_2A_1$, A_4A_3, $D_5(a_1)A_2$, A_6 and A_6A_1 (numbers 7, 20, 27, 29, 30, 34, 36, 42 and 44 respectively).

In these cases we proceed to define the group C_r, which will be of type A_1, as follows. We define two 1-dimensional subgroups of G as the images of maps $x_\beta, x_{-\beta} : k \to G$, where $x_\beta(t)$ and $x_{-\beta}(t)$ for $t \in k$ are expressed as products of root elements of G. For $\gamma \in \{\pm\beta\}$ one checks the following:

(I) x_γ defines a morphism of algebraic groups $\mathbf{G}_a \to G$;

(II) for all $t \in k$ the element $x_\gamma(t)$ fixes e and lies in $C_G(\mathrm{im}(\tau))$, and thus lies in C°;

(III) the subgroup $X_\gamma = \{x_\gamma(t) : t \in k\}$ is normalized by $Z(L)^\circ$, and is therefore a $Z(L)^\circ$-root subgroup of C°;

(IV) the action of $Z(L)^\circ$ on $X_{-\gamma}$ is the inverse of that on X_γ.

We then set $C_r = \langle X_{\pm\beta} \rangle$. Moreover, we observe that in all but two of these cases the roots occurring in the expressions for $x_{\pm\beta}(t)$ are orthogonal to $\bigcup_{i=1}^{r-1} \Phi(G_i)$; in the remaining cases (the orbits labelled $A_2{A_1}^2$ in E_7 and E_8), one checks that $[X_{\pm\beta}, C_i] = 1$ for $i < r$. Thus $C_1 \dots C_{r-1}C_r$ is a commuting product, so that $\dim(C_r \cap C_1 \dots C_{r-1}) = 0$; as $\dim C^\circ = \dim(C_1 \dots C_{r-1}Z(L)^\circ) + 2$, we must have $C^\circ = C_1 \dots C_{r-1}C_rZ(L)^\circ$. Thus if we write $J = C_1 \dots C_{r-1}Z(L)^\circ$, we have $C^\circ = JC_r$, so $\dim JC_r = \dim J + 2$, giving

$$\begin{aligned} 2 &= \dim(JC_r/J) \\ &= \dim(C_r/(C_r \cap J)) \\ &= \dim(C_r/(C_r \cap Z(L)^\circ)) \\ &= \dim(C_rZ(L)^\circ/Z(L)^\circ) \\ &= \dim C_rZ(L)^\circ - \dim Z(L)^\circ. \end{aligned}$$

For $\gamma \in \{\pm\beta\}$ we set $v_\gamma = dx_\gamma(1)$; then v_γ spans the Lie algebra $\mathfrak{L}(X_\gamma)$. Let $\mathfrak{M}$ be the subalgebra of $\mathfrak{L}(G)$ generated by $v_{\pm\beta}$. By calculation we see that $\mathfrak{M} \simeq \mathfrak{sl}_2(k)$. Since $\mathfrak{L}(C_rZ(L)^\circ)$ contains both $\mathfrak{L}(Z(L)^\circ)$ and $\mathfrak{M}$, and its dimension is $\dim \mathfrak{L}(Z(L)^\circ) + 2$ while $\dim(\mathfrak{M} \cap \mathfrak{L}(Z(L)^\circ)) \leq 1$, we must have $\mathfrak{L}(C_rZ(L)^\circ) = \mathfrak{M} + \mathfrak{L}(Z(L)^\circ)$. As this Lie algebra has no non-trivial ideals consisting of nilpotent elements, we see that $C_rZ(L)^\circ$ is reductive; consideration of dimensions now shows that C_r is of type A_1.

We now turn to the orbits excluded above. Note that the orbit $A_2{A_1}^3$ occurs in both E_7 and E_8 (numbers 11 and 9 respectively); however, it will suffice to treat it in E_7 since the difference in dimensions between the centralizers in E_8 and in E_7 is 3, which is accounted for by the fact that the centralizer in E_8 contains $C_{E_8}(E_7) = \langle X_{\pm\delta} \rangle$ for $\delta = {}^{2465432}_{3}$. The other two orbits ${A_2}^2{A_1}^2$ and $D_4(a_1)A_2$ occur only in E_8 (numbers 15 and 22 respectively).

In each of these three cases we shall give four maps $x_{\beta_1}, x_{-\beta_1}, x_{\beta_2}, x_{-\beta_2} : k \to G$. For $\gamma \in \{\pm\beta_1, \pm\beta_2\}$ one may then check that properties (I)–(IV) above hold, and thus we may write $\gamma : Z(L)^\circ \to k^*$ for the root corresponding to the $Z(L)^\circ$-root subgroup X_γ. As above, for each γ we set $v_\gamma = dx_\gamma(1)$; then v_γ spans the Lie algebra $\mathfrak{L}(X_\gamma)$. Let $\mathfrak{M}$ be the subalgebra of $\mathfrak{L}(G)$ generated by $v_{\pm\beta_1}, v_{\pm\beta_2}$; since each of the generating vectors lies in $\mathfrak{L}(C^\circ)$, we have $\mathfrak{M} \subseteq \mathfrak{L}(C^\circ)$. In each case we will find in $\mathfrak{M}$ a set of linearly independent vectors of cardinality equal to $\dim C^\circ$; it follows that $\mathfrak{M} = \mathfrak{L}(C^\circ)$ and the vectors found form a basis. We check that this basis satisfies the relations of a simple Lie algebra of type A_2, B_2 or G_2, having β_1 and β_2 as simple roots. In particular, $\mathfrak{L}(C^\circ)$ has no non-trivial ideal, and so must be simple; hence C° is simple and its type is that of $\mathfrak{M}$, with $\{\beta_1, \beta_2\}$ a simple system in $\Phi(C^\circ)$.

We begin with the orbit labelled ${A_2A_1}^3$ in E_7. Set

$$\begin{aligned}
x_{\beta_1}(t) &= x_{\substack{111000\\0}}(t)x_{\substack{001100\\1}}(-2t)x_{\substack{112100\\1}}(-t^2)x_{\substack{011000\\1}}(-t)x_{\substack{011100\\0}}(t),\\
x_{-\beta_1}(t) &= x_{-\substack{111000\\0}}(2t)x_{-\substack{001100\\1}}(-t)x_{-\substack{112100\\1}}(t^2)x_{-\substack{011000\\1}}(-t)x_{-\substack{011100\\0}}(t),\\
x_{\beta_2}(t) &= x_{\substack{000110\\0}}(t)x_{\substack{000011\\0}}(t),\\
x_{-\beta_2}(t) &= x_{-\substack{000110\\0}}(t)x_{-\substack{000011\\0}}(t).
\end{aligned}$$

We have

$$\begin{aligned}
v_{\beta_1} &= e_{\substack{111000\\0}} - 2e_{\substack{001100\\1}} - e_{\substack{011000\\1}} + e_{\substack{011100\\0}},\\
v_{-\beta_1} &= 2f_{\substack{111000\\0}} - f_{\substack{001100\\1}} - f_{\substack{011000\\1}} + f_{\substack{011100\\0}},\\
v_{\beta_2} &= e_{\substack{000110\\0}} + e_{\substack{000011\\0}},\\
v_{-\beta_2} &= f_{\substack{000110\\0}} + f_{\substack{000011\\0}}.
\end{aligned}$$

Let

$$\begin{aligned}
v_{\beta_1+\beta_2} = [v_{\beta_2}, v_{\beta_1}] &= 2e_{\substack{001111\\1}} - e_{\substack{111110\\0}} + e_{\substack{011110\\1}} - e_{\substack{011111\\0}},\\
v_{-(\beta_1+\beta_2)} = [v_{-\beta_1}, v_{-\beta_2}] &= f_{\substack{001111\\1}} - 2f_{\substack{111110\\0}} + f_{\substack{011110\\1}} - f_{\substack{011111\\0}},\\
v_{2\beta_1+\beta_2} = \tfrac{1}{2}[v_{\beta_1+\beta_2}, v_{\beta_1}] &= e_{\substack{122110\\1}} - 2e_{\substack{012211\\1}} + e_{\substack{112111\\1}} - e_{\substack{112210\\1}},\\
v_{-(2\beta_1+\beta_2)} = \tfrac{1}{2}[v_{-\beta_1}, v_{-(\beta_1+\beta_2)}] &= 2f_{\substack{122110\\1}} - f_{\substack{012211\\1}} + f_{\substack{112111\\1}} - f_{\substack{112210\\1}},\\
v_{3\beta_1+\beta_2} = \tfrac{1}{3}[v_{\beta_1}, v_{2\beta_1+\beta_2}] &= -e_{\substack{123211\\1}} - e_{\substack{123210\\2}},\\
v_{-(3\beta_1+\beta_2)} = \tfrac{1}{3}[v_{-(2\beta_1+\beta_2)}, v_{-\beta_1}] &= -f_{\substack{123211\\1}} - f_{\substack{123210\\2}},\\
v_{3\beta_1+2\beta_2} = [v_{\beta_2}, v_{3\beta_1+\beta_2}] &= -e_{\substack{123321\\1}} + e_{\substack{123221\\2}},\\
v_{-(3\beta_1+2\beta_2)} = [v_{-(3\beta_1+\beta_2)}, v_{-\beta_2}] &= -f_{\substack{123321\\1}} + f_{\substack{123221\\2}},\\
h_{\beta_1} = [v_{\beta_1}, v_{-\beta_1}] &= 2h_{\substack{111000\\0}} + 2h_{\substack{001100\\1}} + h_{\substack{011000\\1}} + h_{\substack{011100\\0}},\\
h_{\beta_2} = [v_{\beta_2}, v_{-\beta_2}] &= h_{\substack{000110\\0}} + h_{\substack{000011\\0}}.
\end{aligned}$$

One checks that the 14 vectors $v_{\pm\beta_1}, v_{\pm\beta_2}, v_{\pm(\beta_1+\beta_2)}, v_{\pm(2\beta_1+\beta_2)}, v_{\pm(3\beta_1+\beta_2)}, v_{\pm(3\beta_1+2\beta_2)}$, h_{β_1}, h_{β_2} satisfy the relations of a Lie algebra of type G_2.

Now consider the orbit labelled ${A_2}^2{A_1}^2$ in E_8. Set

$$\begin{aligned}
x_{\beta_1}(t) &= x_{\substack{1121100\\1}}(t)x_{\substack{1221000\\1}}(t)x_{\substack{0122100\\1}}(-t),\\
x_{-\beta_1}(t) &= x_{-\substack{1121100\\1}}(t)x_{-\substack{1221000\\1}}(t)x_{-\substack{0122100\\1}}(-t),\\
x_{\beta_2}(t) &= x_{\substack{1111110\\0}}(t)x_{\substack{0011111\\1}}(-2t)x_{\substack{1122221\\1}}(t^2)x_{\substack{0111110\\1}}(-t)x_{\substack{0111111\\0}}(t),\\
x_{-\beta_2}(t) &= x_{-\substack{1111110\\0}}(2t)x_{-\substack{0011111\\1}}(-t)x_{-\substack{1122221\\1}}(-t^2)x_{-\substack{0111110\\1}}(-t)x_{-\substack{0111111\\0}}(t).
\end{aligned}$$

We have

$$\begin{aligned}
v_{\beta_1} &= e_{\substack{1121100\\1}} + e_{\substack{1221000\\1}} - e_{\substack{0122100\\1}}\,,\\
v_{-\beta_1} &= f_{\substack{1121100\\1}} + f_{\substack{1221000\\1}} - f_{\substack{0122100\\1}}\,,\\
v_{\beta_2} &= e_{\substack{1111110\\0}} - 2e_{\substack{0011111\\1}} - e_{\substack{0111110\\1}} + e_{\substack{0111111\\0}}\,,\\
v_{-\beta_2} &= 2f_{\substack{1111110\\0}} - f_{\substack{0011111\\1}} - f_{\substack{0111110\\1}} + f_{\substack{0111111\\0}}\,.
\end{aligned}$$

Let

$$\begin{aligned}
v_{\beta_1+\beta_2} = [v_{\beta_1}, v_{\beta_2}] &= e_{\substack{1233210\\1}} - 2e_{\substack{1232111\\2}} - e_{\substack{1232210\\2}} + e_{\substack{1232211\\1}}\,,\\
v_{-(\beta_1+\beta_2)} = [v_{-\beta_2}, v_{-\beta_1}] &= 2f_{\substack{1233210\\1}} - f_{\substack{1232111\\2}} - f_{\substack{1232210\\2}} + f_{\substack{1232211\\1}}\,,\\
v_{\beta_1+2\beta_2} = \tfrac{1}{2}[v_{\beta_1+\beta_2}, v_{\beta_2}] &= e_{\substack{1343321\\2}} - e_{\substack{1244321\\2}} - e_{\substack{2343221\\2}}\,,\\
v_{-(\beta_1+2\beta_2)} = \tfrac{1}{2}[v_{-\beta_2}, v_{-(\beta_1+\beta_2)}] &= f_{\substack{1343321\\2}} - f_{\substack{1244321\\2}} - f_{\substack{2343221\\2}}\,,\\
h_{\beta_1} = [v_{\beta_1}, v_{-\beta_1}] &= h_{\substack{1121100\\1}} + h_{\substack{1221000\\1}} + h_{\substack{0122100\\1}}\,,\\
h_{\beta_2} = [v_{\beta_2}, v_{-\beta_2}] &= 2h_{\substack{1111110\\0}} + 2h_{\substack{0011111\\1}} + h_{\substack{0111110\\1}} + h_{\substack{0111111\\0}}\,.
\end{aligned}$$

One checks that the 10 vectors $v_{\pm\beta_1}, v_{\pm\beta_2}, v_{\pm(\beta_1+\beta_2)}, v_{\pm(\beta_1+2\beta_2)}, h_{\beta_1}, h_{\beta_2}$ satisfy the relations of a Lie algebra of type B_2.

Finally we consider the orbit labelled $D_4(a_1)A_2$ in E_8. Set

$$\begin{aligned}
x_{\beta_1}(t) &= x_{\substack{0001111\\0}}(3t)x_{\substack{0121100\\1}}(t)x_{\substack{0011111\\0}}(-t)x_{\substack{0111100\\1}}(t)x_{\substack{0122211\\1}}(-2t^2)\times\\
&\qquad x_{\substack{0011110\\1}}(2t)x_{\substack{0111110\\0}}(-t),\\
x_{-\beta_1}(t) &= x_{-\substack{0001111\\0}}(t)x_{-\substack{0121100\\1}}(3t)x_{-\substack{0011111\\0}}(-t)x_{-\substack{0111100\\1}}(t)x_{-\substack{0122211\\1}}(2t^2)\times\\
&\qquad x_{-\substack{0011110\\1}}(t)x_{-\substack{0111110\\0}}(-2t),\\
x_{\beta_2}(t) &= x_{\substack{1111111\\1}}(-3t)x_{\substack{1232100\\1}}(-t)x_{\substack{1121111\\1}}(-t)x_{\substack{1222100\\1}}(t)x_{\substack{2343211\\2}}(-2t^2)\times\\
&\qquad x_{\substack{1122110\\1}}(2t)x_{\substack{1221110\\1}}(-t),\\
x_{-\beta_2}(t) &= x_{-\substack{1111111\\1}}(-t)x_{-\substack{1232100\\1}}(-3t)x_{-\substack{1121111\\1}}(-t)x_{-\substack{1222100\\1}}(t)x_{-\substack{2343211\\2}}(2t^2)\times\\
&\qquad x_{-\substack{1122110\\1}}(t)x_{-\substack{1221110\\1}}(-2t).
\end{aligned}$$

We have

$$\begin{aligned}
v_{\beta_1} &= 3e_{\substack{0001111\\0}} + e_{\substack{0121100\\1}} - e_{\substack{0011111\\0}} + e_{\substack{0111100\\1}} + 2e_{\substack{0011110\\1}} - e_{\substack{0111110\\0}}\,,\\
v_{-\beta_1} &= f_{\substack{0001111\\0}} + 3f_{\substack{0121100\\1}} - f_{\substack{0011111\\0}} + f_{\substack{0111100\\1}} + f_{\substack{0011110\\1}} - 2f_{\substack{0111110\\0}}\,,\\
v_{\beta_2} &= -3e_{\substack{1111111\\1}} - e_{\substack{1232100\\1}} - e_{\substack{1121111\\1}} + e_{\substack{1222100\\1}} + 2e_{\substack{1122110\\1}} - e_{\substack{1221110\\1}}\,,\\
v_{-\beta_2} &= -f_{\substack{1111111\\1}} - 3f_{\substack{1232100\\1}} - f_{\substack{1121111\\1}} + f_{\substack{1222100\\1}} + f_{\substack{1122110\\1}} - 2f_{\substack{1221110\\1}}\,.
\end{aligned}$$

Let

$$
\begin{aligned}
v_{\beta_1+\beta_2} &= \tfrac{1}{2}[v_{\beta_1}, v_{\beta_2}] = 2e_{\substack{1233210\\2}} - e_{\substack{1232221\\1}} + e_{\substack{1233211\\1}} + e_{\substack{1232211\\2}}\,,\\
v_{-(\beta_1+\beta_2)} &= \tfrac{1}{2}[v_{-\beta_2}, v_{-\beta_1}] = f_{\substack{1233210\\2}} - 2f_{\substack{1232221\\1}} + f_{\substack{1233211\\1}} + f_{\substack{1232211\\2}}\,,\\
h_{\beta_1} &= \tfrac{1}{2}[v_{\beta_1}, v_{-\beta_1}] = 2h_{\substack{0001111\\0}} + 2h_{\substack{0010000\\0}} + 2h_{\substack{0111100\\1}} + h_{\substack{0011110\\1}} + h_{\substack{0111110\\0}}\,,\\
h_{\beta_2} &= \tfrac{1}{2}[v_{\beta_2}, v_{-\beta_2}] = 2h_{\substack{1111111\\1}} + 2h_{\substack{0010000\\0}} + 2h_{\substack{1222100\\1}} + h_{\substack{1122110\\1}} + h_{\substack{1221110\\1}}\,.
\end{aligned}
$$

One checks that the 8 vectors $v_{\pm\beta_1}, v_{\pm\beta_2}, v_{\pm(\beta_1+\beta_2)}, h_{\beta_1}, h_{\beta_2}$ (suitably scaled) satisfy the relations of a Lie algebra of type A_2.

This completes the determination of C° in all cases.

8 A composition series for the Lie algebra centralizer

Once more, G will be a simple algebraic group of exceptional type defined over an algebraically closed field k whose characteristic is either 0 or a good prime for G, and T will be a fixed maximal torus of G. Let $e \in \mathfrak{L}(G)$ be a non-zero nilpotent orbit representative listed in Table 2, with associated cocharacter $\tau : k^* \to T$ and Levi subgroup L of G as given at the top of the page for e in §11, so that $T \subset L$ and e is distinguished in $\mathfrak{L}(L)$; write $C = C_G(e) \cap C_G(\mathrm{im}(\tau))$ as in §2. In this section, we shall determine the action of C° on $C_{\mathfrak{L}(G)}(e)_+$; this will enable us in §9 to find $(Z(C_{\mathfrak{L}(G)}(e)_+))^C$.

We will in fact give rather more detailed information. As C acts on $C_{\mathfrak{L}(G)}(e)_+$ as Lie algebra automorphisms, it preserves each term $Z_n(C_{\mathfrak{L}(G)}(e)_+)$ in the upper central series of $C_{\mathfrak{L}(G)}(e)_+$. Moreover as C lies in $C_G(\mathrm{im}(\tau))$, it respects the grading $\mathfrak{L}(G) = \bigoplus_{m\in\mathbb{Z}} \mathfrak{L}(G)_m$; thus for each $n > 0$, it preserves each summand in the decomposition

$$Z_n(C_{\mathfrak{L}(G)}(e)_+) = \bigoplus_{m>0} Z_{n,m},$$

where as in §7.1 we write $Z_{n,m} = (Z_n(C_{\mathfrak{L}(G)}(e)_+))_m$. For each n and m, we will describe $Z_{n,m}$ as a sum of indecomposable tilting modules for the group $[C^\circ, C^\circ]$, the vast majority of which will turn out to be irreducible. For those which are irreducible we shall give a high weight vector; for those few which are reducible we shall give generating vectors.

8.1 Parametrizing a maximal torus of $[C^\circ, C^\circ]$

We first explain our parametrization of a maximal torus of $[C^\circ, C^\circ]$; we shall then go on in §8.2 to determine the weights of $[C^\circ, C^\circ]$ on each $Z_{n,m}$. Recall from §7.2 that we have $[C^\circ, C^\circ] = C_1 \dots C_r$, a commuting product of simple groups; for each i we have obtained a simple system $\Pi(C_i)$ and the corresponding root subgroups, and we shall now give an appropriate maximal torus of C_i. For each $\beta \in \Pi(C_i)$, we will define a cocharacter $h_\beta : k^* \to T$ with the following properties: its image lies in C_i, and for all $\beta' \in \Pi(C_i)$, $t \in k$ and $c \in k^*$ we have $h_\beta(c)x_{\beta'}(t)h_\beta(c)^{-1} = x_{\beta'}(c^{n_{\beta\beta'}}t)$, where the $n_{\beta\beta'}$ are the integers given by the Cartan matrix of the root system $\Phi(C_i)$. It follows that h_β is the standard parametrization of the 1-dimensional torus $\langle X_{\pm\beta}\rangle \cap T$.

Consider first those C_i obtained in §7.2 as fixed point subgroups of subsystem subgroups G_i. In these cases, for each $\beta \in \Pi(C_i)$ we obtained subgroups X_β and $X_{-\beta}$, isomorphic to $\mathbf{G}_a$, such that $SL_2(k)$ maps surjectively onto $\langle X_{\pm\beta}\rangle$; on the page for e in §11 we give isomorphisms $x_\gamma : \mathbf{G}_a \to X_\gamma$ for $\gamma \in \{\pm\beta\}$. For all $\beta \in \Pi(C_i)$ and $c \in k^*$, set $n_\beta(c) = x_\beta(c)x_{-\beta}(-c^{-1})x_\beta(c)$ and $h_\beta(c) = n_\beta(c)n_\beta(-1)$. In fact, the expression for each such h_β in terms of the h_α for $\alpha \in \Pi(G)$ can be obtained directly as follows. We have $x_\beta(t) = \prod_{\gamma\in S_\beta} x_\gamma(f_\gamma(t))$ for some non-zero polynomials f_γ, where S_β is a subset of $\Phi(G)$ in which any two roots are orthogonal; moreover, we observe that each such polynomial f_γ is linear. This then implies that $h_\beta = \prod_{\gamma\in S_\beta} h_\gamma$; writing each h_γ in terms of the h_α for $\alpha \in \Pi(G)$ gives the desired expression. It is then straightforward to check that the required properties of the h_β hold.

We now consider the orbits listed under (i)–(iv) in §7.2. Here we constructed C_r of type A_1, again giving subgroups X_γ, and explicit isomorphisms $x_\gamma : \mathbf{G}_a \to X_\gamma$, for $\gamma \in \{\pm\beta\}$. In each case, we note that $\langle X_{\pm\beta}\rangle$ is contained in a subsystem subgroup H, which is a commuting product $H = H_1 \dots H_s$ where each H_j is of type A_{i_j} for some $i_j \in \mathbb{N}$. For each factor H_j of type A_1 or A_2, we let $\pi_j : H \to H_j$ be the projection map and apply the Lemma of [49, pp.301–302] to find a cocharacter $\psi_j : k^* \to T \cap \pi_j(\langle X_{\pm\beta}\rangle)$, such that for all $t \in k$ and $c \in k^*$ we have $\psi_j(c)\pi_j(x_\beta(t))\psi_j(c)^{-1} = \pi_j(x_\beta(c^2t))$. If each H_j is of type A_1 or A_2, we define $h_\beta = \prod_{j=1}^{s} \psi_j$. This leaves just the orbits labelled $A_3A_2A_1$ in E_7 and E_8 (numbers 21 and 20 respectively), A_4A_2 in E_7 and E_8 (numbers 26 and 27 respectively) and $A_4A_2A_1$ in E_8 (number 30). We shall treat together orbits with the same label.

Consider first the $A_3A_2A_1$ orbits. If $G = E_7$, we know from §7.2 that C° is simple of type A_1, and hence is equal to $\langle X_{\pm\beta}\rangle$. We define a cocharacter $h_\beta : k^* \to T$ by

$$h_\beta(c) = h_{\alpha_1}(c^8)h_{\alpha_2}(c^{12})h_{\alpha_3}(c^{16})h_{\alpha_4}(c^{24})h_{\alpha_5}(c^{18})h_{\alpha_6}(c^{12})h_{\alpha_7}(c^6);$$

one then checks that $\mathrm{im}(h_\beta) \subseteq Z(L)^\circ \subseteq C^\circ$, and $h_\beta(c)x_\beta(t)h_\beta(c)^{-1} = x_\beta(c^2t)$ for all $c \in k^*$ and $t \in k$ as required. If instead $G = E_8$, then C° is a commuting product of the group $\langle X_{\pm\beta}\rangle$ above and $C_G(E_7) = \langle X_{\pm\delta}\rangle$ for $\delta = {}^{2465432}_{3}$, which has already been treated; we take the same cocharacter h_β for the factor $\langle X_{\pm\beta}\rangle$.

Next consider the A_4A_2 orbits. If $G = E_7$, again we know from §7.2 that C° is simple of type A_1, and hence is equal to $\langle X_{\pm\beta}\rangle$. Here we define h_β by

$$h_\beta(c) = h_{\alpha_1}(c^6)h_{\alpha_2}(c^9)h_{\alpha_3}(c^{12})h_{\alpha_4}(c^{18})h_{\alpha_5}(c^{15})h_{\alpha_6}(c^{10})h_{\alpha_7}(c^5);$$

one then checks that $\mathrm{im}(h_\beta) \subseteq Z(L)^\circ \subseteq C^\circ$, and $h_\beta(c)x_\beta(t)h_\beta(c)^{-1} = x_\beta(c^2t)$ for all $c \in k^*$ and $t \in k$ as required. If instead $G = E_8$, we proceed exactly as before.

Finally consider the $A_4A_2A_1$ orbit; here $G = E_8$. Once more we know from §7.2 that C° is simple of type A_1, and hence is equal to $\langle X_{\pm\beta}\rangle$. This time we define h_β by

$$h_\beta(c) = h_{\alpha_1}(c^{10})h_{\alpha_2}(c^{15})h_{\alpha_3}(c^{20})h_{\alpha_4}(c^{30})h_{\alpha_5}(c^{24})h_{\alpha_6}(c^{18})h_{\alpha_7}(c^{12})h_{\alpha_8}(c^6);$$

one then checks that $\mathrm{im}(h_\beta) \subseteq Z(L)^\circ \subseteq C^\circ$, and $h_\beta(c)x_\beta(t)h_\beta(c)^{-1} = x_\beta(c^2t)$ for all $c \in k^*$ and $t \in k$ as required.

We now consider the orbits treated separately in §7.2, which are those labelled ${A_2A_1}^3$ in E_7 and E_8 (numbers 11 and 9 respectively), ${A_2}^2{A_1}^2$ in E_8 (number 15) and $D_4(a_1)A_2$ in E_8 (number 22). With the exception of the ${A_2A_1}^3$ orbit in E_8, in each of these cases C° is simple of rank 2; in the exceptional case there is an additional A_1 factor $\langle X_{\pm\delta}\rangle$ for $\delta = {}^{2465432}_{3}$, which we may treat as above. We defined $Z(L)^\circ$-root subgroups X_{β_1} and X_{β_2} of C°, and showed that $\{\beta_1, \beta_2\}$ form a simple system for $\Phi(C^\circ)$. We will now define two cocharacters h_{β_1} and h_{β_2}; one then checks that $\mathrm{im}(h_{\beta_i}) \subseteq Z(L)^\circ \subseteq C^\circ$ and $h_{\beta_i}(c)x_{\beta_j}(t)h_{\beta_i}(c)^{-1} = x_{\beta_j}(c^{n_{\beta_i\beta_j}}t)$ for all $c \in k^*$, $t \in k$ and $i, j \in \{1, 2\}$, where the $n_{\beta_i\beta_j}$ are the integers given by the Cartan matrix, as required.

First consider the ${A_2A_1}^3$ orbit in both E_7 and E_8. For $c \in k^*$, set

$$\begin{aligned} h_{\beta_1}(c) &= h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)h_{\alpha_3}(c^4)h_{\alpha_4}(c^6)h_{\alpha_5}(c^3), \\ h_{\beta_2}(c) &= h_{\alpha_5}(c)h_{\alpha_6}(c^2)h_{\alpha_7}(c). \end{aligned}$$

Now consider the ${A_2}^2{A_1}^2$ orbit in E_8. For $c \in k^*$, set

$$\begin{aligned} h_{\beta_1}(c) &= h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)h_{\alpha_3}(c^4)h_{\alpha_4}(c^6)h_{\alpha_5}(c^4)h_{\alpha_6}(c^2), \\ h_{\beta_2}(c) &= h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)h_{\alpha_3}(c^4)h_{\alpha_4}(c^6)h_{\alpha_5}(c^6)h_{\alpha_6}(c^6)h_{\alpha_7}(c^6)h_{\alpha_8}(c^3). \end{aligned}$$

Finally consider the $D_4(a_1)A_2$ orbit in E_8. For $c \in k^*$, set

$$\begin{aligned} h_{\beta_1}(c) &= h_{\alpha_2}(c^3)h_{\alpha_3}(c^3)h_{\alpha_4}(c^6)h_{\alpha_5}(c^6)h_{\alpha_6}(c^6)h_{\alpha_7}(c^4)h_{\alpha_8}(c^2), \\ h_{\beta_2}(c) &= h_{\alpha_1}(c^6)h_{\alpha_2}(c^6)h_{\alpha_3}(c^9)h_{\alpha_4}(c^{12})h_{\alpha_5}(c^9)h_{\alpha_6}(c^6)h_{\alpha_7}(c^4)h_{\alpha_8}(c^2). \end{aligned}$$

8.2 Composition factors of $C_{\mathfrak{L}(G)}(e)_+$

We now turn to the action of C° on each subspace $Z_{n,m} = (Z_n(C_{\mathfrak{L}(G)}(e)_+))_m$. For each m we treat the subspaces in the order of increasing values of n; note that we have

$$0 \subset Z_{1,m} \subset Z_{2,m} \subset \cdots,$$

while $Z_{n,m} = C_{\mathfrak{L}(G)}(e)_m$ for sufficiently large n. For convenience we write $D = [C^\circ, C^\circ]$ from now on, so that we have the central product $C^\circ = DZ(C^\circ)^\circ$.

We fix a maximal torus and Borel subgroup for the group D as follows. For each of the simple factors C_i of D, we have given a simple system $\Pi(C_i)$; for each $\beta \in \Pi(C_i)$ we have given a simple root subgroup X_β and cocharacter h_β. Write $\Pi(D) = \bigcup_i \Pi(C_i)$, and set $T_0 = \langle h_\beta(c) : \beta \in \Pi(D), c \in k^* \rangle$ and $B_0 = T_0 \langle X_\beta : \beta \in \Pi(D) \rangle$. For $\beta \in \Pi(D)$ let λ_β be the fundamental dominant weight corresponding to the simple root β. For a dominant weight λ, we fix notation for certain kD-modules with high weight λ as follows: we write $V_D(\lambda)$ for the irreducible module, $W_D(\lambda)$ for the Weyl module, and $T_D(\lambda)$ for the tilting module (see [14, pp.183, 458] for the definitions of the second and third of these).

For each n and m, we find the set of T_0-weights on $Z_{n,m}$, with multiplicities, using the basis of this subspace obtained by the procedure described in §7.1. This task is simplified by the fact that $T_0 \subset Z(L)^0$, and hence T_0 is constant on each of the $Z(L)^\circ$-weight spaces $\mathfrak{L}(G)^\chi$. We find that, with a small number of exceptions, all such weights are restricted; these exceptions will be indicated and treated later.

Thus assume all T_0-weights on $Z_{n,m}$ are restricted. Let μ_1 be a maximal weight with respect to the ordering imposed by the choice of base $\Pi(D)$. As μ_1 is maximal, we deduce the existence of a D-composition factor $V_D(\mu_1)$ in $Z_{n,m}$. Moreover, in each of the cases which occur, we observe that $\dim V_D(\mu_1) = \dim W_D(\mu_1)$. In particular, the set of weights of $V_D(\mu_1)$ and their multiplicities are the same as in characteristic 0. (We remark here that the modules which occur are small enough to be treated in [22].)

Now if $\dim Z_{n,m} > \dim V_D(\mu_1)$, we take the list of T_0-weights occurring in $Z_{n,m}$ and remove those found in $V_D(\mu_1)$ according to their multiplicities; we then iterate the above process, choosing at each stage a weight μ_j maximal among the remaining weights. Again we find that the Weyl module $W_D(\mu_j)$ is irreducible. It follows from [14, Proposition II.2.14] that $Z_{n,m}$ decomposes as a direct sum of irreducible submodules, one for each μ_j. We now explain how we find a high weight vector for each of the submodules in this direct sum decomposition.

The basis of $Z_{n,m}$ found by the procedure described in §7.1 is a basis of weight vectors for the torus $Z(L)^\circ$, and hence for T_0. Let μ be the high weight of a D-composition factor of $Z_{n,m}$. We have a basis of the T_0-weight space $(Z_{n,m})_\mu$, which we may use to calculate the fixed point space $V_{n,m,\mu}$ of the subgroup $R_u(B_0)$ in its action on $(Z_{n,m})_\mu$. Since $Z(C^\circ)^\circ$ commutes with the action of D, it stabilizes $V_{n,m,\mu}$; hence there is a basis of $V_{n,m,\mu}$ consisting of $Z(C^\circ)^\circ$-weight vectors v, and if $n > 1$ we may choose this basis to extend that of $V_{n-1,m,\mu}$ already found. We give these vectors in the table on the page for e in §11.

In fact, for each such high weight vector v one can obtain a basis for the irreducible D-submodule which it generates. Firstly, since the Weyl module $W_D(\mu)$ is both irreducible and the universal high weight module with high weight μ, it is the only cyclic D-module of high weight μ (up to isomorphism); thus $\langle Dv \rangle$ is an irreducible D-submodule of high weight μ, and therefore also a C°-submodule. Now as the weight μ is restricted, by [9] we see that $\mathfrak{L}(D)$ acts irreducibly on the D-module $V_D(\mu)$; we thus have $\langle \mathfrak{L}(D)v \rangle = \langle Dv \rangle$. The calculation may therefore be performed within $\mathfrak{L}(G)$, which simplifies matters.

We illustrate the procedure with an example. Let $G = E_8$ and take e to be the representative of the D_6 orbit (number 48). On the page for e in §11 we have identified D as being of type B_2, with simple roots β_1 and β_2, where $\beta_1 = {\substack{2343210\\2}}$ and $x_{\pm\beta_2}(t) = x_{\pm\substack{0011111\\1}}(t)x_{\pm\substack{0111111\\0}}(-t)$. For $\gamma \in \{\pm\beta_1, \pm\beta_2\}$ set $v_\gamma = dx_\gamma(1)$, so that

$$\begin{aligned} v_{\beta_1} &= e_{\substack{2343210\\2}}, & v_{\beta_2} &= e_{\substack{0011111\\1}} - e_{\substack{0111111\\0}}, \\ v_{-\beta_1} &= f_{\substack{2343210\\2}}, & v_{-\beta_2} &= f_{\substack{0011111\\1}} - f_{\substack{0111111\\0}}; \end{aligned}$$

then $\mathfrak{L}(D)$ is the Lie algebra generated by $v_{\pm\beta_1}, v_{\pm\beta_2}$. The table on the page for e lists nine high weight vectors, of which five have T_0-weight 0, one has T_0-weight λ_1 and three have T_0-weight λ_2. Those with T_0-weight 0 span trivial D-submodules. The T_0-weights in $V_D(\lambda_1)$ are λ_1, $\lambda_1 - \beta_1$, $\lambda_1 - \beta_1 - \beta_2$, $\lambda_1 - \beta_1 - 2\beta_2$ and $\lambda_1 - 2\beta_1 - 2\beta_2$. Thus if we let w_1 be the high weight vector of T_0-weight λ_1 lying in $Z_{2,10}$, by successively taking commutators (and scaling where appropriate for convenience) we obtain

$$\begin{aligned} w_1 &= e_{\substack{2465431\\3}}, \\ w_2 = [v_{-\beta_1}, w_1] &= e_{\substack{0122221\\1}}, \\ w_3 = [v_{-\beta_2}, w_2] &= e_{\substack{0011110\\1}} - e_{\substack{0111110\\0}}, \\ w_4 = \tfrac{1}{2}[v_{-\beta_2}, w_3] &= f_{\substack{0000001\\0}}, \\ w_5 = -[v_{-\beta_1}, w_4] &= f_{\substack{2343211\\2}}; \end{aligned}$$

hence the D-submodule is

$$\langle e_{\substack{2465431\\3}}, e_{\substack{0122221\\1}}, e_{\substack{0011110\\1}} - e_{\substack{0111110\\0}}, f_{\substack{0000001\\0}}, f_{\substack{2343211\\2}} \rangle.$$

The T_0-weights in $V_D(\lambda_2)$ are λ_2, $\lambda_2 - \beta_2$, $\lambda_2 - \beta_1 - \beta_2$ and $\lambda_2 - \beta_1 - 2\beta_2$. Thus if we let x_1 be the high weight vector of T_0-weight λ_1 lying in $Z_{3,5}$, by successively taking commutators we obtain

$$\begin{aligned} x_1 &= e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}} + e_{\substack{1233321\\2}}, \\ x_2 = [v_{-\beta_2}, x_1] &= e_{\substack{1232110\\1}} - e_{\substack{1222210\\1}} + e_{\substack{1232100\\2}}, \\ x_3 = [v_{-\beta_1}, x_2] &= f_{\substack{1111100\\1}} + f_{\substack{1121000\\1}} + f_{\substack{1111110\\0}}, \\ x_4 = [v_{-\beta_2}, x_3] &= f_{\substack{1122221\\1}} - f_{\substack{1222211\\1}} + f_{\substack{1232111\\1}}; \end{aligned}$$

hence the D-submodule is

$$\begin{aligned} \langle & e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}} + e_{\substack{1233321\\2}}, e_{\substack{1232110\\1}} - e_{\substack{1222210\\1}} + e_{\substack{1232100\\2}}, \\ & f_{\substack{1111100\\1}} + f_{\substack{1121000\\1}} + f_{\substack{1111110\\0}}, f_{\substack{1122221\\1}} - f_{\substack{1222211\\1}} + f_{\substack{1232111\\1}} \rangle. \end{aligned}$$

Similarly we find that the D-submodules forming $Z_{2,9}$ and $Z_{1,15}$ are respectively

$$\langle e_{\substack{1343321\\2}} - e_{\substack{1244321\\2}}, e_{\substack{1233210\\1}} - e_{\substack{1232210\\2}}, f_{\substack{1111000\\0}} + f_{\substack{1110000\\1}}, f_{\substack{1122111\\1}} - f_{\substack{1221111\\1}} \rangle$$

and

$$\langle e_{\substack{1354321\\3}}, e_{\substack{1343210\\2}}, f_{\substack{1000000\\0}}, f_{\substack{1111111\\0}} \rangle.$$

We now consider the instances of non-restricted T_0-weights in some $C_{\mathfrak{L}(G)}(e)_m$; here we must have $\mathrm{char}(k) = p > 0$. The cases concerned are as follows:

(i) if $G = E_7$ and e lies in the orbit labelled $A_3A_2A_1$ (number 21), the D-modules $Z_{n,2}$ for $n \geq 3$ and $Z_{n,4}$ for $n \geq 2$ are non-restricted when $p \in \{5, 7\}$ and $p = 5$ respectively;

(ii) if $G = E_7$ and e lies in the orbit labelled A_4A_2 (number 26), the D-module $Z_{n,4}$ for $n \geq 3$ is non-restricted when $p = 5$;

(iii) if $G = E_8$ and e lies in the orbit labelled $A_3A_2A_1$ (number 20), the D-module $Z_{n,2}$ for $n \geq 5$ is non-restricted when $p = 7$.

We first note that case (iii) will be covered by the treatment of case (i). In each of cases (i) and (ii), $C^\circ = D$ is of type A_1 and so the T_0-weights may be regarded as integers. Moreover, as the T_0-weights on $\mathfrak{L}(G)$ are all at most $2p - 2$, the A_1 subgroups are 'good' in the sense of [38], which then shows that $\mathfrak{L}(G)$ is a direct sum of tilting modules for D. Such a tilting module $T_D(a)$ of high weight a has a composition series of the form $0 \subset W_1 \subset W_2 \subset W_3$, where $W_1 \simeq W_3/W_2 \simeq V_D(2p - 2 - a)$ and $W_2/W_1 \simeq V_D(a)$, and is generated by the union of the T_0-weight spaces of weights a and $2p - 2 - a$.

In both cases (i) and (ii), we note that the T_0-weights occurring in $Z_{n,4}$ (with multiplicities) are $6, 4, 2, 2, 0, 0, -2, -2, -4, -6$. If $p = 5$, there exists a tilting module summand of high weight 6; as $T_D(6)$ has dimension 10, we must have $Z_{n,4} \simeq T_D(6)$. We give bases for the T_0-weight spaces of weights 2 and 6 in the table on the page for e in §11.

In case (i), we find that the T_0-weights occurring in $Z_{n,2}$ (with multiplicities) are $8, 6, 4, 4, 2, 2, 0, 0, 0, -2, -2, -4, -4, -6, -8$. We deduce therefore that $Z_{n,2}$ has a tilting module summand of high weight 8. We may take as a basis for $Z_{n,2}$ the following vectors, where in each case the subscript denotes the T_0-weight:

$$\begin{aligned}
w_8 &= e_{\substack{234321\\2}}, \\
w_6 &= e_{\substack{123321\\1}} - e_{\substack{123221\\2}}, \\
w_4 &= e_{\substack{122210\\1}} + e_{\substack{122111\\1}}, & w_4{}' &= e_{\substack{012221\\1}} - e_{\substack{122210\\1}} - e_{\substack{112211\\1}}, \\
w_2 &= e_{\substack{001111\\1}} - e_{\substack{011111\\0}} - e_{\substack{111110\\0}} - e_{\substack{111100\\1}}, & w_2{}' &= e_{\substack{001111\\1}} + e_{\substack{011110\\1}} + e_{\substack{111100\\1}}, \\
w_0 &= e_{\substack{000000\\1}}, \qquad w_0{}' = e_{\substack{100000\\0}} + e_{\substack{010000\\0}}, & w_0{}'' &= e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}, \\
w_{-2} &= f_{\substack{111000\\0}} - f_{\substack{011000\\1}} - f_{\substack{001100\\1}} - f_{\substack{001110\\0}}, & w_{-2}{}' &= f_{\substack{111000\\0}} + f_{\substack{011100\\0}} + f_{\substack{001110\\0}}, \\
w_{-4} &= f_{\substack{012111\\1}} + f_{\substack{012210\\1}}, & w_{-4}{}' &= f_{\substack{112110\\1}} + f_{\substack{012111\\1}} + f_{\substack{122100\\1}}, \\
w_{-6} &= f_{\substack{123211\\1}} + f_{\substack{123210\\2}}, \\
w_{-8} &= f_{\substack{124321\\2}}.
\end{aligned}$$

Now if $p = 5$, the T_0-weights occurring in the tilting module $T_D(8)$ (with multiplicities) are $8, 6, 4, 2, 0, 0, -2, -4, -6, -8$. Since the remaining T_0-weights in $Z_{n,2}$ are $4, 2, 0, -2, -4$, we must have $Z_{n,2} \simeq T_D(8) \oplus V_D(4)$ (whereas $Z_{1,2} = Z_{2,2} = \langle e \rangle \simeq V_D(0)$, so that $Z_{2,2}$ is not a direct summand of $Z_{3,2}$; this is the only example of such behaviour in the present work). We observe that in $Z_{n,2}$ the space of fixed points of T_0-weight 4 is 1-dimensional, spanned by $3w_4 + w_4{}'$; we therefore list this vector in the table on the page for e in §11. In order to find generating vectors for $T_D(8)$, we will work upward from the bottom of the composition series; it will in fact be convenient to identify a basis of this summand. It is clear that $e = w_0 + w_0' + w_0''$ spans W_1. To complete a basis for W_2, we apply $x_{-\beta}(t)$ to w_8 and $x_\beta(t)$ to w_{-8} and iterate to obtain

$$\{w_8, w_6, w_4 - 2w_4', w_2 - 3w_2', w_{-8}, w_{-6}, 3w_{-4} - 2w_{-4}', w_{-2} - 3w_{-2}'\}.$$

To complete a basis of W_3, it then suffices to find a vector w of T_0-weight 0 such that the projection of $x_{-\beta}(t)w - w$ into the space of vectors of T_0-weight -2 is a nonzero multiple of $w_{-2} - 3w_{-2}'$; we find that $w = 3w_0 - w_0'$ has this property. We therefore list the vectors w_8, e and w in the table on the page for e in §11.

If instead $p = 7$, the T_0-weights occurring in the tilting module $T_D(8)$ (with multiplicities) are $8, 6, 4, 4, 2, 2, 0, 0, -2, -2, -4, -4, -6, -8$; thus $Z_{n,2}$ decomposes as a direct sum $T_D(8) \oplus V_D(0)$. It is clear that the 1-dimensional irreducible summand is spanned by e. We find a basis for the summand $T_D(8)$ as in the preceding case; we give bases for the T_0-weight spaces of weights 4 and 8 in the table on the page for e in §11.

This completes the consideration of the action of C° on the $Z_{n,m}$.

9 The Lie algebra of the centre of the centralizer

As in the preceding sections, G will be a simple algebraic group of exceptional type defined over an algebraically closed field k whose characteristic is either 0 or a good prime for G, and T will be a fixed maximal torus of G; here, however, for convenience we shall take G of adjoint type. Let $e \in \mathfrak{L}(G)$ be a non-zero nilpotent orbit representative listed in Table 2, with associated cocharacter $\tau : k^* \to T$ and Levi subgroup L of G as given at the top of the page for e in §11, so that $T \subset L$ and e is distinguished in $\mathfrak{L}(L)$; write $C = C_G(e) \cap C_G(\mathrm{im}(\tau))$ as in §2, and $D = [C^\circ, C^\circ]$ as in §8.2. In this section we will complete our calculations by finding a basis for $\mathcal{Z} = (Z(C_{\mathfrak{L}(G)}(e)_+))^C$, which by Theorem 3.9 is equal to $\mathfrak{L}(Z(C_G(e)))$; as explained at the end of §2, by Proposition 2.13 the results obtained will also be valid for G of simply connected type.

In §7.1 we obtained the upper central series

$$0 \subset Z_1(C_{\mathfrak{L}(G)}(e)_+) \subset Z_2(C_{\mathfrak{L}(G)}(e)_+) \subset \cdots ;$$

for each n we wrote

$$Z_n(C_{\mathfrak{L}(G)}(e)_+) = \bigoplus_{m>0} Z_{n,m},$$

where $Z_{n,m} = (Z_n(C_{\mathfrak{L}(G)}(e)_+))_m$. In §8.2 we then wrote each $Z_{n,m}$ as a direct sum of indecomposable tilting modules for D; since the generating vectors were chosen to be $Z(C^\circ)^\circ$-weight vectors, and $C^\circ = DZ(C^\circ)^\circ$, each summand is in fact a C°-submodule. The table on the page for e in §11 is divided vertically into sections, the last n of which give the C°-submodules lying in $Z_n(C_{\mathfrak{L}(G)}(e)_+)$, along with their τ-weights m; in particular the last section of the table lists the C°-submodules lying in $Z(C_{\mathfrak{L}(G)}(e)_+)$.

Now we observe that $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ}$ is simply the sum of the trivial C°-submodules in $Z(C_{\mathfrak{L}(G)}(e)_+)$. Lemma 7.2 shows that $C^\circ = DZ(L)^\circ$; we have

$$(Z(C_{\mathfrak{L}(G)}(e)_+))^{Z(L)^\circ} \subseteq \mathfrak{L}(G)^{Z(L)^\circ} = \mathfrak{L}(G)^0 = \mathfrak{L}(L).$$

Thus a basis for $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ}$ may be formed from the C°-submodules listed in the last section of the table by taking those vectors which both lie in $\mathfrak{L}(L)$ and generate trivial D-submodules. In the fifth column of the table we have written $\mathcal{Z}^\natural$ for $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ}$, and have indicated its basis vectors; for convenience of reference each such vector is renamed z_m, with the subscript being the τ-weight, and vectors with the same value of m being distinguished by superscripts.

Finally we complete the determination of $\mathcal{Z}$, whose basis vectors are indicated in the sixth column of the table, by considering the action of the full reductive complement C on $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ}$ in the cases where $C/C^\circ \neq 1$; it is here that the assumption on the isogeny type of G simplifies matters, since it minimizes the size of the component group C/C°. Each of the cases must be treated separately, except that in the groups of type E_ℓ we take together orbits with the same name. We write $A = C/C^\circ$; given $c \in C$ we denote its image in A by $\bar{c}$. For the structure of the finite group A, we refer to [6, 25].

In each case, we give a set of representatives c of generators of A, which we found by explicit calculation; one verifies in each case that $c \in C_G(e)$. The elements c which we give are all conjugates of elements of $N_G(T)$, and in most cases actually lie in $N_G(T)$. We write elements of $N_G(T)$ as products $n_{\beta_1} \ldots n_{\beta_s} t$, where $t \in T$ and $\beta_1, \ldots, \beta_s \in \Phi$; we write t as a product of elements $h_i(\lambda_i)$, where for brevity we use h_i to denote h_{α_i}. If $s > 0$, for the convenience of the reader we shall give the image of each relevant root vector under $\mathrm{Ad}(c)$. We take $\omega, \zeta, i \in k \setminus \{1\}$ with $\omega^3 = \zeta^5 = -i^2 = 1$, and set $\phi = \zeta^2 + \zeta^3$.

9.1 G of type G_2

If G is of type G_2 there is only a single orbit having $A \neq 1$.

9.1.1 Orbit $G_2(a_1)$ (number 3)

Here $e = e_{01} + e_{31}$, which is regular in the Lie algebra of H, a long root A_2 subsystem subgroup of G; we have $A \cong \mathcal{S}_3$, and $C^\circ = 1$. We take

$$c_1 = h_1(\omega),$$
$$c_2 = n_{10}h_2(-1).$$

Clearly $c_1 \in Z(H)$. The action of $\mathrm{Ad}(c_2)$ on the root vectors concerned is as follows:

$$\text{interchanged}: \ e_{01} \leftrightarrow e_{31}.$$

Thus c_2 acts as a non-trivial graph automorphism of H. Hence $A = \langle \bar{c}_1, \bar{c}_2 \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_4 \rangle$, where $z_2 = e$ and $z_4 = e_{32}$. Since $\mathrm{Ad}(c_2)$ negates z_4, we have $\mathcal{Z} = \langle z_2 \rangle$.

9.2 G of type F_4

If G is of type F_4 there are 7 orbits having $A \neq 1$.

9.2.1 Orbit $\tilde{A}_1$ (number 2)

Here $e = e_{0001}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = A_3$ with $\Pi(D) = \{1000, 0100, 1242\}$. We take

$$c = n_{0121}h_3(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\text{fixed}: \ e_{0001}, \ e_{0100};$$
$$\text{interchanged}: \ e_{1000} \leftrightarrow e_{1242}.$$

Thus c acts as a non-trivial graph automorphism of C°. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2 \rangle$, where $z_2 = e$, so $\mathcal{Z} = \langle z_2 \rangle$.

9.2.2 Orbit A_2 (number 4)

Here $e = e_{1000} + e_{0100}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = \tilde{A}_2$ with $\Pi(D) = \{0001, 1231\}$. We take

$$c = n_{0110}n_{1120}h_1(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\text{interchanged}: \ e_{1000} \leftrightarrow e_{0100}, \ e_{0001} \leftrightarrow e_{1231}.$$

Thus c acts as a non-trivial graph automorphism of C°. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_4 \rangle$, where $z_2 = e$ and $z_4 = e_{1100}$. Since $\mathrm{Ad}(c)$ negates z_4, we have $\mathcal{Z} = \langle z_2 \rangle$.

9.2.3 Orbit B_2 (number 7)

Here $e = e_{0100} + e_{0010}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = {A_1}^2$ with $\Pi(D) = \{0122, 2342\}$. We take

$$c = n_{1110} h_4(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\ e_{0100},\ e_{0010}; \\ \text{interchanged}: &\ e_{0122} \leftrightarrow e_{2342}. \end{aligned}$$

Thus c acts as a non-trivial graph automorphism of C°. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_6 \rangle$, where $z_2 = e$ and $z_6 = e_{0120}$. Since $\mathrm{Ad}(c)$ fixes z_6, we have $\mathcal{Z} = \langle z_2, z_6 \rangle$.

9.2.4 Orbit $C_3(a_1)$ (number 9)

Here $e = e_{0001} + e_{0120} + e_{0100}$, which is non-regular distinguished in the Lie algebra of H, a C_3 subsystem subgroup of G with simple system $\{0001, 0010, 0100\}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = A_1$ with $\Pi(D) = \{2342\}$. We take

$$c = h_4(-1).$$

We find that $C_H(c) \cong C_2 C_1$, and that e is regular in $C_H(c)$. Moreover, as the only element of order 2 in $T \cap C^\circ$ is $h_2(-1)h_4(-1)$ it follows that $c \notin C^\circ$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_6 \rangle$, where $z_2 = e$ and $z_6 = e_{0122}$. Since $\mathrm{Ad}(c)$ fixes z_6, we have $\mathcal{Z} = \langle z_2, z_6 \rangle$.

9.2.5 Orbit $F_4(a_3)$ (number 10)

Here $e = e_{0100} + e_{1120} + e_{1111} + e_{0121}$, which is regular in the Lie algebra of H, an $A_2\tilde{A}_2$ subsystem subgroup of G; we have $A \cong \mathcal{S}_4$, and $C^\circ = 1$. We take

$$\begin{aligned} c_1 &= h_1(\omega) h_3(\omega), \\ c_2 &= n_{1000} n_{0010} h_2(-1) h_3(-1), \\ c_3 &= (n_{0011} h_3(-\tfrac{2}{3}) h_4(\tfrac{2}{3}))^u, \end{aligned}$$

where

$$u = x_{0011}(-\tfrac{1}{2}) x_{0001}(1) x_{0010}(-1).$$

Clearly $c_1 \in Z(H)$. The action of $\mathrm{Ad}(c_2)$ on the root vectors concerned is as follows:

$$\text{interchanged}: \ e_{0100} \leftrightarrow e_{1120},\ e_{1111} \leftrightarrow e_{0121}.$$

Thus c_2 acts as a non-trivial graph automorphism of H. The action of c_3 is rather more complicated. Calculation shows that

$$\mathrm{Ad}(u)e = e_{0100} + e_{0110} + e_{0120} + e_{1120} - \tfrac{1}{2} e_{0111} + e_{1111} + \tfrac{3}{2} e_{0121} + \tfrac{9}{4} e_{0122}.$$

The action of $\mathrm{Ad}({c_3}^{u^{-1}})$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\ e_{0120},\ e_{1120},\ e_{0111},\ e_{1111}; \\ \text{interchanged}: &\ e_{0100} \leftrightarrow \tfrac{9}{4} e_{0122},\ e_{0110} \leftrightarrow \tfrac{3}{2} e_{0121}. \end{aligned}$$

Thus $c_3 \in C_G(e)$. Hence $A = \langle \bar{c}_1, \bar{c}_2, \bar{c}_3 \rangle$. (Indeed, we see that ${c_1}^3 = {c_2}^2 = {c_3}^2 = 1$, with ${c_1}^{c_2} = {c_1}^{-1}$, $[c_2, c_3] = 1$ and $(c_1 c_3)^3 = 1$; we may thus identify c_1, c_2 and c_3 with the elements (1 2 3), (1 2) and (1 2)(3 4) of S_4.) We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, {z_6}^1, {z_6}^2 \rangle$, where $z_2 = e$, ${z_6}^1 = e_{1342}$ and ${z_6}^2 = e_{2342}$. Since $\mathrm{Ad}(c_1)$ multiplies ${z_6}^1$ by ω^2 and ${z_6}^2$ by ω, we have $\mathcal{Z} = \langle z_2 \rangle$.

9.2.6 Orbit $F_4(a_2)$ (number 13)

Here $e = e_{1110} + e_{0001} + e_{0120} + e_{0100}$, which is regular in the Lie algebra of H, a C_3A_1 subsystem subgroup of G; we have $A \cong \mathcal{S}_2$, and $C^\circ = 1$. We take

$$c = h_2(-1).$$

Clearly $c \in Z(H)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, {z_{10}}^1, {z_{10}}^2 \rangle$, where $z_2 = e$, ${z_{10}}^1 = e_{1342}$ and ${z_{10}}^2 = e_{2342}$. Since $\mathrm{Ad}(c)$ negates ${z_{10}}^1$ and fixes ${z_{10}}^2$, we have $\mathcal{Z} = \langle z_2, {z_{10}}^2 \rangle$.

9.2.7 Orbit $F_4(a_1)$ (number 14)

Here $e = e_{0100} + e_{1000} + e_{0120} + e_{0001}$, which is regular in the Lie algebra of H, a B_4 subsystem subgroup of G; we have $A \cong \mathcal{S}_2$, and $C^\circ = 1$. We take

$$c = h_4(-1).$$

Clearly $c \in Z(H)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{10}, z_{14} \rangle$, where $z_2 = e$, $z_{10} = e_{1222} - e_{1242}$ and $z_{14} = e_{2342}$. Since $\mathrm{Ad}(c)$ fixes both z_{10} and z_{14}, we have $\mathcal{Z} = \langle z_2, z_{10}, z_{14} \rangle$.

This completes the consideration of the group F_4.

9.3 G of type E_n

According as G is of type E_6, E_7 or E_8 there are 3, 13 or 32 orbits having $A \neq 1$. As stated above, we treat together orbits in different groups having the same label. Note that we will write all roots as lying in $\Phi(E_8)$.

9.3.1 Orbit A_2 (number 4 in E_6, 5 in E_7, 4 in E_8)

Here $e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = {A_2}^2$, A_5 or E_6 respectively with $\Pi(D) = \{ \substack{0001000\\0}, \substack{0000001\\0}, \substack{0000100\\0}, \substack{0000010\\0}, \substack{1232100\\1}, \substack{0000000\\1} \} \cap \Phi(G)$. We take

$$c = n_{\substack{0111000\\0}} n_{\substack{0110000\\1}} n_{\substack{1121000\\1}} h_1(-1) h_2(-1) h_5(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\ e_{\substack{0000001\\0}},\ e_{\substack{0000010\\0}}; \\ \text{interchanged}: &\ e_{\substack{1000000\\0}} \leftrightarrow e_{\substack{0100000\\0}},\ e_{\substack{0001000\\0}} \leftrightarrow e_{\substack{0000000\\1}},\ e_{\substack{0000100\\0}} \leftrightarrow e_{\substack{1232100\\1}}. \end{aligned}$$

Thus c acts as a non-trivial graph automorphism of C°. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_4 \rangle$, where $z_2 = e$ and $z_4 = e_{\substack{1100000\\0}}$. Since $\mathrm{Ad}(c)$ negates z_4, we have $\mathcal{Z} = \langle z_2 \rangle$.

9.3.2 Orbit A_2A_1 (number 7 in E_7, 6 in E_8)

Here $e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0000000\\1}}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = A_3T_1$ or A_5 respectively with $\Pi(D) = \{\substack{0000001\\0}, \substack{0000010\\0}, \substack{0000100\\0}, \substack{0001000\\0}, \substack{2464321\\3}\} \cap \Phi(G)$. We take

$$c = n_{\substack{1121110\\1}} n_{\substack{1122100\\1}} n_{\substack{1343210\\2}} h_3(-1)h_5(-1)h_7(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\ e_{\substack{0000000\\1}},\ e_{\substack{0000100\\0}};\\ \text{interchanged}: &\ e_{\substack{1000000\\0}} \leftrightarrow e_{\substack{0100000\\0}},\ e_{\substack{0000001\\0}} \leftrightarrow e_{\substack{2464321\\3}},\ e_{\substack{0000010\\0}} \leftrightarrow e_{\substack{0001000\\0}}. \end{aligned}$$

Thus c acts as a non-trivial graph automorphism of C°. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_4 \rangle$, where $z_2 = e$ and $z_4 = e_{\substack{1100000\\0}}$. Since $\mathrm{Ad}(c)$ negates z_4, we have $\mathcal{Z} = \langle z_2 \rangle$.

9.3.3 Orbit ${A_2}^2$ (number 10 in E_8)

Here $e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = {G_2}^2$. In this case C° is the fixed point subgroup under a non-trivial graph automorphism of a ${D_4}^2$ subsystem subgroup K with $\Pi(K) = \{\substack{1110000\\0}, \substack{0000000\\1}, \substack{0111000\\0}, \substack{0011100\\0}, \substack{1221110\\1}, \substack{0000001\\0}, \substack{1122110\\1}, \substack{0122210\\1}\}$. We take

$$c = n_{\substack{0111111\\0}} n_{\substack{0111110\\1}} n_{\substack{1122221\\1}} h_3(-1),$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\ e_{\substack{0001000\\0}},\ e_{\substack{0000100\\0}};\\ \text{interchanged}: &\ e_{\substack{1000000\\0}} \leftrightarrow e_{\substack{0100000\\0}},\ e_{\substack{1110000\\0}} \leftrightarrow e_{\substack{1221110\\1}},\ e_{\substack{0000000\\1}} \leftrightarrow e_{\substack{0000001\\0}},\\ &\ e_{\substack{0111000\\0}} \leftrightarrow -e_{\substack{1122110\\1}},\ e_{\substack{0011100\\0}} \leftrightarrow e_{\substack{0122210\\1}}. \end{aligned}$$

Thus c acts as a non-trivial graph automorphism of K. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2 \rangle$, where $z_2 = e$, so $\mathcal{Z} = \langle z_2 \rangle$.

9.3.4 Orbit $D_4(a_1)$ (number 11 in E_6, 15 in E_7, 13 in E_8)

Here $e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$, which is non-regular distinguished in the Lie algebra of H, a D_4 subsystem subgroup of G with $\Pi(H) = \{\substack{0100000\\0}, \substack{0010000\\0}, \substack{0000000\\1}, \substack{0001000\\0}\}$; we have $A \cong \mathcal{S}_3$, and $C^\circ = T_2$, ${A_1}^3$ or D_4 respectively with $\Pi(D) = \{\substack{0000010\\0}, \substack{0000001\\0}, \substack{0122210\\1}, \substack{2343210\\2}\} \cap \Phi(G)$. We take

$$\begin{aligned} c_1 &= n_{\substack{1110000\\1}} n_{\substack{1111000\\0}} h_2(-1),\\ c_2 &= (n_{\substack{1221100\\1}} n_{\substack{1122100\\1}} h_1(-1)h_2(-1)h_6(-1))^g, \end{aligned}$$

where

$$g = x_{\substack{0010000\\0}}(\tfrac{1}{3}) n_{\substack{0010000\\0}} h_1(4)h_2(-4)h_3(16)h_4(-48)h_5(16)h_6(-8) x_{\substack{0010000\\0}}(-\tfrac{1}{3}).$$

The action of $\mathrm{Ad}(c_1)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\ e_{\substack{0100000\\0}},\ e_{\substack{0000010\\0}},\ e_{\substack{0000001\\0}};\\ \text{negated}: &\ e_{\substack{0010000\\0}};\\ \text{interchanged}: &\ e_{\substack{0010000\\1}} \leftrightarrow e_{\substack{0011000\\0}},\ e_{\substack{0000000\\1}} \leftrightarrow e_{\substack{0001000\\0}},\ e_{\substack{0122210\\1}} \leftrightarrow e_{\substack{2343210\\2}}. \end{aligned}$$

Thus c_1 acts as a non-trivial graph automorphism of C°. The action of c_2 is a little more complicated. Calculation shows that

$$\mathrm{Ad}(g)e = e_{\substack{0000000\\1}} + e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}},$$

which is still non-regular distinguished in $\mathfrak{L}(H)$, and $(C^\circ)^{g^{-1}} = C^\circ$. The action of $\mathrm{Ad}({c_2}^{g^{-1}})$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\ e_{\substack{0000000\\1}},\ e_{\substack{0000001\\0}},\ e_{\substack{0122210\\1}};\\ \text{negated}: &\ e_{\substack{0010000\\0}};\\ \text{interchanged}: &\ e_{\substack{0110000\\0}} \leftrightarrow e_{\substack{0011000\\0}},\ e_{\substack{0100000\\0}} \leftrightarrow e_{\substack{0001000\\0}},\ e_{\substack{0000010\\0}} \leftrightarrow e_{\substack{2343210\\2}}. \end{aligned}$$

Thus c_2 also acts as a non-trivial graph automorphism of C°. Hence $A = \langle \bar{c}_1, \bar{c}_2 \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, {z_6}^1, {z_6}^2 \rangle$, where $z_2 = e$, ${z_6}^1 = e_{\substack{0111000\\1}}$ and ${z_6}^2 = e_{\substack{0121000\\1}}$. Since both $\mathrm{Ad}(c_1)$ and $\mathrm{Ad}({c_2}^{g^{-1}})$ negate ${z_6}^1$ and fix ${z_6}^2$, and $\mathrm{Ad}(g){z_6}^2 = -\frac{3}{2}{z_6}^1 - \frac{1}{2}{z_6}^2$, we have $\mathcal{Z} = \langle z_2 \rangle$.

9.3.5 Orbit $D_4(a_1)A_1$ (number 18 in E_7, 17 in E_8)

Here $e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$, which is non-regular distinguished in the Lie algebra of H, a D_4A_1 subsystem subgroup of G with $\Pi(H) = \{ \substack{0100000\\0}, \substack{0010000\\0}, \substack{0000000\\1}, \substack{0001000\\0}, \substack{0000010\\0} \}$; we have $A \cong \mathcal{S}_2$ if $G = E_7$ or $\mathcal{S}_3$ if $G = E_8$, and $C^\circ = {A_1}^2$ or ${A_1}^3$ respectively, with $\Pi(D) = \{ \substack{0122210\\1}, \substack{2343210\\2}, \substack{2465432\\3} \} \cap \Phi(G)$. We take

$$\begin{aligned} c_1 &= n_{\substack{1110000\\1}} n_{\substack{1111000\\0}} h_2(-1),\\ c_2 &= (n_{\substack{1221111\\1}} n_{\substack{1122111\\1}} h_3(i) h_5(i))^g, \end{aligned}$$

where

$$g = x_{\substack{0010000\\0}}(\tfrac{1}{3}) n_{\substack{0010000\\0}} h_1(4) h_2(-4) h_3(16) h_4(-48) h_5(16) h_6(-8) x_{\substack{0010000\\0}}(-\tfrac{1}{3}).$$

The action of $\mathrm{Ad}(c_1)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\ e_{\substack{0100000\\0}},\ e_{\substack{0000010\\0}},\ e_{\substack{2465432\\3}};\\ \text{negated}: &\ e_{\substack{0010000\\0}};\\ \text{interchanged}: &\ e_{\substack{0010000\\1}} \leftrightarrow e_{\substack{0011000\\0}},\ e_{\substack{0000000\\1}} \leftrightarrow e_{\substack{0001000\\0}},\ e_{\substack{0122210\\1}} \leftrightarrow e_{\substack{2343210\\2}}. \end{aligned}$$

Thus c_1 acts as a non-trivial graph automorphism of C°. The action of c_2 is a little more complicated. Calculation shows that

$$\mathrm{Ad}(g)e = e_{\substack{0000000\\1}} + e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} - \tfrac{1}{8} e_{\substack{0000010\\0}},$$

which is still non-regular distinguished in $\mathfrak{L}(H)$, and $(C^\circ)^{g^{-1}} = C^\circ$. The action of $\mathrm{Ad}({c_2}^{g^{-1}})$ on the root vectors concerned is as follows:

$$\begin{aligned}
\text{fixed}: &\ \ e_{\substack{0000000\\1}},\ e_{\substack{0000010\\0}},\ e_{\substack{2343210\\2}};\\
\text{negated}: &\ \ e_{\substack{0010000\\0}};\\
\text{interchanged}: &\ \ e_{\substack{0110000\\0}} \leftrightarrow e_{\substack{0011000\\0}},\ e_{\substack{0100000\\0}} \leftrightarrow e_{\substack{0001000\\0}},\ e_{\substack{0122210\\1}} \leftrightarrow e_{\substack{2465432\\3}}.
\end{aligned}$$

Thus c_2 also acts as a non-trivial graph automorphism of C° if $G = E_8$. Hence $A = \langle \bar{c}_1 \rangle$ if $G = E_7$ and $\langle \bar{c}_1, \bar{c}_2 \rangle$ if $G = E_8$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, {z_6}^1, {z_6}^2 \rangle$, where $z_2 = e$, ${z_6}^1 = e_{\substack{0111000\\1}}$ and ${z_6}^2 = e_{\substack{0121000\\1}}$. Since both $\mathrm{Ad}(c_1)$ and $\mathrm{Ad}({c_2}^{g^{-1}})$ negate ${z_6}^1$ and fix ${z_6}^2$, and $\mathrm{Ad}(g){z_6}^2 = -\frac{3}{2}{z_6}^1 - \frac{1}{2}{z_6}^2$, we have $\mathcal{Z} = \langle z_2, {z_6}^2 \rangle$ if $G = E_7$ and $\langle z_2 \rangle$ if $G = E_8$.

9.3.6 Orbit A_3A_2 (number 19 in E_7, 18 in E_8)

Here $e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = A_1T_1$ or B_2T_1 respectively. In this case $D = J \cap G$, where J is the fixed point subgroup under a non-trivial graph automorphism of an A_3 subsystem subgroup K with $\Pi(K) = \{ {\substack{0011111\\1}}, {\substack{2343210\\2}}, {\substack{0111111\\0}} \}$. We take

$$c = n_{\substack{0011100\\1}} n_{\substack{0111100\\0}} n_{\substack{0122110\\1}} h_2(-1)h_4(-1)h_5(-1)h_6(-1)h_7(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned}
\text{fixed}: &\ \ e_{\substack{0010000\\0}},\ e_{\substack{2343210\\2}};\\
\text{interchanged}: &\ \ e_{\substack{0100000\\0}} \leftrightarrow e_{\substack{0000000\\1}},\ e_{\substack{0000100\\0}} \leftrightarrow e_{\substack{0000010\\0}},\ e_{\substack{0011111\\1}} \leftrightarrow e_{\substack{0111111\\0}}.
\end{aligned}$$

Thus c acts as a non-trivial graph automorphism of K. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_6 \rangle$, where $z_2 = e$ and $z_6 = e_{\substack{0110000\\1}}$. Since $\mathrm{Ad}(c)$ fixes z_6, we have $\mathcal{Z} = \langle z_2, z_6 \rangle$.

9.3.7 Orbit A_4 (number 20 in E_7, 19 in E_8)

Here $e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = A_2T_1$ or A_4 respectively with $\Pi(D) = \{ {\substack{0000001\\0}}, {\substack{0000010\\0}}, {\substack{0000100\\0}}, {\substack{2465321\\3}} \} \cap \Phi(G)$. We take

$$c = n_{\substack{0111100\\1}} n_{\substack{1111100\\0}} n_{\substack{1222110\\1}} n_{\substack{1243210\\2}} h_2(-1)h_3(-1)h_4(-1)h_8(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned}
\text{interchanged}: &\ \ e_{\substack{1000000\\0}} \leftrightarrow e_{\substack{0000000\\1}},\ e_{\substack{0100000\\0}} \leftrightarrow e_{\substack{0010000\\0}},\ e_{\substack{0000001\\0}} \leftrightarrow e_{\substack{2465321\\3}},\\
&\ \ e_{\substack{0000010\\0}} \leftrightarrow e_{\substack{0000100\\0}}.
\end{aligned}$$

(Note that if $G = E_7$ we replace $h_8(-1)$ by $h_6(-1)$ to obtain an element of G whose effect upon those root vectors listed which lie in $\mathfrak{L}(G)$ is the same.) Thus c acts as a non-trivial graph automorphism of C°. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_6, z_8 \rangle$, where $z_2 = e$, $z_6 = e_{\substack{1110000\\0}} - e_{\substack{0110000\\1}}$ and $z_8 = e_{\substack{1110000\\1}}$. Since $\mathrm{Ad}(c)$ fixes z_6 and negates z_8, we have $\mathcal{Z} = \langle z_2, z_6 \rangle$.

9.3.8 Orbit $D_4(a_1)A_2$ (number 22 in E_8)

Here $e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$, which is non-regular distinguished in the Lie algebra of H, a D_4A_2 subsystem subgroup of G with $\Pi(H) = \{ \substack{0100000\\0}, \substack{0010000\\0}, \substack{0000000\\1}, \substack{0001000\\0}, \substack{0000010\\0}, \substack{0000001\\0} \}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = A_2$. Here C° was found in §7.2; we have $C^\circ = \langle X_{\pm\beta_1}, X_{\pm\beta_2} \rangle$ where

$$\begin{aligned}
x_{\beta_1}(t) &= x_{\substack{0001111\\0}}(3t) x_{\substack{0121100\\1}}(t) x_{\substack{0011111\\0}}(-t) x_{\substack{0111100\\1}}(t) x_{\substack{0122211\\1}}(-2t^2) \times \\
&\qquad x_{\substack{0011110\\1}}(2t) x_{\substack{0111110\\0}}(-t), \\
x_{\beta_2}(t) &= x_{\substack{1111111\\1}}(-3t) x_{\substack{1232100\\1}}(-t) x_{\substack{1121111\\1}}(-t) x_{\substack{1222100\\1}}(t) x_{\substack{2343211\\2}}(-2t^2) \times \\
&\qquad x_{\substack{1122110\\1}}(2t) x_{\substack{1221110\\1}}(-t).
\end{aligned}$$

We take

$$c = n_{\substack{1111000\\0}} n_{\substack{1110000\\1}} h_5(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned}
\text{fixed}: &\quad e_{\substack{0100000\\0}},\ e_{\substack{0000010\\0}},\ e_{\substack{0000001\\0}}; \\
\text{negated}: &\quad e_{\substack{0010000\\0}}; \\
\text{interchanged}: &\quad e_{\substack{0010000\\1}} \leftrightarrow e_{\substack{0011000\\0}},\ e_{\substack{0000000\\1}} \leftrightarrow e_{\substack{0001000\\0}},\ e_{\substack{0001111\\0}} \leftrightarrow -e_{\substack{1111111\\1}}, \\
&\quad e_{\substack{0121100\\1}} \leftrightarrow -e_{\substack{1232100\\1}},\ e_{\substack{0011111\\0}} \leftrightarrow e_{\substack{1121111\\1}},\ e_{\substack{0111100\\1}} \leftrightarrow e_{\substack{1222100\\1}}, \\
&\quad e_{\substack{0011110\\1}} \leftrightarrow e_{\substack{1122110\\1}},\ e_{\substack{0111110\\0}} \leftrightarrow e_{\substack{1221110\\1}},\ e_{\substack{0122211\\1}} \leftrightarrow e_{\substack{2343211\\2}}.
\end{aligned}$$

Thus c acts as a non-trivial graph automorphism of both H and C°; we find that $C_H(c) \cong B_2B_1A_1$, and that e is regular in $C_H(c)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2 \rangle$, where $z_2 = e$, and so $\mathcal{Z} = \langle z_2 \rangle$.

9.3.9 Orbit A_4A_1 (number 24 in E_7, 23 in E_8)

Here $e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = T_2$ or A_2T_1 respectively with $\Pi(D) = \{ \substack{0000001\\0}, \substack{2465431\\3} \} \cap \Phi(G)$. We take

$$c = n_{\substack{0111110\\1}} n_{\substack{1111110\\0}} n_{\substack{1222100\\1}} n_{\substack{1243210\\2}} h_1(-1)h_3(-1)h_4(-1)h_6(-1)h_8(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned}
\text{fixed}: &\quad e_{\substack{0000100\\0}}; \\
\text{interchanged}: &\quad e_{\substack{1000000\\0}} \leftrightarrow e_{\substack{0000000\\1}},\ e_{\substack{0100000\\0}} \leftrightarrow e_{\substack{0010000\\0}},\ e_{\substack{0000001\\0}} \leftrightarrow e_{\substack{2465431\\3}}.
\end{aligned}$$

(Note that if $G = E_7$ we delete the term $h_8(-1)$ to obtain an element of G whose effect upon those root vectors listed which lie in $\mathfrak{L}(G)$ is the same.) Thus c acts as a non-trivial graph automorphism of the A_2 subsystem subgroup containing D. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_8 \rangle$, where $z_2 = e$ and $z_8 = e_{\substack{1110000\\1}}$. Since $\mathrm{Ad}(c)$ negates z_8, we have $\mathcal{Z} = \langle z_2 \rangle$.

9.3.10 Orbit $D_5(a_1)$ (number 25 in E_7, 25 in E_8)

Here $e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$, which is non-regular distinguished in the Lie algebra of H, a D_5 subsystem subgroup of G with $\Pi(H) = \{ {\substack{1000000\\0}}, {\substack{0100000\\0}}, {\substack{0010000\\0}}, {\substack{0000000\\1}}, {\substack{0001000\\0}} \}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = A_1T_1$ or A_3 respectively with $\Pi(D) = \{ {\substack{0000001\\0}}, {\substack{0000010\\0}}, {\substack{2465421\\3}} \} \cap \Phi(G)$. We take

$$c = n_{\substack{1233210\\1}} n_{\substack{1232210\\2}} h_1(-1)h_2(-1)h_4(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

fixed : $e_{\substack{1000000\\0}}$, $e_{\substack{0100000\\0}}$, $e_{\substack{0000010\\0}}$;

negated : $e_{\substack{0010000\\0}}$;

interchanged : $e_{\substack{0010000\\1}} \leftrightarrow e_{\substack{0011000\\0}}$, $e_{\substack{0000000\\1}} \leftrightarrow e_{\substack{0001000\\0}}$, $e_{\substack{0000001\\0}} \leftrightarrow e_{\substack{2465421\\3}}$.

Thus c acts as a non-trivial graph automorphism of both H and the A_3 subsystem subgroup containing C°; we find that $C_H(c) \cong B_3B_1$, and that e is regular in $C_H(c)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_8, z_{10} \rangle$, where $z_2 = e$, $z_8 = e_{\substack{1111000\\1}}$ and $z_{10} = e_{\substack{1221000\\1}}$. Since $\mathrm{Ad}(c)$ negates z_8 and fixes z_{10}, we have $\mathcal{Z} = \langle z_2, z_{10} \rangle$.

9.3.11 Orbit ${A_4A_1}^2$ (number 26 in E_8)

Here $e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = A_1T_1$. In this case D is the fixed point subgroup under a non-trivial graph automorphism of an ${A_1}^2$ subsystem subgroup K with $\Pi(K) = \{ {\substack{0000110\\0}}, {\substack{0000011\\0}} \}$. We take

$$c = n_{\substack{1222210\\1}} n_{\substack{1222111\\1}} n_{\substack{1354321\\3}} n_{\substack{2354321\\2}} h_1(-1)h_3(-1)h_4(-1)h_6(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

interchanged : $e_{\substack{1000000\\0}} \leftrightarrow e_{\substack{0000000\\1}}$, $e_{\substack{0100000\\0}} \leftrightarrow e_{\substack{0010000\\0}}$, $e_{\substack{0000100\\0}} \leftrightarrow e_{\substack{0000001\\0}}$,
$e_{\substack{0000110\\0}} \leftrightarrow e_{\substack{0000011\\0}}$.

Thus c acts as a non-trivial graph automorphism of K. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_8 \rangle$, where $z_2 = e$ and $z_8 = e_{\substack{1110000\\1}}$. Since $\mathrm{Ad}(c)$ negates z_8, we have $\mathcal{Z} = \langle z_2 \rangle$.

9.3.12 Orbit D_4A_2 (number 31 in E_8)

Here $e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = A_2$. In this case C° is the fixed point subgroup under a non-trivial graph automorphism of an ${A_2}^2$ subsystem subgroup K with $\Pi(K) = \{ {\substack{1110000\\1}}, {\substack{1244321\\2}}, {\substack{1111000\\0}}, {\substack{1343321\\2}} \}$. We take

$$c = n_{\substack{0011110\\1}} n_{\substack{0111110\\0}} n_{\substack{0122211\\1}} h_2(-1)h_4(-1)h_5(-1)h_6(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

fixed : $e_{\substack{0010000\\0}}$, $e_{\substack{0001000\\0}}$;

interchanged : $e_{\substack{0100000\\0}} \leftrightarrow e_{\substack{0000000\\1}}$, $e_{\substack{0000010\\0}} \leftrightarrow e_{\substack{0000001\\0}}$, $e_{\substack{1110000\\1}} \leftrightarrow e_{\substack{1343321\\2}}$, $e_{\substack{1244321\\2}} \leftrightarrow -e_{\substack{1111000\\0}}$.

Thus c acts as a non-trivial graph automorphism of K. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{10} \rangle$, where $z_2 = e$ and $z_{10} = e_{\substack{0121000\\1}}$. Since $\mathrm{Ad}(c)$ fixes z_{10}, we have $\mathcal{Z} = \langle z_2, z_{10} \rangle$.

9.3.13 Orbit $E_6(a_3)$ (number 17 in E_6, 31 in E_7, 32 in E_8)

Here $e = e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}}$, which is regular in the Lie algebra of H, an A_5A_1 subsystem subgroup of G; we have $A \cong \mathcal{S}_2$, and $C^\circ = 1$, A_1 or G_2 respectively. In this case $C^\circ = J \cap G$, where J is the fixed point subgroup under a non-trivial graph automorphism of a D_4 subsystem subgroup K with $\Pi(K) = \{ {\substack{1122110\\1}}, {\substack{0000001\\0}}, {\substack{0122210\\1}}, {\substack{1221110\\1}} \}$. We take

$$c = h_4(-1).$$

Clearly $c \in Z(H)$. Moreover, as $c \in E_6$ it follows that $c \notin C^\circ$. Hence $A = \langle \bar{c} \rangle$. If $G = E_7$ or E_8 we have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, {z_{10}}^1, {z_{10}}^2 \rangle$, where ${z_{10}}^1 = e_{\substack{1232100\\1}}$ and ${z_{10}}^2 = e_{\substack{1232100\\2}}$; if $G = E_6$ we have an additional vector $z_8 = e_{\substack{12211\\1}} + e_{\substack{11221\\1}}$. Since $\mathrm{Ad}(c)$ fixes ${z_{10}}^2$ (and z_8 if $G = E_6$) and negates ${z_{10}}^1$, we have $\mathcal{Z} = \langle z_2, z_8, {z_{10}}^2 \rangle$ if $G = E_6$ and $\langle z_2, {z_{10}}^2 \rangle$ if $G = E_7$ or E_8.

9.3.14 Orbit $D_6(a_2)$ (number 37 in E_8)

Here $e = e_{\substack{0000010\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$, which is non-regular distinguished in the Lie algebra of H, a D_6 subsystem subgroup of G with $\Pi(H) = \{ {\substack{0000010\\0}}, {\substack{0000100\\0}}, {\substack{0001000\\0}}, {\substack{0010000\\0}}, {\substack{0000000\\1}}, {\substack{0100000\\0}} \}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = {A_1}^2$ with $\Pi(D) = \{ {\substack{2343210\\2}}, {\substack{2465432\\3}} \}$. We take

$$c = n_{\substack{0011111\\1}} n_{\substack{0111111\\0}} h_4(-1)h_5(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

fixed : $e_{\substack{0000010\\0}}$, $e_{\substack{0001100\\0}}$, $e_{\substack{0011000\\0}}$;

negated : $e_{\substack{0010000\\0}}$, $e_{\substack{0001000\\0}}$, $e_{\substack{0000100\\0}}$;

interchanged : $e_{\substack{0010000\\1}} \leftrightarrow -e_{\substack{0110000\\0}}$, $e_{\substack{0000000\\1}} \leftrightarrow e_{\substack{0100000\\0}}$, $e_{\substack{2343210\\2}} \leftrightarrow e_{\substack{2465432\\3}}$.

Thus c acts as a non-trivial graph automorphism of both H and C°; we find that $C_H(c) \cong B_3B_2$, and that e is regular in $C_H(c)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, {z_{10}}^1, {z_{10}}^2 \rangle$, where $z_2 = e$, ${z_{10}}^1 = e_{\substack{0122110\\1}}$ and ${z_{10}}^2 = e_{\substack{0122210\\1}}$. Since $\mathrm{Ad}(c)$ negates ${z_{10}}^1$ and fixes ${z_{10}}^2$, we have $\mathcal{Z} = \langle z_2, {z_{10}}^2 \rangle$.

9.3.15 Orbit $E_6(a_3)A_1$ (number 38 in E_8)

Here $e = e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0000001\\0}}$, which is regular in the Lie algebra of H, an $A_5{A_1}^2$ subsystem subgroup of G; we have $A \cong \mathcal{S}_2$, and $C^\circ = A_1$. In this case C° is the fixed point subgroup under a non-trivial graph automorphism of an ${A_1}^3$ subsystem subgroup K with $\Pi(K) = \{ \substack{1244321\\2}, \substack{1343321\\2}, \substack{2343221\\2} \}$. We take

$$c = h_4(-1).$$

Clearly $c \in Z(H)$. Moreover, as $c \in E_6$ it follows that $c \notin C^\circ$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, {z_{10}}^1, {z_{10}}^2 \rangle$, where $z_2 = e$, ${z_{10}}^1 = e_{\substack{1232100\\1}}$ and ${z_{10}}^2 = e_{\substack{1232100\\2}}$. Since $\mathrm{Ad}(c)$ negates ${z_{10}}^1$ and fixes ${z_{10}}^2$, we have $\mathcal{Z} = \langle z_2, {z_{10}}^2 \rangle$.

9.3.16 Orbit $E_7(a_5)$ (number 33 in E_7, 39 in E_8)

Here $e = e_{\substack{1111000\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0000010\\0}} + e_{\substack{0111100\\0}} + e_{\substack{1110000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0111000\\1}}$, which is regular in the Lie algebra of H, an A_5A_2 subsystem subgroup of G; we have $A \cong \mathcal{S}_3$, and $C^\circ = 1$ or A_1 respectively with $\Pi(D) = \{ \substack{2465432\\3} \} \cap \Phi(G)$. We take

$$\begin{aligned} c_1 &= h_2(\omega)h_3(\omega)h_5(\omega), \\ c_2 &= n_{\substack{0000000\\1}} n_{\substack{0100000\\0}} n_{\substack{0001000\\0}} h_3(-1)h_4(-1). \end{aligned}$$

Clearly $c_1 \in Z(H)$. The action of $\mathrm{Ad}(c_2)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\quad e_{\substack{0000010\\0}}, \ e_{\substack{2465432\\3}}; \\ \text{interchanged}: &\quad e_{\substack{1111000\\0}} \leftrightarrow e_{\substack{1110000\\1}}, \ e_{\substack{0011100\\1}} \leftrightarrow e_{\substack{0111100\\0}}, \ e_{\substack{0010000\\0}} \leftrightarrow e_{\substack{0111000\\1}}. \end{aligned}$$

Thus c_2 acts as a non-trivial graph automorphism of H. Moreover, as $c_1, c_2 \in E_7$ it follows that $c_1, c_2 \notin C^\circ$. Hence $A = \langle \bar{c}_1, \bar{c}_2 \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, {z_{10}}^1, {z_{10}}^2, {z_{10}}^3 \rangle$, where $z_2 = e$, ${z_{10}}^1 = e_{\substack{1243210\\2}}$, ${z_{10}}^2 = e_{\substack{1343210\\2}}$ and ${z_{10}}^3 = e_{\substack{2343210\\2}}$. Since $\mathrm{Ad}(c_1)$ fixes ${z_{10}}^3$ and multiplies ${z_{10}}^2$ by ω and ${z_{10}}^1$ by ω^2, and $\mathrm{Ad}(c_2)$ fixes ${z_{10}}^3$, we have $\mathcal{Z} = \langle z_2, {z_{10}}^3 \rangle$.

9.3.17 Orbit $E_8(a_7)$ (number 41 in E_8)

Here $e = e_{\substack{1111111\\0}} + e_{\substack{0121110\\1}} + e_{\substack{0001000\\0}} + e_{\substack{1121100\\1}} + e_{\substack{1221000\\1}} + e_{\substack{0011111\\1}} + e_{\substack{0111100\\0}} + e_{\substack{1111110\\1}}$, which is regular in the Lie algebra of H, an ${A_4}^2$ subsystem subgroup of G; we have $A \cong \mathcal{S}_5$, and $C^\circ = 1$. We take

$$\begin{aligned} c_1 &= h_2(\zeta^2)h_3(\zeta^4)h_4(\zeta)h_6(\zeta^4)h_7(\zeta)h_8(\zeta^2), \\ c_2 &= n_{\substack{0100000\\0}} n_{\substack{0010000\\0}} n_{\substack{0000000\\1}} n_{\substack{0000100\\0}} n_{\substack{0000010\\0}} n_{\substack{0000001\\0}} \\ &\quad \times h_1(-1)h_3(-1)h_5(-1)h_6(-1)h_8(-1), \\ c_3 &= (n_{\substack{1110000\\1}} n_{\substack{0110000\\0}} n_{\substack{0000111\\0}} n_{\substack{0000010\\0}} h)^u, \end{aligned}$$

where

$$
\begin{aligned}
h &= h_1(\tfrac{2}{5})h_2(-\tfrac{2}{5})h_3(\tfrac{2(1-3\phi)}{25})h_4(\tfrac{2(1-3\phi)}{25})h_6(\tfrac{3+\phi}{5})h_7(-\tfrac{3+\phi}{5})h_8(-\tfrac{3+\phi}{5}),\\
u &= x_{\substack{1110000\\1}}(-\tfrac{1}{2}+\tfrac{1}{2}\phi)x_{\substack{0110000\\1}}(-3-2\phi)x_{\substack{0010000\\1}}(-3-2\phi)x_{\substack{0000000\\1}}(1)x_{\substack{1110000\\0}}(1+\phi)\\
&\quad\times x_{\substack{0110000\\0}}(5+3\phi)x_{\substack{0010000\\0}}(4+2\phi)x_{\substack{1100000\\0}}(1+\tfrac{1}{2}\phi)x_{\substack{0100000\\0}}(1+\phi)x_{\substack{1000000\\0}}(\phi)\\
&\quad\times x_{\substack{0000111\\0}}(-\phi)x_{\substack{0000100\\0}}(\phi)x_{\substack{0000011\\0}}(-1+\phi)x_{\substack{0000010\\0}}(-\phi)x_{\substack{0000001\\0}}(-\phi).
\end{aligned}
$$

Clearly $c_1 \in Z(H)$. The action of $\mathrm{Ad}(c_2)$ on the root vectors concerned is as follows:

$$
\begin{aligned}
\text{cycled}: \;& e_{\substack{1111111\\0}} \mapsto e_{\substack{1221000\\1}} \mapsto e_{\substack{1121100\\1}} \mapsto e_{\substack{1111110\\1}} \mapsto e_{\substack{1111111\\0}},\\
& e_{\substack{0121110\\1}} \mapsto e_{\substack{0011111\\1}} \mapsto e_{\substack{0001000\\0}} \mapsto e_{\substack{0111100\\0}} \mapsto e_{\substack{0121110\\1}}.
\end{aligned}
$$

Thus c_2 acts as a non-trivial graph automorphism of H. The action of c_3 is considerably more complicated. Calculation shows that

$$
\begin{aligned}
\mathrm{Ad}(u)e = {}& e_{\substack{0001000\\0}} + (4+2\phi)e_{\substack{0011000\\0}} + e_{\substack{0011000\\1}} + (5+3\phi)e_{\substack{0111000\\0}} + (2+\phi)e_{\substack{0111000\\1}}\\
&+ (1+\phi)e_{\substack{1111000\\0}} + (\tfrac{1}{2}+\tfrac{3}{2}\phi)e_{\substack{1111000\\1}} + (5+3\phi)e_{\substack{0121000\\1}} + (4+3\phi)e_{\substack{1121000\\1}}\\
&+ (5+\tfrac{5}{2}\phi)e_{\substack{1221000\\1}} - \phi e_{\substack{0001100\\0}} - (2+2\phi)e_{\substack{0011100\\0}} - \phi e_{\substack{0011100\\1}}\\
&- (2+2\phi)e_{\substack{0111100\\0}} - \phi e_{\substack{0111100\\1}} - (1-\phi)e_{\substack{1111100\\0}} - (\tfrac{3}{2}-2\phi)e_{\substack{1111100\\1}}\\
&+ \phi e_{\substack{0111110\\0}} + \phi e_{\substack{0111110\\1}} + (1-\phi)e_{\substack{1111110\\0}} + (2-\phi)e_{\substack{1111110\\1}}\\
&+ (3+\phi)e_{\substack{0121110\\1}} + (5+5\phi)e_{\substack{1121110\\1}} + (\tfrac{15}{2}+\tfrac{5}{2}\phi)e_{\substack{1221110\\1}} + \phi e_{\substack{0001111\\0}}\\
&+ (2+2\phi)e_{\substack{0011111\\0}} + (1+\phi)e_{\substack{0011111\\1}} + (4+\phi)e_{\substack{0111111\\0}} + (3+\phi)e_{\substack{0111111\\1}}\\
&+ (1+2\phi)e_{\substack{1111111\\0}} + (\tfrac{5}{2}+\tfrac{5}{2}\phi)e_{\substack{1111111\\1}} + (5+5\phi)e_{\substack{0121111\\1}}\\
&+ (15+5\phi)e_{\substack{1121111\\1}} + (\tfrac{25}{2}+\tfrac{25}{2}\phi)e_{\substack{1221111\\1}}.
\end{aligned}
$$

The action of $\mathrm{Ad}({c_3}^{u^{-1}})$ on the root vectors concerned is as follows:

$$
\begin{aligned}
\text{interchanged}: \;& e_{\substack{0001000\\0}} \leftrightarrow (\tfrac{25}{2}+\tfrac{25}{2}\phi)e_{\substack{1221111\\1}},\; e_{\substack{0011000\\0}} \leftrightarrow (5-\tfrac{5}{2}\phi)e_{\substack{1121111\\1}},\\
& e_{\substack{0011000\\1}} \leftrightarrow (5+5\phi)e_{\substack{0121111\\1}},\; e_{\substack{0111000\\0}} \leftrightarrow (-\tfrac{5}{2}+5\phi)e_{\substack{1111111\\1}},\\
& e_{\substack{0111000\\1}} \leftrightarrow (2-\phi)e_{\substack{0111111\\1}},\; e_{\substack{1111000\\0}} \leftrightarrow (2-\phi)e_{\substack{1111111\\0}},\\
& e_{\substack{1111000\\1}} \leftrightarrow (2+2\phi)e_{\substack{0111111\\0}},\; e_{\substack{0121000\\1}} \leftrightarrow (-1+2\phi)e_{\substack{0011111\\1}},\\
& e_{\substack{1121000\\1}} \leftrightarrow (\tfrac{4}{5}-\tfrac{2}{5}\phi)e_{\substack{0011111\\0}},\; e_{\substack{1221000\\1}} \leftrightarrow (-\tfrac{2}{5}+\tfrac{4}{5}\phi)e_{\substack{0001111\\0}},\\
& e_{\substack{0001100\\0}} \leftrightarrow (-10-\tfrac{15}{2}\phi)e_{\substack{1221110\\1}},\; e_{\substack{0011100\\0}} \leftrightarrow -\tfrac{5}{2}e_{\substack{1121110\\1}},\\
& e_{\substack{0011100\\1}} \leftrightarrow (-4-3\phi)e_{\substack{0121110\\1}},\; e_{\substack{0111100\\0}} \leftrightarrow (\tfrac{1}{2}-\tfrac{3}{2}\phi)e_{\substack{1111110\\1}},\\
& e_{\substack{0111100\\1}} \leftrightarrow -e_{\substack{0111110\\1}},\; e_{\substack{1111100\\0}} \leftrightarrow -e_{\substack{1111110\\0}},\\
& e_{\substack{1111100\\1}} \leftrightarrow (-\tfrac{8}{5}-\tfrac{6}{5}\phi)e_{\substack{0111110\\0}}.
\end{aligned}
$$

Thus $c_3 \in C_G(e)$. Hence $A = \langle \bar{c}_1, \bar{c}_2, \bar{c}_3 \rangle$. (Indeed, we see that ${c_1}^5 = {c_2}^4 = {c_3}^2 = 1$, with ${c_1}^{c_2} = {c_1}^3$, ${c_2}^{c_3} = {c_2}^{-1}$ and $(c_1c_3)^3 = 1$; we may thus identify c_1, c_2 and c_3 with the elements (1 2 3 4 5), (2 3 5 4) and (2 3)(4 5) of S_5.) We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, {z_{10}}^1, {z_{10}}^2, {z_{10}}^3, {z_{10}}^4 \rangle$, where $z_2 = e$, ${z_{10}}^1 = e_{\substack{2465321\\3}}$, ${z_{10}}^2 = e_{\substack{2465421\\3}}$, ${z_{10}}^3 = e_{\substack{2465431\\3}}$ and ${z_{10}}^4 = e_{\substack{2465432\\3}}$. Since $\mathrm{Ad}(c_1)$ multiplies ${z_{10}}^1$ by ζ, ${z_{10}}^2$ by ζ^3, ${z_{10}}^3$ by ζ^4 and ${z_{10}}^4$ by ζ^2, we have $\mathcal{Z} = \langle z_2 \rangle$.

9.3.18 Orbit $D_6(a_1)$ (number 43 in E_8)

Here $e = e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$, which is non-regular distinguished in the Lie algebra of H, a D_6 subsystem subgroup of G with $\Pi(H) = \{ {}^{0000010}_{\ \ \ 0}, {}^{0000100}_{\ \ \ 0}, {}^{0001000}_{\ \ \ 0}, {}^{0010000}_{\ \ \ 0}, {}^{0000000}_{\ \ \ 1}, {}^{0100000}_{\ \ \ 0} \}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = {A_1}^2$ with $\Pi(D) = \{ {}^{2343210}_{\ \ \ 2}, {}^{2465432}_{\ \ \ 3} \}$. We take

$$c = n_{\substack{0011111\\1}} n_{\substack{0111111\\0}} h_2(i) h_3(i).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

fixed : $e_{\substack{0000010\\0}}$, $e_{\substack{0000100\\0}}$, $e_{\substack{0001000\\0}}$;

negated : $e_{\substack{0010000\\0}}$;

interchanged : $e_{\substack{0010000\\1}} \leftrightarrow -e_{\substack{0110000\\0}}$, $e_{\substack{0000000\\1}} \leftrightarrow e_{\substack{0100000\\0}}$, $e_{\substack{2343210\\2}} \leftrightarrow e_{\substack{2465432\\3}}$.

Thus c acts as a non-trivial graph automorphism of both H and C°; we find that $C_H(c) \cong B_4B_1$, and that e is regular in $C_H(c)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, {z_{10}}^1, {z_{10}}^2, z_{14} \rangle$, where $z_2 = e$, ${z_{10}}^1 = e_{\substack{0121110\\1}} + e_{\substack{0122100\\1}}$, ${z_{10}}^2 = e_{\substack{0111110\\1}}$ and $z_{14} = e_{\substack{0122210\\1}}$. Since $\mathrm{Ad}(c)$ negates ${z_{10}}^2$ and fixes ${z_{10}}^1$ and z_{14}, we have $\mathcal{Z} = \langle z_2, {z_{10}}^1, z_{14} \rangle$.

9.3.19 Orbit $E_7(a_4)$ (number 37 in E_7, 45 in E_8)

Here $e = e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0000010\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}}$, which is non-regular distinguished in the Lie algebra of H, a D_6A_1 subsystem subgroup of G with $\Pi(H) = \{ {}^{0110000}_{\ \ \ 1}, {}^{1000000}_{\ \ \ 0}, {}^{0111000}_{\ \ \ 0}, {}^{0000100}_{\ \ \ 0}, {}^{0000010}_{\ \ \ 0}, {}^{0011000}_{\ \ \ 1}, {}^{0010000}_{\ \ \ 0} \}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = 1$ or A_1 respectively with $\Pi(D) = \{ {}^{2465432}_{\ \ \ 3} \} \cap \Phi(G)$. We take

$$c = h_4(-1).$$

Clearly $c \in Z(H)$. Moreover, as $c \in E_7$ it follows that $c \notin C^\circ$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{10}, z_{14} \rangle$, where $z_2 = e$, $z_{10} = 5e_{\substack{1232110\\2}} + e_{\substack{1232210\\2}} + e_{\substack{1233210\\1}}$ and $z_{14} = e_{\substack{2343210\\2}}$. Since $\mathrm{Ad}(c)$ fixes z_{10} and z_{14}, we have $\mathcal{Z} = \langle z_2, z_{10}, z_{14} \rangle$.

9.3.20 Orbit $E_6(a_1)$ (number 39 in E_7, 46 in E_8)

Here $e = e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}}$, which is non-regular distinguished in the Lie algebra of H, an E_6 subsystem subgroup of G with

$\Pi(H) = \{ {\substack{1000000\\0}}, {\substack{0000000\\1}}, {\substack{0100000\\0}}, {\substack{0010000\\0}}, {\substack{0001000\\0}}, {\substack{0000100\\0}} \}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = T_1$ or A_2 respectively with $\Pi(D) = \{ {\substack{0000001\\0}}, {\substack{2465431\\3}} \} \cap \Phi(G)$. We take

$$c = n_{\substack{0122210\\1}} n_{\substack{1122110\\1}} n_{\substack{1221110\\1}} h_1(-1)h_2(-1)h_3(-1)h_5(-1)h_6(-1)h_8(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned}
\text{fixed}: &\quad e_{\substack{0000000\\1}};\\
\text{negated}: &\quad e_{\substack{0010000\\0}};\\
\text{interchanged}: &\quad e_{\substack{1000000\\0}} \leftrightarrow e_{\substack{0000100\\0}},\ e_{\substack{0100000\\0}} \leftrightarrow e_{\substack{0001000\\0}},\ e_{\substack{0011000\\0}} \leftrightarrow e_{\substack{0110000\\0}},\\
&\quad e_{\substack{0000001\\0}} \leftrightarrow e_{\substack{2465431\\3}}.
\end{aligned}$$

(Note that if $G = E_7$ we delete the term $h_8(-1)$ to obtain an element of G whose effect upon those root vectors listed which lie in $\mathfrak{L}(G)$ is the same.) Thus c acts as a non-trivial graph automorphism of both H and the A_2 subsystem subgroup containing C°; we find that $C_H(c) \cong C_4$, and that e is regular in $C_H(c)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{10}, z_{14}, z_{16} \rangle$, where $z_2 = e$, $z_{10} = e_{\substack{1121100\\1}} + e_{\substack{1221000\\1}} - e_{\substack{0122100\\1}}$, $z_{14} = e_{\substack{1222100\\1}}$ and $z_{16} = e_{\substack{1232100\\2}}$. Since $\mathrm{Ad}(c)$ fixes z_{10} and z_{14} and negates z_{16}, we have $\mathcal{Z} = \langle z_2, z_{10}, z_{14} \rangle$.

9.3.21 Orbit D_5A_2 (number 47 in E_8)

Here $e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = T_1$. We take

$$c = n_{\substack{1233211\\1}} n_{\substack{1232211\\2}} n_{\substack{2465431\\3}} h_1(-1)h_2(-1)h_4(-1)h_5(-1)h_7(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned}
\text{fixed}: &\quad e_{\substack{1000000\\0}},\ e_{\substack{0100000\\0}},\ e_{\substack{0010000\\0}};\\
\text{interchanged}: &\quad e_{\substack{0000000\\1}} \leftrightarrow e_{\substack{0001000\\0}},\ e_{\substack{0000010\\0}} \leftrightarrow e_{\substack{0000001\\0}}.
\end{aligned}$$

Thus c acts as a non-trivial graph automorphism of the D_5A_2 subsystem subgroup in whose Lie algebra e is regular. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14} \rangle$, where $z_2 = e$ and $z_{14} = e_{\substack{1221000\\1}}$. Since $\mathrm{Ad}(c)$ fixes z_{14}, we have $\mathcal{Z} = \langle z_2, z_{14} \rangle$.

9.3.22 Orbit $D_7(a_2)$ (number 50 in E_8)

Here $e = e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$, which is non-regular distinguished in the Lie algebra of H, a D_7 subsystem subgroup of G with $\Pi(H) = \{ {\substack{0000001\\0}}, {\substack{0000010\\0}}, {\substack{0000100\\0}}, {\substack{0001000\\0}}, {\substack{0010000\\0}}, {\substack{0000000\\1}}, {\substack{0100000\\0}} \}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = T_1$. We take

$$c = n_{\substack{2354321\\3}} n_{\substack{2454321\\2}} h_4(-1)h_5(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\ e_{\substack{0000001\\0}},\ e_{\substack{0000010\\0}},\ e_{\substack{0001100\\0}},\ e_{\substack{0011000\\0}};\\ \text{negated}: &\ e_{\substack{0010000\\0}},\ e_{\substack{0001000\\0}},\ e_{\substack{0000100\\0}};\\ \text{interchanged}: &\ e_{\substack{0010000\\1}} \leftrightarrow -e_{\substack{0110000\\0}},\ e_{\substack{0000000\\1}} \leftrightarrow e_{\substack{0100000\\0}}. \end{aligned}$$

Thus c acts as a non-trivial graph automorphism of H; we find that $C_H(c) \cong B_4B_2$, and that e is regular in $C_H(c)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14} \rangle$, where $z_2 = e$ and $z_{14} = e_{\substack{0122221\\1}}$. Since $\mathrm{Ad}(c)$ fixes z_{14}, we have $\mathcal{Z} = \langle z_2, z_{14} \rangle$.

9.3.23 Orbit $E_6(a_1)A_1$ (number 52 in E_8)

Here $e = e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000001\\0}}$, which is non-regular distinguished in the Lie algebra of H, an E_6A_1 subsystem subgroup of G with $\Pi(H) = \{\substack{1000000\\0}, \substack{0000000\\1}, \substack{0100000\\0}, \substack{0010000\\0}, \substack{0001000\\0}, \substack{0000100\\0}, \substack{0000001\\0}\}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = T_1$. We take

$$c = n_{\substack{1244321\\2}} n_{\substack{1343321\\2}} n_{\substack{2343221\\2}} h_1(-1)h_2(-1)h_3(-1)h_5(-1)h_6(-1)h_8(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}: &\ e_{\substack{0000000\\1}},\ e_{\substack{0000001\\0}};\\ \text{negated}: &\ e_{\substack{0010000\\0}};\\ \text{interchanged}: &\ e_{\substack{1000000\\0}} \leftrightarrow e_{\substack{0000100\\0}},\ e_{\substack{0100000\\0}} \leftrightarrow e_{\substack{0001000\\0}},\ e_{\substack{0011000\\0}} \leftrightarrow e_{\substack{0110000\\0}}. \end{aligned}$$

Thus c acts as a non-trivial graph automorphism of H; we find that $C_H(c) \cong C_4A_1$, and that e is regular in $C_H(c)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14}, z_{16} \rangle$, where $z_2 = e$, $z_{14} = e_{\substack{1222100\\1}}$ and $z_{16} = e_{\substack{1232100\\2}}$. Since $\mathrm{Ad}(c)$ fixes z_{14} and negates z_{16}, we have $\mathcal{Z} = \langle z_2, z_{14} \rangle$.

9.3.24 Orbit $E_7(a_3)$ (number 41 in E_7, 53 in E_8)

Here $e = e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}}$, which is regular in the Lie algebra of H, a D_6A_1 subsystem subgroup of G; we have $A \cong \mathcal{S}_2$, and $C^\circ = 1$ or A_1 respectively with $\Pi(D) = \{\substack{2465432\\3}\} \cap \Phi(G)$. We take

$$c = h_4(-1).$$

Clearly $c \in Z(H)$. Moreover, as $c \in E_7$ it follows that $c \notin C^\circ$. Hence $A = \langle \bar{c} \rangle$. If $G = E_8$ we have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14}, z_{16}, z_{18} \rangle$, where $z_2 = e$, $z_{14} = e_{\substack{1232210\\2}} + e_{\substack{1233210\\1}}$, $z_{16} = e_{\substack{1243210\\2}}$ and $z_{18} = e_{\substack{2343210\\2}}$; if $G = E_7$ we have an additional vector $z_{10} = 3e_{\substack{122111\\1}} + 2e_{\substack{123210\\2}} + e_{\substack{112211\\1}} - e_{\substack{012221\\1}}$. Since $\mathrm{Ad}(c)$ fixes z_{14} and z_{18} (and z_{10} if $G = E_7$) and negates z_{16}, we have $\mathcal{Z} = \langle z_2, z_{10}, z_{14}, z_{18} \rangle$ if $G = E_7$ and $\langle z_2, z_{14}, z_{18} \rangle$ if $G = E_8$.

9.3.25 Orbit $E_8(b_6)$ (number 54 in E_8)

Here $e = e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{1111100\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0111110\\1}} - e_{\substack{1111110\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1110000\\1}}$, which is non-regular distinguished in the Lie algebra of H, an E_6A_2 subsystem subgroup of G with $\Pi(H) = \{\substack{0011000\\1}, \substack{0000001\\0}, \substack{1111100\\0}, \substack{0000010\\0}, \substack{0111100\\1}, \substack{0010000\\0}, \substack{0111000\\0}, \substack{1110000\\1}\}$ (with the projection of e on the Lie algebra of the E_6 factor lying in the $E_6(a_1)$ orbit — see the comments at the end of §5); we have $A \cong \mathcal{S}_3$, and $C^\circ = 1$. We take

$$c_1 = h_1(\omega)h_2(\omega)h_5(\omega^2),$$
$$c_2 = n_{\substack{1000000\\0}} n_{\substack{0000000\\1}} n_{\substack{0001000\\0}} h_2(-1)h_3(-1)h_4(-1)h_5(-1)h_8(-1).$$

Clearly $c_1 \in Z(H)$. The action of $\mathrm{Ad}(c_2)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}:&\ e_{\substack{0000001\\0}};\\ \text{negated}:&\ e_{\substack{0000010\\0}};\\ \text{interchanged}:&\ e_{\substack{0011000\\1}} \leftrightarrow e_{\substack{0010000\\0}},\ e_{\substack{1111100\\0}} \leftrightarrow e_{\substack{0111100\\1}},\ e_{\substack{0111110\\1}} \leftrightarrow -e_{\substack{1111110\\0}},\\ &\ e_{\substack{0111000\\0}} \leftrightarrow e_{\substack{1110000\\1}}. \end{aligned}$$

Thus c_2 acts as a non-trivial graph automorphism of H; we find that $C_H(c_2) \cong C_4A_1$, and that e is regular in $C_H(c_2)$. Hence $A = \langle \bar{c}_1, \bar{c}_2 \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14}, z_{16} \rangle$, where $z_2 = e$, $z_{14} = e_{\substack{2465421\\3}}$ and $z_{16} = e_{\substack{2465432\\3}}$. Since $\mathrm{Ad}(c_1)$ fixes both z_{14} and z_{16}, and $\mathrm{Ad}(c_2)$ fixes z_{14} but negates z_{16}, we have $\mathcal{Z} = \langle z_2, z_{14} \rangle$.

9.3.26 Orbit $D_7(a_1)$ (number 55 in E_8)

Here $e = e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$, which is non-regular distinguished in the Lie algebra of H, a D_7 subsystem subgroup of G with $\Pi(H) = \{\substack{0000001\\0}, \substack{0000010\\0}, \substack{0000100\\0}, \substack{0001000\\0}, \substack{0010000\\0}, \substack{0000000\\1}, \substack{0100000\\0}\}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = T_1$. We take

$$c = n_{\substack{2354321\\3}} n_{\substack{2454321\\2}} h_2(-1)h_4(-1)h_6(-1)h_8(-1).$$

The action of $\mathrm{Ad}(c)$ on the root vectors concerned is as follows:

$$\begin{aligned} \text{fixed}:&\ e_{\substack{0000001\\0}},\ e_{\substack{0000010\\0}},\ e_{\substack{0000100\\0}},\ e_{\substack{0001000\\0}};\\ \text{negated}:&\ e_{\substack{0010000\\0}};\\ \text{interchanged}:&\ e_{\substack{0010000\\1}} \leftrightarrow -e_{\substack{0110000\\0}},\ e_{\substack{0000000\\1}} \leftrightarrow e_{\substack{0100000\\0}}. \end{aligned}$$

Thus c acts as a non-trivial graph automorphism of H; we find that $C_H(c) \cong B_5B_1$, and that e is regular in $C_H(c)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14}, z_{18} \rangle$, where $z_2 = e$, $z_{14} = e_{\substack{0122111\\1}} + e_{\substack{0122210\\1}}$ and $z_{18} = e_{\substack{0122221\\1}}$. Since $\mathrm{Ad}(c)$ fixes z_{14} and z_{18}, we have $\mathcal{Z} = \langle z_2, z_{14}, z_{18} \rangle$.

9.3.27 Orbit $E_8(a_6)$ (number 58 in E_8)

Here $e = e_{\substack{0011000\\1}} + e_{\substack{1111100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001111\\0}} + e_{\substack{1110000\\1}} + e_{\substack{0111000\\0}}$, which is regular in the Lie algebra of H, an A_8 subsystem subgroup of G; we have $A \cong \mathcal{S}_3$, and $C^\circ = 1$. We take

$$c_1 = h_2(\omega)h_3(\omega)h_5(\omega)h_6(\omega)h_8(\omega^2),$$
$$c_2 = n_{\substack{0000000\\1}} n_{\substack{0100000\\0}} n_{\substack{0000001\\0}} n_{\substack{0001100\\0}} h_2(-1)h_3(-1)h_4(-1)h_5(-1)h_7(-1)h_8(-1).$$

Clearly $c_1 \in Z(H)$. The action of $\mathrm{Ad}(c_2)$ on the root vectors concerned is as follows:

interchanged : $e_{\substack{0011000\\1}} \leftrightarrow e_{\substack{0111000\\0}}$, $e_{\substack{1111100\\0}} \leftrightarrow e_{\substack{1110000\\1}}$, $e_{\substack{0000010\\0}} \leftrightarrow e_{\substack{0001111\\0}}$, $e_{\substack{0111100\\1}} \leftrightarrow e_{\substack{0010000\\0}}$.

Thus c_2 acts as a non-trivial graph automorphism of H. Hence $A = \langle \bar{c}_1, \bar{c}_2 \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14}, {z_{18}}^1, {z_{18}}^2 \rangle$, where $z_2 = e$, $z_{14} = e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$, ${z_{18}}^1 = e_{\substack{2465431\\3}}$ and ${z_{18}}^2 = e_{\substack{2465432\\3}}$. Since $\mathrm{Ad}(c_1)$ multiplies ${z_{18}}^1$ by ω and ${z_{18}}^2$ by ω^2 while fixing z_{14}, and $\mathrm{Ad}(c_2)$ fixes z_{14}, we have $\mathcal{Z} = \langle z_2, z_{14} \rangle$.

9.3.28 Orbit $E_8(b_5)$ (number 60 in E_8)

Here $e = e_{\substack{0011000\\1}} + e_{\substack{0000001\\0}} + e_{\substack{1111100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1110000\\1}}$, which is regular in the Lie algebra of H, an E_6A_2 subsystem subgroup of G; we have $A \cong \mathcal{S}_3$, and $C^\circ = 1$. We take

$$c_1 = h_1(\omega)h_2(\omega)h_5(\omega^2),$$
$$c_2 = n_{\substack{1000000\\0}} n_{\substack{0000000\\1}} n_{\substack{0001000\\0}} h_1(-1)h_3(-1)h_4(-1).$$

Clearly $c_1 \in Z(H)$. The action of $\mathrm{Ad}(c_2)$ on the root vectors concerned is as follows:

fixed : $e_{\substack{0000001\\0}}$, $e_{\substack{0000010\\0}}$;

interchanged : $e_{\substack{0011000\\1}} \leftrightarrow e_{\substack{0010000\\0}}$, $e_{\substack{1111100\\0}} \leftrightarrow e_{\substack{0111100\\1}}$, $e_{\substack{0111000\\0}} \leftrightarrow e_{\substack{1110000\\1}}$.

Thus c_2 acts as a non-trivial graph automorphism of H. Hence $A = \langle \bar{c}_1, \bar{c}_2 \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14}, {z_{18}}^1, {z_{18}}^2, z_{22} \rangle$, where $z_2 = e$, $z_{14} = e_{\substack{1243221\\2}} + e_{\substack{1343321\\2}} + e_{\substack{2344321\\2}}$, ${z_{18}}^1 = e_{\substack{2464321\\3}}$, ${z_{18}}^2 = e_{\substack{2465321\\3}}$ and $z_{22} = e_{\substack{2465432\\3}}$. Since $\mathrm{Ad}(c_1)$ multiplies ${z_{18}}^1$ by ω and ${z_{18}}^2$ by ω^2 while fixing z_{14} and z_{22}, and $\mathrm{Ad}(c_2)$ fixes both z_{14} and z_{22}, we have $\mathcal{Z} = \langle z_2, z_{14}, z_{22} \rangle$.

9.3.29 Orbit $E_8(a_5)$ (number 62 in E_8)

Here $e = e_{\substack{0011000\\1}} + e_{\substack{0000111\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} - e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000010\\0}}$, which is non-regular distinguished in the Lie algebra of H, a D_8 subsystem subgroup of G with $\Pi(H) = \{ \substack{0011000\\1}, \substack{0000111\\0}, \substack{0111000\\0}, \substack{1000000\\0}, \substack{0110000\\1}, \substack{0001100\\0}, \substack{0010000\\0}, \substack{0000010\\0} \}$; we have $A \cong \mathcal{S}_2$, and $C^\circ = 1$. We take

$$c = h_4(-1)h_7(-1).$$

Clearly $c \in Z(H)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14}, {z_{22}}^1, {z_{22}}^2 \rangle$, where $z_2 = e$, $z_{14} = 7e_{\substack{1243221\\2}} - e_{\substack{1244321\\2}} - e_{\substack{1343321\\2}} - e_{\substack{2343210\\2}}$, ${z_{22}}^1 = e_{\substack{2465431\\3}}$ and ${z_{22}}^2 = e_{\substack{2465432\\3}}$. Since $\mathrm{Ad}(c)$ negates ${z_{22}}^1$ and fixes the remaining basis vectors, we have $\mathcal{Z} = \langle z_2, z_{14}, {z_{22}}^2 \rangle$.

9.3.30 Orbit $E_8(b_4)$ (number 63 in E_8)

Here $e = e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0010000\\0}}$, which is non-regular distinguished in the Lie algebra of H, an E_7A_1 subsystem subgroup of G with $\Pi(H) = \{ \substack{0000001\\0}, \substack{0011000\\1}, \substack{0000010\\0}, \substack{0000100\\0}, \substack{0111000\\0}, \substack{1000000\\0}, \substack{0110000\\1}, \substack{0010000\\0} \}$ (with the projection of e on the Lie algebra of the E_7 factor lying in the $E_7(a_1)$ orbit — see the comments at the end of §5); we have $A \cong \mathcal{S}_2$, and $C^\circ = 1$. We take

$$c = h_4(-1).$$

Clearly $c \in Z(H)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14}, z_{22}, z_{26} \rangle$, where $z_2 = e$, $z_{14} = e_{\substack{1232221\\2}} - e_{\substack{1233321\\1}} - e_{\substack{1343211\\2}} - e_{\substack{2343210\\2}}$, $z_{22} = e_{\substack{2465321\\3}} + e_{\substack{2465421\\3}}$ and $z_{26} = e_{\substack{2465432\\3}}$. Since $\mathrm{Ad}(c)$ fixes each of the basis vectors, we have $\mathcal{Z} = \langle z_2, z_{14}, z_{22}, z_{26} \rangle$.

9.3.31 Orbit $E_8(a_4)$ (number 65 in E_8)

Here $e = e_{\substack{0011000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0001110\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000001\\0}}$, which is regular in the Lie algebra of H, a D_8 subsystem subgroup of G; we have $A \cong \mathcal{S}_2$, and $C^\circ = 1$. We take

$$c = h_4(-1)h_8(-1).$$

Clearly $c \in Z(H)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14}, z_{22}, z_{26}, z_{28} \rangle$, where $z_2 = e$, $z_{14} = 4e_{\substack{1243210\\2}} + 3e_{\substack{1233211\\2}} - e_{\substack{1232211\\1}} - e_{\substack{1233221\\1}} - e_{\substack{1232221\\2}}$, $z_{22} = e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$, $z_{26} = e_{\substack{2465421\\3}}$ and $z_{28} = e_{\substack{2465432\\3}}$. Since $\mathrm{Ad}(c)$ negates z_{28} and fixes the remaining basis vectors, we have $\mathcal{Z} = \langle z_2, z_{14}, z_{22}, z_{26} \rangle$.

9.3.32 Orbit $E_8(a_3)$ (number 66 in E_8)

Here $e = e_{\substack{0000001\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0010000\\0}}$, which is regular in the Lie algebra of H, an E_7A_1 subsystem subgroup of G; we have $A \cong \mathcal{S}_2$, and $C^\circ = 1$. We take

$$c = h_4(-1).$$

Clearly $c \in Z(H)$. Hence $A = \langle \bar{c} \rangle$. We have $(Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ} = \langle z_2, z_{14}, z_{22}, z_{26}, z_{28}, z_{34} \rangle$, where $z_2 = e$, $z_{14} = e_{\substack{0122221\\1}} - e_{\substack{1122211\\1}} - e_{\substack{1232210\\2}} - 2e_{\substack{1232111\\2}} - e_{\substack{1233210\\1}}$, $z_{22} = e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}} - e_{\substack{1244321\\2}}$, $z_{26} = e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$, $z_{28} = e_{\substack{2464321\\3}}$ and $z_{34} = e_{\substack{2465432\\3}}$. Since $\mathrm{Ad}(c)$ negates z_{28} and fixes the remaining basis vectors, we have $\mathcal{Z} = \langle z_2, z_{14}, z_{22}, z_{26}, z_{34} \rangle$.

This completes the consideration of the groups E_n.

10 Proofs of the main theorems for exceptional groups

In this section we use the information obtained in §§5–9 to provide proofs of the main results stated in §1, for groups G of exceptional type. In each case the conclusion is immediate if the nilpotent element e is 0; we therefore fix a non-zero nilpotent orbit representative $e \in \mathfrak{L}(G)$ listed in Table 2, and the associated cocharacter τ as given at the top of the page for e in §11.

We begin with Theorems 1 and 2; recall that we proved the second equality in each of Theorems 1(i) and 2 at the beginning of §3. The sixth column of the table on the page for e in §11 identifies the subalgebra $\mathcal{Z} = (Z(C_{\mathfrak{L}(G)}(e)_+))^C$, which by Theorem 3.9 is equal to $\mathfrak{L}(Z(C_G(e)))$. We first consider those orbits for which all labels of Δ are even, and thus are either 0 or 2. It is a straightforward check to see that in each of these cases we have $\dim \mathcal{Z} = n_2(\Delta)$. As each distinguished orbit in $\mathfrak{L}(G)$ has an even diagram, this establishes the remainder of Theorems 1(i) and 2. For the proof of Theorem 1(ii), we first recall as in §4.1 the result of Kostant in [16], which states that if e is regular nilpotent then the τ-weights in $C_{\mathfrak{L}(G)}(e)$ are $2d_1 - 2, \ldots, 2d_\ell - 2$, where $d_1 < \cdots < d_\ell$ are the degrees of the invariant polynomials of the Weyl group of G. It is now straightforward to consider the non-regular distinguished orbits in $\mathfrak{L}(G)$ and observe that the τ-weights in $\mathcal{Z}$ are the first $n_2(\Delta)$ of these integers; for the convenience of the reader we have in fact listed in Table 4 both the dimension of $\mathcal{Z}$ and the τ-weights, with multiplicities, occurring in $\mathcal{Z}$ for each nilpotent orbit representative.

We therefore turn to the proofs of Theorems 3 and 4. In some cases these involve information on certain nilpotent orbits in simple Lie algebras of classical type, which were treated in §4; for convenience we have listed the relevant data in Table 5.

For each group G of exceptional type, Table 6 has one row for each orbit whose labelled diagram has at least one label equal to 2. In each case we have specified the orbit name and labelled diagram Δ, and recorded the dimensions of $C_G(e)$ and $\mathcal{Z}$. Recall from §1 the definitions of the 2-free core Δ_0 of Δ and the semisimple group G_0; the next entry in the row is the type of the group G_0. Reference to [2] reveals that the labelled diagram Δ_0 corresponds to an orbit $\mathrm{Ad}(G_0)e_0$ of nilpotent elements of $\mathfrak{L}(G_0)$, whose name is listed in the next entry; this proves the existence statement of Theorem 3. The last two entries in the row are the dimensions of $C_{G_0}(e_0)$ and $\mathcal{Z}_0 = \mathfrak{L}(Z(C_{G_0}(e_0)))$. In each case it is then immediate that $\dim Z(C_G(e)) - \dim Z(C_{G_0}(e_0)) = n_2(\Delta)$ (and indeed it is also apparent that $\dim C_G(e) - \dim C_{G_0}(e_0) = n_2(\Delta)$, as proved in Proposition 3.1). This completes the proof of Theorem 3. Finally the proof of Theorem 4 is a case-by-case verification.

Table 4: τ-weights of vectors in $\mathcal{Z}$ for exceptional groups

$G = G_2$					
Orbit	dim $\mathcal{Z}$	Weights	Orbit	dim $\mathcal{Z}$	Weights
A_1	1	2	$G_2(a_1)$	1	2
$\tilde{A}_1$	1	2	G_2	2	$2, 10$

$G = F_4$								
Orbit	dim $\mathcal{Z}$	Weights	Orbit	dim $\mathcal{Z}$	Weights	Orbit	dim $\mathcal{Z}$	Weights
A_1	1	2	$A_2\tilde{A}_1$	1	2	B_3	2	$2, 10$
$\tilde{A}_1$	1	2	B_2	2	$2, 6$	C_3	2	$2, 10$
$A_1\tilde{A}_1$	1	2	$\tilde{A}_2A_1$	1	2	$F_4(a_2)$	2	$2, 10$
A_2	1	2	$C_3(a_1)$	2	$2, 6$	$F_4(a_1)$	3	$2, 10, 14$
$\tilde{A}_2$	1	2	$F_4(a_3)$	1	2	F_4	4	$2, 10, 14, 22$

$G = E_6$								
Orbit	dim $\mathcal{Z}$	Weights	Orbit	dim $\mathcal{Z}$	Weights	Orbit	dim $\mathcal{Z}$	Weights
A_1	1	2	A_3	2	$2, 6$	A_5	3	$2, 8, 10$
${A_1}^2$	1	2	${A_2}^2A_1$	1	2	$D_5(a_1)$	3	$2, 8, 10$
${A_1}^3$	1	2	A_3A_1	2	$2, 6$	$E_6(a_3)$	3	$2, 8, 10$
A_2	1	2	$D_4(a_1)$	1	2	D_5	4	$2, 8, 10, 14$
A_2A_1	2	$2, 4$	A_4	3	$2, 6, 8$	$E_6(a_1)$	5	$2, 8, 10, 14, 16$
${A_2}^2$	2	$2, 4$	D_4	2	$2, 10$	E_6	6	$2, 8, 10, 14, 16, 22$
$A_2{A_1}^2$	1	2	A_4A_1	2	$2, 8$			

$G = E_7$								
Orbit	dim $\mathcal{Z}$	Weights	Orbit	dim $\mathcal{Z}$	Weights	Orbit	dim $\mathcal{Z}$	Weights
A_1	1	2	$A_3{A_1}^2$	2	$2, 6$	$E_6(a_3)$	2	$2, 10$
${A_1}^2$	1	2	D_4	2	$2, 10$	D_5	3	$2, 10, 14$
$({A_1}^3)''$	1	2	$D_4(a_1)A_1$	2	$2, 6$	$E_7(a_5)$	2	$2, 10$
$({A_1}^3)'$	1	2	A_3A_2	2	$2, 6$	A_6	2	$2, 10$
A_2	1	2	A_4	2	$2, 6$	D_5A_1	3	$2, 10, 14$
${A_1}^4$	1	2	$A_3A_2A_1$	1	2	$D_6(a_1)$	4	$2, 10, 10, 14$
A_2A_1	1	2	$(A_5)''$	3	$2, 6, 10$	$E_7(a_4)$	3	$2, 10, 14$
$A_2{A_1}^2$	1	2	D_4A_1	2	$2, 10$	D_6	4	$2, 10, 14, 18$
A_3	2	$2, 6$	A_4A_1	1	2	$E_6(a_1)$	3	$2, 10, 14$
${A_2}^2$	1	2	$D_5(a_1)$	2	$2, 10$	E_6	4	$2, 10, 14, 22$
$A_2{A_1}^3$	1	2	A_4A_2	1	2	$E_7(a_3)$	4	$2, 10, 14, 18$
$(A_3A_1)''$	2	$2, 6$	$(A_5)'$	2	$2, 10$	$E_7(a_2)$	5	$2, 10, 14, 18, 22$
${A_2}^2A_1$	1	2	A_5A_1	2	$2, 10$	$E_7(a_1)$	6	$2, 10, 14, 18, 22, 26$
$(A_3A_1)'$	2	$2, 6$	$D_5(a_1)A_1$	2	$2, 10$	E_7	7	$2, 10, 14, 18, 22, 26, 34$
$D_4(a_1)$	1	2	$D_6(a_2)$	3	$2, 10, 10$			

Table 4: τ-weights of vectors in $\mathcal{Z}$ for exceptional groups (continued)

$G = E_8$					
Orbit	$\dim \mathcal{Z}$	Weights	Orbit	$\dim \mathcal{Z}$	Weights
A_1	1	2	$D_5(a_1)A_2$	2	$2, 10$
${A_1}^2$	1	2	$D_6(a_2)$	2	$2, 10$
${A_1}^3$	1	2	$E_6(a_3)A_1$	2	$2, 10$
A_2	1	2	$E_7(a_5)$	2	$2, 10$
${A_1}^4$	1	2	D_5A_1	3	$2, 10, 14$
A_2A_1	1	2	$E_8(a_7)$	1	2
$A_2{A_1}^2$	1	2	A_6	2	$2, 10$
A_3	2	$2, 6$	$D_6(a_1)$	3	$2, 10, 14$
$A_2{A_1}^3$	1	2	A_6A_1	1	2
${A_2}^2$	1	2	$E_7(a_4)$	3	$2, 10, 14$
${A_2}^2A_1$	1	2	$E_6(a_1)$	3	$2, 10, 14$
A_3A_1	2	$2, 6$	D_5A_2	2	$2, 14$
$D_4(a_1)$	1	2	D_6	3	$2, 14, 18$
D_4	2	$2, 10$	E_6	4	$2, 10, 14, 22$
${A_2}^2{A_1}^2$	1	2	$D_7(a_2)$	2	$2, 14$
$A_3{A_1}^2$	2	$2, 6$	A_7	2	$2, 14$
$D_4(a_1)A_1$	1	2	$E_6(a_1)A_1$	2	$2, 14$
A_3A_2	2	$2, 6$	$E_7(a_3)$	3	$2, 14, 18$
A_4	2	$2, 6$	$E_8(b_6)$	2	$2, 14$
$A_3A_2A_1$	1	2	$D_7(a_1)$	3	$2, 14, 18$
D_4A_1	2	$2, 10$	E_6A_1	3	$2, 14, 22$
$D_4(a_1)A_2$	1	2	$E_7(a_2)$	4	$2, 14, 18, 22$
A_4A_1	1	2	$E_8(a_6)$	2	$2, 14$
${A_3}^2$	1	2	D_7	3	$2, 14, 22$
$D_5(a_1)$	2	$2, 10$	$E_8(b_5)$	3	$2, 14, 22$
$A_4{A_1}^2$	1	2	$E_7(a_1)$	5	$2, 14, 18, 22, 26$
A_4A_2	1	2	$E_8(a_5)$	3	$2, 14, 22$
A_5	2	$2, 10$	$E_8(b_4)$	4	$2, 14, 22, 26$
$D_5(a_1)A_1$	2	$2, 10$	E_7	5	$2, 14, 22, 26, 34$
$A_4A_2A_1$	1	2	$E_8(a_4)$	4	$2, 14, 22, 26$
D_4A_2	2	$2, 10$	$E_8(a_3)$	5	$2, 14, 22, 26, 34$
$E_6(a_3)$	2	$2, 10$	$E_8(a_2)$	6	$2, 14, 22, 26, 34, 38$
D_5	3	$2, 10, 14$	$E_8(a_1)$	7	$2, 14, 22, 26, 34, 38, 46$
A_4A_3	1	2	E_8	8	$2, 14, 22, 26, 34, 38, 46, 58$
A_5A_1	2	$2, 10$			

Table 5: Some nilpotent orbits in classical Lie algebras

G	$\mathrm{Ad}(G)e$	Δ	$\dim C_G(e)$	$\dim Z(C_G(e))$
A_5	A_1	10001	25	1
A_5	A_2A_1	11011	13	2
B_3	B_1A_1	101	9	1
C_3	C_1	001	15	1
D_4	A_1D_2	$10\,^{1}_{1}$	12	1
D_5	A_1D_2	$101\,^{0}_{0}$	21	1
D_5	A_2A_1	$010\,^{1}_{1}$	17	2
D_6	A_1	$0100\,^{0}_{0}$	48	1
D_6	${A_1}^2D_2$	$1000\,^{1}_{1}$	30	1
D_6	A_2A_1	$0101\,^{0}_{0}$	26	1
D_6	A_3D_2	$0110\,^{1}_{1}$	18	2
D_7	A_1D_2	$10100\,^{0}_{0}$	51	1
D_7	A_3D_3	$10110\,^{1}_{1}$	21	2

Table 6: 2-free cores for exceptional groups

G	$\mathrm{Ad}(G)e$	Δ	$\dim C_G(e)$	$\dim \mathcal{Z}$	G_0	$\mathrm{Ad}(G_0)e_0$	$\dim C_{G_0}(e_0)$	$\dim \mathcal{Z}_0$
G_2	$G_2(a_1)$	02	4	1	A_1	$\emptyset$	3	0
	G_2	22	2	2	1	$\emptyset$	0	0
F_4	A_2	2000	22	1	C_3	$\emptyset$	21	0
	$\tilde{A}_2$	0002	22	1	B_3	$\emptyset$	21	0
	B_2	2001	16	2	C_3	C_1	15	1
	$F_4(a_3)$	0200	12	1	$A_1\tilde{A}_2$	$\emptyset$	11	0
	B_3	2200	10	2	$\tilde{A}_2$	$\emptyset$	8	0
	C_3	1012	10	2	B_3	B_1A_1	9	1
	$F_4(a_2)$	0202	8	2	$A_1\tilde{A}_1$	$\emptyset$	6	0
	$F_4(a_1)$	2202	6	3	$\tilde{A}_1$	$\emptyset$	3	0
	F_4	2222	4	4	1	$\emptyset$	0	0
E_6	A_2	$\substack{00000\\2}$	36	1	A_5	$\emptyset$	35	0
	${A_2}^2$	$\substack{20002\\0}$	30	2	D_4	$\emptyset$	28	0
	A_3	$\substack{10001\\2}$	26	2	A_5	A_1	25	1
	$D_4(a_1)$	$\substack{00200\\0}$	20	1	${A_2}^2A_1$	$\emptyset$	19	0
	A_4	$\substack{20002\\2}$	18	3	A_3	$\emptyset$	15	0
	D_4	$\substack{00200\\2}$	18	2	${A_2}^2$	$\emptyset$	16	0
	A_5	$\substack{21012\\1}$	14	3	D_4	A_1D_2	12	1
	$D_5(a_1)$	$\substack{11011\\2}$	14	3	A_5	A_2A_1	13	2
	$E_6(a_3)$	$\substack{20202\\0}$	12	3	${A_1}^3$	$\emptyset$	9	0
	D_5	$\substack{20202\\2}$	10	4	${A_1}^2$	$\emptyset$	6	0
	$E_6(a_1)$	$\substack{22022\\2}$	8	5	A_1	$\emptyset$	3	0
	E_6	$\substack{22222\\2}$	6	6	1	$\emptyset$	0	0

Table 6: 2-free cores for exceptional groups (continued)

G	$\mathrm{Ad}(G)e$	Δ	$\dim C_G(e)$	$\dim \mathcal{Z}$	G_0	$\mathrm{Ad}(G_0)e_0$	$\dim C_{G_0}(e_0)$	$\dim \mathcal{Z}_0$
E_7	$({A_1}^3)''$	$\substack{000002\\0}$	79	1	E_6	$\emptyset$	78	0
	A_2	$\substack{200000\\0}$	67	1	D_6	$\emptyset$	66	0
	A_3	$\substack{200010\\0}$	49	2	D_6	A_1	48	1
	${A_2}^2$	$\substack{000020\\0}$	49	1	D_5A_1	$\emptyset$	48	0
	$A_2{A_1}^3$	$\substack{000000\\2}$	49	1	A_6	$\emptyset$	48	0
	$(A_3A_1)''$	$\substack{200002\\0}$	47	2	D_5	$\emptyset$	45	0
	$D_4(a_1)$	$\substack{020000\\0}$	39	1	A_5A_1	$\emptyset$	38	0
	D_4	$\substack{220000\\0}$	37	2	A_5	$\emptyset$	35	0
	A_4	$\substack{200020\\0}$	33	2	D_4A_1	$\emptyset$	31	0
	$A_3A_2A_1$	$\substack{000200\\0}$	33	1	A_4A_2	$\emptyset$	32	0
	$(A_5)''$	$\substack{200022\\0}$	31	3	D_4	$\emptyset$	28	0
	D_4A_1	$\substack{210001\\1}$	31	2	D_6	${A_1}^2D_2$	30	1
	$D_5(a_1)$	$\substack{201010\\0}$	27	2	D_6	A_2A_1	26	1
	A_4A_2	$\substack{002000\\0}$	27	1	$A_3A_2A_1$	$\emptyset$	26	0
	$(A_5)'$	$\substack{101020\\0}$	25	2	D_5A_1	A_1D_2	24	1
	A_5A_1	$\substack{101012\\0}$	25	2	E_6	${A_2}^2A_1$	24	1
	$D_5(a_1)A_1$	$\substack{200200\\0}$	25	2	A_3A_2	$\emptyset$	23	0
	$D_6(a_2)$	$\substack{010102\\1}$	23	3	E_6	A_3A_1	22	2
	$E_6(a_3)$	$\substack{020020\\0}$	23	2	$A_3{A_1}^2$	$\emptyset$	21	0
	D_5	$\substack{220020\\0}$	21	3	A_3A_1	$\emptyset$	18	0
	$E_7(a_5)$	$\substack{002002\\0}$	21	2	${A_2}^2A_1$	$\emptyset$	19	0
	A_6	$\substack{002020\\0}$	19	2	$A_2{A_1}^3$	$\emptyset$	17	0
	D_5A_1	$\substack{210110\\1}$	19	3	D_6	A_3D_2	18	2
	$D_6(a_1)$	$\substack{210102\\1}$	19	4	D_5	A_2A_1	17	2
	$E_7(a_4)$	$\substack{202002\\0}$	17	3	$A_2{A_1}^2$	$\emptyset$	14	0
	D_6	$\substack{210122\\1}$	15	4	D_4	A_1D_2	12	1
	$E_6(a_1)$	$\substack{202020\\0}$	15	3	${A_1}^4$	$\emptyset$	12	0
	E_6	$\substack{222020\\0}$	13	4	${A_1}^3$	$\emptyset$	9	0
	$E_7(a_3)$	$\substack{202022\\0}$	13	4	${A_1}^3$	$\emptyset$	9	0
	$E_7(a_2)$	$\substack{220202\\2}$	11	5	${A_1}^2$	$\emptyset$	6	0
	$E_7(a_1)$	$\substack{220222\\2}$	9	6	A_1	$\emptyset$	3	0
	E_7	$\substack{222222\\2}$	7	7	1	$\emptyset$	0	0

Table 6: 2-free cores for exceptional groups (continued)

G	$\mathrm{Ad}(G)e$	Δ	$\dim C_G(e)$	$\dim \mathcal{Z}$	G_0	$\mathrm{Ad}(G_0)e_0$	$\dim C_{G_0}(e_0)$	$\dim \mathcal{Z}_0$
E_8	A_2	$\begin{smallmatrix}0000002\\0\end{smallmatrix}$	134	1	E_7	$\emptyset$	133	0
	A_3	$\begin{smallmatrix}1000002\\0\end{smallmatrix}$	100	2	E_7	A_1	99	1
	${A_2}^2$	$\begin{smallmatrix}2000000\\0\end{smallmatrix}$	92	1	D_7	$\emptyset$	91	0
	$D_4(a_1)$	$\begin{smallmatrix}0000020\\0\end{smallmatrix}$	82	1	E_6A_1	$\emptyset$	81	0
	D_4	$\begin{smallmatrix}0000022\\0\end{smallmatrix}$	80	2	E_6	$\emptyset$	78	0
	A_4	$\begin{smallmatrix}2000002\\0\end{smallmatrix}$	68	2	D_6	$\emptyset$	66	0
	D_4A_1	$\begin{smallmatrix}0000012\\1\end{smallmatrix}$	64	2	E_7	${A_1}^4$	63	1
	$D_4(a_1)A_2$	$\begin{smallmatrix}0000000\\2\end{smallmatrix}$	64	1	A_7	$\emptyset$	63	0
	$D_5(a_1)$	$\begin{smallmatrix}1000102\\0\end{smallmatrix}$	58	2	E_7	A_2A_1	57	1
	A_4A_2	$\begin{smallmatrix}0000200\\0\end{smallmatrix}$	54	1	D_5A_2	$\emptyset$	53	0
	A_5	$\begin{smallmatrix}2000101\\0\end{smallmatrix}$	52	2	D_7	A_1D_2	51	1
	$D_5(a_1)A_1$	$\begin{smallmatrix}0010002\\0\end{smallmatrix}$	52	2	E_7	$A_2{A_1}^2$	51	1
	D_4A_2	$\begin{smallmatrix}0000002\\2\end{smallmatrix}$	50	2	A_6	$\emptyset$	48	0
	$E_6(a_3)$	$\begin{smallmatrix}2000020\\0\end{smallmatrix}$	50	2	D_5A_1	$\emptyset$	48	0
	D_5	$\begin{smallmatrix}2000022\\0\end{smallmatrix}$	48	3	D_5	$\emptyset$	45	0
	D_5A_1	$\begin{smallmatrix}1001012\\0\end{smallmatrix}$	40	3	E_7	$A_3{A_1}^2$	39	2
	$E_8(a_7)$	$\begin{smallmatrix}0002000\\0\end{smallmatrix}$	40	1	A_4A_3	$\emptyset$	39	0
	A_6	$\begin{smallmatrix}2000200\\0\end{smallmatrix}$	38	2	D_4A_2	$\emptyset$	36	0
	$D_6(a_1)$	$\begin{smallmatrix}0100012\\1\end{smallmatrix}$	38	3	E_7	$D_4(a_1)A_1$	37	2
	$E_7(a_4)$	$\begin{smallmatrix}0010102\\0\end{smallmatrix}$	36	3	E_7	A_3A_2	35	2
	$E_6(a_1)$	$\begin{smallmatrix}2000202\\0\end{smallmatrix}$	34	3	D_4A_1	$\emptyset$	31	0
	D_5A_2	$\begin{smallmatrix}0002002\\0\end{smallmatrix}$	34	2	A_4A_2	$\emptyset$	32	0
	D_6	$\begin{smallmatrix}2100012\\1\end{smallmatrix}$	32	3	D_6	${A_1}^2D_2$	30	1
	E_6	$\begin{smallmatrix}2000222\\0\end{smallmatrix}$	32	4	D_4	$\emptyset$	28	0
	$E_6(a_1)A_1$	$\begin{smallmatrix}1010102\\0\end{smallmatrix}$	30	2	E_7	A_4A_1	29	1
	$E_7(a_3)$	$\begin{smallmatrix}2010102\\0\end{smallmatrix}$	28	3	D_6	A_2A_1	26	1
	$E_8(b_6)$	$\begin{smallmatrix}0020002\\0\end{smallmatrix}$	28	2	$A_3A_2A_1$	$\emptyset$	26	0
	$D_7(a_1)$	$\begin{smallmatrix}2002002\\0\end{smallmatrix}$	26	3	A_3A_2	$\emptyset$	23	0
	E_6A_1	$\begin{smallmatrix}1010122\\0\end{smallmatrix}$	26	3	E_6	${A_2}^2A_1$	24	1
	$E_7(a_2)$	$\begin{smallmatrix}0101022\\1\end{smallmatrix}$	24	4	E_6	A_3A_1	22	2
	$E_8(a_6)$	$\begin{smallmatrix}0020020\\0\end{smallmatrix}$	24	2	${A_2}^2{A_1}^2$	$\emptyset$	22	0
	D_7	$\begin{smallmatrix}2101101\\1\end{smallmatrix}$	22	3	D_7	A_3D_3	21	2
	$E_8(b_5)$	$\begin{smallmatrix}0020022\\0\end{smallmatrix}$	22	3	${A_2}^2A_1$	$\emptyset$	19	0
	$E_7(a_1)$	$\begin{smallmatrix}2101022\\1\end{smallmatrix}$	20	5	D_5	A_2A_1	17	2
	$E_8(a_5)$	$\begin{smallmatrix}2020020\\0\end{smallmatrix}$	20	3	$A_2{A_1}^3$	$\emptyset$	17	0
	$E_8(b_4)$	$\begin{smallmatrix}2020022\\0\end{smallmatrix}$	18	4	$A_2{A_1}^2$	$\emptyset$	14	0
	E_7	$\begin{smallmatrix}2101222\\1\end{smallmatrix}$	16	5	D_4	A_1D_2	12	1
	$E_8(a_4)$	$\begin{smallmatrix}2020202\\0\end{smallmatrix}$	16	4	${A_1}^4$	$\emptyset$	12	0
	$E_8(a_3)$	$\begin{smallmatrix}2020222\\0\end{smallmatrix}$	14	5	${A_1}^3$	$\emptyset$	9	0
	$E_8(a_2)$	$\begin{smallmatrix}2202022\\2\end{smallmatrix}$	12	6	${A_1}^2$	$\emptyset$	6	0
	$E_8(a_1)$	$\begin{smallmatrix}2202222\\2\end{smallmatrix}$	10	7	A_1	$\emptyset$	3	0
	E_8	$\begin{smallmatrix}2222222\\2\end{smallmatrix}$	8	8	1	$\emptyset$	0	0

11 Detailed results

This final section contains the detailed information for exceptional groups obtained as described in the preceding sections. We begin by explaining the manner in which this is presented.

We take the non-zero nilpotent orbits in the order of Table 2. Each such orbit has its own 'page' (which may or may not coincide with a physical page); horizontal lines are used to separate pages where necessary. The page may be regarded as falling into three sections.

The first of the three sections concerns the orbit itself. It begins with the algebraic group G, and the number and name of the orbit; it concludes with the chosen orbit representative e. In between we specify the Levi subgroup L of G with e distinguished in $\mathfrak{L}(L)$, the cocharacter τ associated to e with image in $[L, L]$, and the labelled diagram Δ corresponding to e. We represent both L and τ diagrammatically: for the former, as explained in §6, we provide a Dynkin diagram of G in which the nodes corresponding to the simple roots of L are coloured black; for the latter, as explained in §3, we give the τ-weights on simple roots, arranged to occupy the positions of the corresponding nodes in the Dynkin diagram (so that Δ is obtained by similarly representing the W-conjugate τ^+ of τ; indeed if $L = G$ then $\tau^+ = \tau$ and so the representation of τ is simply Δ).

The second section concerns the reductive part C of the centralizer $C_G(e)$; recall that $C = C_G(e) \cap C_G(\mathrm{im}(\tau))$ and that $Z(L)^\circ$ is a maximal torus of C. We first list the isomorphism types of both the connected component C° and the component group C/C°; the former is given as in §7.2 as a product of simple factors $C_1, \dots, C_r$, possibly with a central torus T_j of dimension $j \in \{1, 2\}$, while generators for the latter are named as in §9. We then specify both groups completely. For C°, we let $\{\beta_1, \beta_2, \dots\}$ be a set of simple roots of $C_1 \dots C_r$, numbered so that those in each simple factor are taken together and occur in the order of [4]. We give root subgroups corresponding to these simple roots and their negatives, using semi-colons to separate each simple factor from the others: for each i, if the $Z(L)^\circ$-root subgroup X_{β_i} corresponding to β_i is in fact a T-root subgroup U_α, we merely write '$\beta_i = \alpha$'; if however X_{β_i} is embedded in a product of two or more T-root subgroups, we give explicit expressions for the root elements $x_{\beta_i}(t)$ and $x_{-\beta_i}(t)$. If there is a 1- or 2-dimensional central torus, we give its elements as products of terms $h_i(\mu^{m_i})$ or $h_i(\mu^{m_i}\nu^{n_i})$, where we write h_i for h_{α_i}. For C/C° we give coset representatives for each of the generators as described in §9.

The third and final section concerns the Lie algebra of the unipotent radical R of $C_G(e)$. Recall that $\mathfrak{L}(R) = C_{\mathfrak{L}(G)}(e)_+$, and we have a decomposition

$$C_{\mathfrak{L}(G)}(e)_+ = \bigoplus_{m>0} C_{\mathfrak{L}(G)}(e)_m$$

into τ-weight spaces. In §7.1 we obtained the upper central series

$$0 \subset Z_1(C_{\mathfrak{L}(G)}(e)_+) \subset Z_2(C_{\mathfrak{L}(G)}(e)_+) \subset \cdots;$$

for each n we wrote

$$Z_n(C_{\mathfrak{L}(G)}(e)_+) = \bigoplus_{m>0} Z_{n,m},$$

where we set $Z_{n,m} = (Z_n(C_{\mathfrak{L}(G)}(e)_+))_m$. In §8.2 we then considered the action of C° on each $Z_{n,m}$; in each case we obtained a decomposition into a direct sum of indecomposable tilting modules for $D = [C^\circ, C^\circ]$, the vast majority of which were irreducible. We provide a table with six columns, usually having one row for each summand; the exceptions are the few cases treated separately in §8.2 involving non-restricted weights when $\mathrm{char}(k) = p \in \{5, 7\}$, where a footnote explains how the entries of two rows must be combined and modified if there is a reducible tilting module.

The first column of the table contains the least value of n in which the summand lies. Summands are listed in decreasing order of n, and different values of n are separated by horizontal lines; thus the table is divided into sections, of which the nth (reading upwards) lists the summands lying in $Z_n(C_{\mathfrak{L}(G)}(e)_+)$ but not in $Z_{n-1}(C_{\mathfrak{L}(G)}(e)_+)$. The second column contains the τ-weight m of the summand. The third column contains the high weight λ of the summand, regarded as a module for D; we write λ_i for the fundamental dominant weight corresponding to the simple root β_i. If in fact $D = 1$, we write a dash in place of λ. The fourth column contains the high weight vector v of the summand. The final two columns are of relevance only to the bottom section of the table, which lists the summands lying in $Z_1(C_{\mathfrak{L}(G)}(e)_+) = Z(C_{\mathfrak{L}(G)}(e)_+)$. The fifth column indicates the summands lying in $\mathcal{Z}^\natural = (Z(C_{\mathfrak{L}(G)}(e)_+))^{C^\circ}$, found by taking those contained in $Z(C_{\mathfrak{L}(G)}(e)_+)$ which lie in $\mathfrak{L}(L)$ and are trivial D-submodules; as explained in §9, for convenience of reference each such high weight vector v is renamed z_m, with the subscript being the τ-weight, and vectors with the same value of m being distinguished by superscripts. The sixth column indicates the summands lying in $\mathcal{Z} = (Z(C_{\mathfrak{L}(G)}(e)_+))^C$, found by taking the vectors in $\mathcal{Z}^\natural$ fixed by the representatives of the generators of the component group C/C°; by Theorem 3.9 we have $\mathfrak{L}(Z(C_G(e))) = \mathcal{Z}$.

G_2, orbit 1: A_1

L : τ : -1 2 $\Delta =$ 0 1

$e = e_{01}$

$C^\circ = \tilde{A}_1$ $C/C^\circ = 1$

$\beta_1 = 21$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	1	$3\lambda_1$	e_{32}		
1	2	0	e	z_2	z_2

G_2, orbit 2: $\tilde{A}_1$

L : τ : $2\ -3$ $\Delta = 1\ 0$

$e = e_{10}$

$C^\circ = A_1$ $C/C^\circ = 1$

$\beta_1 = 32$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	0	e	z_2	z_2
	3	λ_1	e_{31}		

G_2, orbit 3: $G_2(a_1)$

L : τ : $0\ 2$ $\Delta = 0\ 2$

$e = e_{01} + e_{31}$

$C^\circ = 1$ $C/C^\circ = \langle c_1C^\circ, c_2C^\circ \rangle \cong S_3$

$c_1 = h_1(\omega)$,
$c_2 = n_{10}h_2(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	2	–	e_{11}		
	2	–	e_{21}		
1	2	–	e	z_2	z_2
	4	–	e_{32}	z_4	

G_2, orbit 4: G_2

L : τ : $2\ 2$ $\Delta = 2\ 2$

$e = e_{10} + e_{01}$

$C^\circ = 1$ $C/C^\circ = 1$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	–	e	z_2	z_2
	10	–	e_{32}	z_{10}	z_{10}

F_4, orbit 1: A_1

L : τ : 2 −1 0 0 Δ = 1 0 0 0

$e = e_{1000}$

$C^\circ = C_3$ $C/C^\circ = 1$

$\beta_1 = 0010$, $\beta_2 = 0001$, $\beta_3 = 1220$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	1	λ_3	e_{2342}		
1	2	0	e	z_2	z_2

F_4, orbit 2: $\tilde{A}_1$

L : τ : 0 0 −1 2 Δ = 0 0 0 1

$e = e_{0001}$

$C^\circ = A_3$ $C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = 1000$, $\beta_2 = 0100$, $\beta_3 = 1242$

$c = n_{0121} h_3(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	1	λ_1	e_{1111}		
	1	λ_3	e_{1232}		
1	2	λ_2	e_{1222}		
	2	0	e	z_2	z_2

F_4, orbit 3: $A_1\tilde{A}_1$

L : τ : 2 −1 −1 2 Δ = 0 1 0 0

$e = e_{1000} + e_{0001}$

$C^\circ = {A_1}^2$ $C/C^\circ = 1$

$\beta_1 = 1242$; $x_{\beta_2}(t) = x_{1110}(t)x_{0111}(t)x_{1221}(\frac{1}{2}t^2)$,
$x_{-\beta_2}(t) = x_{-1110}(2t)x_{-0111}(2t)x_{-1221}(-2t^2)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	1	$\lambda_1 + 4\lambda_2$	e_{2342}		
2	2	$4\lambda_2$	e_{1222}		
1	2	0	e	z_2	z_2
	3	λ_1	e_{1122}		

F_4, orbit 4: A_2

L : τ : 2 2 −2 0 Δ = 2 0 0 0

$e = e_{1000} + e_{0100}$

$C^\circ = \tilde{A}_2 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = 0001$, $\beta_2 = 1231$

$c = n_{0110} n_{1120} h_1(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	2	$2\lambda_1$	e_{1222}		
	2	$2\lambda_2$	e_{2342}		
1	2	0	e	z_2	z_2
	4	0	e_{1100}	z_4	

F_4, orbit 5: $\tilde{A}_2$

L : τ : 0 −4 2 2 Δ = 0 0 0 2

$e = e_{0010} + e_{0001}$

$C^\circ = G_2 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm 0111}(t) x_{\pm 0120}(-t)$, $\beta_2 = 1000$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	0	e	z_2	z_2
	4	λ_1	e_{1242}		

F_4, orbit 6: $A_2\tilde{A}_1$

L : τ : 2 2 −3 2 Δ = 0 0 1 0

$e = e_{1000} + e_{0100} + e_{0001}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\beta_1} = x_{0122}(2t) x_{1220}(t) x_{1342}(-t^2) x_{1121}(-t)$,
$x_{-\beta_1} = x_{-0122}(t) x_{-1220}(2t) x_{-1342}(t^2) x_{-1121}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	$3\lambda_1$	e_{1232}		
3	2	$4\lambda_1$	e_{2342}		
2	3	λ_1	e_{1111}		
1	2	0	e	z_2	z_2
	4	$2\lambda_1$	e_{1222}		

F_4, orbit 7: B_2

L : τ : $-4\ 2\ \ 2\ -3$ Δ = $2\ \ 0\ \ 0\ \ 1$

$e = e_{0100} + e_{0010}$

$C^\circ = {A_1}^2 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = 0122$; $\beta_2 = 2342$

$c = n_{1110} h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	3	λ_2	e_{1231}		
	3	λ_1	e_{0121}		
1	2	0	e	z_2	z_2
	4	$\lambda_1 + \lambda_2$	e_{1342}		
	6	0	e_{0120}	z_6	z_6

F_4, orbit 8: $\tilde{A}_2 A_1$

L : τ : $2\ -5\ 2\ \ 2$ Δ = $0\ \ 1\ \ 0\ \ 1$

$e = e_{0010} + e_{0001} + e_{1000}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm 1222}(t) x_{\pm 1231}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	1	$3\lambda_1$	e_{2342}		
	3	λ_1	$e_{1121} - 2e_{0122}$		
2	2	0	e_{1000}		
	4	$2\lambda_1$	e_{1242}		
1	2	0	e	z_2	z_2
	5	λ_1	e_{1122}		

F_4, orbit 9: $C_3(a_1)$

L : τ : $-5\ 2\ 0\ 2$ $\Delta = 1\ 0\ 1\ 0$

$e = e_{0001} + e_{0120} + e_{0100}$

$C^\circ = A_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = 2342$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	2	0	$e_{0110} + e_{0011}$		
	2	0	e_{0100}		
2	3	λ_1	$e_{1242} - e_{1222}$		
	3	λ_1	e_{1232}		
	4	0	e_{0111}		
1	2	0	e	z_2	z_2
	5	λ_1	e_{1342}		
	6	0	e_{0122}	z_6	z_6

F_4, orbit 10: $F_4(a_3)$

L : τ : $0\ 2\ 0\ 0$ $\Delta = 0\ 2\ 0\ 0$

$e = e_{0100} + e_{1120} + e_{1111} + e_{0121}$

$C^\circ = 1 \qquad C/C^\circ = \langle c_1C^\circ, c_2C^\circ, c_3C^\circ \rangle \cong S_4$

$c_1 = h_1(\omega)h_3(\omega)$,
$c_2 = n_{1000}n_{0010}h_2(-1)h_3(-1)$,
$c_3 = (n_{0011}h_3(-\frac{2}{3})h_4(\frac{2}{3}))^u$,
$u = x_{0011}(-\frac{1}{2})x_{0001}(1)x_{0010}(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	2	–	$e_{0110} + e_{1121} - 2e_{0122}$		
	2	–	$e_{1110} + e_{0111} - 2e_{1122}$		
	2	–	$e_{1100} + e_{0110}$		
	2	–	$e_{0120} + e_{1110}$		
	2	–	$e_{0100} + e_{1120}$		
2	4	–	$e_{1221} + 2e_{1242}$		
	4	–	$e_{1231} - 2e_{1222}$		
	4	–	e_{1220}		
	4	–	e_{1232}		
1	2	–	e	z_2	z_2
	6	–	e_{1342}	${z_6}^1$	
	6	–	e_{2342}	${z_6}^2$	

F_4, orbit 11: B_3

L : τ : 2 2 2 −6 Δ = 2 2 0 0

$e = e_{1000} + e_{0100} + e_{0010}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\beta_1} = x_{1111}(t)x_{0121}(-t)x_{1232}(-\frac{1}{2}t^2)$, $x_{-\beta_1} = x_{-1111}(2t)x_{-0121}(-2t)x_{-1232}(2t^2)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	0	e	z_2	z_2
	6	$4\lambda_1$	e_{2342}		
	10	0	e_{1220}	z_{10}	z_{10}

F_4, orbit 12: C_3

L : τ : −9 2 2 2 Δ = 1 0 1 2

$e = e_{0001} + e_{0010} + e_{0100}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$\beta_1 = 2342$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	3	λ_1	$e_{1231} - e_{1222}$		
2	6	0	$e_{0120} - e_{0111}$		
1	2	0	e	z_2	z_2
	9	λ_1	e_{1342}		
	10	0	e_{0122}	z_{10}	z_{10}

F_4, orbit 13: $F_4(a_2)$

L : τ : 0 2 0 2 Δ = 0 2 0 2

$e = e_{1110} + e_{0001} + e_{0120} + e_{0100}$

$C^\circ = 1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$c = h_2(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	–	$e_{0110} + e_{0011} + e_{1120} - 3e_{1100}$		
	2	–	e_{0100}		
4	4	–	$e_{0111} + e_{1220}$		
3	6	–	$e_{0122} - e_{1231}$		
2	8	–	$e_{1222} - e_{1242}$		
1	2	–	e	z_2	z_2
	10	–	e_{1342}	${z_{10}}^1$	
	10	–	e_{2342}	${z_{10}}^2$	${z_{10}}^2$

F_4, orbit 14: $F_4(a_1)$

L : τ : 2 2 0 2 Δ = 2 2 0 2

$e = e_{0100} + e_{1000} + e_{0120} + e_{0001}$

$C^\circ = 1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	4	–	$e_{1110} + e_{0111}$		
	6	–	$e_{1220} - e_{1121} + 2e_{0122}$		
2	10	–	e_{1232}		
1	2	–	e	z_2	z_2
	10	–	$e_{1222} - e_{1242}$	z_{10}	z_{10}
	14	–	e_{2342}	z_{14}	z_{14}

F_4, orbit 15: F_4

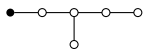
L : τ : 2 2 2 2 Δ = 2 2 2 2

$e = e_{1000} + e_{0100} + e_{0010} + e_{0001}$

$C^\circ = 1 \qquad C/C^\circ = 1$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	–	e	z_2	z_2
	10	–	$e_{1220} - e_{1121} + 2e_{0122}$	z_{10}	z_{10}
	14	–	$e_{1231} - e_{1222}$	z_{14}	z_{14}
	22	–	e_{2342}	z_{22}	z_{22}

E_6, orbit 1: A_1

L :  $\qquad \tau \ : \ \begin{smallmatrix} 2 & -1 & 0 & 0 & 0 \\ & & 0 & & \end{smallmatrix} \qquad \Delta \ = \ \begin{smallmatrix} 0 & 0 & 0 & 0 & 0 \\ & & 1 & & \end{smallmatrix}$

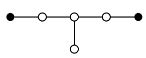
$e = e_{\substack{10000\\0}}$

$C^\circ = A_5 \qquad C/C^\circ = 1$

$\beta_1 = {}^{00000}_{\ \ 1}$, $\beta_2 = {}^{00100}_{\ \ 0}$, $\beta_3 = {}^{00010}_{\ \ 0}$, $\beta_4 = {}^{00001}_{\ \ 0}$, $\beta_5 = {}^{12210}_{\ \ 1}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	1	λ_3	$e_{\substack{11221\\1}}$		
1	2	0	e	z_2	z_2

E_6, orbit 2: ${A_1}^2$

L :  $\qquad \tau \ : \ \begin{smallmatrix} 2 & -1 & 0 & -1 & 2 \\ & & 0 & & \end{smallmatrix} \qquad \Delta \ = \ \begin{smallmatrix} 1 & 0 & 0 & 0 & 1 \\ & & 0 & & \end{smallmatrix}$

$e = e_{\substack{10000\\0}} + e_{\substack{00001\\0}}$

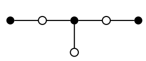
$C^\circ = B_3T_1 \qquad C/C^\circ = 1$

$\beta_1 = {}^{00100}_{\ \ 0}$, $\beta_2 = {}^{00000}_{\ \ 1}$, $x_{\pm\beta_3}(t) = x_{\pm \substack{11110\\0}}(t)x_{\pm \substack{01111\\0}}(t)$

$T_1 = \{h_1(\mu)h_3(\mu^2)h_5(\mu^{-2})h_6(\mu^{-1}) : \mu \in k^*\}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	1	λ_3	$e_{\substack{12211\\1}}$		
	1	λ_3	$e_{\substack{11221\\1}}$		
1	2	λ_1	$e_{\substack{11211\\1}}$		
	2	0	e	z_2	z_2

E_6, orbit 3: ${A_1}^3$

L :  $\qquad \tau \ : \ \begin{smallmatrix} 2 & -2 & 2 & -2 & 2 \\ & & -1 & & \end{smallmatrix} \qquad \Delta \ = \ \begin{smallmatrix} 0 & 0 & 1 & 0 & 0 \\ & & 0 & & \end{smallmatrix}$

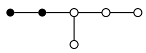
$e = e_{\substack{10000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00001\\0}}$

$C^\circ = A_2A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{11000\\0}}(t)x_{\pm \substack{01100\\0}}(t)$, $x_{\pm\beta_2}(t) = x_{\pm \substack{00110\\0}}(t)x_{\pm \substack{00011\\0}}(t)$; $\beta_3 = {}^{12321}_{\ \ 2}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	1	$\lambda_1 + \lambda_2 + \lambda_3$	$e_{\substack{12321\\1}}$		
2	2	$\lambda_1 + \lambda_2$	$e_{\substack{11111\\0}}$		
1	2	0	e	z_2	z_2
	3	λ_3	$e_{\substack{11211\\1}}$		

E_6, orbit 4: A_2

L :  τ : $\begin{smallmatrix}2 & 2 & -2 & 0 & 0\\ & & 0 & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}0 & 0 & 0 & 0 & 0\\ & & 2 & & \end{smallmatrix}$

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}}$

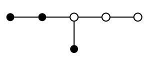
$C^\circ = {A_2}^2 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = {}^{\substack{00010\\0}}, \beta_2 = {}^{\substack{00001\\0}}; \beta_3 = {}^{\substack{00000\\1}}, \beta_4 = {}^{\substack{12321\\1}}$

$c = n_{\substack{01110\\0}} n_{\substack{01100\\1}} n_{\substack{11210\\1}} h_1(-1)h_2(-1)h_5(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	2	$\lambda_2+\lambda_3$	$e_{\substack{11111\\1}}$		
	2	$\lambda_1+\lambda_4$	$e_{\substack{12221\\1}}$		
1	2	0	e	z_2	z_2
	4	0	$e_{\substack{11000\\0}}$	z_4	

E_6, orbit 5: A_2A_1

L :  τ : $\begin{smallmatrix}2 & 2 & -3 & 0 & 0\\ & & 2 & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}1 & 0 & 0 & 0 & 1\\ & & 1 & & \end{smallmatrix}$

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00000\\1}}$

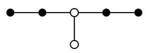
$C^\circ = A_2T_1 \qquad C/C^\circ = 1$

$\beta_1 = {}^{\substack{00010\\0}}, \beta_2 = {}^{\substack{00001\\0}}$

$T_1 = \{h_1(\mu^2)h_2(\mu^3)h_3(\mu^4)h_4(\mu^6)h_5(\mu^4)h_6(\mu^2) : \mu \in k^*\}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	λ_1	$f_{\substack{01100\\0}} - f_{\substack{00100\\1}}$		
	1	λ_2	$e_{\substack{11111\\0}} - e_{\substack{01111\\1}}$		
	1	0	$f_{\substack{12321\\1}}$		
	1	0	$e_{\substack{12321\\2}}$		
3	2	λ_2	$f_{\substack{01210\\1}}$		
	2	λ_1	$e_{\substack{12221\\1}}$		
	2	0	$e_{\substack{00000\\1}}$		
2	3	λ_1	$f_{\substack{00100\\0}}$		
	3	λ_2	$e_{\substack{11111\\1}}$		
1	2	0	e	z_2	z_2
	4	0	$e_{\substack{11000\\0}}$	z_4	z_4

E_6, orbit 6: ${A_2}^2$

L :  τ : $\begin{smallmatrix}2&2&-4&2&2\\&&0&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}2&0&0&0&2\\&&0&&\end{smallmatrix}$

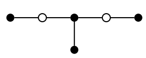
$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$

$C^\circ = G_2 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{11100\\0}}(t) x_{\pm \substack{01110\\0}}(t) x_{\pm \substack{00111\\0}}(t)$, $\beta_2 = \substack{00000\\1}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	2	λ_1	$e_{\substack{12211\\1}} - e_{\substack{11221\\1}}$		
1	2	0	e	z_2	z_2
	4	λ_1	$e_{\substack{12221\\1}}$		
	4	0	$e_{\substack{11000\\0}} + e_{\substack{00011\\0}}$	z_4	z_4

E_6, orbit 7: $A_2{A_1}^2$

L :  τ : $\begin{smallmatrix}2&-3&2&-3&2\\&&2&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}$

$e = e_{\substack{00000\\1}} + e_{\substack{00100\\0}} + e_{\substack{10000\\0}} + e_{\substack{00001\\0}}$

$C^\circ = A_1T_1 \qquad C/C^\circ = 1$

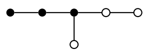
$x_{\beta_1}(t) = x_{\substack{11111\\0}}(2t) x_{\substack{01210\\1}}(-t) x_{\substack{12321\\1}}(-t^2) x_{\substack{11110\\1}}(t) x_{\substack{01111\\1}}(-t)$,

$x_{-\beta_1}(t) = x_{-\substack{11111\\0}}(t) x_{-\substack{01210\\1}}(-2t) x_{-\substack{12321\\1}}(t^2) x_{-\substack{11110\\1}}(t) x_{-\substack{01111\\1}}(-t)$

$T_1 = \{h_1(\mu)h_3(\mu^2)h_5(\mu^{-2})h_6(\mu^{-1}) : \mu \in k^*\}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	$3\lambda_1$	$e_{\substack{12211\\1}}$		
	1	$3\lambda_1$	$e_{\substack{11221\\1}}$		
3	2	$4\lambda_1$	$e_{\substack{12321\\2}}$		
	2	$2\lambda_1$	$e_{\substack{01211\\1}} + e_{\substack{11210\\1}}$		
2	3	λ_1	$e_{\substack{11100\\1}}$		
	3	λ_1	$e_{\substack{00111\\1}}$		
1	2	0	e	z_2	z_2
	4	$2\lambda_1$	$e_{\substack{11211\\1}}$		

E_6, orbit 8: A_3

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & -3 & 0 \\ & & -3 & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 0 & 0 & 1 \\ & & 2 & & \end{smallmatrix}$

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}}$

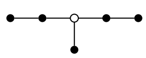
$C^\circ = B_2T_1 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{00001\\0}}$, $x_{\pm\beta_2}(t) = x_{\pm\substack{11110\\1}}(t)x_{\pm\substack{01210\\1}}(-t)$

$T_1 = \{h_1(\mu)h_3(\mu^2)h_4(\mu^3)h_5(\mu^4)h_6(\mu^2) : \mu \in k^*\}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	3	λ_2	$e_{\substack{11100\\1}}$		
	3	λ_2	$e_{\substack{12321\\1}}$		
1	2	0	e	z_2	z_2
	4	λ_1	$e_{\substack{12211\\1}}$		
	6	0	$e_{\substack{11100\\0}}$	z_6	z_6

E_6, orbit 9: $A_2{}^2A_1$

L :  τ : $\begin{smallmatrix} 2 & 2 & -5 & 2 & 2 \\ & & 2 & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 1 & 0 & 1 \\ & & 0 & & \end{smallmatrix}$

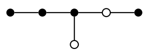
$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}} + e_{\substack{00000\\1}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm\substack{12210\\1}}(t)x_{\pm\substack{11211\\1}}(t)x_{\pm\substack{01221\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	$3\lambda_1$	$e_{\substack{12321\\2}}$		
	1	λ_1	$e_{\substack{11100\\1}} + e_{\substack{11110\\0}} + e_{\substack{01111\\0}} - e_{\substack{00111\\1}}$		
4	2	$2\lambda_1$	$e_{\substack{12211\\1}} - e_{\substack{11221\\1}}$		
	2	0	$e_{\substack{00000\\1}}$		
3	3	λ_1	$e_{\substack{11110\\1}} + e_{\substack{01111\\1}}$		
	3	λ_1	$e_{\substack{11110\\1}} + e_{\substack{11111\\0}}$		
2	4	$2\lambda_1$	$e_{\substack{12221\\1}}$		
	4	0	$e_{\substack{11000\\0}} + e_{\substack{00011\\0}}$		
1	2	0	e	z_2	z_2
	5	λ_1	$e_{\substack{11111\\1}}$		

E_6, orbit 10: A_3A_1

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & -4 & 2 \\ & & -3 & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 0 & 1 & 0 & 1 & 0 \\ & & 1 & & \end{smallmatrix}$

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00001\\0}}$

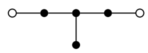
$C^\circ = A_1T_1 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{12321\\2}}$

$T_1 = \{h_1(\mu)h_3(\mu^2)h_4(\mu^3)h_5(\mu^4)h_6(\mu^2) : \mu \in k^*\}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	λ_1	$e_{\substack{11111\\1}} - e_{\substack{01211\\1}}$		
	2	0	$f_{\substack{00110\\0}} + f_{\substack{00011\\0}}$		
	2	0	$e_{\substack{01111\\0}} + e_{\substack{11110\\0}}$		
3	2	0	$e_{\substack{00001\\0}}$		
	3	λ_1	$e_{\substack{11100\\1}}$		
	3	λ_1	$e_{\substack{12210\\1}} + e_{\substack{11211\\1}}$		
	3	λ_1	$e_{\substack{12321\\1}}$		
2	4	0	$f_{\substack{00010\\0}}$		
	4	0	$e_{\substack{11111\\0}}$		
	4	0	$e_{\substack{11000\\0}} + e_{\substack{01100\\0}}$		
1	2	0	e	z_2	z_2
	5	λ_1	$e_{\substack{12211\\1}}$		
	6	0	$e_{\substack{11100\\0}}$	z_6	z_6

E_6, orbit 11: $D_4(a_1)$

L :  τ : $\begin{smallmatrix}-4 & 2 & 0 & 2 & -4\\ & & 2 & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}0 & 0 & 2 & 0 & 0\\ & & 0 & & \end{smallmatrix}$

$e = e_{\substack{01000\\0}} + e_{\substack{00100\\1}} + e_{\substack{00110\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$

$C^\circ = T_2 \qquad C/C^\circ = \langle c_1 C^\circ, c_2 C^\circ \rangle \cong S_3$

$T_2 = \{h_1(\mu^2)h_2(\mu\nu)h_3(\mu^2\nu)h_4(\mu^2\nu^2)h_5(\mu\nu^2)h_6(\nu^2) : \mu, \nu \in k^*\}$

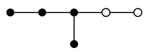
$c_1 = n_{\substack{11100\\1}} n_{\substack{11110\\0}} h_2(-1)$,

$c_2 = (n_{\substack{12211\\1}} n_{\substack{11221\\1}} h_1(-1)h_2(-1)h_6(-1))^g$,

$g = x_{\substack{00100\\0}}(\frac{1}{3}) n_{\substack{00100\\0}} h_1(4)h_2(-4)h_3(16)h_4(-48)h_5(16)h_6(-8) x_{\substack{00100\\0}}(-\frac{1}{3})$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	2	–	$f_{\substack{11111\\1}} + f_{\substack{11211\\1}}$		
	2	–	$e_{\substack{12221\\1}} - e_{\substack{12321\\1}}$		
	2	–	$f_{\substack{00011\\0}} - f_{\substack{00111\\0}}$		
	2	–	$e_{\substack{01111\\1}} + e_{\substack{01211\\1}}$		
	2	–	$f_{\substack{11100\\0}}$		
	2	–	$e_{\substack{11110\\1}}$		
	2	–	$2e_{\substack{00000\\1}} - e_{\substack{00100\\1}} + e_{\substack{01100\\0}} + e_{\substack{00110\\0}}$		
	2	–	$e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$		
2	4	–	$f_{\substack{11111\\0}}$		
	4	–	$e_{\substack{12321\\2}}$		
	4	–	$f_{\substack{00001\\0}}$		
	4	–	$e_{\substack{01221\\1}}$		
	4	–	$f_{\substack{10000\\0}}$		
	4	–	$e_{\substack{12210\\1}}$		
	4	–	$e_{\substack{01100\\1}} + 2e_{\substack{00110\\1}} - e_{\substack{01110\\0}}$		
1	2	–	e	z_2	z_2
	6	–	$e_{\substack{01110\\1}}$	${z_6}^1$	
	6	–	$e_{\substack{01210\\1}}$	${z_6}^2$	

E_6, orbit 12: A_4

L :  $\qquad \tau : \begin{smallmatrix}2&2&2&-6&0\\&&2&&\end{smallmatrix} \qquad \Delta = \begin{smallmatrix}2&0&0&0&2\\&&2&&\end{smallmatrix}$

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}}$

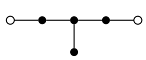
$C^\circ = A_1T_1 \qquad C/C^\circ = 1$

$\beta_1 = \substack{00001\\0}$

$T_1 = \{h_1(\mu^4)h_2(\mu^6)h_3(\mu^8)h_4(\mu^{12})h_5(\mu^{10})h_6(\mu^5) : \mu \in k^*\}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	2	λ_1	$f_{\substack{01110\\0}} - f_{\substack{00110\\1}}$		
	2	λ_1	$e_{\substack{11111\\1}} - e_{\substack{01211\\1}}$		
3	4	0	$f_{\substack{01221\\1}}$		
	4	0	$e_{\substack{12321\\2}}$		
	4	0	$e_{\substack{11000\\0}} + e_{\substack{01100\\0}} - e_{\substack{00100\\1}}$		
2	6	λ_1	$f_{\substack{00010\\0}}$		
	6	λ_1	$e_{\substack{12211\\1}}$		
1	2	0	e	z_2	z_2
	6	0	$e_{\substack{11100\\0}} - e_{\substack{01100\\1}}$	z_6	z_6
	8	0	$e_{\substack{11100\\1}}$	z_8	z_8

E_6, orbit 13: D_4

L :  $\qquad \tau : \begin{smallmatrix}-6&2&2&2&-6\\&&2&&\end{smallmatrix} \qquad \Delta = \begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}$

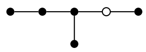
$e = e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$

$C^\circ = A_2 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm\substack{11100\\1}}(t)x_{\pm\substack{11110\\0}}(t)$, $x_{\pm\beta_2}(t) = x_{\pm\substack{00111\\1}}(t)x_{\pm\substack{01111\\0}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	0	e	z_2	z_2
	6	$\lambda_1 + \lambda_2$	$e_{\substack{12321\\2}}$		
	10	0	$e_{\substack{01210\\1}}$	z_{10}	z_{10}

E_6, orbit 14: A_4A_1

L :  $\qquad \tau$: $\begin{smallmatrix} 2 & 2 & 2 & -7 & 2 \\ & & 2 & & \end{smallmatrix}$ $\qquad \Delta = \begin{smallmatrix} 1 & 1 & 0 & 1 & 1 \\ & & 1 & & \end{smallmatrix}$

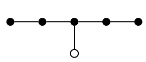
$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}} + e_{\substack{00001\\0}}$

$C^\circ = T_1 \qquad C/C^\circ = 1$

$T_1 = \{h_1(\mu^4)h_2(\mu^6)h_3(\mu^8)h_4(\mu^{12})h_5(\mu^{10})h_6(\mu^5) : \mu \in k^*\}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
8	1	–	$f_{\substack{01110\\1}} + f_{\substack{00111\\1}} - f_{\substack{01111\\0}} - 2f_{\substack{11110\\0}}$		
	1	–	$e_{\substack{01111\\1}} + e_{\substack{01210\\1}} - e_{\substack{11110\\1}} - 2e_{\substack{11111\\0}}$		
7	2	–	$e_{\substack{00001\\0}}$		
6	3	–	$f_{\substack{01110\\0}} - f_{\substack{00110\\1}}$		
	3	–	$e_{\substack{11111\\1}} - e_{\substack{01211\\1}}$		
5	4	–	$f_{\substack{01221\\1}}$		
	4	–	$e_{\substack{12321\\2}}$		
	4	–	$e_{\substack{11000\\0}} + e_{\substack{01100\\0}} - e_{\substack{00100\\1}}$		
4	5	–	$f_{\substack{00110\\0}} + f_{\substack{00011\\0}}$		
	5	–	$e_{\substack{12210\\1}} + e_{\substack{11211\\1}}$		
3	6	–	$e_{\substack{11100\\0}} - e_{\substack{01100\\1}}$		
2	7	–	$f_{\substack{00010\\0}}$		
	7	–	$e_{\substack{12211\\1}}$		
1	2	–	e	z_2	z_2
	8	–	$e_{\substack{11100\\1}}$	z_8	z_8

E_6, orbit 15: A_5

L :  $\qquad \tau$: $\begin{smallmatrix} 2 & 2 & 2 & 2 & 2 \\ & & -9 & & \end{smallmatrix}$ $\qquad \Delta = \begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}$

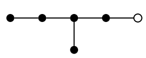
$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{12321\\2}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	3	λ_1	$e_{\substack{12210\\1}} + e_{\substack{11211\\1}} - e_{\substack{01221\\1}}$		
2	4	0	$e_{\substack{11000\\0}} + e_{\substack{01100\\0}} + e_{\substack{00110\\0}} + e_{\substack{00011\\0}}$		
	5	λ_1	$e_{\substack{12211\\1}} - e_{\substack{11221\\1}}$		
	6	0	$e_{\substack{11100\\0}} + e_{\substack{01110\\0}} + e_{\substack{00111\\0}}$		
1	2	0	e	z_2	z_2
	8	0	$e_{\substack{11110\\0}} + e_{\substack{01111\\0}}$	z_8	z_8
	9	λ_1	$e_{\substack{12321\\1}}$		
	10	0	$e_{\substack{11111\\0}}$	z_{10}	z_{10}

E_6, orbit 16: $D_5(a_1)$

L :  τ : $\begin{smallmatrix} 2 & 2 & 0 & 2 & -7 \\ & & 2 & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 1 & 0 & 1 & 1 \\ & & 2 & & \end{smallmatrix}$

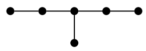
$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\1}} + e_{\substack{00110\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$

$C^\circ = T_1 \qquad C/C^\circ = 1$

$T_1 = \{h_1(\mu^2)h_2(\mu^3)h_3(\mu^4)h_4(\mu^6)h_5(\mu^5)h_6(\mu^4) : \mu \in k^*\}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	–	$f_{\substack{11111\\0}} + f_{\substack{01211\\1}}$		
	1	–	$e_{\substack{11111\\1}} + e_{\substack{01221\\1}}$		
	4	–	$e_{\substack{11100\\0}} + e_{\substack{01110\\0}} - e_{\substack{01100\\1}} - 2e_{\substack{00110\\1}}$		
4	2	–	$e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$		
	5	–	$f_{\substack{00011\\0}} - f_{\substack{00111\\0}}$		
	5	–	$e_{\substack{12221\\1}} - e_{\substack{12321\\1}}$		
3	6	–	$e_{\substack{11110\\0}} - e_{\substack{01110\\1}} + e_{\substack{01210\\1}}$		
	6	–	$e_{\substack{11100\\1}} + e_{\substack{01110\\1}} + e_{\substack{01210\\1}}$		
2	7	–	$f_{\substack{00001\\0}}$		
	7	–	$e_{\substack{12321\\2}}$		
1	2	–	e	z_2	z_2
	8	–	$e_{\substack{11110\\1}}$	z_8	z_8
	10	–	$e_{\substack{12210\\1}}$	z_{10}	z_{10}

E_6, orbit 17: $E_6(a_3)$

L :  τ : $\begin{smallmatrix} 2 & 0 & 2 & 0 & 2 \\ & & 0 & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 0 & 2 & 0 & 2 \\ & & 0 & & \end{smallmatrix}$

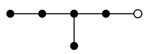
$e = e_{\substack{01100\\1}} + e_{\substack{10000\\0}} + e_{\substack{01110\\0}} + e_{\substack{00001\\0}} + e_{\substack{00110\\1}} + e_{\substack{00100\\0}}$

$C^\circ = 1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	$-$	$e_{\substack{11000\\0}} + e_{\substack{01100\\0}} + e_{\substack{00110\\0}} + e_{\substack{00011\\0}} - 3e_{\substack{00100\\1}} + e_{\substack{01110\\1}}$		
	2	$-$	$e_{\substack{00100\\0}}$		
4	4	$-$	$e_{\substack{11100\\0}} - e_{\substack{00111\\0}} - e_{\substack{01210\\1}}$		
3	4	$-$	$e_{\substack{11110\\0}} + e_{\substack{01111\\0}} - e_{\substack{11100\\1}} - e_{\substack{00111\\1}}$		
	4	$-$	$e_{\substack{11100\\0}} + e_{\substack{00111\\0}} - e_{\substack{11110\\1}} - e_{\substack{01111\\1}}$		
	6	$-$	$e_{\substack{11111\\0}} - e_{\substack{12210\\1}} - e_{\substack{01221\\1}}$		
2	6	$-$	$e_{\substack{11210\\1}} + e_{\substack{01211\\1}}$		
	8	$-$	$e_{\substack{11211\\1}} + e_{\substack{12221\\1}}$		
1	2	$-$	e	z_2	z_2
	8	$-$	$e_{\substack{12211\\1}} + e_{\substack{11221\\1}}$	z_8	z_8
	10	$-$	$e_{\substack{12321\\1}}$	${z_{10}}^1$	
	10	$-$	$e_{\substack{12321\\2}}$	${z_{10}}^2$	${z_{10}}^2$

E_6, orbit 18: D_5

L :  $\qquad \tau$: $\begin{smallmatrix}2&2&2&2&-10\\&&2&&\end{smallmatrix} \qquad \Delta = \begin{smallmatrix}2&0&2&0&2\\&&2&&\end{smallmatrix}$

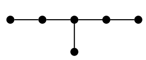
$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$

$C^\circ = T_1 \qquad C/C^\circ = 1$

$T_1 = \{h_1(\mu^2)h_2(\mu^3)h_3(\mu^4)h_4(\mu^6)h_5(\mu^5)h_6(\mu^4) : \mu \in k^*\}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	4	–	$f_{\substack{01111\\0}} - f_{\substack{00111\\1}}$		
	4	–	$e_{\substack{12211\\1}} - e_{\substack{11221\\1}}$		
	6	–	$e_{\substack{11100\\0}} - e_{\substack{01100\\1}} - 2e_{\substack{00110\\1}} + e_{\substack{01110\\0}}$		
2	10	–	$f_{\substack{00001\\0}}$		
	10	–	$e_{\substack{12321\\2}}$		
1	2	–	e	z_2	z_2
	8	–	$e_{\substack{11110\\0}} + e_{\substack{11100\\1}}$	z_8	z_8
	10	–	$e_{\substack{11110\\1}} - e_{\substack{01210\\1}}$	z_{10}	z_{10}
	14	–	$e_{\substack{12210\\1}}$	z_{14}	z_{14}

E_6, orbit 19: $E_6(a_1)$

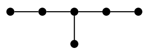
L :  $\qquad \tau$: $\begin{smallmatrix}2&2&0&2&2\\&&2&&\end{smallmatrix} \qquad \Delta = \begin{smallmatrix}2&2&0&2&2\\&&2&&\end{smallmatrix}$

$e = e_{\substack{10000\\0}} + e_{\substack{00001\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00110\\0}} + e_{\substack{01100\\0}} + e_{\substack{00000\\1}}$

$C^\circ = 1 \qquad C/C^\circ = 1$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	4	–	$e_{\substack{11100\\0}} - 3e_{\substack{11000\\0}} - e_{\substack{01100\\1}} + e_{\substack{00110\\1}} - 3e_{\substack{00011\\0}} - 2e_{\substack{01110\\0}} + e_{\substack{00111\\0}}$		
	6	–	$e_{\substack{11100\\1}} + e_{\substack{11110\\0}} - e_{\substack{00111\\1}} + e_{\substack{01111\\0}} + e_{\substack{01210\\1}}$		
2	10	–	$e_{\substack{11111\\1}} + e_{\substack{12210\\1}} + e_{\substack{01221\\1}}$		
1	2	–	e	z_2	z_2
	8	–	$e_{\substack{11110\\1}} - e_{\substack{11210\\1}} - e_{\substack{01111\\1}} + 2e_{\substack{11111\\0}} - e_{\substack{01211\\1}}$	z_8	z_8
	10	–	$e_{\substack{11211\\1}} + e_{\substack{12210\\1}} - e_{\substack{01221\\1}}$	z_{10}	z_{10}
	14	–	$e_{\substack{12221\\1}}$	z_{14}	z_{14}
	16	–	$e_{\substack{12321\\2}}$	z_{16}	z_{16}

E_6, orbit 20: E_6

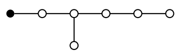
L :  $\qquad \tau$: $\begin{smallmatrix} 2 & 2 & 2 & 2 & 2 \\ & & 2 & & \end{smallmatrix}$ $\qquad \Delta = \begin{smallmatrix} 2 & 2 & 2 & 2 & 2 \\ & & 2 & & \end{smallmatrix}$

$e = e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$

$C^\circ = 1 \qquad C/C^\circ = 1$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	–	e	z_2	z_2
	8	–	$e_{\substack{11100\\1}} + e_{\substack{11110\\0}} + e_{\substack{01111\\0}} - e_{\substack{00111\\1}}$	z_8	z_8
	10	–	$e_{\substack{11110\\1}} + 2e_{\substack{11111\\0}} - e_{\substack{01111\\1}} - e_{\substack{01210\\1}}$	z_{10}	z_{10}
	14	–	$e_{\substack{12210\\1}} + e_{\substack{11211\\1}} - e_{\substack{01221\\1}}$	z_{14}	z_{14}
	16	–	$e_{\substack{12211\\1}} - e_{\substack{11221\\1}}$	z_{16}	z_{16}
	22	–	$e_{\substack{12321\\2}}$	z_{22}	z_{22}

E_7, orbit 1: A_1

L :  τ : $\begin{smallmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$

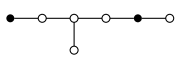
$e = e_{\substack{100000\\0}}$

$C^\circ = D_6 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{000000\\1}}$, $\beta_2 = {\substack{001000\\0}}$, $\beta_3 = {\substack{000100\\0}}$, $\beta_4 = {\substack{000010\\0}}$, $\beta_5 = {\substack{000001\\0}}$, $\beta_6 = {\substack{122100\\1}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	1	λ_6	$e_{\substack{234321\\2}}$		
1	2	0	e	z_2	z_2

E_7, orbit 2: ${A_1}^2$

L :  τ : $\begin{smallmatrix} 2 & -1 & 0 & -1 & 2 & -1 \\ & & 0 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ & & 0 & & & \end{smallmatrix}$

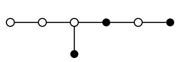
$e = e_{\substack{100000\\0}} + e_{\substack{000010\\0}}$

$C^\circ = B_4 A_1 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{000111\\0}}$, $\beta_2 = {\substack{001000\\0}}$, $\beta_3 = {\substack{000000\\1}}$, $x_{\pm\beta_4}(t) = x_{\pm \substack{111100\\0}}(t) x_{\pm \substack{011110\\0}}(t)$; $\beta_5 = {\substack{122111\\1}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	1	$\lambda_4 + \lambda_5$	$e_{\substack{234321\\2}}$		
1	2	λ_1	$e_{\substack{112221\\1}}$		
	2	0	e	z_2	z_2

E_7, orbit 3: $({A_1}^3)''$

L :  τ : $\begin{smallmatrix} 0 & 0 & -2 & 2 & -2 & 2 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$

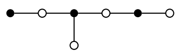
$C^\circ = F_4 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{100000\\0}}$, $\beta_2 = {\substack{010000\\0}}$, $x_{\pm\beta_3}(t) = x_{\pm \substack{001000\\1}}(t) x_{\pm \substack{001100\\0}}(t)$,

$x_{\pm\beta_4}(t) = x_{\pm \substack{000110\\0}}(t) x_{\pm \substack{000011\\0}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	λ_4	$e_{\substack{123321\\2}}$		
	2	0	e	z_2	z_2

E_7, orbit 4: $(A_1{}^3)'$

L :  $\qquad \tau : \begin{smallmatrix} 2 & -2 & 2 & -2 & 2 & -1 \\ & & -1 & & & \end{smallmatrix} \qquad \Delta = \begin{smallmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$

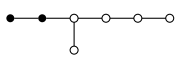
$e = e_{\substack{100000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000010\\0}}$

$C^\circ = C_3A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{001100\\0}}(t)x_{\pm \substack{000110\\0}}(t)$, $x_{\pm\beta_2}(t) = x_{\pm \substack{110000\\0}}(t)x_{\pm \substack{011000\\0}}(t)$, $\beta_3 = {\substack{001111\\1}}$; $\beta_4 = {\substack{123210\\2}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	1	$\lambda_2+\lambda_4$	$e_{\substack{234321\\2}}$		
2	2	λ_2	$e_{\substack{123221\\1}}$		
1	2	0	e	z_2	z_2
	3	λ_4	$e_{\substack{112110\\1}}$		

E_7, orbit 5: A_2

L :  $\qquad \tau : \begin{smallmatrix} 2 & 2 & -2 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix} \qquad \Delta = \begin{smallmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}}$

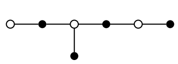
$C^\circ = A_5 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = {\substack{000100\\0}}$, $\beta_2 = {\substack{000010\\0}}$, $\beta_3 = {\substack{000001\\0}}$, $\beta_4 = {\substack{123210\\1}}$, $\beta_5 = {\substack{000000\\1}}$

$c = n_{\substack{011100\\0}} n_{\substack{011000\\1}} n_{\substack{112100\\1}} h_1(-1)h_2(-1)h_5(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	2	λ_2	$e_{\substack{122221\\1}}$		
	2	λ_4	$e_{\substack{234321\\2}}$		
1	2	0	e	z_2	z_2
	4	0	$e_{\substack{110000\\0}}$	z_4	

E_7, orbit 6: $A_1{}^4$

L :  $\qquad \tau : \begin{smallmatrix} -1 & 2 & -3 & 2 & -2 & 2 \\ & & 2 & & & \end{smallmatrix} \qquad \Delta = \begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ & & 1 & & & \end{smallmatrix}$

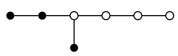
$e = e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$

$C^\circ = C_3 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{111000\\1}}(t)x_{\pm \substack{111100\\0}}(t)$, $x_{\pm\beta_2}(t) = x_{\pm \substack{000110\\0}}(t)x_{\pm \substack{000011\\0}}(t)$, $\beta_3 = {\substack{012100\\1}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	1	λ_3	$e_{\substack{134321\\2}}$		
	1	λ_1	$e_{\substack{122210\\1}} - e_{\substack{122111\\1}} + 2e_{\substack{112211\\1}}$		
2	2	λ_2	$e_{\substack{123321\\2}}$		
	2	0	$e_{\substack{010000\\0}}$		
1	2	0	e	z_2	z_2
	3	λ_1	$e_{\substack{122211\\1}}$		

E_7, orbit 7: A_2A_1

L :  τ : $\begin{smallmatrix}2 & 2 & -3 & 0 & 0 & 0\\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}1 & 0 & 0 & 0 & 1 & 0\\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000000\\1}}$

$C^\circ = A_3T_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

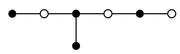
$\beta_1 = {\substack{000100\\0}}$, $\beta_2 = {\substack{000010\\0}}$, $\beta_3 = {\substack{000001\\0}}$

$T_1 = \{h_1(\mu^4)h_2(\mu^6)h_3(\mu^8)h_4(\mu^{12})h_5(\mu^9)h_6(\mu^6)h_7(\mu^3) : \mu \in k^*\}$

$c = n_{\substack{112111\\1}} n_{\substack{112210\\1}} n_{\substack{134321\\2}} h_3(-1)h_5(-1)h_7(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	λ_1	$f_{\substack{011000\\0}} - f_{\substack{001000\\1}}$		
	1	λ_3	$e_{\substack{111111\\0}} - e_{\substack{011111\\1}}$		
	1	λ_3	$f_{\substack{123210\\1}}$		
	1	λ_1	$e_{\substack{123321\\2}}$		
3	2	λ_2	$f_{\substack{012100\\1}}$		
	2	λ_2	$e_{\substack{122221\\1}}$		
	2	0	$f_{\substack{124321\\2}}$		
	2	0	$e_{\substack{234321\\2}}$		
	2	0	$e_{\substack{000000\\1}}$		
2	3	λ_1	$f_{\substack{001000\\0}}$		
	3	λ_3	$e_{\substack{111111\\1}}$		
1	2	0	e	z_2	z_2
	4	0	$e_{\substack{110000\\0}}$	z_4	

E_7, orbit 8: $A_2{A_1}^2$

L :  τ : $\begin{smallmatrix} 2 & -3 & 2 & -3 & 2 & -1 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{000000\\1}} + e_{\substack{001000\\0}} + e_{\substack{100000\\0}} + e_{\substack{000010\\0}}$

$C^\circ = {A_1}^3 \qquad C/C^\circ = 1$

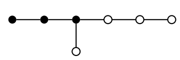
$\beta_1 = {}^{122111}_{\;\;1}$; $x_{\pm\beta_2}(t) = x_{\pm\substack{112211\\1}}(t)x_{\pm\substack{012221\\1}}(-t)$;

$x_{\beta_3}(t) = x_{\substack{111110\\0}}(2t)x_{\substack{012100\\1}}(-t)x_{\substack{123210\\1}}(-t^2)x_{\substack{111100\\1}}(t)x_{\substack{011110\\1}}(-t)$,

$x_{-\beta_3}(t) = x_{-\substack{111110\\0}}(t)x_{-\substack{012100\\1}}(-2t)x_{-\substack{123210\\1}}(t^2)x_{-\substack{111100\\1}}(t)x_{-\substack{011110\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	$\lambda_1 + \lambda_2 + 3\lambda_3$	$e_{\substack{234321\\2}}$		
3	2	$2\lambda_2 + 2\lambda_3$	$e_{\substack{124321\\2}}$		
	2	$4\lambda_3$	$e_{\substack{123210\\2}}$		
2	3	$\lambda_1 + \lambda_2 + \lambda_3$	$e_{\substack{123221\\2}}$		
1	2	0	e	z_2	z_2
	4	$2\lambda_3$	$e_{\substack{112110\\1}}$		

E_7, orbit 9: A_3

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & -3 & 0 & 0 \\ & & -3 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 0 & 0 & 0 & 1 & 0 \\ & & 0 & & & \end{smallmatrix}$

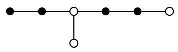
$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}}$

$C^\circ = B_3A_1 \qquad C/C^\circ = 1$

$\beta_1 = {}^{000001}_{\;\;0}$, $\beta_2 = {}^{000010}_{\;\;0}$, $x_{\pm\beta_3}(t) = x_{\pm\substack{111100\\1}}(t)x_{\pm\substack{012100\\1}}(-t)$; $\beta_4 = {}^{123321}_{\;\;1}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	3	$\lambda_3 + \lambda_4$	$e_{\substack{234321\\2}}$		
1	2	0	e	z_2	z_2
	4	λ_1	$e_{\substack{122111\\1}}$		
	6	0	$e_{\substack{111000\\0}}$	z_6	z_6

E_7, orbit 10: ${A_2}^2$

L :  τ : $\begin{smallmatrix} 2 & 2 & -4 & 2 & 2 & -2 \\ & & 0 & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 0 & 0 & 0 & 0 & 2 & 0 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}}$

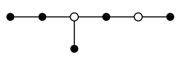
$C^\circ = G_2A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm\substack{111000\\0}}(t)x_{\pm\substack{011100\\0}}(t)x_{\pm\substack{001110\\0}}(t)$, $\beta_2 = \substack{000000\\1}$;

$x_{\pm\beta_3}(t) = x_{\pm\substack{122111\\1}}(t)x_{\pm\substack{112211\\1}}(-t)x_{\pm\substack{012221\\1}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	2	$\lambda_1 + 2\lambda_3$	$e_{\substack{234321\\2}}$		
1	2	0	e	z_2	z_2
	4	λ_1	$e_{\substack{122210\\1}}$		
	4	$2\lambda_3$	$e_{\substack{122221\\1}}$		

E_7, orbit 11: $A_2{A_1}^3$

L :  τ : $\begin{smallmatrix} 2 & 2 & -4 & 2 & -2 & 2 \\ & & 2 & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & & 2 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$

$C^\circ = G_2 \qquad C/C^\circ = 1$

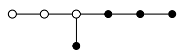
$x_{\beta_1}(t) = x_{\substack{111000\\0}}(t)x_{\substack{001100\\1}}(-2t)x_{\substack{112100\\1}}(-t^2)x_{\substack{011000\\1}}(-t)x_{\substack{011100\\0}}(t)$,

$x_{-\beta_1}(t) = x_{-\substack{111000\\0}}(2t)x_{-\substack{001100\\1}}(-t)x_{-\substack{112100\\1}}(t^2)x_{-\substack{011000\\1}}(-t)x_{-\substack{011100\\0}}(t)$,

$x_{\pm\beta_2}(t) = x_{\pm\substack{000110\\0}}(t)x_{\pm\substack{000011\\0}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	2	$2\lambda_1$	$e_{\substack{234321\\2}}$		
1	2	0	e	z_2	z_2
	4	λ_1	$e_{\substack{122211\\1}}$		

E_7, orbit 12: $(A_3A_1)''$

L :  τ : $\begin{smallmatrix} 0 & 0 & -4 & 2 & 2 & 2 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 0 & 0 & 0 & 0 & 2 \\ & & 0 & & & \end{smallmatrix}$

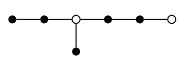
$e = e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{000000\\1}}$

$C^\circ = B_3 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{010000\\0}}$, $\beta_2 = {\substack{100000\\0}}$, $x_{\pm\beta_3}(t) = x_{\pm \substack{012111\\1}}(t)x_{\pm \substack{012210\\1}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	2	λ_3	$e_{\substack{123221\\2}} - e_{\substack{123321\\1}}$		
	2	0	$e_{\substack{000000\\1}}$		
2	4	λ_3	$e_{\substack{123321\\2}}$		
1	2	0	e	z_2	z_2
	4	λ_1	$e_{\substack{122221\\1}}$		
	6	0	$e_{\substack{000111\\0}}$	z_6	z_6

E_7, orbit 13: ${A_2}^2A_1$

L :  τ : $\begin{smallmatrix} 2 & 2 & -5 & 2 & 2 & -2 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ & & 0 & & & \end{smallmatrix}$

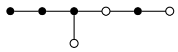
$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000000\\1}}$

$C^\circ = {A_1}^2 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{122100\\1}}(t)x_{\pm \substack{112110\\1}}(t)x_{\pm \substack{012210\\1}}(-t)$;
$x_{\pm\beta_2}(t) = x_{\pm \substack{122111\\1}}(t)x_{\pm \substack{112211\\1}}(-t)x_{\pm \substack{012221\\1}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	$\lambda_1 + 2\lambda_2$	$e_{\substack{123221\\2}} - e_{\substack{123321\\1}}$		
	1	$3\lambda_1$	$e_{\substack{123210\\2}}$		
4	2	$2\lambda_1 + 2\lambda_2$	$e_{\substack{234321\\2}}$		
	2	0	$e_{\substack{000000\\1}}$		
3	3	$\lambda_1 + 2\lambda_2$	$e_{\substack{123321\\2}}$		
	3	λ_1	$e_{\substack{111100\\1}} + 2e_{\substack{111110\\0}} - e_{\substack{011110\\1}}$		
2	4	$2\lambda_2$	$e_{\substack{122221\\1}}$		
	4	$2\lambda_1$	$e_{\substack{122210\\1}}$		
1	2	0	e	z_2	z_2
	5	λ_1	$e_{\substack{111110\\1}}$		

E_7, orbit 14: $(A_3A_1)'$

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & -4 & 2 & -1 \\ & & -3 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$

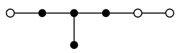
$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000010\\0}}$

$C^\circ = {A_1}^3 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{123210\\2}}$; $\beta_2 = {\substack{123321\\1}}$; $x_{\pm\beta_3}(t) = x_{\pm \substack{111111\\1}}(t)x_{\pm \substack{012111\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	$\lambda_1 + 2\lambda_3$	$e_{\substack{123221\\2}}$		
	2	$\lambda_2 + \lambda_3$	$e_{\substack{123211\\1}} - e_{\substack{122221\\1}}$		
3	2	0	$e_{\substack{000010\\0}}$		
	3	$\lambda_1 + \lambda_2 + \lambda_3$	$e_{\substack{234321\\2}}$		
	3	λ_1	$e_{\substack{122100\\1}} + e_{\substack{112110\\1}}$		
2	4	$\lambda_2 + \lambda_3$	$e_{\substack{123221\\1}}$		
	4	$2\lambda_3$	$e_{\substack{122111\\1}}$		
1	2	0	e	z_2	z_2
	5	λ_1	$e_{\substack{122110\\1}}$		
	6	0	$e_{\substack{111000\\0}}$	z_6	z_6

E_7, orbit 15: $D_4(a_1)$

L :  τ : $\begin{smallmatrix}-4 & 2 & 0 & 2 & -4 & 0\\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}0 & 2 & 0 & 0 & 0 & 0\\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$

$C^\circ = {A_1}^3 \qquad C/C^\circ = \langle c_1C^\circ, c_2C^\circ\rangle \cong S_3$

$\beta_1 = {\substack{000001\\0}}$; $\beta_2 = {\substack{012221\\1}}$; $\beta_3 = {\substack{234321\\2}}$

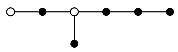
$c_1 = n_{\substack{111000\\1}} n_{\substack{111100\\0}} h_2(-1)$,

$c_2 = (n_{\substack{122110\\1}} n_{\substack{112210\\1}} h_1(-1)h_2(-1)h_6(-1))^g$,

$g = x_{\substack{001000\\0}}(\frac{1}{3}) n_{\substack{001000\\0}} h_1(4)h_2(-4)h_3(16)h_4(-48)h_5(16)h_6(-8) x_{\substack{001000\\0}}(-\frac{1}{3})$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	2	$\lambda_2+\lambda_3$	$e_{\substack{123321\\2}}$		
	2	$\lambda_1+\lambda_3$	$e_{\substack{122211\\1}} - e_{\substack{123211\\1}}$		
	2	$\lambda_1+\lambda_2$	$e_{\substack{011111\\1}} + e_{\substack{012111\\1}}$		
	2	0	$2e_{\substack{000000\\1}} - e_{\substack{001000\\1}} + e_{\substack{011000\\0}} + e_{\substack{001100\\0}}$		
	2	0	$e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$		
2	4	$\lambda_2+\lambda_3$	$e_{\substack{134321\\2}}$		
	4	$\lambda_1+\lambda_3$	$e_{\substack{123211\\2}}$		
	4	$\lambda_1+\lambda_2$	$e_{\substack{012211\\1}}$		
	4	0	$e_{\substack{011000\\1}} + 2e_{\substack{001100\\1}} - e_{\substack{011100\\0}}$		
1	2	0	e	z_2	z_2
	6	0	$e_{\substack{011100\\1}}$	${z_6}^1$	
	6	0	$e_{\substack{012100\\1}}$	${z_6}^2$	

E_7, orbit 16: $A_3{A_1}^2$

L :  τ : $\begin{smallmatrix} -1 & 2 & -5 & 2 & 2 & 2 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ & & 0 & & & \end{smallmatrix}$

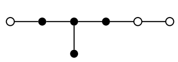
$e = e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}}$

$C^\circ = {A_1}^2 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{234321\\2}}$; $x_{\pm\beta_2}(t) = x_{\pm \substack{012111\\1}}(t) x_{\pm \substack{012210\\1}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
6	1	$\lambda_1 + 2\lambda_2$	$e_{\substack{134321\\2}}$		
	1	λ_2	$e_{\substack{011110\\0}} + e_{\substack{001110\\1}} + e_{\substack{001111\\0}} + 2e_{\substack{011100\\1}}$		
5	2	$\lambda_1 + \lambda_2$	$e_{\substack{123221\\2}} - e_{\substack{123321\\1}}$		
	2	0	$e_{\substack{000000\\1}}$		
	2	0	$e_{\substack{010000\\0}}$		
4	3	λ_1	$e_{\substack{122211\\1}} - e_{\substack{112221\\1}}$		
	3	λ_2	$e_{\substack{011110\\1}} + e_{\substack{001111\\1}}$		
	3	λ_2	$e_{\substack{011110\\1}} + e_{\substack{011111\\0}}$		
3	4	$\lambda_1 + \lambda_2$	$e_{\substack{123321\\2}}$		
	4	$2\lambda_2$	$e_{\substack{012221\\1}}$		
2	5	λ_2	$e_{\substack{011111\\1}}$		
1	2	0	e	z_2	z_2
	5	λ_1	$e_{\substack{122221\\1}}$		
	6	0	$e_{\substack{000111\\0}}$	z_6	z_6

E_7, orbit 17: D_4

L :  τ : $\begin{smallmatrix} -6 & 2 & 2 & 2 & -6 & 0 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$

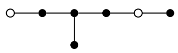
$e = e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$

$C^\circ = C_3 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{111000\\1}}(t) x_{\pm \substack{111100\\0}}(t)$, $x_{\pm\beta_2}(t) = x_{\pm \substack{001110\\1}}(t) x_{\pm \substack{011110\\0}}(-t)$, $\beta_3 = {\substack{000001\\0}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	0	e	z_2	z_2
	6	λ_2	$e_{\substack{134321\\2}}$		
	10	0	$e_{\substack{012100\\1}}$	z_{10}	z_{10}

E_7, orbit 18: $D_4(a_1)A_1$

L :  τ : $\begin{smallmatrix} -4 & 2 & 0 & 2 & -5 & 2 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ & & 1 & & & \end{smallmatrix}$

$e = e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$

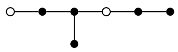
$C^\circ = {A_1}^2 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = {}^{\substack{012221\\1}}$; $\beta_2 = {}^{\substack{234321\\2}}$

$c = n_{\substack{111000\\1}} n_{\substack{111100\\0}} h_2(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
6	1	λ_2	$e_{\substack{122111\\1}} + e_{\substack{123210\\1}} - 2e_{\substack{112211\\1}} - e_{\substack{122210\\1}}$		
	1	λ_1	$e_{\substack{011110\\1}} + e_{\substack{012110\\1}} + 2e_{\substack{001111\\1}} - e_{\substack{011111\\0}}$		
5	2	$\lambda_1+\lambda_2$	$e_{\substack{123321\\2}}$		
	2	0	$2e_{\substack{000000\\1}} - e_{\substack{001000\\1}} + e_{\substack{011000\\0}} + e_{\substack{001100\\0}}$		
	2	0	$e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$		
	2	0	$e_{\substack{000001\\0}}$		
4	3	λ_2	$e_{\substack{123210\\2}} + e_{\substack{123211\\1}}$		
	3	λ_2	$e_{\substack{122211\\1}} - e_{\substack{123211\\1}}$		
	3	λ_1	$e_{\substack{012111\\1}} + e_{\substack{012210\\1}}$		
	3	λ_1	$e_{\substack{012111\\1}} + e_{\substack{011111\\1}}$		
3	4	$\lambda_1+\lambda_2$	$e_{\substack{134321\\2}}$		
	4	0	$e_{\substack{011000\\1}} + 2e_{\substack{001100\\1}} - e_{\substack{011100\\0}}$		
2	5	λ_2	$e_{\substack{123211\\2}}$		
	5	λ_1	$e_{\substack{012211\\1}}$		
1	2	0	e	z_2	z_2
	6	0	$e_{\substack{011100\\1}}$	${z_6}^1$	
	6	0	$e_{\substack{012100\\1}}$	${z_6}^2$	${z_6}^2$

E_7, orbit 19: A_3A_2

L :  τ : $\begin{smallmatrix} -3 & 2 & 2 & -6 & 2 & 2 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$

$C^\circ = A_1T_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

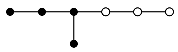
$\beta_1 = {\substack{234321\\2}}$

$T_1 = \{h_2(\mu^3)h_3(\mu^3)h_4(\mu^6)h_5(\mu^6)h_6(\mu^4)h_7(\mu^2) : \mu \in k^*\}$

$c = n_{\substack{001110\\1}} n_{\substack{011110\\0}} n_{\substack{012211\\1}} h_2(-1)h_4(-1)h_5(-1)h_6(-1)h_7(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
6	1	λ_1	$e_{\substack{122100\\1}} + e_{\substack{112110\\1}} + e_{\substack{111111\\1}}$		
	1	λ_1	$e_{\substack{123210\\2}} + e_{\substack{123211\\1}} - e_{\substack{122221\\1}}$		
5	2	0	$f_{\substack{012210\\1}}$		
	2	0	$e_{\substack{012221\\1}}$		
	2	0	$f_{\substack{011100\\0}} + f_{\substack{001110\\0}} + f_{\substack{000111\\0}}$		
	2	0	$e_{\substack{012100\\1}} + e_{\substack{011110\\1}} + e_{\substack{011111\\0}}$		
	2	0	$f_{\substack{011100\\0}} - f_{\substack{001100\\1}}$		
	2	0	$e_{\substack{011111\\0}} - e_{\substack{001111\\1}}$		
	2	0	$e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$		
4	3	λ_1	$e_{\substack{111000\\1}}$		
	3	λ_1	$e_{\substack{134321\\2}}$		
	3	λ_1	$e_{\substack{122110\\1}} + e_{\substack{112111\\1}}$		
	3	λ_1	$e_{\substack{123211\\2}} - e_{\substack{123221\\1}}$		
3	4	0	$f_{\substack{001100\\0}} + f_{\substack{000110\\0}}$		
	4	0	$e_{\substack{012110\\1}} + e_{\substack{011111\\1}}$		
	4	0	$e_{\substack{000011\\0}}$		
2	4	0	$e_{\substack{011000\\0}} - e_{\substack{001000\\1}} + e_{\substack{000011\\0}}$		
	5	λ_1	$e_{\substack{122111\\1}}$		
	5	λ_1	$e_{\substack{123221\\2}}$		
1	2	0	e	z_2	z_2
	6	0	$f_{\substack{000100\\0}}$		
	6	0	$e_{\substack{012111\\1}}$		
	6	0	$e_{\substack{011000\\1}}$	z_6	z_6

E_7, orbit 20: A_4

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & -6 & 0 & 0 \\ & & 2 & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 2 & 0 & 0 & 0 & 2 & 0 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}}$

$C^\circ = A_2T_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

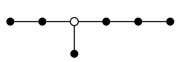
$\beta_1 = {\substack{000010\\0}}$, $\beta_2 = {\substack{000001\\0}}$

$T_1 = \{h_1(\mu^6)h_2(\mu^9)h_3(\mu^{12})h_4(\mu^{18})h_5(\mu^{15})h_6(\mu^{10})h_7(\mu^5) : \mu \in k^*\}$

$c = n_{\substack{011110\\1}} n_{\substack{111110\\0}} n_{\substack{122211\\1}} n_{\substack{124321\\2}} h_2(-1)h_3(-1)h_4(-1)h_6(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	2	λ_1	$f_{\substack{011100\\0}} - f_{\substack{001100\\1}}$		
	2	λ_2	$e_{\substack{111111\\1}} - e_{\substack{012111\\1}}$		
3	4	λ_2	$f_{\substack{012210\\1}}$		
	4	λ_1	$e_{\substack{123221\\2}}$		
	4	0	$e_{\substack{110000\\0}} + e_{\substack{011000\\0}} - e_{\substack{001000\\1}}$		
2	4	0	$f_{\substack{123321\\1}}$		
	4	0	$e_{\substack{234321\\2}}$		
	6	λ_1	$f_{\substack{000100\\0}}$		
	6	λ_2	$e_{\substack{122111\\1}}$		
1	2	0	e	z_2	z_2
	6	0	$e_{\substack{111000\\0}} - e_{\substack{011000\\1}}$	z_6	z_6
	8	0	$e_{\substack{111000\\1}}$	z_8	

E_7, orbit 21: $A_3A_2A_1$

L :  τ : $\begin{smallmatrix} 2 & 2 & -6 & 2 & 2 & 2 \\ & & 2 & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000000\\1}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\beta_1}(t) = x_{\substack{011100\\1}}(2t)x_{\substack{001111\\0}}(t)x_{\substack{111000\\1}}(3t)x_{\substack{011110\\0}}(t)x_{\substack{012211\\1}}(t^2)x_{\substack{112111\\1}}(\frac{3}{2}t^2) \times$
$x_{\substack{122110\\1}}(-\frac{3}{2}t^2)x_{\substack{123211\\2}}(-2t^3)x_{\substack{123221\\1}}(-t^3)x_{\substack{134321\\2}}(\frac{9}{4}t^4)x_{\substack{111100\\0}}(t)x_{\substack{001110\\1}}(t) \times$
$x_{\substack{112210\\1}}(-\frac{1}{2}t^2)$,

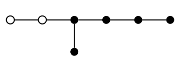
$x_{-\beta_1}(t) = x_{-\substack{011100\\1}}(2t)x_{-\substack{001111\\0}}(6t)x_{-\substack{111000\\1}}(2t)x_{-\substack{011110\\0}}(4t)x_{-\substack{012211\\1}}(-6t^2) \times$
$x_{-\substack{112111\\1}}(-6t^2)x_{-\substack{122110\\1}}(4t^2)x_{-\substack{123211\\2}}(-8t^3)x_{-\substack{123221\\1}}(-16t^3)x_{-\substack{134321\\2}}(-36t^4) \times$
$x_{-\substack{111100\\0}}(2t)x_{-\substack{001110\\1}}(2t)x_{-\substack{112210\\1}}(2t^2)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	2	$8\lambda_1$	$e_{\substack{234321\\2}}$		
	2	$4\lambda_1$	$3e_{\substack{122111\\1}} + 2e_{\substack{122210\\1}} + e_{\substack{012221\\1}} - e_{\substack{112211\\1}}$		
2	4	$6\lambda_1$	$e_{\substack{123321\\2}}$		
	4	$2\lambda_1$	$e_{\substack{111110\\1}} - e_{\substack{111111\\0}} + 2e_{\substack{011111\\1}}$		
1	2	0	e	z_2	z_2
	6	$4\lambda_1$	$e_{\substack{122221\\1}}$		

If $p = 5$, the two summands with $m = 2$ and high weights $8\lambda_1$ and 0 are replaced by a single reducible tilting module $T_{A_1}(8\lambda_1)$ generated by the two given vectors together with $3e_{\substack{000000\\1}} - e_{\substack{100000\\0}} - e_{\substack{010000\\0}}$; also the two summands with $m = 4$ and high weights $6\lambda_1$ and $2\lambda_1$ are replaced by a single reducible tilting module $T_{A_1}(6\lambda_1)$ generated by the two given vectors together with $e_{\substack{111110\\1}} + e_{\substack{111111\\0}}$.

If $p = 7$, the two summands with $m = 2$ and high weights $8\lambda_1$ and $4\lambda_1$ are replaced by a single reducible tilting module $T_{A_1}(8\lambda_1)$ generated by the two given vectors together with $e_{\substack{122111\\1}} + e_{\substack{122210\\1}}$.

E_7, orbit 22: $(A_5)''$

L :  τ : $\substack{0\ -8\ 2\ 2\ 2\ 2\\2}$ $\Delta = \substack{2\ 0\ 0\ 0\ 2\ 2\\0}$

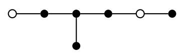
$e = e_{\substack{000000\\1}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$

$C^\circ = G_2 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{011110\\1}}(t)x_{\pm \substack{011111\\0}}(t)x_{\pm \substack{012100\\1}}(t)$, $\beta_2 = \substack{100000\\0}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	4	λ_1	$e_{\substack{123221\\2}} - e_{\substack{123321\\1}}$		
1	2	0	e	z_2	z_2
	6	0	$e_{\substack{001100\\1}} + e_{\substack{001110\\0}} + e_{\substack{000111\\0}}$	z_6	z_6
	8	λ_1	$e_{\substack{124321\\2}}$		
	10	0	$e_{\substack{001111\\1}}$	z_{10}	z_{10}

E_7, orbit 23: D_4A_1

L :  τ : $\begin{smallmatrix}-6&2&2&2&-7&2\\&&2&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}2&1&0&0&0&1\\&&&1&&\end{smallmatrix}$

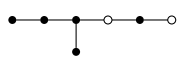
$e = e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$

$C^\circ = B_2 \qquad C/C^\circ = 1$

$\beta_1 = \substack{012221\\1}$, $x_{\pm\beta_2}(t) = x_{\pm\substack{111000\\1}}(t)x_{\pm\substack{111100\\0}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	1	λ_2	$e_{\substack{122111\\1}} - e_{\substack{112211\\1}}$		
	5	λ_2	$e_{\substack{123210\\2}} + e_{\substack{123211\\1}}$		
2	2	0	$e_{\substack{000001\\0}}$		
	6	λ_1	$e_{\substack{134321\\2}}$		
	6	0	$e_{\substack{011000\\1}} + 2e_{\substack{001100\\1}} - e_{\substack{011100\\0}}$		
1	2	0	e	z_2	z_2
	7	λ_2	$e_{\substack{123211\\2}}$		
	10	0	$e_{\substack{012100\\1}}$	z_{10}	z_{10}

E_7, orbit 24: A_4A_1

L :  τ : $\begin{smallmatrix}2&2&2&-7&2&-1\\&&2&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}1&0&1&0&1&0\\&&&0&&\end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000010\\0}}$

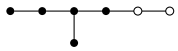
$C^\circ = T_2 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$T_2 = \{h_1(\mu^2)h_2(\mu^3)h_3(\mu^4)h_4(\mu^6)h_5(\mu^5)h_6(\mu^3\nu)h_7(\mu\nu^2) : \mu, \nu \in k^*\}$

$c = n_{\substack{011111\\1}} n_{\substack{111111\\0}} n_{\substack{122210\\1}} n_{\substack{124321\\2}} h_1(-1)h_3(-1)h_4(-1)h_6(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
8	1	–	$f_{\substack{011100\\1}} + f_{\substack{001110\\1}} - f_{\substack{011110\\0}} - 2f_{\substack{111100\\0}}$		
	1	–	$e_{\substack{011110\\1}} + e_{\substack{012100\\1}} - e_{\substack{111100\\1}} - 2e_{\substack{111110\\0}}$		
	1	–	$f_{\substack{000001\\0}}$		
	1	–	$e_{\substack{000011\\0}}$		
7	2	–	$f_{\substack{011111\\0}} - f_{\substack{001111\\1}}$		
	2	–	$e_{\substack{111111\\1}} - e_{\substack{012111\\1}}$		
	2	–	$e_{\substack{000010\\0}}$		
6	3	–	$f_{\substack{112211\\1}} - f_{\substack{012221\\1}}$		
	3	–	$e_{\substack{123221\\1}} - e_{\substack{123211\\2}}$		
	3	–	$f_{\substack{011100\\0}} - f_{\substack{001100\\1}}$		
	3	–	$e_{\substack{111110\\1}} - e_{\substack{012110\\1}}$		
5	4	–	$f_{\substack{123321\\1}}$		
	4	–	$e_{\substack{234321\\2}}$		
	4	–	$f_{\substack{012210\\1}}$		
	4	–	$e_{\substack{123210\\2}}$		
	4	–	$e_{\substack{110000\\0}} + e_{\substack{011000\\0}} - e_{\substack{001000\\1}}$		
4	5	–	$f_{\substack{012211\\1}}$		
	5	–	$e_{\substack{123221\\2}}$		
	5	–	$f_{\substack{001100\\0}} + f_{\substack{000110\\0}}$		
	5	–	$e_{\substack{122100\\1}} + e_{\substack{112110\\1}}$		
3	6	–	$f_{\substack{000111\\0}}$		
	6	–	$e_{\substack{122111\\1}}$		
	6	–	$e_{\substack{111000\\0}} - e_{\substack{011000\\1}}$		
2	7	–	$f_{\substack{000100\\0}}$		
	7	–	$e_{\substack{122110\\1}}$		
1	2	–	e	z_2	z_2
	8	–	$e_{\substack{111000\\1}}$	z_8	

E_7, orbit 25: $D_5(a_1)$

L :  τ : $\begin{smallmatrix}2 & 2 & 0 & 2 & -7 & 0\\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}2 & 0 & 1 & 0 & 1 & 0\\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$

$C^\circ = A_1T_1 \qquad C/C^\circ = \langle cC^\circ\rangle \cong S_2$

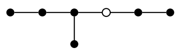
$\beta_1 = {\substack{000001\\0}}$

$T_1 = \{h_1(\mu^2)h_2(\mu^3)h_3(\mu^4)h_4(\mu^6)h_5(\mu^5)h_6(\mu^4)h_7(\mu^2) : \mu \in k^*\}$

$c = n_{\substack{123321\\1}}\, n_{\substack{123221\\2}}\, h_1(-1)h_2(-1)h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	λ_1	$f_{\substack{111110\\0}} + f_{\substack{012110\\1}}$		
	1	λ_1	$e_{\substack{111111\\1}} + e_{\substack{012211\\1}}$		
	4	0	$e_{\substack{111000\\0}} + e_{\substack{011100\\0}} - e_{\substack{011000\\1}} - 2e_{\substack{001100\\1}}$		
4	2	0	$f_{\substack{123221\\1}}$		
	2	0	$e_{\substack{123321\\2}}$		
	2	0	$e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$		
	5	λ_1	$f_{\substack{000110\\0}} - f_{\substack{001110\\0}}$		
	5	λ_1	$e_{\substack{122211\\1}} - e_{\substack{123211\\1}}$		
3	6	0	$f_{\substack{012221\\1}}$		
	6	0	$e_{\substack{234321\\2}}$		
	6	0	$e_{\substack{111100\\0}} - e_{\substack{011100\\1}} + e_{\substack{012100\\1}}$		
	6	0	$e_{\substack{111000\\1}} + e_{\substack{011100\\1}} + e_{\substack{012100\\1}}$		
2	7	λ_1	$f_{\substack{000010\\0}}$		
	7	λ_1	$e_{\substack{123211\\2}}$		
1	2	0	e	z_2	z_2
	8	0	$e_{\substack{111100\\1}}$	z_8	
	10	0	$e_{\substack{122100\\1}}$	z_{10}	z_{10}

E_7, orbit 26: A_4A_2

L :  $\quad \tau$: $\begin{smallmatrix} 2 & 2 & 2 & -8 & 2 & 2 \\ & & 2 & & & \end{smallmatrix}$ $\quad \Delta = \begin{smallmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$

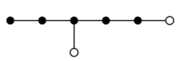
$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\beta_1}(t) = x_{\substack{001111\\1}}(3t)x_{\substack{111110\\0}}(-2t)x_{\substack{012100\\1}}(t)x_{\substack{112221\\1}}(3t^2)x_{\substack{123210\\1}}(-t^2)x_{\substack{124321\\2}}(-2t^3) \times$
$x_{\substack{111100\\1}}(-t)x_{\substack{011111\\0}}(-2t)x_{\substack{122211\\1}}(t^2)x_{\substack{011110\\1}}(t)$,
$x_{-\beta_1}(t) = x_{-\substack{001111\\1}}(t)x_{-\substack{111110\\0}}(-2t)x_{-\substack{012100\\1}}(3t)x_{-\substack{112221\\1}}(-t^2)x_{-\substack{123210\\1}}(3t^2) \times$
$x_{-\substack{124321\\2}}(-2t^3)x_{-\substack{111100\\1}}(-2t)x_{-\substack{011111\\0}}(-t)x_{-\substack{122211\\1}}(-t^2)x_{-\substack{011110\\1}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	2	$4\lambda_1$	$e_{\substack{123210\\2}} + e_{\substack{123211\\1}} - e_{\substack{122221\\1}}$		
3	4	$6\lambda_1$	$e_{\substack{234321\\2}}$		
	4	$2\lambda_1$	$e_{\substack{122100\\1}} + e_{\substack{112110\\1}} + 2e_{\substack{111111\\1}} - e_{\substack{012111\\1}}$		
2	6	$4\lambda_1$	$e_{\substack{123221\\2}}$		
1	2	0	e	z_2	z_2
	8	$2\lambda_1$	$e_{\substack{122111\\1}}$		

If $p = 5$, the two summands with $m = 4$ and high weights $6\lambda_1$ and $2\lambda_1$ are replaced by a single reducible tilting module $T_{A_1}(6\lambda_1)$ generated by the two given vectors together with $e_{\substack{111111\\1}} - e_{\substack{012111\\1}}$.

E_7, orbit 27: $(A_5)'$

L :  $\quad \tau$: $\begin{smallmatrix} 2 & 2 & 2 & 2 & 2 & -5 \\ & & -9 & & & \end{smallmatrix}$ $\quad \Delta = \begin{smallmatrix} 1 & 0 & 1 & 0 & 2 & 0 \\ & & 0 & & & \end{smallmatrix}$

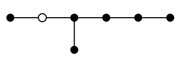
$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}}$

$C^\circ = {A_1}^2 \qquad C/C^\circ = 1$

$\beta_1 = {}^{\substack{123210\\2}}$; $x_{\pm\beta_2}(t) = x_{\pm\substack{122111\\1}}(t)x_{\pm\substack{112211\\1}}(-t)x_{\pm\substack{012221\\1}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	3	λ_1	$e_{\substack{122100\\1}} + e_{\substack{112110\\1}} - e_{\substack{012210\\1}}$		
2	4	$2\lambda_2$	$e_{\substack{122221\\1}} - e_{\substack{123211\\1}}$		
	5	$\lambda_1 + 2\lambda_2$	$e_{\substack{234321\\2}}$		
	6	0	$e_{\substack{111000\\0}} + e_{\substack{011100\\0}} + e_{\substack{001110\\0}}$		
1	2	0	e	z_2	z_2
	8	$2\lambda_2$	$e_{\substack{123321\\1}}$		
	9	λ_1	$e_{\substack{123210\\1}}$		
	10	0	$e_{\substack{111110\\0}}$	z_{10}	z_{10}

E_7, orbit 28: A_5A_1

L :  τ : $\begin{smallmatrix} 2 & -9 & 2 & 2 & 2 & 2 \\ & & 2 & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 1 & 0 & 1 & 0 & 1 & 2 \\ & & 0 & & & \end{smallmatrix}$

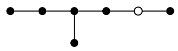
$e = e_{\substack{000000\\1}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{100000\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{123210\\2}}(t) x_{\pm \substack{123211\\1}}(t) x_{\pm \substack{122221\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	$3\lambda_1$	$e_{\substack{234321\\2}}$		
	3	λ_1	$e_{\substack{112110\\1}} - e_{\substack{012210\\1}} - e_{\substack{012111\\1}} + 2e_{\substack{111111\\1}}$		
4	2	0	$e_{\substack{100000\\0}}$		
	4	$2\lambda_1$	$e_{\substack{123321\\1}} - e_{\substack{123221\\2}}$		
3	5	λ_1	$e_{\substack{112210\\1}} + e_{\substack{112111\\1}}$		
	7	λ_1	$e_{\substack{112211\\1}} - e_{\substack{012221\\1}}$		
2	6	0	$e_{\substack{001100\\1}} + e_{\substack{001110\\0}} + e_{\substack{000111\\0}}$		
	8	$2\lambda_1$	$e_{\substack{124321\\2}}$		
1	2	0	e	z_2	z_2
	9	λ_1	$e_{\substack{112221\\1}}$		
	10	0	$e_{\substack{001111\\1}}$	z_{10}	z_{10}

E_7, orbit 29: $D_5(a_1)A_1$

L :  τ : $\begin{smallmatrix} 2 & 2 & 0 & 2 & -8 & 2 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$

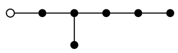
$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\beta_1}(t) = x_{\substack{111110\\1}}(t)x_{\substack{012111\\1}}(t)x_{\substack{123221\\2}}(\frac{1}{2}t^2)x_{\substack{012210\\1}}(t)x_{\substack{111111\\0}}(t)x_{\substack{123321\\1}}(\frac{1}{2}t^2)$,

$x_{-\beta_1}(t) = x_{-\substack{111110\\1}}(2t)x_{-\substack{012111\\1}}(2t)x_{-\substack{123221\\2}}(-2t^2)x_{-\substack{012210\\1}}(2t)x_{-\substack{111111\\0}}(2t) \times$
$x_{-\substack{123321\\1}}(-2t^2)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	2	$4\lambda_1$	$e_{\substack{123321\\2}}$		
	2	0	$e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$		
	4	$2\lambda_1$	$e_{\substack{122111\\1}} + e_{\substack{123210\\1}} - e_{\substack{122210\\1}} - 2e_{\substack{112211\\1}}$		
2	6	$4\lambda_1$	$e_{\substack{234321\\2}}$		
	6	$2\lambda_1$	$e_{\substack{123210\\2}} - e_{\substack{123211\\1}} + 2e_{\substack{122211\\1}}$		
1	2	0	e	z_2	z_2
	8	$2\lambda_1$	$e_{\substack{123211\\2}}$		
	10	0	$e_{\substack{122100\\1}}$	z_{10}	z_{10}

E_7, orbit 30: $D_6(a_2)$

L :  τ : $\begin{smallmatrix}-9&2&0&2&0&2\\&&2&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}0&1&0&1&0&2\\&&1&&&\end{smallmatrix}$

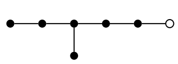
$e = e_{\substack{000001\\0}} + e_{\substack{000110\\0}} + e_{\substack{001000\\1}} - e_{\substack{011000\\0}} + e_{\substack{001100\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{234321\\2}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	0	$e_{\substack{000000\\1}} - e_{\substack{010000\\0}} + e_{\substack{000100\\0}} - e_{\substack{001000\\1}} - e_{\substack{011000\\0}} - e_{\substack{000011\\0}} - e_{\substack{001110\\0}}$		
	2	0	$e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001100\\0}}$		
4	3	λ_1	$e_{\substack{122211\\1}} + e_{\substack{123221\\1}} - e_{\substack{122221\\1}} - e_{\substack{123211\\1}}$		
	3	λ_1	$e_{\substack{122211\\1}} + e_{\substack{123210\\2}}$		
	4	0	$e_{\substack{001100\\1}} + 2e_{\substack{011000\\1}} + e_{\substack{001110\\1}} - e_{\substack{011100\\0}} + e_{\substack{011110\\0}} + e_{\substack{001111\\0}}$		
3	5	λ_1	$e_{\substack{123221\\2}} - e_{\substack{123211\\2}} + e_{\substack{123321\\1}}$		
	6	0	$e_{\substack{011110\\1}} + e_{\substack{011111\\0}} - e_{\substack{012100\\1}} - e_{\substack{012110\\1}}$		
	6	0	$e_{\substack{011100\\1}}$		
2	6	0	$e_{\substack{011111\\0}} - e_{\substack{001111\\1}} - 2e_{\substack{012110\\1}} - 2e_{\substack{011100\\1}}$		
	7	λ_1	$e_{\substack{123321\\2}} + e_{\substack{124321\\2}}$		
	8	0	$e_{\substack{011111\\1}} - e_{\substack{012210\\1}}$		
1	2	0	e	z_2	z_2
	9	λ_1	$e_{\substack{134321\\2}}$		
	10	0	$e_{\substack{012211\\1}}$	${z_{10}}^1$	${z_{10}}^1$
	10	0	$e_{\substack{012221\\1}}$	${z_{10}}^2$	${z_{10}}^2$

E_7, orbit 31: $E_6(a_3)$

L :  τ : $\begin{smallmatrix}2&0&2&0&2&-8\\&&0&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}0&2&0&0&2&0\\&&0&&&\end{smallmatrix}$

$e = e_{\substack{011000\\1}} + e_{\substack{100000\\0}} + e_{\substack{011100\\0}} + e_{\substack{000010\\0}} + e_{\substack{001100\\1}} + e_{\substack{001000\\0}}$

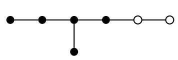
$C^\circ = A_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$x_{\pm\beta_1}(t) = x_{\pm \substack{122111\\1}}(t)x_{\pm \substack{112211\\1}}(t)x_{\pm \substack{012221\\1}}(-t)$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	0	$e_{\substack{110000\\0}} + e_{\substack{011000\\0}} + e_{\substack{001100\\0}} + e_{\substack{000110\\0}} - 3e_{\substack{001000\\1}} + e_{\substack{011100\\1}}$		
	2	0	$e_{\substack{001000\\0}}$		
4	4	0	$e_{\substack{111000\\0}} - e_{\substack{001110\\0}} - e_{\substack{012100\\1}}$		
3	4	$2\lambda_1$	$e_{\substack{123221\\2}} + e_{\substack{123321\\1}}$		
	4	$2\lambda_1$	$e_{\substack{123221\\1}} + e_{\substack{123321\\2}}$		
	6	0	$e_{\substack{111110\\0}} - e_{\substack{122100\\1}} - e_{\substack{012210\\1}}$		
2	6	$2\lambda_1$	$e_{\substack{124321\\2}}$		
	8	0	$e_{\substack{112110\\1}} + e_{\substack{122210\\1}}$		
1	2	0	e	z_2	z_2
	8	$2\lambda_1$	$e_{\substack{234321\\2}}$		
	10	0	$e_{\substack{123210\\1}}$	${z_{10}}^1$	
	10	0	$e_{\substack{123210\\2}}$	${z_{10}}^2$	${z_{10}}^2$

E_7, orbit 32: D_5

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & 2 & -10 & 0 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 2 & 0 & 0 & 2 & 0 \\ & & 0 & & & \end{smallmatrix}$

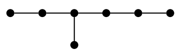
$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}}$

$C^\circ = {A_1}^2 \qquad C/C^\circ = 1$

$\beta_1 = {}^{\substack{000001\\0}}$; $x_{\pm\beta_2}(t) = x_{\pm \substack{123221\\2}}(t) x_{\pm \substack{123321\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	4	$\lambda_1 + \lambda_2$	$e_{\substack{122111\\1}} - e_{\substack{112211\\1}}$		
	6	0	$e_{\substack{111000\\0}} - e_{\substack{011000\\1}} - 2e_{\substack{001100\\1}} + e_{\substack{011100\\0}}$		
2	10	$\lambda_1 + \lambda_2$	$e_{\substack{123211\\2}}$		
1	2	0	e	z_2	z_2
	8	$2\lambda_2$	$e_{\substack{234321\\2}}$		
	10	0	$e_{\substack{111100\\1}} - e_{\substack{012100\\1}}$	z_{10}	z_{10}
	14	0	$e_{\substack{122100\\1}}$	z_{14}	z_{14}

E_7, orbit 33: $E_7(a_5)$

L :  τ : $\begin{smallmatrix}0&0&2&0&0&2\\&&0&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}0&0&2&0&0&2\\&&0&&&\end{smallmatrix}$

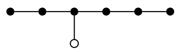
$e = e_{\substack{111100\\0}} + e_{\substack{001110\\1}} + e_{\substack{000001\\0}} + e_{\substack{011110\\0}} + e_{\substack{111000\\1}} + e_{\substack{001000\\0}} + e_{\substack{011100\\1}}$

$C^\circ = 1 \qquad C/C^\circ = \langle c_1C^\circ, c_2C^\circ \rangle \cong S_3$

$c_1 = h_2(\omega)h_3(\omega)h_5(\omega)$,
$c_2 = n_{\substack{000000\\1}} n_{\substack{010000\\0}} n_{\substack{000100\\0}} h_3(-1)h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	–	$e_{\substack{111110\\0}} + e_{\substack{011110\\1}} - 2e_{\substack{111100\\1}} + 2e_{\substack{001100\\0}} - e_{\substack{011000\\0}} + e_{\substack{000011\\0}}$		
	2	–	$e_{\substack{111110\\1}} + e_{\substack{001110\\0}} - 2e_{\substack{111000\\0}} + 2e_{\substack{011000\\1}} - e_{\substack{001100\\1}} + e_{\substack{000111\\0}}$		
	2	–	$e_{\substack{011000\\1}} + e_{\substack{001100\\1}} - e_{\substack{011100\\0}}$		
	2	–	$e_{\substack{001100\\0}} + e_{\substack{011000\\0}} - e_{\substack{001000\\1}}$		
	2	–	$e_{\substack{001000\\0}} + e_{\substack{011100\\1}}$		
4	4	–	$e_{\substack{001111\\0}} + e_{\substack{112110\\1}} - 2e_{\substack{122100\\1}} - e_{\substack{012210\\1}}$		
	4	–	$e_{\substack{011111\\1}} + e_{\substack{122210\\1}} - 2e_{\substack{112100\\1}} + e_{\substack{012110\\1}}$		
	4	–	$e_{\substack{011111\\0}} - e_{\substack{001111\\1}} + e_{\substack{122110\\1}} - e_{\substack{112210\\1}}$		
	4	–	$e_{\substack{012100\\1}}$		
3	6	–	$e_{\substack{122211\\1}} + e_{\substack{112221\\1}} - e_{\substack{012111\\1}}$		
	6	–	$e_{\substack{112111\\1}} - e_{\substack{122221\\1}} + e_{\substack{012211\\1}}$		
	6	–	$e_{\substack{122111\\1}} + e_{\substack{112211\\1}} + e_{\substack{012221\\1}}$		
	6	–	$e_{\substack{012111\\1}} + e_{\substack{123210\\1}}$		
	6	–	$e_{\substack{012211\\1}} + e_{\substack{123210\\2}}$		
2	8	–	$e_{\substack{123221\\1}} + e_{\substack{123211\\2}}$		
	8	–	$e_{\substack{123321\\2}} + e_{\substack{123211\\1}}$		
	8	–	$e_{\substack{123221\\2}} + e_{\substack{123321\\1}}$		
1	2	–	e	z_2	z_2
	10	–	$e_{\substack{124321\\2}}$	$z_{10}{}^1$	
	10	–	$e_{\substack{134321\\2}}$	$z_{10}{}^2$	
	10	–	$e_{\substack{234321\\2}}$	$z_{10}{}^3$	$z_{10}{}^3$

E_7, orbit 34: A_6

L :  $\quad \tau$: $\begin{smallmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ & & -12 & & & \end{smallmatrix}$ $\quad \Delta = \begin{smallmatrix} 0 & 0 & 2 & 0 & 2 & 0 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$

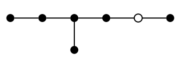
$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\beta_1}(t) = x_{\substack{012210\\1}}(-t)x_{\substack{111111\\1}}(2t)x_{\substack{123321\\2}}(-\frac{1}{2}t^2)x_{\substack{112110\\1}}(t)x_{\substack{012111\\1}}(-t)x_{\substack{122100\\1}}(t),$

$x_{-\beta_1}(t) = x_{-\substack{012210\\1}}(-2t)x_{-\substack{111111\\1}}(t)x_{-\substack{123321\\2}}(\frac{1}{2}t^2)x_{-\substack{112110\\1}}(t)x_{-\substack{012111\\1}}(-t) \times$
$x_{-\substack{122100\\1}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	4	$2\lambda_1$	$e_{\substack{122111\\1}} - e_{\substack{112211\\1}} + e_{\substack{012221\\1}}$		
2	6	$4\lambda_1$	$e_{\substack{234321\\2}}$		
	8	$2\lambda_1$	$e_{\substack{123211\\1}} - e_{\substack{122221\\1}}$		
1	2	0	e	z_2	z_2
	10	0	$e_{\substack{111110\\0}} + e_{\substack{011111\\0}}$	z_{10}	z_{10}
	12	$2\lambda_1$	$e_{\substack{123321\\1}}$		

E_7, orbit 35: D_5A_1

L :  $\quad \tau$: $\begin{smallmatrix} 2 & 2 & 2 & 2 & -11 & 2 \\ & & 2 & & & \end{smallmatrix}$ $\quad \Delta = \begin{smallmatrix} 2 & 1 & 0 & 1 & 1 & 0 \\ & & 1 & & & \end{smallmatrix}$

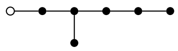
$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm\substack{123221\\2}}(t)x_{\pm\substack{123321\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	0	$e_{\substack{000001\\0}}$		
	3	λ_1	$e_{\substack{122110\\1}} + e_{\substack{112111\\1}} - e_{\substack{112210\\1}} - 2e_{\substack{012211\\1}}$		
4	5	λ_1	$e_{\substack{122111\\1}} - e_{\substack{112211\\1}}$		
	6	0	$e_{\substack{111000\\0}} - e_{\substack{011000\\1}} - 2e_{\substack{001100\\1}} + e_{\substack{011100\\0}}$		
3	8	$2\lambda_1$	$e_{\substack{234321\\2}}$		
	9	λ_1	$e_{\substack{123210\\2}} + e_{\substack{123211\\1}}$		
2	11	λ_1	$e_{\substack{123211\\2}}$		
1	2	0	e	z_2	z_2
	10	0	$e_{\substack{111100\\1}} - e_{\substack{012100\\1}}$	z_{10}	z_{10}
	14	0	$e_{\substack{122100\\1}}$	z_{14}	z_{14}

E_7, orbit 36: $D_6(a_1)$

L :  τ : $\begin{smallmatrix} -11 & 2 & 0 & 2 & 2 & 2 \\ & & 2 & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 1 & 0 & 1 & 0 & 2 \\ & & 1 & & & \end{smallmatrix}$

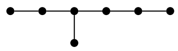
$e = e_{\substack{000001\\0}} + e_{\substack{000010\\0}} + e_{\substack{000100\\0}} + e_{\substack{001000\\1}} - e_{\substack{011000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$\beta_1 = {}^{\substack{234321\\2}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	0	$e_{\substack{000000\\1}} + e_{\substack{010000\\0}}$		
	3	λ_1	$e_{\substack{112221\\1}} - e_{\substack{122211\\1}} - e_{\substack{123211\\1}} - 2e_{\substack{123210\\2}}$		
4	5	λ_1	$e_{\substack{122221\\1}} - e_{\substack{123211\\2}}$		
	6	0	$2e_{\substack{011100\\1}} + e_{\substack{001110\\1}} + e_{\substack{011110\\0}} + e_{\substack{001111\\0}}$		
	6	0	$2e_{\substack{012100\\1}} + e_{\substack{001110\\1}} - e_{\substack{011110\\0}} + e_{\substack{000111\\0}}$		
3	8	0	$2e_{\substack{011110\\1}} + e_{\substack{011111\\0}} + e_{\substack{001111\\1}}$		
	9	λ_1	$e_{\substack{123321\\2}} + e_{\substack{124321\\2}}$		
2	11	λ_1	$e_{\substack{134321\\2}}$		
1	2	0	e	z_2	z_2
	10	0	$e_{\substack{012111\\1}} + e_{\substack{012210\\1}}$	${z_{10}}^1$	${z_{10}}^1$
	10	0	$e_{\substack{011111\\1}}$	${z_{10}}^2$	${z_{10}}^2$
	14	0	$e_{\substack{012221\\1}}$	z_{14}	z_{14}

E_7, orbit 37: $E_7(a_4)$

L :  τ : $\begin{smallmatrix}2&0&2&0&0&2\\&&0&&&\end{smallmatrix}$ Δ = $\begin{smallmatrix}2&0&2&0&0&2\\&&0&&&\end{smallmatrix}$

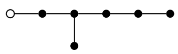
$e = e_{\substack{011000\\1}} + e_{\substack{100000\\0}} + e_{\substack{011100\\0}} + e_{\substack{000011\\0}} + e_{\substack{001110\\1}} + e_{\substack{000001\\0}} + e_{\substack{001100\\1}} + e_{\substack{001000\\0}}$

$C^\circ = 1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
7	2	–	$e_{\substack{110000\\0}} + 4e_{\substack{001100\\0}} + e_{\substack{011000\\0}} - 2e_{\substack{001110\\0}} + e_{\substack{011100\\1}} + 2e_{\substack{000111\\0}} - 6e_{\substack{001000\\1}} + e_{\substack{011110\\1}}$		
	2	–	$e_{\substack{000001\\0}} + e_{\substack{001100\\1}}$		
	2	–	$e_{\substack{001000\\0}}$		
6	4	–	$e_{\substack{111000\\0}} - e_{\substack{111100\\1}} - e_{\substack{012110\\1}} - e_{\substack{011111\\1}}$		
	4	–	$e_{\substack{111100\\1}} - e_{\substack{012100\\1}} - 2e_{\substack{001111\\0}} + e_{\substack{011111\\1}}$		
5	6	–	$e_{\substack{122110\\1}} + e_{\substack{112210\\1}} + 2e_{\substack{012211\\1}} - e_{\substack{111111\\0}}$		
	6	–	$e_{\substack{122100\\1}} - e_{\substack{112210\\1}} - 2e_{\substack{012221\\1}} - e_{\substack{111111\\0}}$		
	6	–	$e_{\substack{112100\\1}} + e_{\substack{012111\\1}}$		
4	8	–	$2e_{\substack{112111\\1}} + e_{\substack{122211\\1}} - e_{\substack{122221\\1}} - e_{\substack{123210\\1}}$		
	8	–	$2e_{\substack{112211\\1}} + e_{\substack{122111\\1}} - e_{\substack{123210\\2}}$		
3	10	–	$e_{\substack{123211\\1}} - e_{\substack{123221\\1}}$		
	10	–	$e_{\substack{123211\\1}} + e_{\substack{123321\\2}}$		
	10	–	$e_{\substack{123211\\2}}$		
2	12	–	$e_{\substack{124321\\2}}$		
1	2	–	e	z_2	z_2
	10	–	$5e_{\substack{123211\\2}} + e_{\substack{123221\\2}} + e_{\substack{123321\\1}}$	z_{10}	z_{10}
	14	–	$e_{\substack{234321\\2}}$	z_{14}	z_{14}

E_7, orbit 38: D_6

L :  τ : $\begin{smallmatrix} -15 & 2 & 2 & 2 & 2 & 2 \\ & & 2 & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 & 2 \\ & & 1 & & & \end{smallmatrix}$

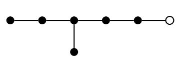
$e = e_{\substack{000001\\0}} + e_{\substack{000010\\0}} + e_{\substack{000100\\0}} + e_{\substack{001000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$\beta_1 = {}^{\substack{234321\\2}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	5	λ_1	$e_{\substack{123210\\2}} + e_{\substack{123211\\1}} - e_{\substack{122221\\1}}$		
2	6	0	$2e_{\substack{011000\\1}} + e_{\substack{011100\\0}} + e_{\substack{001100\\1}} + e_{\substack{001110\\0}} + e_{\substack{000111\\0}}$		
	9	λ_1	$e_{\substack{123321\\1}} - e_{\substack{123221\\2}}$		
	10	0	$e_{\substack{001111\\1}} - e_{\substack{011111\\0}}$		
1	2	0	e	z_2	z_2
	10	0	$e_{\substack{011110\\1}} + e_{\substack{012100\\1}} + 3e_{\substack{001111\\1}} - 2e_{\substack{011111\\0}}$	z_{10}	z_{10}
	14	0	$e_{\substack{012210\\1}} + e_{\substack{012111\\1}}$	z_{14}	z_{14}
	15	λ_1	$e_{\substack{134321\\2}}$		
	18	0	$e_{\substack{012221\\1}}$	z_{18}	z_{18}

E_7, orbit 39: $E_6(a_1)$

L :  τ : $\begin{smallmatrix} 2 & 2 & 0 & 2 & 2 & -12 \\ & & 2 & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 2 & 0 & 2 & 0 & 2 & 0 \\ & & 0 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{000010\\0}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{001100\\0}} + e_{\substack{011000\\0}} + e_{\substack{000000\\1}}$

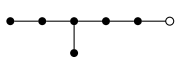
$C^\circ = T_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$T_1 = \{h_1(\mu^2)h_2(\mu^3)h_3(\mu^4)h_4(\mu^6)h_5(\mu^5)h_6(\mu^4)h_7(\mu^3) : \mu \in k^*\}$

$c = n_{\substack{012221\\1}} n_{\substack{112211\\1}} n_{\substack{122111\\1}} h_1(-1)h_2(-1)h_3(-1)h_5(-1)h_6(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	4	–	$f_{\substack{111111\\0}} - f_{\substack{011111\\1}} - f_{\substack{012111\\1}}$		
	4	–	$e_{\substack{123221\\1}} + e_{\substack{122221\\1}} - e_{\substack{123211\\2}}$		
	4	–	$e_{\substack{111000\\0}} - 3e_{\substack{110000\\0}} - e_{\substack{011000\\1}} + e_{\substack{001100\\1}} - 3e_{\substack{000110\\0}} - 2e_{\substack{011100\\0}} + e_{\substack{001110\\0}}$		
3	6	–	$e_{\substack{111000\\1}} + e_{\substack{111100\\0}} - e_{\substack{001110\\1}} + e_{\substack{011110\\0}} + e_{\substack{012100\\1}}$		
	8	–	$f_{\substack{001111\\0}} - f_{\substack{000111\\0}}$		
	8	–	$e_{\substack{123321\\2}} - e_{\substack{124321\\2}}$		
	8	–	$e_{\substack{111100\\1}} - e_{\substack{112100\\1}} - e_{\substack{011110\\1}} + 2e_{\substack{111110\\0}} - e_{\substack{012110\\1}}$		
2	10	–	$e_{\substack{111110\\1}} + e_{\substack{122100\\1}} + e_{\substack{012210\\1}}$		
	12	–	$f_{\substack{000001\\0}}$		
	12	–	$e_{\substack{234321\\2}}$		
1	2	–	e	z_2	z_2
	10	–	$e_{\substack{112110\\1}} + e_{\substack{122100\\1}} - e_{\substack{012210\\1}}$	z_{10}	z_{10}
	14	–	$e_{\substack{122210\\1}}$	z_{14}	z_{14}
	16	–	$e_{\substack{123210\\2}}$	z_{16}	

E_7, orbit 40: E_6

L :  τ : $\begin{smallmatrix}2&2&2&2&2&-16\\&&2&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}2&2&2&0&2&0\\&&0&&&\end{smallmatrix}$

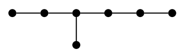
$e = e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm\substack{122111\\1}}(t)x_{\pm\substack{112211\\1}}(-t)x_{\pm\substack{012221\\1}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	8	$2\lambda_1$	$e_{\substack{123221\\2}} - e_{\substack{123321\\1}}$		
1	2	0	e	z_2	z_2
	10	0	$e_{\substack{111100\\1}} + 2e_{\substack{111110\\0}} - e_{\substack{011110\\1}} - e_{\substack{012100\\1}}$	z_{10}	z_{10}
	14	0	$e_{\substack{122100\\1}} + e_{\substack{112110\\1}} - e_{\substack{012210\\1}}$	z_{14}	z_{14}
	16	$2\lambda_1$	$e_{\substack{234321\\2}}$		
	22	0	$e_{\substack{123210\\2}}$	z_{22}	z_{22}

E_7, orbit 41: $E_7(a_3)$

L :  $\quad \tau : \begin{smallmatrix}2&0&2&0&2&2\\ &&0&&&\end{smallmatrix} \quad \Delta = \begin{smallmatrix}2&0&2&0&2&2\\ &&0&&&\end{smallmatrix}$

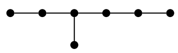
$e = e_{\substack{011000\\1}} + e_{\substack{100000\\0}} + e_{\substack{011100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{001100\\1}} + e_{\substack{001000\\0}}$

$C^\circ = 1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	–	$e_{\substack{001000\\0}}$		
	4	–	$e_{\substack{001110\\0}} + e_{\substack{111100\\1}} + e_{\substack{000111\\0}} - 3e_{\substack{111000\\0}} + e_{\substack{011110\\1}} + 2e_{\substack{012100\\1}}$		
4	6	–	$e_{\substack{001111\\0}} + e_{\substack{112100\\1}} + e_{\substack{012110\\1}}$		
3	6	–	$e_{\substack{111110\\0}} + e_{\substack{011111\\0}} - 2e_{\substack{001111\\1}} - e_{\substack{122100\\1}} - e_{\substack{012210\\1}}$		
	8	–	$e_{\substack{111111\\1}} - e_{\substack{112110\\1}} - 2e_{\substack{012111\\1}} - e_{\substack{122210\\1}}$		
	10	–	$e_{\substack{122111\\1}} + e_{\substack{123210\\2}}$		
2	10	–	$e_{\substack{112111\\1}} - e_{\substack{123210\\1}}$		
	14	–	$e_{\substack{123221\\1}} + e_{\substack{123321\\2}}$		
1	2	–	e	z_2	z_2
	10	–	$3e_{\substack{122111\\1}} + 2e_{\substack{123210\\2}} + e_{\substack{112211\\1}} - e_{\substack{012221\\1}}$	z_{10}	z_{10}
	14	–	$e_{\substack{123221\\2}} + e_{\substack{123321\\1}}$	z_{14}	z_{14}
	16	–	$e_{\substack{124321\\2}}$	z_{16}	
	18	–	$e_{\substack{234321\\2}}$	z_{18}	z_{18}

E_7, orbit 42: $E_7(a_2)$

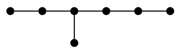
L :  τ : $\begin{smallmatrix} 2 & 2 & 0 & 2 & 0 & 2 \\ & & 2 & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 2 & 2 & 0 & 2 & 0 & 2 \\ & & 2 & & & \end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000110\\0}} + e_{\substack{000011\\0}}$

$C^\circ = 1 \qquad C/C^\circ = 1$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	–	$e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$		
	6	–	$2e_{\substack{111000\\1}} + 3e_{\substack{111110\\0}} - e_{\substack{111100\\0}} + 3e_{\substack{012110\\1}} + e_{\substack{012100\\1}} - e_{\substack{011110\\1}} + e_{\substack{011100\\1}} - e_{\substack{011111\\0}} + 2e_{\substack{001111\\1}}$		
4	8	–	$e_{\substack{111110\\1}} - e_{\substack{111100\\1}} - e_{\substack{111111\\0}} + e_{\substack{012210\\1}} - e_{\substack{012111\\1}}$		
3	10	–	$e_{\substack{111111\\1}} + e_{\substack{012211\\1}}$		
	14	–	$e_{\substack{123210\\2}} + e_{\substack{123221\\1}}$		
2	16	–	$e_{\substack{123221\\2}} - e_{\substack{123211\\2}} + e_{\substack{123321\\1}}$		
1	2	–	e	z_2	z_2
	10	–	$e_{\substack{111111\\1}} + e_{\substack{012211\\1}} + e_{\substack{122100\\1}} + e_{\substack{112210\\1}} + e_{\substack{112111\\1}} + 2e_{\substack{012221\\1}}$	z_{10}	z_{10}
	14	–	$e_{\substack{123210\\2}} + e_{\substack{123211\\1}} + e_{\substack{122221\\1}} - e_{\substack{122211\\1}}$	z_{14}	z_{14}
	18	–	$e_{\substack{123321\\2}}$	z_{18}	z_{18}
	22	–	$e_{\substack{234321\\2}}$	z_{22}	z_{22}

E_7, orbit 43: $E_7(a_1)$

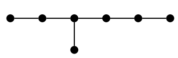
L :  τ : $\begin{smallmatrix}2&2&0&2&2&2\\&&2&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}2&2&0&2&2&2\\&&2&&&\end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{011000\\0}} + e_{\substack{001000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$

$C^\circ = 1 \qquad C/C^\circ = 1$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	6	–	$2e_{\substack{111000\\1}} - e_{\substack{111100\\0}} - e_{\substack{012100\\1}} + 2e_{\substack{011100\\1}} - e_{\substack{011110\\0}} + 2e_{\substack{001110\\1}} - 3e_{\substack{001111\\0}} + 2e_{\substack{000111\\0}}$		
	10	–	$e_{\substack{111110\\1}} + e_{\substack{011111\\1}} + e_{\substack{122100\\1}}$		
2	16	–	$e_{\substack{123210\\2}} - e_{\substack{123211\\1}} + e_{\substack{122211\\1}} - e_{\substack{112221\\1}}$		
1	2	–	e	z_2	z_2
	10	–	$2e_{\substack{111110\\1}} + 2e_{\substack{011111\\1}} + e_{\substack{122100\\1}} + e_{\substack{012111\\1}} + e_{\substack{012210\\1}} - e_{\substack{112110\\1}} + 2e_{\substack{111111\\0}}$	z_{10}	z_{10}
	14	–	$e_{\substack{123210\\1}} + e_{\substack{122111\\1}} - e_{\substack{112211\\1}} + e_{\substack{012221\\1}}$	z_{14}	z_{14}
	18	–	$e_{\substack{123211\\2}} - e_{\substack{122221\\1}}$	z_{18}	z_{18}
	22	–	$e_{\substack{124321\\2}} - e_{\substack{123321\\2}}$	z_{22}	z_{22}
	26	–	$e_{\substack{234321\\2}}$	z_{26}	z_{26}

E_7, orbit 44: E_7

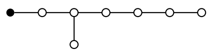
L :  τ : $\begin{smallmatrix}2&2&2&2&2&2\\&&2&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}2&2&2&2&2&2\\&&2&&&\end{smallmatrix}$

$e = e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$

$C^\circ = 1 \qquad C/C^\circ = 1$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	–	e	z_2	z_2
	10	–	$e_{\substack{111100\\1}} + 2e_{\substack{111110\\0}} - e_{\substack{011110\\1}} - 3e_{\substack{001111\\1}} + 2e_{\substack{011111\\0}} - e_{\substack{012100\\1}}$	z_{10}	z_{10}
	14	–	$e_{\substack{122100\\1}} + e_{\substack{112110\\1}} + 2e_{\substack{111111\\1}} - e_{\substack{012111\\1}} - e_{\substack{012210\\1}}$	z_{14}	z_{14}
	18	–	$e_{\substack{122111\\1}} - e_{\substack{112211\\1}} + e_{\substack{012221\\1}}$	z_{18}	z_{18}
	22	–	$e_{\substack{123210\\2}} + e_{\substack{123211\\1}} - e_{\substack{122221\\1}}$	z_{22}	z_{22}
	26	–	$e_{\substack{123221\\2}} - e_{\substack{123321\\1}}$	z_{26}	z_{26}
	34	–	$e_{\substack{234321\\2}}$	z_{34}	z_{34}

E_8, orbit 1: A_1

L :  τ : $\begin{smallmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & 0 & & & & \end{smallmatrix}$

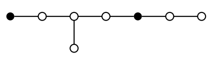
$e = e_{\substack{1000000\\0}}$

$C^\circ = E_7 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{0000001\\0}}$, $\beta_2 = {\substack{1221000\\1}}$, $\beta_3 = {\substack{0000010\\0}}$, $\beta_4 = {\substack{0000100\\0}}$, $\beta_5 = {\substack{0001000\\0}}$, $\beta_6 = {\substack{0010000\\0}}$,
$\beta_7 = {\substack{0000000\\1}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	1	λ_7	$e_{\substack{2354321\\3}}$		
1	2	0	e	z_2	z_2

E_8, orbit 2: ${A_1}^2$

L :  τ : $\begin{smallmatrix} 2 & -1 & 0 & -1 & 2 & -1 & 0 \\ & & 0 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$

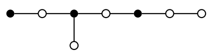
$e = e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}}$

$C^\circ = B_6 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{1221110\\1}}$, $\beta_2 = {\substack{0000001\\0}}$, $\beta_3 = {\substack{0001110\\0}}$, $\beta_4 = {\substack{0010000\\0}}$, $\beta_5 = {\substack{0000000\\1}}$,
$x_{\pm\beta_6}(t) = x_{\pm \substack{1111000\\0}}(t) x_{\pm \substack{0111100\\0}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	1	λ_6	$e_{\substack{2465421\\3}}$		
1	2	λ_1	$e_{\substack{2343321\\2}}$		
	2	0	e	z_2	z_2

E_8, orbit 3: ${A_1}^3$

L :  τ : $\begin{smallmatrix}2 & -2 & 2 & -2 & 2 & -1 & 0\\ & & -1 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}0 & 0 & 0 & 0 & 0 & 1 & 0\\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000100\\0}}$

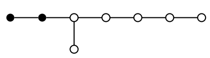
$C^\circ = F_4A_1 \qquad C/C^\circ = 1$

$\beta_1 = \substack{0000001\\0}$, $\beta_2 = \substack{0011110\\1}$, $x_{\pm\beta_3}(t) = x_{\pm\substack{1100000\\0}}(t)x_{\pm\substack{0110000\\0}}(t)$,

$x_{\pm\beta_4}(t) = x_{\pm\substack{0011000\\0}}(t)x_{\pm\substack{0001100\\0}}(t)$; $\beta_5 = \substack{1232100\\2}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	1	$\lambda_4+\lambda_5$	$e_{\substack{2465421\\3}}$		
2	2	λ_4	$e_{\substack{2354321\\2}}$		
1	2	0	e	z_2	z_2
	3	λ_5	$e_{\substack{1121100\\1}}$		

E_8, orbit 4: A_2

L :  τ : $\begin{smallmatrix}2 & 2 & -2 & 0 & 0 & 0 & 0\\ & & 0 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}0 & 0 & 0 & 0 & 0 & 0 & 2\\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}}$

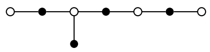
$C^\circ = E_6 \qquad C/C^\circ = \langle cC^\circ\rangle \cong S_2$

$\beta_1 = \substack{0001000\\0}$, $\beta_2 = \substack{0000001\\0}$, $\beta_3 = \substack{0000100\\0}$, $\beta_4 = \substack{0000010\\0}$, $\beta_5 = \substack{1232100\\1}$, $\beta_6 = \substack{0000000\\1}$

$c = n_{\substack{0111000\\0}}n_{\substack{0110000\\1}}n_{\substack{1121000\\1}}h_1(-1)h_2(-1)h_5(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	2	λ_1	$e_{\substack{2344321\\2}}$		
	2	λ_6	$e_{\substack{2454321\\3}}$		
1	2	0	e	z_2	z_2
	4	0	$e_{\substack{1100000\\0}}$	z_4	

E_8, orbit 5: $A_1{}^4$

L :  τ : $\substack{-1\ \ 2\ \ -3\ \ 2\ \ -2\ \ 2\ \ -1\\ 2}$ $\Delta = \substack{0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\\ 1}$

$e = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$

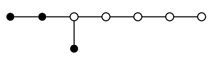
$C^\circ = C_4 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{0011111\\1}}(t) x_{\pm \substack{0111111\\0}}(-t)$, $x_{\pm\beta_2}(t) = x_{\pm \substack{1110000\\1}}(t) x_{\pm \substack{1111000\\0}}(t)$,

$x_{\pm\beta_3}(t) = x_{\pm \substack{0001100\\0}}(t) x_{\pm \substack{0000110\\0}}(t)$, $\beta_4 = \substack{0121000\\1}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	1	λ_3	$e_{\substack{2465431\\3}}$		
2	2	λ_2	$e_{\substack{2454321\\3}}$		
1	2	0	e	z_2	z_2
	3	λ_1	$e_{\substack{1233221\\2}}$		

E_8, orbit 6: $A_2 A_1$

L :  τ : $\substack{2\ \ 2\ \ -3\ \ 0\ \ 0\ \ 0\ \ 0\\ 2}$ $\Delta = \substack{1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\\ 0}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0000000\\1}}$

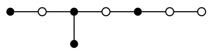
$C^\circ = A_5 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = \substack{0000001\\0}$, $\beta_2 = \substack{0000010\\0}$, $\beta_3 = \substack{0000100\\0}$, $\beta_4 = \substack{0001000\\0}$, $\beta_5 = \substack{2464321\\3}$

$c = n_{\substack{1121110\\1}} n_{\substack{1122100\\1}} n_{\substack{1343210\\2}} h_3(-1) h_5(-1) h_7(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	λ_5	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
	1	λ_1	$e_{\substack{1111111\\0}} - e_{\substack{0111111\\1}}$		
	1	λ_3	$e_{\substack{1233321\\2}}$		
3	2	λ_2	$e_{\substack{1222221\\1}}$		
	2	λ_4	$e_{\substack{2344321\\2}}$		
	2	0	$e_{\substack{0000000\\1}}$		
2	3	λ_5	$e_{\substack{2454321\\3}}$		
	3	λ_1	$e_{\substack{1111111\\1}}$		
1	2	0	e	z_2	z_2
	4	0	$e_{\substack{1100000\\0}}$	z_4	

E_8, orbit 7: $A_2{A_1}^2$

L :  τ : $\begin{smallmatrix} 2 & -3 & 2 & -3 & 2 & -1 & 0 \\ & & 2 & & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}}$

$C^\circ = B_3A_1 \qquad C/C^\circ = 1$

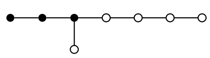
$\beta_1 = {\substack{1221110\\1}}$, $\beta_2 = {\substack{0000001\\0}}$, $x_{\pm\beta_3}(t) = x_{\pm \substack{1122110\\1}}(t)x_{\pm \substack{0122210\\1}}(-t)$;

$x_{\beta_4}(t) = x_{\substack{1111100\\0}}(2t)x_{\substack{0121000\\1}}(-t)x_{\substack{1232100\\1}}(-t^2)x_{\substack{1111000\\1}}(t)x_{\substack{0111100\\1}}(-t)$,

$x_{-\beta_4}(t) = x_{-\substack{1111100\\0}}(t)x_{-\substack{0121000\\1}}(-2t)x_{-\substack{1232100\\1}}(t^2)x_{-\substack{1111000\\1}}(t)x_{-\substack{0111100\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	$\lambda_3 + 3\lambda_4$	$e_{\substack{2465421\\3}}$		
3	2	$\lambda_1 + 2\lambda_4$	$e_{\substack{2464321\\3}}$		
	2	$4\lambda_4$	$e_{\substack{1232100\\2}}$		
2	3	$\lambda_3 + \lambda_4$	$e_{\substack{2354321\\3}}$		
1	2	0	e	z_2	z_2
	4	$2\lambda_4$	$e_{\substack{1121100\\1}}$		

E_8, orbit 8: A_3

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & -3 & 0 & 0 & 0 \\ & & -3 & & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}}$

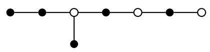
$C^\circ = B_5 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{1233210\\1}}$, $\beta_2 = {\substack{0000001\\0}}$, $\beta_3 = {\substack{0000010\\0}}$, $\beta_4 = {\substack{0000100\\0}}$,

$x_{\pm\beta_5}(t) = x_{\pm \substack{1111000\\1}}(t)x_{\pm \substack{0121000\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	3	λ_5	$e_{\substack{2464321\\3}}$		
1	2	0	e	z_2	z_2
	4	λ_1	$e_{\substack{2454321\\2}}$		
	6	0	$e_{\substack{1110000\\0}}$	z_6	z_6

E_8, orbit 9: ${A_2A_1}^3$

L :  τ : $\begin{smallmatrix} 2 & 2 & -4 & 2 & -2 & 2 & -1 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$

$C^\circ = G_2A_1 \qquad C/C^\circ = 1$

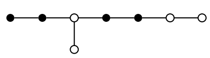
$x_{\beta_1}(t) = x_{\substack{1110000\\0}}(t)x_{\substack{0011000\\1}}(-2t)x_{\substack{1121000\\1}}(-t^2)x_{\substack{0110000\\1}}(-t)x_{\substack{0111000\\0}}(t),$

$x_{-\beta_1}(t) = x_{-\substack{1110000\\0}}(2t)x_{-\substack{0011000\\1}}(-t)x_{-\substack{1121000\\1}}(t^2)x_{-\substack{0110000\\1}}(-t)x_{-\substack{0111000\\0}}(t),$

$x_{\pm\beta_2}(t) = x_{\pm\substack{0001100\\0}}(t)x_{\pm\substack{0000110\\0}}(t)$; $\beta_3 = \substack{2465432\\3}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	$\lambda_2+\lambda_3$	$e_{\substack{2465431\\3}}$		
3	2	$2\lambda_1$	$e_{\substack{2343210\\2}}$		
2	3	$\lambda_1+\lambda_3$	$e_{\substack{2454321\\3}}$		
1	2	0	e	z_2	z_2
	4	λ_1	$e_{\substack{1222110\\1}}$		

E_8, orbit 10: ${A_2}^2$

L :  τ : $\begin{smallmatrix} 2 & 2 & -4 & 2 & 2 & -2 & 0 \\ & & 0 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}}$

$C^\circ = {G_2}^2 \qquad C/C^\circ = \langle cC^\circ\rangle \cong S_2$

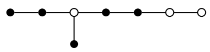
$x_{\pm\beta_1}(t) = x_{\pm\substack{1110000\\0}}(t)x_{\pm\substack{0111000\\0}}(t)x_{\pm\substack{0011100\\0}}(t)$, $\beta_2 = \substack{0000000\\1}$;

$x_{\pm\beta_3}(t) = x_{\pm\substack{1221110\\1}}(t)x_{\pm\substack{1122110\\1}}(-t)x_{\pm\substack{0122210\\1}}(t)$, $\beta_4 = \substack{0000001\\0}$

$c = n_{\substack{0111111\\0}}n_{\substack{0111110\\1}}n_{\substack{1122221\\1}}h_3(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	2	$\lambda_1+\lambda_3$	$e_{\substack{2465421\\3}}$		
1	2	0	e	z_2	z_2
	4	λ_1	$e_{\substack{1222100\\1}}$		
	4	λ_3	$e_{\substack{2344321\\2}}$		

E_8, orbit 11: $A_2{}^2A_1$

L :  τ : $\begin{smallmatrix} 2 & 2 & -5 & 2 & 2 & -2 & 0 \\ & & 2 & & & & \end{smallmatrix}$ Δ = $\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000000\\1}}$

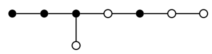
$C^\circ = G_2A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{1221110\\1}}(t)x_{\pm \substack{1122110\\1}}(-t)x_{\pm \substack{0122210\\1}}(t)$, $\beta_2 = \substack{0000001\\0}$;

$x_{\pm\beta_3}(t) = x_{\pm \substack{1221000\\1}}(t)x_{\pm \substack{1121100\\1}}(t)x_{\pm \substack{0122100\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	$\lambda_1+\lambda_3$	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
	1	$3\lambda_3$	$e_{\substack{1232100\\2}}$		
4	2	$\lambda_1+2\lambda_3$	$e_{\substack{2465421\\3}}$		
	2	0	$e_{\substack{0000000\\1}}$		
3	3	$\lambda_1+\lambda_3$	$e_{\substack{2454321\\3}}$		
	3	λ_3	$e_{\substack{1111000\\1}} + 2e_{\substack{1111100\\0}} - e_{\substack{0111100\\1}}$		
2	4	λ_1	$e_{\substack{2344321\\2}}$		
	4	$2\lambda_3$	$e_{\substack{1222100\\1}}$		
1	2	0	e	z_2	z_2
	5	λ_3	$e_{\substack{1111100\\1}}$		

E_8, orbit 12: A_3A_1

L :  τ : $\begin{smallmatrix}2 & 2 & 2 & -4 & 2 & -1 & 0\\ & & -3 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}0 & 0 & 0 & 0 & 1 & 0 & 1\\ & & 0 & & & & \end{smallmatrix}$

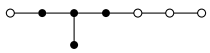
$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000100\\0}}$

$C^\circ = B_3A_1 \qquad C/C^\circ = 1$

$\beta_1 = \substack{1233210\\1}$, $\beta_2 = \substack{0000001\\0}$, $x_{\pm\beta_3}(t) = x_{\pm \substack{1111110\\1}}(t)x_{\pm \substack{0121110\\1}}(-t)$; $\beta_4 = \substack{1232100\\2}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	1	$\lambda_1+\lambda_4$	$e_{\substack{2465421\\3}}$		
	2	λ_3	$e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}}$		
3	2	0	$e_{\substack{0000100\\0}}$		
	3	$\lambda_3+\lambda_4$	$e_{\substack{2464321\\3}}$		
	3	λ_4	$e_{\substack{1221000\\1}} + e_{\substack{1121100\\1}}$		
2	4	λ_1	$e_{\substack{2454321\\2}}$		
	4	λ_3	$e_{\substack{2343321\\2}}$		
1	2	0	e	z_2	z_2
	5	λ_4	$e_{\substack{1221100\\1}}$		
	6	0	$e_{\substack{1110000\\0}}$	z_6	z_6

E_8, orbit 13: $D_4(a_1)$

L :  τ : $\begin{smallmatrix} -4 & 2 & 0 & 2 & -4 & 0 & 0 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$

$C^\circ = D_4 \qquad C/C^\circ = \langle c_1 C^\circ, c_2 C^\circ \rangle \cong S_3$

$\beta_1 = {\substack{0000010\\0}}$, $\beta_2 = {\substack{0000001\\0}}$, $\beta_3 = {\substack{0122210\\1}}$, $\beta_4 = {\substack{2343210\\2}}$

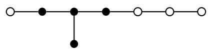
$c_1 = n_{\substack{1110000\\1}} n_{\substack{1111000\\0}} h_2(-1)$,

$c_2 = (n_{\substack{1221100\\1}} n_{\substack{1122100\\1}} h_1(-1) h_2(-1) h_6(-1))^g$,

$g = x_{\substack{0010000\\0}}(\frac{1}{3}) n_{\substack{0010000\\0}} h_1(4) h_2(-4) h_3(16) h_4(-48) h_5(16) h_6(-8) x_{\substack{0010000\\0}}(-\frac{1}{3})$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	2	λ_4	$e_{\substack{2464321\\3}} + e_{\substack{2454321\\3}}$		
	2	λ_3	$e_{\substack{1354321\\2}} - e_{\substack{1344321\\2}}$		
	2	λ_1	$e_{\substack{1233221\\2}}$		
	2	0	$2e_{\substack{0000000\\1}} - e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}}$		
	2	0	$e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$		
2	4	λ_4	$e_{\substack{2465321\\3}}$		
	4	λ_3	$e_{\substack{1354321\\3}}$		
	4	λ_1	$e_{\substack{1343221\\2}}$		
	4	0	$e_{\substack{0110000\\1}} + 2e_{\substack{0011000\\1}} - e_{\substack{0111000\\0}}$		
1	2	0	e	z_2	z_2
	6	0	$e_{\substack{0111000\\1}}$	${z_6}^1$	
	6	0	$e_{\substack{0121000\\1}}$	${z_6}^2$	

E_8, orbit 14: D_4

L :  τ : $\begin{smallmatrix} -6 & 2 & 2 & 2 & -6 & 0 & 0 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$

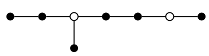
$C^\circ = F_4 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{0000001\\0}}$, $\beta_2 = {\substack{0000010\\0}}$, $x_{\pm\beta_3}(t) = x_{\pm \substack{0011100\\1}}(t) x_{\pm \substack{0111100\\0}}(-t)$,

$x_{\pm\beta_4}(t) = x_{\pm \substack{1110000\\1}}(t) x_{\pm \substack{1111000\\0}}(t)$,

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	0	e	z_2	z_2
	6	λ_4	$e_{\substack{2465321\\3}}$		
	10	0	$e_{\substack{0121000\\1}}$	z_{10}	z_{10}

E_8, orbit 15: ${A_2}^2{A_1}^2$

L :  τ : $\begin{smallmatrix} 2 & 2 & -5 & 2 & 2 & -3 & 2 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000001\\0}}$

$C^\circ = B_2 \qquad C/C^\circ = 1$

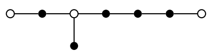
$x_{\pm\beta_1}(t) = x_{\pm \substack{1121100\\1}}(t) x_{\pm \substack{1221000\\1}}(t) x_{\pm \substack{0122100\\1}}(-t)$,

$x_{\beta_2}(t) = x_{\substack{1111110\\0}}(t) x_{\substack{0011111\\1}}(-2t) x_{\substack{1122221\\1}}(t^2) x_{\substack{0111110\\1}}(-t) x_{\substack{0111111\\0}}(t)$,

$x_{-\beta_2}(t) = x_{-\substack{1111110\\0}}(2t) x_{-\substack{0011111\\1}}(-t) x_{-\substack{1122221\\1}}(-t^2) x_{-\substack{0111110\\1}}(-t) x_{-\substack{0111111\\0}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	$3\lambda_2$	$e_{\substack{2465432\\3}}$		
4	2	$2\lambda_1$	$e_{\substack{2465421\\3}}$		
	2	λ_1	$e_{\substack{1233210\\2}} + e_{\substack{1233211\\1}}$		
3	3	$\lambda_1 + \lambda_2$	$e_{\substack{2454321\\3}}$		
2	4	$2\lambda_2$	$e_{\substack{2344321\\2}}$		
1	2	0	e	z_2	z_2
	5	λ_2	$e_{\substack{1222211\\1}}$		

E_8, orbit 16: $A_3{A_1}^2$

L :  τ : $\substack{-1\ 2\ -5\ 2\ 2\ 2\ -3\\2}$ $\Delta = \substack{0\ 1\ 0\ 0\ 0\ 0\ 1\\0}$

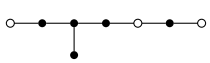
$e = e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$

$C^\circ = B_2A_1 \qquad C/C^\circ = 1$

$\beta_1 = \substack{2343210\\2}$, $x_{\pm\beta_2}(t) = x_{\pm\substack{0011111\\1}}(t)x_{\pm\substack{0111111\\0}}(-t)$; $x_{\pm\beta_3}(t) = x_{\pm\substack{0121110\\1}}(t)x_{\pm\substack{0122100\\1}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
6	1	$\lambda_2+2\lambda_3$	$e_{\substack{1354321\\3}}$		
	1	λ_3	$e_{\substack{0111100\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0011110\\0}} + 2e_{\substack{0111000\\1}}$		
5	2	λ_1	$e_{\substack{2454321\\3}}$		
	2	$\lambda_2+\lambda_3$	$e_{\substack{1343321\\2}} - e_{\substack{1244321\\2}}$		
	2	0	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$		
4	3	$\lambda_1+\lambda_3$	$e_{\substack{2465431\\3}}$		
	3	λ_2	$e_{\substack{1233221\\2}} - e_{\substack{1233321\\1}}$		
	3	λ_3	$2e_{\substack{0111100\\1}} + e_{\substack{0111110\\0}} + e_{\substack{0011110\\1}}$		
3	4	$\lambda_2+\lambda_3$	$e_{\substack{1344321\\2}}$		
	4	$2\lambda_3$	$e_{\substack{0122210\\1}}$		
2	5	λ_3	$e_{\substack{0111110\\1}}$		
1	2	0	e	z_2	z_2
	5	λ_2	$e_{\substack{1233321\\2}}$		
	6	0	$e_{\substack{0001110\\0}}$	z_6	z_6

E_8, orbit 17: $D_4(a_1)A_1$

L :  τ : $\substack{-4\ 2\ 0\ 2\ -5\ 2\ -1\\2}$ $\Delta = \substack{0\ 0\ 0\ 0\ 0\ 1\ 0\\1}$

$e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$

$C^\circ = {A_1}^3 \qquad C/C^\circ = \langle c_1C^\circ, c_2C^\circ\rangle \cong S_3$

$\beta_1 = \substack{0122210\\1}$; $\beta_2 = \substack{2343210\\2}$; $\beta_3 = \substack{2465432\\3}$

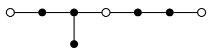
$c_1 = n_{\substack{1110000\\1}} n_{\substack{1111000\\0}} h_2(-1)$,

$c_2 = (n_{\substack{1221111\\1}} n_{\substack{1122111\\1}} h_3(i)h_5(i))^g$,

$g = x_{\substack{0010000\\0}}(\frac{1}{3}) n_{\substack{0010000\\0}} h_1(4)h_2(-4)h_3(16)h_4(-48)h_5(16)h_6(-8)x_{\substack{0010000\\0}}(-\frac{1}{3})$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
6	1	$\lambda_1+\lambda_2+\lambda_3$	$e_{\substack{2465431\\3}}$		
	1	λ_3	$e_{\substack{1232221\\2}} - 2e_{\substack{1233211\\2}} + e_{\substack{1233221\\1}}$		
	1	λ_2	$e_{\substack{1221110\\1}} + e_{\substack{1232100\\1}} - 2e_{\substack{1122110\\1}} - e_{\substack{1222100\\1}}$		
	1	λ_1	$e_{\substack{0111100\\1}} + e_{\substack{0121100\\1}} + 2e_{\substack{0011110\\1}} - e_{\substack{0111110\\0}}$		
5	2	$\lambda_2+\lambda_3$	$e_{\substack{2454321\\3}} + e_{\substack{2464321\\3}}$		
	2	$\lambda_1+\lambda_3$	$e_{\substack{1344321\\2}} - e_{\substack{1354321\\2}}$		
	2	$\lambda_1+\lambda_2$	$e_{\substack{1233210\\2}}$		
	2	0	$2e_{\substack{0000000\\1}} - e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}}$		
	2	0	$e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$		
	2	0	$e_{\substack{0000010\\0}}$		
4	3	λ_3	$e_{\substack{1233221\\2}}$		
	3	λ_3	$e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}}$		
	3	λ_2	$e_{\substack{1232100\\2}} + e_{\substack{1232110\\1}}$		
	3	λ_2	$e_{\substack{1222110\\1}} - e_{\substack{1232110\\1}}$		
	3	λ_1	$e_{\substack{0121110\\1}} + e_{\substack{0122100\\1}}$		
	3	λ_1	$e_{\substack{0121110\\1}} + e_{\substack{0111110\\1}}$		
3	4	$\lambda_2+\lambda_3$	$e_{\substack{2465321\\3}}$		
	4	$\lambda_1+\lambda_3$	$e_{\substack{1354321\\3}}$		
	4	$\lambda_1+\lambda_2$	$e_{\substack{1343210\\2}}$		
	4	0	$e_{\substack{0110000\\1}} + 2e_{\substack{0011000\\1}} - e_{\substack{0111000\\0}}$		
2	5	λ_3	$e_{\substack{1343221\\2}}$		
	5	λ_2	$e_{\substack{1232110\\2}}$		
	5	λ_1	$e_{\substack{0122110\\1}}$		
1	2	0	e	z_2	z_2
	6	0	$e_{\substack{0111000\\1}}$	${z_6}^1$	
	6	0	$e_{\substack{0121000\\1}}$	${z_6}^2$	

E_8, orbit 18: A_3A_2

L :  τ : $\begin{smallmatrix} -3 & 2 & 2 & -6 & 2 & 2 & -2 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$

$C^\circ = B_2T_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

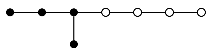
$\beta_1 = {}^{2343210}_{\ \ \ 2}$, $x_{\pm\beta_2}(t) = x_{\pm \substack{0011111\\1}}(t)x_{\pm \substack{0111111\\0}}(-t)$

$T_1 = \{h_2(\mu^3)h_3(\mu^3)h_4(\mu^6)h_5(\mu^6)h_6(\mu^4)h_7(\mu^2) : \mu \in k^*\}$

$c = n_{\substack{0011100\\1}} n_{\substack{0111100\\0}} n_{\substack{0122110\\1}} h_2(-1)h_4(-1)h_5(-1)h_6(-1)h_7(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
6	1	λ_2	$e_{\substack{1232111\\2}} - e_{\substack{1232211\\1}} + e_{\substack{1222221\\1}}$		
	1	λ_2	$e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}} + e_{\substack{1233321\\2}}$		
5	2	λ_1	$e_{\substack{2343321\\2}}$		
	2	λ_1	$e_{\substack{2465431\\3}}$		
	2	0	$f_{\substack{0122100\\1}}$		
	2	0	$e_{\substack{0122210\\1}}$		
	2	0	$2f_{\substack{0011100\\0}} + 2f_{\substack{0001110\\0}} + f_{\substack{0111000\\0}} + f_{\substack{0011000\\1}}$		
	2	0	$2e_{\substack{0121000\\1}} + 2e_{\substack{0111100\\1}} + e_{\substack{0111110\\0}} + e_{\substack{0011110\\1}}$		
	2	0	$e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$		
4	3	λ_2	$e_{\substack{1221111\\1}}$		
	3	λ_2	$e_{\substack{1354321\\3}}$		
	3	λ_2	$e_{\substack{1232211\\2}} - e_{\substack{1232221\\1}}$		
	3	λ_2	$e_{\substack{1343221\\2}} - e_{\substack{1243321\\2}}$		
3	4	λ_1	$e_{\substack{2464321\\3}}$		
	4	0	$f_{\substack{0011000\\0}} + f_{\substack{0001100\\0}}$		
	4	0	$e_{\substack{0121100\\1}} + e_{\substack{0111110\\1}}$		
	4	0	$e_{\substack{0000110\\0}}$		
2	5	λ_2	$e_{\substack{1232221\\2}}$		
	5	λ_2	$e_{\substack{1343321\\2}}$		
1	2	0	e	z_2	z_2
	6	0	$f_{\substack{0001000\\0}}$		
	6	0	$e_{\substack{0121110\\1}}$		
	6	0	$e_{\substack{0110000\\1}}$	z_6	z_6

E_8, orbit 19: A_4

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & -6 & 0 & 0 & 0 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}}$

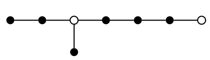
$C^\circ = A_4 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = {\substack{0000001\\0}}$, $\beta_2 = {\substack{0000010\\0}}$, $\beta_3 = {\substack{0000100\\0}}$, $\beta_4 = {\substack{2465321\\3}}$

$c = n_{\substack{0111100\\1}} n_{\substack{1111100\\0}} n_{\substack{1222110\\1}} n_{\substack{1243210\\2}} h_2(-1)h_3(-1)h_4(-1)h_8(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	2	λ_4	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
	2	λ_1	$e_{\substack{1111111\\1}} - e_{\substack{0121111\\1}}$		
3	4	λ_2	$e_{\substack{1232221\\2}}$		
	4	λ_3	$e_{\substack{2343321\\2}}$		
	4	0	$e_{\substack{1100000\\0}} + e_{\substack{0110000\\0}} - e_{\substack{0010000\\1}}$		
2	6	λ_4	$e_{\substack{2464321\\3}}$		
	6	λ_1	$e_{\substack{1221111\\1}}$		
1	2	0	e	z_2	z_2
	6	0	$e_{\substack{1110000\\0}} - e_{\substack{0110000\\1}}$	z_6	z_6
	8	0	$e_{\substack{1110000\\1}}$	z_8	

E_8, orbit 20: $A_3A_2A_1$

L :  τ : $\begin{smallmatrix} 2 & 2 & -6 & 2 & 2 & 2 & -3 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0000000\\1}}$

$C^\circ = {A_1}^2 \qquad C/C^\circ = 1$

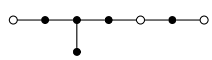
$\beta_1 = {\substack{2465432\\3}}$; $x_{\beta_2}(t) = x_{\substack{0111000\\1}}(2t)x_{\substack{0011110\\0}}(t)x_{\substack{1110000\\1}}(3t)x_{\substack{0111100\\0}}(t)x_{\substack{0122110\\1}}(t^2) \times$
$x_{\substack{1121110\\1}}(\frac{3}{2}t^2)x_{\substack{1221100\\1}}(-\frac{3}{2}t^2)x_{\substack{1232110\\2}}(-2t^3)x_{\substack{1232210\\1}}(-t^3)x_{\substack{1343210\\2}}(\frac{9}{4}t^4)x_{\substack{1111000\\0}}(t) \times$
$x_{\substack{0011100\\1}}(t)x_{\substack{1122100\\1}}(-\frac{1}{2}t^2)$,

$x_{-\beta_2}(t) = x_{-\substack{0111000\\1}}(2t)x_{-\substack{0011110\\0}}(6t)x_{-\substack{1110000\\1}}(2t)x_{-\substack{0111100\\0}}(4t)x_{-\substack{0122110\\1}}(-6t^2) \times$
$x_{-\substack{1121110\\1}}(-6t^2)x_{-\substack{1221100\\1}}(4t^2)x_{-\substack{1232110\\2}}(-8t^3)x_{-\substack{1232210\\1}}(-16t^3)x_{-\substack{1343210\\2}}(-36t^4) \times$
$x_{-\substack{1111000\\0}}(2t)x_{-\substack{0011100\\1}}(2t)x_{-\substack{1122100\\1}}(2t^2)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
6	1	$\lambda_1+4\lambda_2$	$e_{\substack{2454321\\2}}-e_{\substack{2354321\\3}}$		
5	2	$8\lambda_2$	$e_{\substack{2343210\\2}}$		
	2	$4\lambda_2$	$3e_{\substack{1221110\\1}}+2e_{\substack{1222100\\1}}+e_{\substack{0122210\\1}}-e_{\substack{1122110\\1}}$		
4	3	$\lambda_1+6\lambda_2$	$e_{\substack{2465431\\3}}$		
3	4	$6\lambda_2$	$e_{\substack{1233210\\2}}$		
	4	$2\lambda_2$	$e_{\substack{1111100\\1}}-e_{\substack{1111110\\0}}+2e_{\substack{0111110\\1}}$		
2	5	$\lambda_1+2\lambda_2$	$e_{\substack{2344321\\2}}$		
1	2	0	e	z_2	z_2
	6	$4\lambda_2$	$e_{\substack{1222210\\1}}$		

If $p=7$, the two summands with $m=2$ and high weights $8\lambda_2$ and $4\lambda_2$ are replaced by a single reducible tilting module $T_{A_1{}^2}(8\lambda_2)$ generated by the two given vectors together with $e_{\substack{1221110\\1}}+e_{\substack{1222100\\1}}$.

E_8, orbit 21: D_4A_1

L :  τ : $\substack{-6\;\;2\;\;2\;\;2\;\;-7\;\;2\;\;-1\\2}$ $\Delta = \substack{0\;\;0\;\;0\;\;0\;\;0\;\;1\;\;2\\1}$

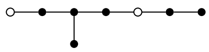
$e=e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0000000\\1}}+e_{\substack{0001000\\0}}+e_{\substack{0000010\\0}}$

$C^\circ=C_3\qquad C/C^\circ=1$

$x_{\pm\beta_1}(t)=x_{\pm\substack{0011111\\1}}(t)x_{\pm\substack{0111111\\0}}(-t)$, $x_{\pm\beta_2}(t)=x_{\pm\substack{1110000\\1}}(t)x_{\pm\substack{1111000\\0}}(t)$, $\beta_3=\substack{0122210\\1}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	1	λ_3	$e_{\substack{2465431\\3}}$		
	5	λ_1	$e_{\substack{1343211\\2}}-e_{\substack{1243221\\2}}$		
2	2	0	$e_{\substack{0000010\\0}}$		
	6	λ_2	$e_{\substack{2465321\\3}}$		
1	2	0	e	z_2	z_2
	7	λ_1	$e_{\substack{1343221\\2}}$		
	10	0	$e_{\substack{0121000\\1}}$	z_{10}	z_{10}

E_8, orbit 22: $D_4(a_1)A_2$

L :  τ : $\begin{smallmatrix}-4&2&0&2&-6&2&2\\&&2&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}0&0&0&0&0&0&0\\&&2&&&&\end{smallmatrix}$

$e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

$C^\circ = A_2 \qquad C/C^\circ = \langle cC^\circ\rangle \cong S_2$

$x_{\beta_1}(t) = x_{\substack{0001111\\0}}(3t)x_{\substack{0121100\\1}}(t)x_{\substack{0011111\\0}}(-t)x_{\substack{0111100\\1}}(t)x_{\substack{0122211\\1}}(-2t^2)x_{\substack{0011110\\1}}(2t)\times$
$x_{\substack{0111110\\0}}(-t)$,

$x_{-\beta_1}(t) = x_{-\substack{0001111\\0}}(t)x_{-\substack{0121100\\1}}(3t)x_{-\substack{0011111\\0}}(-t)x_{-\substack{0111100\\1}}(t)x_{-\substack{0122211\\1}}(2t^2)\times$
$x_{-\substack{0011110\\1}}(t)x_{-\substack{0111110\\0}}(-2t)$,

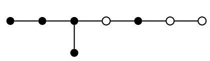
$x_{\beta_2}(t) = x_{\substack{1111111\\1}}(-3t)x_{\substack{1232100\\1}}(-t)x_{\substack{1121111\\1}}(-t)x_{\substack{1222100\\1}}(t)x_{\substack{2343211\\2}}(-2t^2)x_{\substack{1122110\\1}}(2t)\times$
$x_{\substack{1221110\\1}}(-t)$,

$x_{-\beta_2}(t) = x_{-\substack{1111111\\1}}(-t)x_{-\substack{1232100\\1}}(-3t)x_{-\substack{1121111\\1}}(-t)x_{-\substack{1222100\\1}}(t)x_{-\substack{2343211\\2}}(2t^2)\times$
$x_{-\substack{1122110\\1}}(t)x_{-\substack{1221110\\1}}(-2t)$

$c = n_{\substack{1111000\\0}}n_{\substack{1110000\\1}}h_5(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	2	$2\lambda_1+2\lambda_2$	$e_{\substack{2465432\\3}}$		
2	4	$3\lambda_1$	$e_{\substack{1354321\\3}}$		
	4	$3\lambda_2$	$e_{\substack{2465321\\3}}$		
1	2	0	e	z_2	z_2
	6	$\lambda_1+\lambda_2$	$e_{\substack{1343221\\2}}$		

E_8, orbit 23: A_4A_1

L :  τ : $\begin{smallmatrix}2&2&2&-7&2&-1&0\\&&2&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}1&0&0&0&1&0&1\\&&0&&&&\end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}}$

$C^\circ = A_2T_1 \qquad C/C^\circ = \langle cC^\circ\rangle \cong S_2$

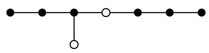
$\beta_1 = {\substack{0000001\\0}}$, $\beta_2 = {\substack{2465431\\3}}$

$T_1 = \{h_1(\mu^4)h_2(\mu^6)h_3(\mu^8)h_4(\mu^{12})h_5(\mu^{10})h_6(\mu^5) : \mu \in k^*\}$

$c = n_{\substack{0111110\\1}}n_{\substack{1111110\\0}}n_{\substack{1222100\\1}}n_{\substack{1243210\\2}}h_1(-1)h_3(-1)h_4(-1)h_6(-1)h_8(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
8	1	λ_1	$e_{\substack{0000111\\0}}$		
	1	λ_2	$e_{\substack{2465421\\3}}$		
	1	0	$f_{\substack{0111000\\1}} + f_{\substack{0011100\\1}} - f_{\substack{0111100\\0}} - 2f_{\substack{1111000\\0}}$		
	1	0	$e_{\substack{0111100\\1}} + e_{\substack{0121000\\1}} - e_{\substack{1111000\\1}} - 2e_{\substack{1111100\\0}}$		
7	2	λ_1	$e_{\substack{1111111\\1}} - e_{\substack{0121111\\1}}$		
	2	λ_2	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
	2	0	$e_{\substack{0000100\\0}}$		
6	3	λ_1	$e_{\substack{1232211\\1}} - e_{\substack{1232111\\2}}$		
	3	λ_2	$e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}}$		
	3	0	$f_{\substack{0111000\\0}} - f_{\substack{0011000\\1}}$		
	3	0	$e_{\substack{1111100\\1}} - e_{\substack{0121100\\1}}$		
5	4	λ_2	$e_{\substack{1232221\\2}}$		
	4	λ_1	$e_{\substack{2343211\\2}}$		
	4	0	$f_{\substack{0122100\\1}}$		
	4	0	$e_{\substack{1232100\\2}}$		
	4	0	$e_{\substack{1100000\\0}} + e_{\substack{0110000\\0}} - e_{\substack{0010000\\1}}$		
4	5	λ_1	$e_{\substack{1232211\\2}}$		
	5	λ_2	$e_{\substack{2343321\\2}}$		
	5	0	$f_{\substack{0011000\\0}} + f_{\substack{0001100\\0}}$		
	5	0	$e_{\substack{1221000\\1}} + e_{\substack{1121100\\1}}$		
3	6	λ_1	$e_{\substack{1221111\\1}}$		
	6	λ_2	$e_{\substack{2464321\\3}}$		
	6	0	$e_{\substack{1110000\\0}} - e_{\substack{0110000\\1}}$		
2	7	0	$f_{\substack{0001000\\0}}$		
	7	0	$e_{\substack{1221100\\1}}$		
1	2	0	e	z_2	z_2
	8	0	$e_{\substack{1110000\\1}}$	z_8	

E_8, orbit 24: ${A_3}^2$

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & -6 & 2 & 2 & 2 \\ & & -3 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

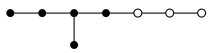
$C^\circ = B_2 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{1232210\\2}}(t)x_{\pm \substack{1232111\\2}}(t)$,

$x_{\pm\beta_2}(t) = x_{\pm \substack{1111000\\0}}(t)x_{\pm \substack{0111100\\0}}(t)x_{\pm \substack{0011110\\0}}(t)x_{\pm \substack{0001111\\0}}(t)$,

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
7	1	λ_2	$2e_{\substack{0122221\\1}} - 2e_{\substack{1122211\\1}} + e_{\substack{1232110\\1}} - e_{\substack{1222210\\1}} + e_{\substack{1222111\\1}}$		
6	2	λ_1	$e_{\substack{2343211\\2}} - e_{\substack{1343221\\2}} + e_{\substack{1243321\\2}}$		
5	3	$\lambda_1 + \lambda_2$	$e_{\substack{2465432\\3}}$		
4	4	$2\lambda_2$	$e_{\substack{2454321\\2}}$		
3	5	λ_2	$e_{\substack{1232211\\1}} - e_{\substack{1222221\\1}}$		
2	6	λ_1	$e_{\substack{2343321\\2}}$		
	6	0	$e_{\substack{1110000\\0}} + e_{\substack{0000111\\0}}$		
1	2	0	e	z_2	z_2
	7	λ_2	$e_{\substack{1232221\\1}}$		

E_8, orbit 25: $D_5(a_1)$

L :  τ : $\begin{smallmatrix} 2 & 2 & 0 & 2 & -7 & 0 & 0 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 2 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$

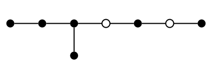
$C^\circ = A_3 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = {\substack{0000001\\0}}$, $\beta_2 = {\substack{0000010\\0}}$, $\beta_3 = {\substack{2465421\\3}}$

$c = n_{\substack{1233210\\1}} n_{\substack{1232210\\2}} h_1(-1)h_2(-1)h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	λ_1	$e_{\substack{1111111\\1}} + e_{\substack{0122111\\1}}$		
	1	λ_3	$e_{\substack{2344321\\2}} - e_{\substack{1354321\\3}}$		
	4	0	$e_{\substack{1110000\\0}} + e_{\substack{0111000\\0}} - e_{\substack{0110000\\1}} - 2e_{\substack{0011000\\1}}$		
4	2	λ_2	$e_{\substack{1233221\\2}}$		
	2	0	$e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$		
	5	λ_1	$e_{\substack{1222111\\1}} - e_{\substack{1232111\\1}}$		
	5	λ_3	$e_{\substack{2454321\\3}} + e_{\substack{2464321\\3}}$		
3	6	λ_2	$e_{\substack{2343221\\2}}$		
	6	0	$e_{\substack{1111000\\0}} - e_{\substack{0111000\\1}} + e_{\substack{0121000\\1}}$		
	6	0	$e_{\substack{1110000\\1}} + e_{\substack{0111000\\1}} + e_{\substack{0121000\\1}}$		
2	7	λ_1	$e_{\substack{1232111\\2}}$		
	7	λ_3	$e_{\substack{2465321\\3}}$		
1	2	0	e	z_2	z_2
	8	0	$e_{\substack{1111000\\1}}$	z_8	
	10	0	$e_{\substack{1221000\\1}}$	z_{10}	z_{10}

E_8, orbit 26: $A_4{A_1}^2$

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & -7 & 2 & -2 & 2 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$

$C^\circ = A_1T_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

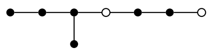
$x_{\pm\beta_1}(t) = x_{\pm \substack{0000110\\0}}(t) x_{\pm \substack{0000011\\0}}(t)$

$T_1 = \{h_1(\mu^8)h_2(\mu^{12})h_3(\mu^{16})h_4(\mu^{24})h_5(\mu^{20})h_6(\mu^{15})h_7(\mu^{10})h_8(\mu^5) : \mu \in k^*\}$

$c = n_{\substack{1222210\\1}} n_{\substack{1222111\\1}} n_{\substack{1354321\\3}} n_{\substack{2354321\\2}} h_1(-1)h_3(-1)h_4(-1)h_6(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
8	1	λ_1	$f_{\substack{2465321\\3}}$		
	1	λ_1	$e_{\substack{2465432\\3}}$		
	1	λ_1	$2f_{\substack{1111000\\0}} - f_{\substack{0111000\\1}} + f_{\substack{0111100\\0}} - f_{\substack{0011100\\1}}$		
	1	λ_1	$2e_{\substack{1111111\\0}} - e_{\substack{0111111\\1}} + e_{\substack{1111110\\1}} - e_{\substack{0121110\\1}}$		
7	2	$2\lambda_1$	$e_{\substack{0000111\\0}}$		
	2	0	$f_{\substack{2344321\\2}} - f_{\substack{1354321\\2}}$		
	2	0	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
	2	0	$2f_{\substack{1222110\\1}} - f_{\substack{1122210\\1}} + f_{\substack{1122111\\1}} - f_{\substack{0122211\\1}}$		
	2	0	$2e_{\substack{1222211\\1}} - e_{\substack{1232111\\1}} + e_{\substack{1232210\\1}} - e_{\substack{1232110\\2}}$		
	2	0	$e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$		
6	3	λ_1	$f_{\substack{1233210\\2}} + f_{\substack{1233211\\1}}$		
	3	λ_1	$e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}}$		
	3	λ_1	$f_{\substack{0111000\\0}} - f_{\substack{0011000\\1}}$		
	3	λ_1	$e_{\substack{0121111\\1}} - e_{\substack{1111111\\1}}$		
5	4	$2\lambda_1$	$f_{\substack{0122100\\1}}$		
	4	$2\lambda_1$	$e_{\substack{1232221\\2}}$		
	4	0	$f_{\substack{0122111\\1}} - f_{\substack{0122210\\1}} + 2f_{\substack{1122110\\1}}$		
	4	0	$e_{\substack{1232210\\2}} - e_{\substack{1232111\\2}} + 2e_{\substack{1232211\\1}}$		
	4	0	$e_{\substack{1100000\\0}} + e_{\substack{0110000\\0}} - e_{\substack{0010000\\1}}$		
4	5	λ_1	$f_{\substack{1233210\\1}}$		
	5	λ_1	$e_{\substack{2343321\\2}}$		
	5	λ_1	$f_{\substack{0011000\\0}} + f_{\substack{0001100\\0}}$		
	5	λ_1	$e_{\substack{1221110\\1}} + e_{\substack{1121111\\1}}$		
3	6	0	$f_{\substack{1244321\\2}}$		
	6	0	$e_{\substack{2464321\\3}}$		
	6	0	$f_{\substack{0122110\\1}}$		
	6	0	$e_{\substack{1232211\\2}}$		
	6	0	$e_{\substack{1110000\\0}} - e_{\substack{0110000\\1}}$		
2	7	λ_1	$f_{\substack{0001000\\0}}$		
	7	λ_1	$e_{\substack{1221111\\1}}$		
1	2	0	e	z_2	z_2
	8	0	$e_{\substack{1110000\\1}}$	z_8	

E_8, orbit 27: A_4A_2

L :  τ : $\begin{array}{ccccccc} 2 & 2 & 2 & -8 & 2 & 2 & -2 \\ & & 2 & & & & \end{array}$ $\Delta = \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ & & 0 & & & & \end{array}$

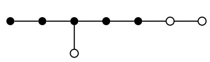
$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$

$C^\circ = {A_1}^2 \qquad C/C^\circ = 1$

$\beta_1 = {}^{2465432}_{3}$; $x_{\beta_2}(t) = x_{\substack{0011110\\1}}(3t)x_{\substack{1111100\\0}}(-2t)x_{\substack{0121000\\1}}(t)x_{\substack{1122210\\1}}(3t^2)x_{\substack{1232100\\1}}(-t^2)\times$
$x_{\substack{1243210\\2}}(-2t^3)x_{\substack{1111000\\1}}(-t)x_{\substack{0111110\\0}}(-2t)x_{\substack{1222110\\1}}(t^2)x_{\substack{0111100\\1}}(t)$,
$x_{-\beta_2}(t) = x_{-\substack{0011110\\1}}(t)x_{-\substack{1111100\\0}}(-2t)x_{-\substack{0121000\\1}}(3t)x_{-\substack{1122210\\1}}(-t^2)x_{-\substack{1232100\\1}}(3t^2)\times$
$x_{-\substack{1243210\\2}}(-2t^3)x_{-\substack{1111000\\1}}(-2t)x_{-\substack{0111110\\0}}(-t)x_{-\substack{1222110\\1}}(-t^2)x_{-\substack{0111100\\1}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	2	$\lambda_1+5\lambda_2$	$e_{\substack{2465431\\3}}$		
	2	$4\lambda_2$	$e_{\substack{1232100\\2}} + e_{\substack{1232110\\1}} - e_{\substack{1222210\\1}}$		
3	4	$\lambda_1+\lambda_2$	$e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}}$		
	4	$6\lambda_2$	$e_{\substack{2343210\\2}}$		
	4	$2\lambda_2$	$e_{\substack{1221000\\1}} + e_{\substack{1121100\\1}} + 2e_{\substack{1111110\\1}} - e_{\substack{0121110\\1}}$		
2	6	$\lambda_1+3\lambda_2$	$e_{\substack{2464321\\3}}$		
	6	$4\lambda_2$	$e_{\substack{1232210\\2}}$		
1	2	0	e	z_2	z_2
	8	$2\lambda_2$	$e_{\substack{1221110\\1}}$		

E_8, orbit 28: A_5

L :  τ : $\begin{array}{ccccccc} 2 & 2 & 2 & 2 & 2 & -5 & 0 \\ & & -9 & & & & \end{array}$ $\Delta = \begin{array}{ccccccc} 2 & 0 & 0 & 0 & 1 & 0 & 1 \\ & & 0 & & & & \end{array}$

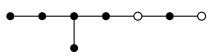
$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}}$

$C^\circ = G_2A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm\substack{1221110\\1}}(t)x_{\pm\substack{1122110\\1}}(-t)x_{\pm\substack{0122210\\1}}(t)$, $\beta_2 = {}^{0000001}_{0}$; $\beta_3 = {}^{1232100}_{2}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	3	λ_3	$e_{\substack{1221000\\1}} + e_{\substack{1121100\\1}} - e_{\substack{0122100\\1}}$		
2	4	λ_1	$e_{\substack{2344321\\2}} - e_{\substack{1354321\\2}}$		
	5	$\lambda_1 + \lambda_3$	$e_{\substack{2465421\\3}}$		
	6	0	$e_{\substack{1110000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0011100\\0}}$		
1	2	0	e	z_2	z_2
	8	λ_1	$e_{\substack{2454321\\2}}$		
	9	λ_3	$e_{\substack{1232100\\1}}$		
	10	0	$e_{\substack{1111100\\0}}$	z_{10}	z_{10}

E_8, orbit 29: $D_5(a_1)A_1$

L :  τ : $\substack{2\ \ 2\ \ 0\ \ 2\ \ -8\ \ 2\ \ -1\\2}$ $\Delta = \substack{0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 2\\0}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$

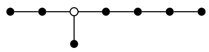
$C^\circ = {A_1}^2 \qquad C/C^\circ = 1$

$\beta_1 = \substack{2465432\\3}$; $x_{\beta_2}(t) = x_{\substack{1111100\\1}}(t)x_{\substack{0121110\\1}}(t)x_{\substack{1232210\\2}}(\frac{1}{2}t^2)x_{\substack{0122100\\1}}(t)x_{\substack{1111110\\0}}(t) \times$
$x_{\substack{1233210\\1}}(\frac{1}{2}t^2)$,

$x_{-\beta_2}(t) = x_{-\substack{1111100\\1}}(2t)x_{-\substack{0121110\\1}}(2t)x_{-\substack{1232210\\2}}(-2t^2)x_{-\substack{0122100\\1}}(2t)x_{-\substack{1111110\\0}}(2t) \times$
$x_{-\substack{1233210\\1}}(-2t^2)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	$\lambda_1 + 4\lambda_2$	$e_{\substack{2465431\\3}}$		
	4	$2\lambda_2$	$e_{\substack{1221110\\1}} + e_{\substack{1232100\\1}} - e_{\substack{1222100\\1}} - 2e_{\substack{1122110\\1}}$		
4	2	$4\lambda_2$	$e_{\substack{1233210\\2}}$		
	2	0	$e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$		
	5	$\lambda_1 + 2\lambda_2$	$e_{\substack{2454321\\3}} + e_{\substack{2464321\\3}}$		
3	3	λ_1	$e_{\substack{1233221\\2}}$		
	6	$4\lambda_2$	$e_{\substack{2343210\\2}}$		
	6	$2\lambda_2$	$e_{\substack{1232100\\2}} - e_{\substack{1232110\\1}} + 2e_{\substack{1222110\\1}}$		
2	7	$\lambda_1 + 2\lambda_2$	$e_{\substack{2465321\\3}}$		
1	2	0	e	z_2	z_2
	8	$2\lambda_2$	$e_{\substack{1232110\\2}}$		
	10	0	$e_{\substack{1221000\\1}}$	z_{10}	z_{10}

E_8, orbit 30: $A_4A_2A_1$

L :  τ : $\begin{smallmatrix}2 & 2 & -7 & 2 & 2 & 2 & 2\\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}0 & 1 & 0 & 0 & 1 & 0 & 0\\ & & 0 & & & & \end{smallmatrix}$

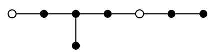
$e = e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0000000\\1}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\beta_1}(t) = x_{\substack{0122210\\1}}(t)x_{\substack{1121111\\1}}(-2t)x_{\substack{1222100\\1}}(-3t)x_{\substack{1243321\\2}}(t^2)x_{\substack{2343211\\2}}(3t^2)\times$
$x_{\substack{2465421\\3}}(2t^3)x_{\substack{1221110\\1}}(-2t)x_{\substack{0122111\\1}}(t)x_{\substack{1343221\\2}}(t^2)x_{\substack{1122110\\1}}(-t)$,
$x_{-\beta_1}(t) = x_{-\substack{0122210\\1}}(3t)x_{-\substack{1121111\\1}}(-2t)x_{-\substack{1222100\\1}}(-t)x_{-\substack{1243321\\2}}(-3t^2)x_{-\substack{2343211\\2}}(-t^2)\times$
$x_{-\substack{2465421\\3}}(2t^3)x_{-\substack{1221110\\1}}(-t)x_{-\substack{0122111\\1}}(2t)x_{-\substack{1343221\\2}}(-t^2)x_{-\substack{1122110\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
8	1	$5\lambda_1$	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
7	2	$4\lambda_1$	$e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}} + e_{\substack{1244321\\2}}$		
	2	0	$e_{\substack{0000000\\1}}$		
6	3	$5\lambda_1$	$e_{\substack{2454321\\3}}$		
	3	λ_1	$3e_{\substack{1111100\\1}} + 2e_{\substack{1111110\\0}} + 2e_{\substack{0111111\\0}} + e_{\substack{0111110\\1}} - e_{\substack{0011111\\1}}$		
5	4	$6\lambda_1$	$e_{\substack{2465432\\3}}$		
	4	$2\lambda_1$	$e_{\substack{1222210\\1}} + 2e_{\substack{1222111\\1}} - e_{\substack{1122211\\1}} + e_{\substack{0122221\\1}}$		
4	5	$3\lambda_1$	$e_{\substack{1233221\\2}} - e_{\substack{1233321\\1}}$		
	5	λ_1	$e_{\substack{1111110\\1}} + e_{\substack{0111111\\1}}$		
3	6	$4\lambda_1$	$e_{\substack{2344321\\2}}$		
2	7	$3\lambda_1$	$e_{\substack{1233321\\2}}$		
1	2	0	e	z_2	z_2
	8	$2\lambda_1$	$e_{\substack{1222221\\1}}$		

E_8, orbit 31: D_4A_2

L :  τ : $\substack{-6\ 2\ 2\ 2\ -8\ 2\ 2\\ 2}$ $\Delta = \substack{0\ 0\ 0\ 0\ 0\ 0\ 2\\ 2}$

$e = e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

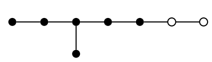
$C^\circ = A_2 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$x_{\pm\beta_1}(t) = x_{\pm \substack{1110000\\1}}(t)x_{\pm \substack{1111000\\0}}(t)$, $x_{\pm\beta_2}(t) = x_{\pm \substack{1343321\\2}}(t)x_{\pm \substack{1244321\\2}}(-t)$

$c = n_{\substack{0011110\\1}} n_{\substack{0111110\\0}} n_{\substack{0122211\\1}} h_2(-1)h_4(-1)h_5(-1)h_6(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	2	$2\lambda_1$	$e_{\substack{2343221\\2}}$		
	2	$2\lambda_2$	$e_{\substack{2465432\\3}}$		
	2	0	$e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$		
	4	λ_1	$e_{\substack{1232100\\2}} + e_{\substack{1232110\\1}} + e_{\substack{1222111\\1}}$		
	4	λ_2	$e_{\substack{1343210\\2}} + e_{\substack{1243211\\2}} - e_{\substack{1233221\\2}}$		
2	4	0	$e_{\substack{0000011\\0}}$		
	6	$\lambda_1+\lambda_2$	$e_{\substack{2465321\\3}}$		
	6	λ_1	$e_{\substack{1232110\\2}} + e_{\substack{1232111\\1}}$		
	6	λ_2	$e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}}$		
1	2	0	e	z_2	z_2
	8	λ_1	$e_{\substack{1232111\\2}}$		
	8	λ_2	$e_{\substack{1343221\\2}}$		
	10	0	$e_{\substack{0121000\\1}}$	z_{10}	z_{10}

E_8, orbit 32: $E_6(a_3)$

L :  τ : $\substack{2\ 0\ 2\ 0\ 2\ -8\ 0\\ 0}$ $\Delta = \substack{2\ 0\ 0\ 0\ 0\ 2\ 0\\ 0}$

$e = e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}}$

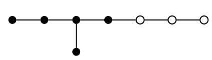
$C^\circ = G_2 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$x_{\pm\beta_1}(t) = x_{\pm \substack{1221110\\1}}(t)x_{\pm \substack{1122110\\1}}(t)x_{\pm \substack{0122210\\1}}(-t)$, $\beta_2 = \substack{0000001\\0}$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	0	$e_{\substack{1100000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}} - 3e_{\substack{0010000\\1}} + e_{\substack{0111000\\1}}$		
	2	0	$e_{\substack{0010000\\0}}$		
4	4	0	$e_{\substack{1110000\\0}} - e_{\substack{0011100\\0}} - e_{\substack{0121000\\1}}$		
3	4	λ_1	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$		
	4	λ_1	$e_{\substack{2354321\\2}} - e_{\substack{2454321\\3}}$		
	6	0	$e_{\substack{1111100\\0}} - e_{\substack{1221000\\1}} - e_{\substack{0122100\\1}}$		
2	6	λ_1	$e_{\substack{2464321\\3}}$		
	8	0	$e_{\substack{1121100\\1}} + e_{\substack{1222100\\1}}$		
1	2	0	e	z_2	z_2
	8	λ_1	$e_{\substack{2465421\\3}}$		
	10	0	$e_{\substack{1232100\\1}}$	${z_{10}}^1$	
	10	0	$e_{\substack{1232100\\2}}$	${z_{10}}^2$	${z_{10}}^2$

E_8, orbit 33: D_5

L :  τ : $\begin{matrix} 2 & 2 & 2 & 2 & -10 & 0 & 0 \\ & & 2 & & & & \end{matrix}$ $\Delta = \begin{matrix} 2 & 0 & 0 & 0 & 0 & 2 & 2 \\ & & 0 & & & & \end{matrix}$

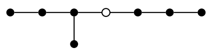
$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}}$

$C^\circ = B_3 \qquad C/C^\circ = 1$

$\beta_1 = {}^{0000010}_{\quad 0}$, $\beta_2 = {}^{0000001}_{\quad 0}$, $x_{\pm\beta_3}(t) = x_{\pm \substack{1232210\\2}}(t)x_{\pm \substack{1233210\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	4	λ_3	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
	6	0	$e_{\substack{1110000\\0}} - e_{\substack{0110000\\1}} - 2e_{\substack{0011000\\1}} + e_{\substack{0111000\\0}}$		
2	10	λ_3	$e_{\substack{2465321\\3}}$		
1	2	0	e	z_2	z_2
	8	λ_1	$e_{\substack{2343221\\2}}$		
	10	0	$e_{\substack{1111000\\1}} - e_{\substack{0121000\\1}}$	z_{10}	z_{10}
	14	0	$e_{\substack{1221000\\1}}$	z_{14}	z_{14}

E_8, orbit 34: A_4A_3

L :  τ : $\substack{2\ \ 2\ \ 2\ {-9}\ \ 2\ \ 2\ \ 2\\2}$ $\Delta = \substack{0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0\\0}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

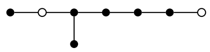
$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\beta_1}(t) = x_{\substack{1122211\\1}}(2t)x_{\substack{1232110\\1}}(-t)x_{\substack{2354321\\2}}(-t^2)x_{\substack{0122221\\1}}(-2t)x_{\substack{1232100\\2}}(-t)x_{\substack{1354321\\3}}(t^2)\times$
$x_{\substack{1222210\\1}}(t)x_{\substack{1222111\\1}}(-t)$,

$x_{-\beta_1}(t) = x_{-\substack{1122211\\1}}(t)x_{-\substack{1232110\\1}}(-2t)x_{-\substack{2354321\\2}}(t^2)x_{-\substack{0122221\\1}}(-t)x_{-\substack{1232100\\2}}(-2t)\times$
$x_{-\substack{1354321\\3}}(-t^2)x_{-\substack{1222210\\1}}(t)x_{-\substack{1222111\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
9	1	$3\lambda_1$	$e_{\substack{2343210\\2}} + e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}} + e_{\substack{1233321\\2}}$		
8	2	$4\lambda_1$	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
7	3	$5\lambda_1$	$e_{\substack{2465432\\3}}$		
	3	λ_1	$e_{\substack{1221000\\1}} - e_{\substack{0121110\\1}} + e_{\substack{1121100\\1}} - e_{\substack{0111111\\1}} + 3e_{\substack{1111111\\0}} + 2e_{\substack{1111110\\1}}$		
6	4	$2\lambda_1$	$e_{\substack{1232210\\2}} + e_{\substack{1232111\\2}}$		
	4	$2\lambda_1$	$e_{\substack{1232210\\2}} + e_{\substack{1232211\\1}} - e_{\substack{1222221\\1}}$		
5	5	$3\lambda_1$	$e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}}$		
	5	λ_1	$e_{\substack{1221100\\1}} + e_{\substack{1121110\\1}} - e_{\substack{1111111\\1}} + 2e_{\substack{0121111\\1}}$		
4	6	$4\lambda_1$	$e_{\substack{2464321\\3}}$		
	6	0	$e_{\substack{1110000\\0}} - e_{\substack{0110000\\1}} - 2e_{\substack{0000111\\0}}$		
3	7	$3\lambda_1$	$e_{\substack{2343321\\2}}$		
2	8	$2\lambda_1$	$e_{\substack{1232221\\2}}$		
1	2	0	e	z_2	z_2
	9	λ_1	$e_{\substack{1221111\\1}}$		

E_8, orbit 35: A_5A_1

L :  τ : $\begin{smallmatrix} 2 & -9 & 2 & 2 & 2 & 2 & -5 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ & & 0 & & & & \end{smallmatrix}$

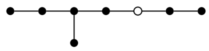
$e = e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{1000000\\0}}$

$C^\circ = {A_1}^2 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{2465432\\3}}$, $x_{\pm\beta_2}(t) = x_{\pm \substack{1232100\\2}}(t)x_{\pm \substack{1232110\\1}}(t)x_{\pm \substack{1222210\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	$3\lambda_2$	$e_{\substack{2343210\\2}}$		
	3	λ_2	$e_{\substack{1121100\\1}} - e_{\substack{0122100\\1}} - e_{\substack{0121110\\1}} + 2e_{\substack{1111110\\1}}$		
4	2	0	$e_{\substack{1000000\\0}}$		
	3	λ_1	$e_{\substack{1243211\\2}} - e_{\substack{1233221\\2}} + e_{\substack{1233321\\1}}$		
	4	$\lambda_1+\lambda_2$	$e_{\substack{2354321\\2}} - e_{\substack{1354321\\3}}$		
	4	$2\lambda_2$	$e_{\substack{1233210\\1}} - e_{\substack{1232210\\2}}$		
3	5	$\lambda_1+2\lambda_2$	$e_{\substack{2465431\\3}}$		
	5	λ_2	$e_{\substack{1122100\\1}} + e_{\substack{1121110\\1}}$		
	7	λ_2	$e_{\substack{1122110\\1}} - e_{\substack{0122210\\1}}$		
2	6	$\lambda_1+\lambda_2$	$e_{\substack{2354321\\3}}$		
	6	0	$e_{\substack{0011000\\1}} + e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}}$		
	8	$2\lambda_2$	$e_{\substack{1243210\\2}}$		
1	2	0	e	z_2	z_2
	9	λ_1	$e_{\substack{1244321\\2}}$		
	9	λ_2	$e_{\substack{1122210\\1}}$		
	10	0	$e_{\substack{0011110\\1}}$	z_{10}	z_{10}

E_8, orbit 36: $D_5(a_1)A_2$

L :  $\quad\tau$: $\begin{smallmatrix}2&2&0&2&-9&2&2\\&&2&&&&\end{smallmatrix}$ $\quad\Delta = \begin{smallmatrix}0&1&0&0&1&0&1\\&&0&&&&\end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

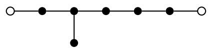
$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\beta_1}(t) = x_{\substack{1232221\\1}}(t)x_{\substack{1233210\\2}}(-2t)x_{\substack{2465431\\3}}(t^2)x_{\substack{1232211\\2}}(-t)x_{\substack{1233211\\1}}(-t),$

$x_{-\beta_1}(t) = x_{-\substack{1232221\\1}}(2t)x_{-\substack{1233210\\2}}(-t)x_{-\substack{2465431\\3}}(-t^2)x_{-\substack{1232211\\2}}(-t)x_{-\substack{1233211\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
7	1	$3\lambda_1$	$e_{\substack{2344321\\2}} - e_{\substack{1354321\\3}}$		
	3	λ_1	$e_{\substack{1221110\\1}} + e_{\substack{1121111\\1}} - e_{\substack{1222100\\1}} - 2e_{\substack{1122110\\1}} + 3e_{\substack{1111111\\1}} + e_{\substack{1232100\\1}}$		
6	2	$4\lambda_1$	$e_{\substack{2465432\\3}}$		
	2	0	$e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$		
	4	$2\lambda_1$	$e_{\substack{2343210\\2}} + e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}}$		
5	3	λ_1	$e_{\substack{1111111\\1}} + e_{\substack{0122111\\1}}$		
	5	$3\lambda_1$	$e_{\substack{2454321\\3}} + e_{\substack{2464321\\3}}$		
	5	λ_1	$e_{\substack{1221111\\1}} - 3e_{\substack{1122111\\1}} - 2e_{\substack{1222110\\1}} + e_{\substack{1232110\\1}} - e_{\substack{1232100\\2}}$		
4	4	$2\lambda_1$	$e_{\substack{1233221\\2}}$		
	6	$2\lambda_1$	$e_{\substack{1343221\\2}} - e_{\substack{2343211\\2}}$		
	6	0	$e_{\substack{1110000\\1}} + e_{\substack{1111000\\0}} + 2e_{\substack{0121000\\1}}$		
3	7	$3\lambda_1$	$e_{\substack{2465321\\3}}$		
	7	λ_1	$2e_{\substack{1232110\\2}} - e_{\substack{1232111\\1}} + 3e_{\substack{1222111\\1}}$		
2	8	$2\lambda_1$	$e_{\substack{2343221\\2}}$		
1	2	0	e	z_2	z_2
	9	λ_1	$e_{\substack{1232111\\2}}$		
	10	0	$e_{\substack{1221000\\1}}$	z_{10}	z_{10}

E_8, orbit 37: $D_6(a_2)$

L :  τ : $\begin{smallmatrix} -9 & 2 & 0 & 2 & 0 & 2 & -6 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ & & 1 & & & & \end{smallmatrix}$

$e = e_{\substack{0000010\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$

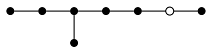
$C^\circ = {A_1}^2 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = {\substack{2343210\\2}}$, $\beta_2 = {\substack{2465432\\3}}$

$c = n_{\substack{0011111\\1}} n_{\substack{0111111\\0}} h_4(-1)h_5(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	0	$e_{\substack{0000000\\1}} - e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} - e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} - e_{\substack{0000110\\0}} - e_{\substack{0011100\\0}}$		
	2	0	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0011000\\0}}$		
4	3	λ_2	$e_{\substack{1233221\\2}} + e_{\substack{1233321\\2}} + e_{\substack{1243221\\2}} + e_{\substack{1243321\\2}}$		
	3	λ_2	$e_{\substack{1233221\\2}} + e_{\substack{1343211\\2}}$		
	3	λ_1	$e_{\substack{1222110\\1}} + e_{\substack{1232210\\1}} - e_{\substack{1222210\\1}} - e_{\substack{1232110\\1}}$		
	3	λ_1	$e_{\substack{1222110\\1}} + e_{\substack{1232100\\2}}$		
	4	0	$e_{\substack{0011000\\1}} + 2e_{\substack{0110000\\1}} + e_{\substack{0011100\\1}} - e_{\substack{0111000\\0}} + e_{\substack{0111100\\0}} + e_{\substack{0011110\\0}}$		
3	4	$\lambda_1 + \lambda_2$	$e_{\substack{2465321\\3}}$		
	5	λ_2	$e_{\substack{1343221\\2}} + e_{\substack{1343321\\2}} - e_{\substack{1244321\\2}}$		
	5	λ_1	$e_{\substack{1232210\\2}} - e_{\substack{1232110\\2}} + e_{\substack{1233210\\1}}$		
	6	0	$e_{\substack{0111100\\1}} + e_{\substack{0111110\\0}} - e_{\substack{0121000\\1}} - e_{\substack{0121100\\1}}$		
	6	0	$e_{\substack{0111000\\1}}$		
2	6	$\lambda_1 + \lambda_2$	$e_{\substack{2465431\\3}}$		
	6	0	$e_{\substack{0111110\\0}} - e_{\substack{0011110\\1}} - 2e_{\substack{0121100\\1}} - 2e_{\substack{0111000\\1}}$		
	7	λ_2	$e_{\substack{1344321\\2}} - e_{\substack{1354321\\2}}$		
	7	λ_1	$e_{\substack{1233210\\2}} + e_{\substack{1243210\\2}}$		
	8	0	$e_{\substack{0111110\\1}} - e_{\substack{0122100\\1}}$		
1	2	0	e	z_2	z_2
	9	λ_2	$e_{\substack{1354321\\3}}$		
	9	λ_1	$e_{\substack{1343210\\2}}$		
	10	0	$e_{\substack{0122110\\1}}$	${z_{10}}^1$	
	10	0	$e_{\substack{0122210\\1}}$	${z_{10}}^2$	${z_{10}}^2$

E_8, orbit 38: $E_6(a_3)A_1$

L :  τ : $\begin{smallmatrix}2&0&2&0&2&-9&2\\&&0&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}1&0&0&1&0&1&0\\&&0&&&&\end{smallmatrix}$

$e = e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0000001\\0}}$

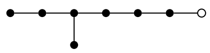
$C^\circ = A_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$x_{\pm\beta_1}(t) = x_{\pm\substack{1244321\\2}}(t)x_{\pm\substack{1343321\\2}}(t)x_{\pm\substack{2343221\\2}}(-t)$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	$3\lambda_1$	$e_{\substack{2465432\\3}}$		
	2	0	$e_{\substack{1100000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}} - 3e_{\substack{0010000\\1}} + e_{\substack{0111000\\1}}$		
	2	0	$e_{\substack{0010000\\0}}$		
	3	λ_1	$e_{\substack{1233210\\2}} + e_{\substack{1232210\\1}} + 2e_{\substack{1232111\\1}} - e_{\substack{1222211\\1}}$		
	3	λ_1	$e_{\substack{1233210\\1}} + e_{\substack{1232210\\2}} + 2e_{\substack{1232111\\2}} + e_{\substack{1122211\\1}}$		
4	2	0	$e_{\substack{0000001\\0}}$		
	4	$2\lambda_1$	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$		
	4	$2\lambda_1$	$e_{\substack{2354321\\2}} - e_{\substack{2454321\\3}}$		
	4	0	$e_{\substack{1110000\\0}} - e_{\substack{0011100\\0}} - e_{\substack{0121000\\1}}$		
	5	λ_1	$e_{\substack{1232211\\1}} - e_{\substack{1243210\\2}}$		
3	5	λ_1	$e_{\substack{1232211\\1}} + e_{\substack{1233211\\2}}$		
	5	λ_1	$e_{\substack{1232211\\2}} + e_{\substack{1233211\\1}}$		
	6	$2\lambda_1$	$e_{\substack{2464321\\3}}$		
	6	0	$e_{\substack{1111100\\0}} - e_{\substack{1221000\\1}} - e_{\substack{0122100\\1}}$		
	7	λ_1	$e_{\substack{2343210\\2}} + e_{\substack{1343211\\2}}$		
2	7	λ_1	$e_{\substack{1243211\\2}}$		
	8	$2\lambda_1$	$e_{\substack{2465421\\3}}$		
	8	0	$e_{\substack{1121100\\1}} + e_{\substack{1222100\\1}}$		
1	2	0	e	z_2	z_2
	9	λ_1	$e_{\substack{2343211\\2}}$		
	10	0	$e_{\substack{1232100\\1}}$	${z_{10}}^1$	
	10	0	$e_{\substack{1232100\\2}}$	${z_{10}}^2$	${z_{10}}^2$

E_8, orbit 39: $E_7(a_5)$

L :  τ : $\begin{smallmatrix}0&0&2&0&0&2&-9\\&&0&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}0&0&1&0&1&0&0\\&&0&&&&\end{smallmatrix}$

$e = e_{\substack{1111000\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0000010\\0}} + e_{\substack{0111100\\0}} + e_{\substack{1110000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0111000\\1}}$

$C^\circ = A_1 \qquad C/C^\circ = \langle c_1C^\circ, c_2C^\circ\rangle \cong S_3$

$\beta_1 = {\substack{2465432\\3}}$

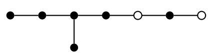
$c_1 = h_2(\omega)h_3(\omega)h_5(\omega)$,
$c_2 = n_{\substack{0000000\\1}} n_{\substack{0100000\\0}} n_{\substack{0001000\\0}} h_3(-1)h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	0	$e_{\substack{1111100\\0}} + e_{\substack{0111100\\1}} - 2e_{\substack{1111000\\1}} + 2e_{\substack{0011000\\0}} - e_{\substack{0110000\\0}} + e_{\substack{0000110\\0}}$		
	2	0	$e_{\substack{1111100\\1}} + e_{\substack{0011100\\0}} - 2e_{\substack{1110000\\0}} + 2e_{\substack{0110000\\1}} - e_{\substack{0011000\\1}} + e_{\substack{0001110\\0}}$		
	2	0	$e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} - e_{\substack{0111000\\0}}$		
	2	0	$e_{\substack{0011000\\0}} + e_{\substack{0110000\\0}} - e_{\substack{0010000\\1}}$		
	2	0	$e_{\substack{0010000\\0}} + e_{\substack{0111000\\1}}$		
4	3	λ_1	$e_{\substack{2343221\\2}} + e_{\substack{1343321\\2}} + e_{\substack{1244321\\2}}$		
	3	λ_1	$e_{\substack{2343321\\2}} + e_{\substack{1344321\\2}} + e_{\substack{1243221\\2}}$		
	3	λ_1	$e_{\substack{2344321\\2}} + e_{\substack{1243321\\2}} + e_{\substack{1343221\\2}}$		
	4	0	$e_{\substack{0011110\\0}} + e_{\substack{1121100\\1}} - 2e_{\substack{1221000\\1}} - e_{\substack{0122100\\1}}$		
	4	0	$e_{\substack{0111110\\1}} + e_{\substack{1222100\\1}} - 2e_{\substack{1121000\\1}} + e_{\substack{0121100\\1}}$		
	4	0	$e_{\substack{0111110\\0}} - e_{\substack{0011110\\1}} + e_{\substack{1221100\\1}} - e_{\substack{1122100\\1}}$		
	4	0	$e_{\substack{0121000\\1}}$		
3	5	λ_1	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$		
	5	λ_1	$e_{\substack{1354321\\3}} - e_{\substack{2354321\\2}}$		
	5	λ_1	$e_{\substack{1354321\\2}} + e_{\substack{2454321\\3}}$		
	6	0	$e_{\substack{1222110\\1}} + e_{\substack{1122210\\1}} - e_{\substack{0121110\\1}}$		
	6	0	$e_{\substack{1121110\\1}} - e_{\substack{1222210\\1}} + e_{\substack{0122110\\1}}$		
	6	0	$e_{\substack{1221110\\1}} + e_{\substack{1122110\\1}} + e_{\substack{0122210\\1}}$		
	6	0	$e_{\substack{0121110\\1}} + e_{\substack{1232100\\1}}$		
	6	0	$e_{\substack{0122110\\1}} + e_{\substack{1232100\\2}}$		
2	7	λ_1	$e_{\substack{2464321\\3}}$		
	7	λ_1	$e_{\substack{2465321\\3}}$		
	8	0	$e_{\substack{1232210\\1}} + e_{\substack{1232110\\2}}$		
	8	0	$e_{\substack{1233210\\2}} + e_{\substack{1232110\\1}}$		
	8	0	$e_{\substack{1232210\\2}} + e_{\substack{1233210\\1}}$		

(table continues on next page)

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	0	e	z_2	z_2
	9	λ_1	$e_{\substack{2465431\\3}}$		
	10	0	$e_{\substack{1243210\\2}}$	${z_{10}}^1$	
	10	0	$e_{\substack{1343210\\2}}$	${z_{10}}^2$	
	10	0	$e_{\substack{2343210\\2}}$	${z_{10}}^3$	${z_{10}}^3$

E_8, orbit 40: D_5A_1

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & 2 & -11 & 2 & -1 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 2 \\ & & 0 & & & & \end{smallmatrix}$

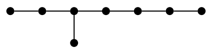
$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$

$C^\circ = {A_1}^2 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{2465432\\3}}$, $x_{\pm\beta_2}(t) = x_{\pm \substack{1232210\\2}}(t)x_{\pm \substack{1233210\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
6	1	$\lambda_1 + 2\lambda_2$	$e_{\substack{2465431\\3}}$		
	3	λ_2	$e_{\substack{1221100\\1}} + e_{\substack{1121110\\1}} - e_{\substack{1122100\\1}} - 2e_{\substack{0122110\\1}}$		
5	2	0	$e_{\substack{0000010\\0}}$		
	4	$\lambda_1 + \lambda_2$	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
	6	0	$e_{\substack{1110000\\0}} - e_{\substack{0110000\\1}} - 2e_{\substack{0011000\\1}} + e_{\substack{0111000\\0}}$		
4	5	λ_2	$e_{\substack{1221110\\1}} - e_{\substack{1122110\\1}}$		
	7	λ_1	$e_{\substack{2343211\\2}} - e_{\substack{1343221\\2}}$		
	9	λ_2	$e_{\substack{1232100\\2}} + e_{\substack{1232110\\1}}$		
3	8	$2\lambda_2$	$e_{\substack{2343210\\2}}$		
	10	$\lambda_1 + \lambda_2$	$e_{\substack{2465321\\3}}$		
2	11	λ_2	$e_{\substack{1232110\\2}}$		
1	2	0	e	z_2	z_2
	9	λ_1	$e_{\substack{2343221\\2}}$		
	10	0	$e_{\substack{1111000\\1}} - e_{\substack{0121000\\1}}$	z_{10}	z_{10}
	14	0	$e_{\substack{1221000\\1}}$	z_{14}	z_{14}

E_8, orbit 41: $E_8(a_7)$

L :  τ : $\substack{0\ 0\ 0\ 2\ 0\ 0\ 0\\0}$ $\Delta = \substack{0\ 0\ 0\ 2\ 0\ 0\ 0\\0}$

$e = e_{\substack{1111111\\0}} + e_{\substack{0121110\\1}} + e_{\substack{0001000\\0}} + e_{\substack{1121100\\1}} + e_{\substack{1221000\\1}} + e_{\substack{0011111\\1}} + e_{\substack{0111100\\0}} + e_{\substack{1111110\\1}}$

$C^\circ = 1 \qquad C/C^\circ = \langle c_1C^\circ, c_2C^\circ, c_3C^\circ\rangle \cong S_5$

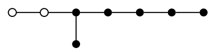
$c_1 = h_2(\zeta^2)h_3(\zeta^4)h_4(\zeta)h_6(\zeta^4)h_7(\zeta)h_8(\zeta^2)$,
$c_2 = n_{\substack{0100000\\0}} n_{\substack{0010000\\0}} n_{\substack{0000000\\1}} n_{\substack{0000100\\0}} n_{\substack{0000010\\0}} n_{\substack{0000001\\0}}$
$\quad \times\ h_1(-1)h_3(-1)h_5(-1)h_6(-1)h_8(-1)$,
$c_3 = (n_{\substack{1110000\\1}} n_{\substack{0110000\\0}} n_{\substack{0000111\\0}} n_{\substack{0000010\\0}} h)^u$,
$h = h_1(\frac{2}{5})h_2(-\frac{2}{5})h_3(\frac{2(1-3\phi)}{25})h_4(\frac{2(1-3\phi)}{25})h_6(\frac{3+\phi}{5})h_7(-\frac{3+\phi}{5})h_8(-\frac{3+\phi}{5})$,
$u = x_{\substack{1110000\\1}}(-\frac{1}{2}+\frac{1}{2}\phi)x_{\substack{0110000\\1}}(-3-2\phi)x_{\substack{0010000\\1}}(-3-2\phi)x_{\substack{0000000\\1}}(1)x_{\substack{1110000\\0}}(1+\phi)$
$\quad \times\ x_{\substack{0110000\\0}}(5+3\phi)x_{\substack{0010000\\0}}(4+2\phi)x_{\substack{1100000\\0}}(1+\frac{1}{2}\phi)x_{\substack{0100000\\0}}(1+\phi)x_{\substack{1000000\\0}}(\phi)$
$\quad \times\ x_{\substack{0000111\\0}}(-\phi)x_{\substack{0000100\\0}}(\phi)x_{\substack{0000011\\0}}(-1+\phi)x_{\substack{0000010\\0}}(-\phi)x_{\substack{0000001\\0}}(-\phi)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	$-$	$e_{\substack{0011110\\1}} + e_{\substack{0121000\\1}} + e_{\substack{0011111\\0}} + e_{\substack{0111111\\1}} + e_{\substack{0001100\\0}} - e_{\substack{1221100\\1}} - 2e_{\substack{1111110\\0}}$		
	2	$-$	$e_{\substack{0001111\\0}} + e_{\substack{0011100\\1}} - e_{\substack{0111000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0111110\\0}} - e_{\substack{1121110\\1}} - 2e_{\substack{1221111\\1}}$		
	2	$-$	$e_{\substack{0111000\\0}} - e_{\substack{0001110\\0}} + e_{\substack{0011100\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0121111\\1}} - e_{\substack{1111111\\1}} - 2e_{\substack{1121000\\1}}$		
	2	$-$	$e_{\substack{0121100\\1}} - e_{\substack{0111111\\0}} + e_{\substack{0111110\\1}} - e_{\substack{0011110\\0}} + e_{\substack{0011000\\1}} - e_{\substack{1111000\\0}} - 2e_{\substack{1111100\\1}}$		
	2	$-$	$e_{\substack{0111000\\1}} + e_{\substack{0011000\\0}} - e_{\substack{0011100\\1}} + e_{\substack{1111100\\0}} - e_{\substack{1121110\\1}} + e_{\substack{1221111\\1}}$		
	2	$-$	$e_{\substack{0011100\\0}} - e_{\substack{0111100\\1}} - e_{\substack{0001110\\0}} + e_{\substack{1221110\\1}} + e_{\substack{1111111\\1}} - e_{\substack{1121000\\1}}$		
	2	$-$	$e_{\substack{0111110\\1}} + e_{\substack{0011110\\0}} - e_{\substack{0111111\\0}} + e_{\substack{1121111\\1}} + e_{\substack{1111000\\0}} - e_{\substack{1111100\\1}}$		
	2	$-$	$e_{\substack{0011111\\0}} - e_{\substack{0111111\\1}} + e_{\substack{0121000\\1}} - e_{\substack{1111000\\1}} + e_{\substack{1221100\\1}} - e_{\substack{1111110\\0}}$		
	2	$-$	$e_{\substack{0001000\\0}} + e_{\substack{1121100\\1}} + e_{\substack{1111111\\0}} + e_{\substack{0121110\\1}}$		

(table continues on next page)

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	4	–	$e_{\substack{0122221\\1}} - e_{\substack{1122110\\1}} - e_{\substack{1232100\\2}} - e_{\substack{1222111\\1}} + e_{\substack{1232210\\1}}$		
	4	–	$e_{\substack{0122111\\1}} + e_{\substack{1222211\\1}} + e_{\substack{1122210\\1}} + e_{\substack{1232100\\1}} - e_{\substack{1232221\\2}}$		
	4	–	$e_{\substack{0122100\\1}} + e_{\substack{1232110\\1}} - e_{\substack{1222221\\1}} - e_{\substack{1232210\\2}} - e_{\substack{1122111\\1}}$		
	4	–	$e_{\substack{0122210\\1}} + e_{\substack{1232211\\2}} - e_{\substack{1232111\\1}} - e_{\substack{1122221\\1}} - e_{\substack{1222100\\1}}$		
	4	–	$e_{\substack{1122110\\1}} - e_{\substack{1232100\\2}} + e_{\substack{1122211\\1}} - e_{\substack{1232210\\1}}$		
	4	–	$e_{\substack{1222211\\1}} - e_{\substack{1122210\\1}} - e_{\substack{1222110\\1}} - e_{\substack{1232221\\2}}$		
	4	–	$e_{\substack{1232110\\1}} + e_{\substack{1222221\\1}} + e_{\substack{1232211\\1}} - e_{\substack{1122111\\1}}$		
	4	–	$e_{\substack{1232211\\2}} + e_{\substack{1232111\\1}} - e_{\substack{1232110\\2}} - e_{\substack{1222100\\1}}$		
	4	–	$e_{\substack{0122211\\1}} + e_{\substack{1222210\\1}} + e_{\substack{1232111\\2}}$		
	4	–	$e_{\substack{0122110\\1}} - e_{\substack{1122100\\1}} + e_{\substack{1232221\\1}}$		
3	6	–	$e_{\substack{1233211\\1}} - e_{\substack{1233210\\2}} + e_{\substack{1243221\\2}}$		
	6	–	$e_{\substack{1343210\\2}} - e_{\substack{1233321\\1}} + e_{\substack{1233211\\2}}$		
	6	–	$e_{\substack{1243321\\2}} + e_{\substack{1343221\\2}} + e_{\substack{1233210\\1}}$		
	6	–	$e_{\substack{1233221\\2}} + e_{\substack{1243211\\2}} + e_{\substack{1343321\\2}}$		
	6	–	$e_{\substack{1243211\\2}} - e_{\substack{2343210\\2}}$		
	6	–	$e_{\substack{1233210\\2}} - e_{\substack{2343321\\2}}$		
	6	–	$e_{\substack{1233321\\1}} - e_{\substack{2343221\\2}}$		
	6	–	$e_{\substack{1343221\\2}} + e_{\substack{2343211\\2}}$		
	6	–	$e_{\substack{1343211\\2}} + e_{\substack{1233321\\2}}$		
	6	–	$e_{\substack{1233221\\1}} - e_{\substack{1243210\\2}}$		
2	8	–	$e_{\substack{1354321\\2}} - e_{\substack{2344321\\2}}$		
	8	–	$e_{\substack{1354321\\3}} + e_{\substack{2454321\\2}}$		
	8	–	$e_{\substack{1244321\\2}} - e_{\substack{2464321\\3}}$		
	8	–	$e_{\substack{1344321\\2}} + e_{\substack{2354321\\3}}$		
	8	–	$e_{\substack{2354321\\2}}$		
	8	–	$e_{\substack{2454321\\3}}$		
1	2	–	e	z_2	z_2
	10	–	$e_{\substack{2465321\\3}}$	${z_{10}}^1$	
	10	–	$e_{\substack{2465421\\3}}$	${z_{10}}^2$	
	10	–	$e_{\substack{2465431\\3}}$	${z_{10}}^3$	
	10	–	$e_{\substack{2465432\\3}}$	${z_{10}}^4$	

E_8, orbit 42: A_6

L :  $\quad \tau$: $\begin{smallmatrix} 0\ -10 & 2 & 2 & 2 & 2 & 2 \\ & 2 & & & & \end{smallmatrix}$ $\quad \Delta = \begin{smallmatrix} 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

$C^\circ = {A_1}^2 \qquad C/C^\circ = 1$

$\beta_1 = {}^{\substack{1000000\\0}}$;

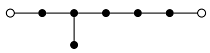
$x_{\beta_2}(t) = x_{\substack{1232111\\2}}(2t) x_{\substack{1233210\\1}}(-t) x_{\substack{2465321\\3}}(-t^2) x_{\substack{1232210\\2}}(t) x_{\substack{1222221\\1}}(t) x_{\substack{1232211\\1}}(-t)$,

$x_{-\beta_2}(t) = x_{-\substack{1232111\\2}}(t) x_{-\substack{1233210\\1}}(-2t) x_{-\substack{2465321\\3}}(t^2) x_{-\substack{1232210\\2}}(t) x_{-\substack{1222221\\1}}(t) \times$

$x_{-\substack{1232211\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
6	2	$\lambda_1 + \lambda_2$	$e_{\substack{1122100\\1}} + e_{\substack{1121110\\1}} + e_{\substack{1111111\\1}}$		
5	4	$2\lambda_2$	$e_{\substack{1243210\\2}} + e_{\substack{1233211\\2}} - e_{\substack{1232221\\2}}$		
4	6	$\lambda_1 + 3\lambda_2$	$e_{\substack{2354321\\3}}$		
3	8	$2\lambda_2$	$e_{\substack{1243221\\2}} - e_{\substack{1233321\\2}}$		
2	6	$4\lambda_2$	$e_{\substack{2465432\\3}}$		
	10	$\lambda_1 + \lambda_2$	$e_{\substack{1122221\\1}}$		
1	2	0	e	z_2	z_2
	10	0	$e_{\substack{0011110\\1}} + e_{\substack{0011111\\0}}$	z_{10}	z_{10}
	12	$2\lambda_2$	$e_{\substack{1244321\\2}}$		

E_8, orbit 43: $D_6(a_1)$

L :  τ : $\begin{smallmatrix}-11 & 2 & 0 & 2 & 2 & 2 & -8\\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}0 & 1 & 0 & 0 & 0 & 1 & 2\\ & & 1 & & & & \end{smallmatrix}$

$e = e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$

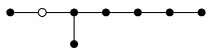
$C^\circ = {A_1}^2 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = {}^{2343210}_{\ \ \ \ 2}$, $\beta_2 = {}^{2465432}_{\ \ \ \ 3}$

$c = n_{\substack{0011111\\1}} n_{\substack{0111111\\0}} h_2(i) h_3(i)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	$\lambda_1+\lambda_2$	$e_{\substack{2454321\\3}}$		
	2	0	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$		
	3	λ_2	$e_{\substack{1233221\\2}} - e_{\substack{1233321\\1}} + 2e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}}$		
	3	λ_1	$e_{\substack{1122210\\1}} - e_{\substack{1222110\\1}} - e_{\substack{1232110\\1}} - 2e_{\substack{1232100\\2}}$		
4	5	λ_2	$e_{\substack{1233321\\2}} + e_{\substack{1343221\\2}}$		
	5	λ_1	$e_{\substack{1222210\\1}} - e_{\substack{1232110\\2}}$		
	6	0	$2e_{\substack{0111000\\1}} + e_{\substack{0011100\\1}} + e_{\substack{0111100\\0}} + e_{\substack{0011110\\0}}$		
	6	0	$2e_{\substack{0121000\\1}} + e_{\substack{0011100\\1}} - e_{\substack{0111100\\0}} + e_{\substack{0001110\\0}}$		
3	8	$\lambda_1+\lambda_2$	$e_{\substack{2465431\\3}}$		
	8	0	$2e_{\substack{0111100\\1}} + e_{\substack{0111110\\0}} + e_{\substack{0011110\\1}}$		
	9	λ_2	$e_{\substack{1344321\\2}} - e_{\substack{1354321\\2}}$		
	9	λ_1	$e_{\substack{1233210\\2}} + e_{\substack{1243210\\2}}$		
2	11	λ_2	$e_{\substack{1354321\\3}}$		
	11	λ_1	$e_{\substack{1343210\\2}}$		
1	2	0	e	z_2	z_2
	10	0	$e_{\substack{0121110\\1}} + e_{\substack{0122100\\1}}$	${z_{10}}^1$	${z_{10}}^1$
	10	0	$e_{\substack{0111110\\1}}$	${z_{10}}^2$	
	14	0	$e_{\substack{0122210\\1}}$	z_{14}	z_{14}

E_8, orbit 44: A_6A_1

L :  τ : $\begin{smallmatrix} 2 & -11 & 2 & 2 & 2 & 2 & 2 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}} + e_{\substack{1000000\\0}}$

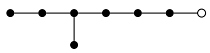
$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\beta_1}(t) = x_{\substack{1232111\\2}}(2t)x_{\substack{1233210\\1}}(-t)x_{\substack{2465321\\3}}(-t^2)x_{\substack{1232210\\2}}(t)x_{\substack{1222221\\1}}(t)x_{\substack{1232211\\1}}(-t)$,

$x_{-\beta_1}(t) = x_{-\substack{1232111\\2}}(t)x_{-\substack{1233210\\1}}(-2t)x_{-\substack{2465321\\3}}(t^2)x_{-\substack{1232210\\2}}(t)x_{-\substack{1222221\\1}}(t) \times$
$x_{-\substack{1232211\\1}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
12	1	λ_1	$e_{\substack{1121100\\1}} - e_{\substack{0111111\\1}} - e_{\substack{0122100\\1}} - e_{\substack{0121110\\1}} + 2e_{\substack{1111110\\1}} + 3e_{\substack{1111111\\0}}$		
11	2	0	$e_{\substack{1000000\\0}}$		
10	3	λ_1	$e_{\substack{1122100\\1}} + e_{\substack{1121110\\1}} + e_{\substack{1111111\\1}}$		
9	4	$2\lambda_1$	$e_{\substack{1243210\\2}} + e_{\substack{1233211\\2}} - e_{\substack{1232221\\2}}$		
8	5	$3\lambda_1$	$e_{\substack{2354321\\2}} - e_{\substack{1354321\\3}}$		
7	6	$4\lambda_1$	$e_{\substack{2465432\\3}}$		
6	7	$3\lambda_1$	$e_{\substack{2354321\\3}}$		
5	8	$2\lambda_1$	$e_{\substack{1243221\\2}} - e_{\substack{1233321\\2}}$		
4	9	λ_1	$e_{\substack{1122211\\1}} - e_{\substack{0122221\\1}}$		
3	10	0	$e_{\substack{0011110\\1}} + e_{\substack{0011111\\0}}$		
2	11	λ_1	$e_{\substack{1122221\\1}}$		
1	2	0	e	z_2	z_2
	12	$2\lambda_1$	$e_{\substack{1244321\\2}}$		

E_8, orbit 45: $E_7(a_4)$

L :  τ : $\substack{2\ 0\ 2\ 0\ 0\ 2\ {-11}\\0}$ $\Delta = \substack{0\ 0\ 1\ 0\ 1\ 0\ 2\\0}$

$e = e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0000010\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}}$

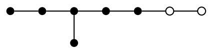
$C^\circ = A_1 \qquad C/C^\circ = \langle cC^\circ\rangle \cong S_2$

$\beta_1 = \substack{2465432\\3}$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
10	1	λ_1	$e_{\substack{1232221\\1}} + 2e_{\substack{1233221\\2}} - e_{\substack{1243211\\2}}$		
	2	0	$e_{\substack{1100000\\0}} + 4e_{\substack{0011000\\0}} + e_{\substack{0110000\\0}} - 2e_{\substack{0011100\\0}} + e_{\substack{0111000\\1}} + 2e_{\substack{0001110\\0}} - 6e_{\substack{0010000\\1}} + e_{\substack{0111100\\1}}$		
9	2	0	$e_{\substack{0000010\\0}} + e_{\substack{0011000\\1}}$		
	2	0	$e_{\substack{0010000\\0}}$		
	3	λ_1	$e_{\substack{1343321\\2}} + e_{\substack{1343221\\2}} - e_{\substack{2343211\\2}} + 2e_{\substack{1244321\\2}}$		
8	3	λ_1	$e_{\substack{1243221\\2}}$		
	4	0	$e_{\substack{1110000\\0}} - e_{\substack{1111000\\1}} - e_{\substack{0121100\\1}} - e_{\substack{0111110\\1}}$		
	4	0	$e_{\substack{1111000\\1}} - e_{\substack{0121000\\1}} - 2e_{\substack{0011110\\0}} + e_{\substack{0111110\\1}}$		
7	5	λ_1	$e_{\substack{2343221\\2}} - e_{\substack{1354321\\3}}$		
	6	0	$e_{\substack{1221100\\1}} + e_{\substack{1122100\\1}} + 2e_{\substack{0122110\\1}} - e_{\substack{1111110\\0}}$		
	6	0	$e_{\substack{1221000\\1}} - e_{\substack{1122100\\1}} - 2e_{\substack{0122210\\1}} - e_{\substack{1111110\\0}}$		
6	6	0	$e_{\substack{1121000\\1}} + e_{\substack{0121110\\1}}$		
	7	λ_1	$e_{\substack{2454321\\3}} - e_{\substack{2354321\\2}}$		
	8	0	$2e_{\substack{1121110\\1}} + e_{\substack{1222110\\1}} - e_{\substack{1222210\\1}} - e_{\substack{1232100\\1}}$		
5	8	0	$2e_{\substack{1122110\\1}} + e_{\substack{1221110\\1}} - e_{\substack{1232100\\2}}$		
	9	λ_1	$e_{\substack{2465421\\3}} + e_{\substack{2465321\\3}}$		
4	9	λ_1	$e_{\substack{2464321\\3}}$		
	10	0	$e_{\substack{1232110\\1}} - e_{\substack{1232210\\1}}$		
3	10	0	$3e_{\substack{1232110\\1}} + 2e_{\substack{1233210\\2}} - e_{\substack{1232210\\1}}$		
	10	0	$e_{\substack{1232110\\2}}$		
	11	λ_1	$e_{\substack{2465431\\3}}$		
2	12	0	$e_{\substack{1243210\\2}}$		
1	2	0	e	z_2	z_2
	10	0	$5e_{\substack{1232110\\2}} + e_{\substack{1232210\\2}} + e_{\substack{1233210\\1}}$	z_{10}	z_{10}
	14	0	$e_{\substack{2343210\\2}}$	z_{14}	z_{14}

E_8, orbit 46: $E_6(a_1)$

L :  τ : $\begin{smallmatrix} 2 & 2 & 0 & 2 & 2 & -12 & 0 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 0 & 0 & 0 & 2 & 0 & 2 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}}$

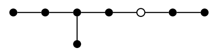
$C^\circ = A_2 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = {\substack{0000001\\0}}$, $\beta_2 = {\substack{2465431\\3}}$

$c = n_{\substack{0122210\\1}} n_{\substack{1122110\\1}} n_{\substack{1221110\\1}} h_1(-1)h_2(-1)h_3(-1)h_5(-1)h_6(-1)h_8(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
4	4	λ_1	$e_{\substack{1232211\\1}} + e_{\substack{1222211\\1}} - e_{\substack{1232111\\2}}$		
	4	λ_2	$e_{\substack{1354321\\3}} + e_{\substack{2344321\\2}} - e_{\substack{2354321\\2}}$		
	4	0	$e_{\substack{1110000\\0}} - 3e_{\substack{1100000\\0}} - e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} - 3e_{\substack{0001100\\0}} - 2e_{\substack{0111000\\0}} + e_{\substack{0011100\\0}}$		
3	6	0	$e_{\substack{1110000\\1}} + e_{\substack{1111000\\0}} - e_{\substack{0011100\\1}} + e_{\substack{0111100\\0}} + e_{\substack{0121000\\1}}$		
	8	λ_1	$e_{\substack{1233211\\2}} - e_{\substack{1243211\\2}}$		
	8	λ_2	$e_{\substack{2454321\\3}} + e_{\substack{2464321\\3}}$		
	8	0	$e_{\substack{1111000\\1}} - e_{\substack{1121000\\1}} - e_{\substack{0111100\\1}} + 2e_{\substack{1111100\\0}} - e_{\substack{0121100\\1}}$		
2	10	0	$e_{\substack{1111100\\1}} + e_{\substack{1221000\\1}} + e_{\substack{0122100\\1}}$		
	12	λ_1	$e_{\substack{2343211\\2}}$		
	12	λ_2	$e_{\substack{2465421\\3}}$		
1	2	0	e	z_2	z_2
	10	0	$e_{\substack{1121100\\1}} + e_{\substack{1221000\\1}} - e_{\substack{0122100\\1}}$	z_{10}	z_{10}
	14	0	$e_{\substack{1222100\\1}}$	z_{14}	z_{14}
	16	0	$e_{\substack{1232100\\2}}$	z_{16}	

E_8, orbit 47: D_5A_2

L :  τ : $\begin{smallmatrix}2&2&2&2&-12&2&2\\&&2&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}0&0&0&2&0&0&2\\&&0&&&&\end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

$C^\circ = T_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$T_1 = \{h_1(\mu^6)h_2(\mu^9)h_3(\mu^{12})h_4(\mu^{18})h_5(\mu^{15})h_6(\mu^{12})h_7(\mu^8)h_8(\mu^4) : \mu \in k^*\}$

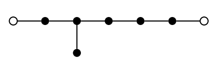
$c = n_{\substack{1233211\\1}} n_{\substack{1232211\\2}} n_{\substack{2465431\\3}} h_1(-1)h_2(-1)h_4(-1)h_5(-1)h_7(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
7	2	–	$f_{\substack{2465421\\3}}$		
	2	–	$e_{\substack{2465432\\3}}$		
	2	–	$f_{\substack{1232210\\2}} - f_{\substack{1233210\\1}}$		
	2	–	$e_{\substack{1232221\\2}} - e_{\substack{1233221\\1}}$		
	2	–	$f_{\substack{0011111\\1}} - f_{\substack{0111111\\0}} + f_{\substack{0111110\\1}} + 3f_{\substack{0121100\\1}} - 2f_{\substack{1111110\\0}} - 2f_{\substack{1111100\\1}}$		
	2	–	$e_{\substack{1221100\\1}} - e_{\substack{1122100\\1}} + e_{\substack{1121110\\1}} + 3e_{\substack{1111111\\1}} - 2e_{\substack{0122110\\1}} - 2e_{\substack{0121111\\1}}$		
	2	–	$e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$		
6	4	–	$f_{\substack{1343321\\2}} - f_{\substack{1244321\\2}}$		
	4	–	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
	4	–	$f_{\substack{0011110\\1}} - f_{\substack{0111110\\0}} + f_{\substack{0111100\\1}} - 2f_{\substack{1111100\\0}}$		
	4	–	$e_{\substack{1221110\\1}} - e_{\substack{1122110\\1}} + e_{\substack{1121111\\1}} - 2e_{\substack{0122111\\1}}$		
	4	–	$e_{\substack{0000011\\0}}$		
5	6	–	$f_{\substack{0122221\\1}} - f_{\substack{1222210\\1}} - f_{\substack{1122211\\1}}$		
	6	–	$e_{\substack{1343211\\2}} + e_{\substack{2343210\\2}} - e_{\substack{1243221\\2}}$		
	6	–	$f_{\substack{0111100\\0}} - f_{\substack{0011100\\1}}$		
	6	–	$e_{\substack{1221111\\1}} - e_{\substack{1122111\\1}}$		
	6	–	$e_{\substack{0110000\\1}} - e_{\substack{0111000\\0}} - e_{\substack{1110000\\0}} + 2e_{\substack{0011000\\1}}$		
4	8	–	$f_{\substack{1122210\\1}} + f_{\substack{0122211\\1}}$		
	8	–	$e_{\substack{1343221\\2}} - e_{\substack{2343211\\2}}$		
	8	–	$f_{\substack{0011100\\0}} + f_{\substack{0001110\\0}} + f_{\substack{0000111\\0}}$		
	8	–	$e_{\substack{1222111\\1}} + e_{\substack{1232110\\1}} + e_{\substack{1232100\\2}}$		
	8	–	$e_{\substack{1111000\\0}} + e_{\substack{1110000\\1}}$		

(table continues on next page)

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	10	–	$f_{\substack{1233321\\1}}$		
	10	–	$e_{\substack{2465321\\3}}$		
	10	–	$f_{\substack{0122210\\1}}$		
	10	–	$e_{\substack{2343221\\2}}$		
	10	–	$f_{\substack{0001100\\0}} + f_{\substack{0000110\\0}}$		
	10	–	$e_{\substack{1232110\\2}} + e_{\substack{1232111\\1}}$		
	10	–	$e_{\substack{1111000\\1}} - e_{\substack{0121000\\1}}$		
2	12	–	$f_{\substack{0000100\\0}}$		
	12	–	$e_{\substack{1232111\\2}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{1221000\\1}}$	z_{14}	z_{14}

E_8, orbit 48: D_6

L :  τ : $\begin{smallmatrix}-15&2&2&2&2&2&-10\\&&2&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}2&1&0&0&0&1&2\\&&1&&&&\end{smallmatrix}$

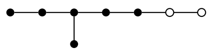
$e = e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$

$C^\circ = B_2 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{2343210\\2}}$, $x_{\pm\beta_2}(t) = x_{\pm \substack{0011111\\1}}(t)x_{\pm \substack{0111111\\0}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	5	λ_2	$e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}} + e_{\substack{1233321\\2}}$		
2	6	0	$2e_{\substack{0110000\\1}} + e_{\substack{0111000\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}}$		
	9	λ_2	$e_{\substack{1343321\\2}} - e_{\substack{1244321\\2}}$		
	10	λ_1	$e_{\substack{2465431\\3}}$		
	10	0	$2e_{\substack{0121000\\1}} + 2e_{\substack{0111100\\1}} + e_{\substack{0111110\\0}} + e_{\substack{0011110\\1}}$		
1	2	0	e	z_2	z_2
	14	0	$e_{\substack{0122100\\1}} + e_{\substack{0121110\\1}}$	z_{14}	z_{14}
	15	λ_2	$e_{\substack{1354321\\3}}$		
	18	0	$e_{\substack{0122210\\1}}$	z_{18}	z_{18}

E_8, orbit 49: E_6

L :  $\quad \tau : \begin{smallmatrix} 2 & 2 & 2 & 2 & 2 & -16 & 0 \\ & & 2 & & & & \end{smallmatrix} \quad \Delta = \begin{smallmatrix} 2 & 0 & 0 & 0 & 2 & 2 & 2 \\ & & 0 & & & & \end{smallmatrix}$

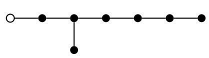
$e = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}}$

$C^\circ = G_2 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{1221110\\1}}(t)x_{\pm \substack{1122110\\1}}(-t)x_{\pm \substack{0122210\\1}}(t),\ \beta_2 = \substack{0000001\\0}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
2	8	λ_1	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
1	2	0	e	z_2	z_2
	10	0	$e_{\substack{1111000\\1}} + 2e_{\substack{1111100\\0}} - e_{\substack{0111100\\1}} - e_{\substack{0121000\\1}}$	z_{10}	z_{10}
	14	0	$e_{\substack{1221000\\1}} + e_{\substack{1121100\\1}} - e_{\substack{0122100\\1}}$	z_{14}	z_{14}
	16	λ_1	$e_{\substack{2465421\\3}}$		
	22	0	$e_{\substack{1232100\\2}}$	z_{22}	z_{22}

E_8, orbit 50: $D_7(a_2)$

L :  $\quad \tau : \begin{smallmatrix} -13 & 2 & 0 & 2 & 0 & 2 & 2 \\ & & 2 & & & & \end{smallmatrix} \quad \Delta = \begin{smallmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$

$C^\circ = T_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$T_1 = \{h_1(\mu^4)h_2(\mu^5)h_3(\mu^7)h_4(\mu^{10})h_5(\mu^8)h_6(\mu^6)h_7(\mu^4)h_8(\mu^2) : \mu \in k^*\}$

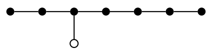
$c = n_{\substack{2354321\\3}} n_{\substack{2454321\\2}} h_4(-1)h_5(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
14	1	–	$2f_{\substack{1221111\\1}} - 2f_{\substack{1122111\\1}} - f_{\substack{1222110\\1}} - 3f_{\substack{1222210\\1}} - f_{\substack{1232110\\1}} - f_{\substack{1232100\\2}} + 2f_{\substack{1122211\\1}} + 5f_{\substack{1232210\\1}}$		
	1	–	$2e_{\substack{1232110\\2}} - 2e_{\substack{1232210\\2}} + e_{\substack{1122221\\1}} - 3e_{\substack{1232111\\1}} - e_{\substack{1222211\\1}} - e_{\substack{1232211\\1}} - 2e_{\substack{1233210\\1}} + 5e_{\substack{1222111\\1}}$		
13	2	–	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0011000\\0}}$		
12	3	–	$2f_{\substack{1111111\\1}} - f_{\substack{1221110\\1}} - f_{\substack{1122110\\1}} + 2f_{\substack{1121111\\1}} - f_{\substack{1232100\\1}} - 3f_{\substack{1122210\\1}}$		
	3	–	$2e_{\substack{1233210\\2}} + e_{\substack{1232211\\2}} - e_{\substack{1222221\\1}} + 2e_{\substack{1243210\\2}} - e_{\substack{1233211\\1}} + 3e_{\substack{1232111\\2}}$		

(table continues on next page)

n	m	λ	v	$\mathcal{Z}^{\natural}$	$\mathcal{Z}$
11	4	–	$f_{\substack{2343321\\2}}$		
	4	–	$e_{\substack{2465321\\3}}$		
	4	–	$e_{\substack{0011000\\1}} + 2e_{\substack{0110000\\1}} + e_{\substack{0011100\\1}} + e_{\substack{0000111\\0}} - e_{\substack{0111000\\0}} + e_{\substack{0111100\\0}} + e_{\substack{0011110\\0}}$		
10	5	–	$f_{\substack{1111110\\1}} + 2f_{\substack{1111111\\0}} - f_{\substack{1221100\\1}} - f_{\substack{1121110\\1}}$		
	5	–	$e_{\substack{1233221\\1}} - e_{\substack{1233211\\2}} + 2e_{\substack{1343210\\2}} + e_{\substack{1243211\\2}}$		
9	6	–	$2e_{\substack{0111100\\1}} + e_{\substack{0111110\\0}} + e_{\substack{0011110\\1}} + e_{\substack{0011111\\0}} - 2e_{\substack{0121000\\1}}$		
	6	–	$e_{\substack{0001111\\0}} - e_{\substack{0111110\\0}} + e_{\substack{0011110\\1}} + 2e_{\substack{0121100\\1}}$		
	6	–	$e_{\substack{0111000\\1}}$		
8	7	–	$f_{\substack{1111100\\1}} - f_{\substack{1111000\\1}} + f_{\substack{1111110\\0}} - f_{\substack{1121000\\1}}$		
	7	–	$e_{\substack{1233221\\2}} + e_{\substack{1233321\\2}} + e_{\substack{1243221\\2}} + e_{\substack{1243321\\2}}$		
	7	–	$f_{\substack{1111110\\0}} - f_{\substack{1121100\\1}}$		
	7	–	$e_{\substack{1233221\\2}} + e_{\substack{1343211\\2}}$		
7	8	–	$f_{\substack{2343210\\2}}$		
	8	–	$e_{\substack{2465432\\3}}$		
	8	–	$2e_{\substack{0111110\\1}} + e_{\substack{0011111\\1}} + e_{\substack{0111111\\0}} - 2e_{\substack{0122100\\1}}$		
6	9	–	$f_{\substack{1110000\\1}} - f_{\substack{1111000\\0}} + f_{\substack{1111100\\0}}$		
	9	–	$e_{\substack{1343221\\2}} + e_{\substack{1343321\\2}} - e_{\substack{1244321\\2}}$		
5	10	–	$e_{\substack{0111111\\1}} - e_{\substack{0122110\\1}}$		
	10	–	$e_{\substack{0121111\\1}} - e_{\substack{0122210\\1}}$		
4	11	–	$f_{\substack{1100000\\0}} + f_{\substack{1110000\\0}}$		
	11	–	$e_{\substack{1344321\\2}} - e_{\substack{1354321\\2}}$		
3	12	–	$e_{\substack{0122111\\1}}$		
2	13	–	$f_{\substack{1000000\\0}}$		
	13	–	$e_{\substack{1354321\\3}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{0122221\\1}}$	z_{14}	z_{14}

E_8, orbit 51: A_7

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ & & -15 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ & & 0 & & & & \end{smallmatrix}$

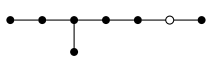
$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{1343211\\2}}(t)x_{\pm \substack{2343210\\2}}(t)x_{\pm \substack{1243221\\2}}(-t)x_{\pm \substack{1233321\\2}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	3	λ_1	$e_{\substack{1232100\\1}} + e_{\substack{1222110\\1}} - e_{\substack{1122210\\1}} + 2e_{\substack{1221111\\1}} - e_{\substack{1122111\\1}}$		
4	4	$2\lambda_1$	$e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}} + e_{\substack{1244321\\2}}$		
	5	λ_1	$e_{\substack{1232110\\1}} - e_{\substack{1222210\\1}} + e_{\substack{1222111\\1}} - 2e_{\substack{1122211\\1}} + 2e_{\substack{0122221\\1}}$		
	6	0	$e_{\substack{1110000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}} + e_{\substack{0000111\\0}}$		
3	7	$3\lambda_1$	$e_{\substack{2465432\\3}}$		
	8	$2\lambda_1$	$e_{\substack{2344321\\2}} - e_{\substack{1354321\\2}}$		
	9	λ_1	$e_{\substack{1233210\\1}} + e_{\substack{1232211\\1}} - e_{\substack{1222221\\1}}$		
2	10	0	$e_{\substack{1111100\\0}} + e_{\substack{0111110\\0}} + e_{\substack{0011111\\0}}$		
	11	λ_1	$e_{\substack{1233211\\1}} - e_{\substack{1232221\\1}}$		
	12	$2\lambda_1$	$e_{\substack{2454321\\2}}$		
1	2	0	e	z_2	z_2
	14	0	$e_{\substack{1111111\\0}}$	z_{14}	z_{14}
	15	λ_1	$e_{\substack{1233321\\1}}$		

E_8, orbit 52: $E_6(a_1)A_1$

L :  τ : $\begin{smallmatrix} 2 & 2 & 0 & 2 & 2 & -13 & 2 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 2 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0000001\\0}}$

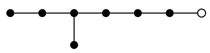
$C^\circ = T_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$T_1 = \{h_1(\mu^4)h_2(\mu^6)h_3(\mu^8)h_4(\mu^{12})h_5(\mu^{10})h_6(\mu^8)h_7(\mu^6)h_8(\mu^3) : \mu \in k^*\}$

$c = n_{\substack{1244321\\2}} n_{\substack{1343321\\2}} n_{\substack{2343221\\2}} h_1(-1)h_2(-1)h_3(-1)h_5(-1)h_6(-1)h_8(-1)$

n	m	λ	v	$\mathcal{Z}^{\natural}$	$\mathcal{Z}$
8	1	–	$f_{\substack{2465431\\3}}$		
	1	–	$e_{\substack{2465432\\3}}$		
	3	–	$f_{\substack{1111110\\1}} - f_{\substack{0111111\\1}} + f_{\substack{1111111\\0}} - f_{\substack{0121111\\1}} + 2f_{\substack{0122110\\1}} - 3f_{\substack{1121110\\1}}$		
	3	–	$e_{\substack{1232110\\2}} - e_{\substack{1222210\\1}} + e_{\substack{1232111\\1}} - e_{\substack{1232210\\1}} + 2e_{\substack{1122211\\1}} - 3e_{\substack{1222111\\1}}$		
7	2	–	$e_{\substack{0000001\\0}}$		
	4	–	$f_{\substack{1233321\\1}} - f_{\substack{1233221\\2}} + f_{\substack{1243221\\2}}$		
	4	–	$e_{\substack{1354321\\3}} + e_{\substack{2344321\\2}} - e_{\substack{2354321\\2}}$		
	4	–	$e_{\substack{1110000\\0}} - 3e_{\substack{1100000\\0}} - e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} - 3e_{\substack{0001100\\0}} - 2e_{\substack{0111000\\0}}$ $+ e_{\substack{0011100\\0}}$		
6	5	–	$f_{\substack{0111110\\1}} - f_{\substack{1111110\\0}} + f_{\substack{0121110\\1}}$		
	5	–	$e_{\substack{1222211\\1}} - e_{\substack{1232111\\2}} + e_{\substack{1232211\\1}}$		
	7	–	$f_{\substack{0001111\\0}} + f_{\substack{0111110\\0}} - 2f_{\substack{0011110\\1}} - f_{\substack{0011111\\0}}$		
	7	–	$e_{\substack{1233210\\2}} + 2e_{\substack{1233211\\1}} - e_{\substack{1232211\\2}} - e_{\substack{1243210\\2}}$		
5	6	–	$e_{\substack{1110000\\1}} + e_{\substack{1111000\\0}} - e_{\substack{0011100\\1}} + e_{\substack{0111100\\0}} + e_{\substack{0121000\\1}}$		
	8	–	$f_{\substack{1222221\\1}} + f_{\substack{1232221\\1}}$		
	8	–	$e_{\substack{2454321\\3}} + e_{\substack{2464321\\3}}$		
	8	–	$e_{\substack{1111000\\1}} - e_{\substack{1121000\\1}} - e_{\substack{0111100\\1}} + 2e_{\substack{1111100\\0}} - e_{\substack{0121100\\1}}$		
4	9	–	$f_{\substack{0001110\\0}} - f_{\substack{0011110\\0}}$		
	9	–	$e_{\substack{1233211\\2}} - e_{\substack{1243211\\2}}$		
	11	–	$f_{\substack{0000110\\0}} + f_{\substack{0000011\\0}}$		
	11	–	$e_{\substack{2343210\\2}} + e_{\substack{1343211\\2}}$		
3	10	–	$e_{\substack{1111100\\1}} + e_{\substack{1221000\\1}} + e_{\substack{0122100\\1}}$		
	10	–	$e_{\substack{1121100\\1}} + e_{\substack{1221000\\1}} - e_{\substack{0122100\\1}}$		
	12	–	$f_{\substack{0122221\\1}}$		
	12	–	$e_{\substack{2465421\\3}}$		
2	13	–	$f_{\substack{0000010\\0}}$		
	13	–	$e_{\substack{2343211\\2}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{1222100\\1}}$	z_{14}	z_{14}
	16	–	$e_{\substack{1232100\\2}}$	z_{16}	

E_8, orbit 53: $E_7(a_3)$

L :  τ : $\begin{smallmatrix}2&0&2&0&2&2&-15\\&&0&&&&\end{smallmatrix}$ Δ = $\begin{smallmatrix}2&0&1&0&1&0&2\\&&0&&&&\end{smallmatrix}$

$e = e_{\substack{0110000\\1}} + e_{\substack{1000000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}}$

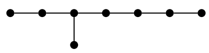
$C^\circ = A_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$\beta_1 = {\substack{2465432\\3}}$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
7	1	λ_1	$e_{\substack{1232221\\1}} + e_{\substack{1243211\\2}}$		
	4	0	$e_{\substack{0011100\\0}} + e_{\substack{1111000\\1}} + e_{\substack{0001110\\0}} - 3e_{\substack{1110000\\0}} + e_{\substack{0111100\\1}} + 2e_{\substack{0121000\\1}}$		
6	2	0	$e_{\substack{0010000\\0}}$		
	5	λ_1	$e_{\substack{1244321\\2}} + e_{\substack{1343321\\2}} - e_{\substack{2343221\\2}}$		
5	6	0	$e_{\substack{0011110\\0}} + e_{\substack{1121000\\1}} + e_{\substack{0121100\\1}}$		
	9	λ_1	$e_{\substack{2454321\\3}} - e_{\substack{2354321\\2}}$		
4	6	0	$e_{\substack{1111100\\0}} + e_{\substack{0111110\\0}} - 2e_{\substack{0011110\\1}} - e_{\substack{1221000\\1}} - e_{\substack{0122100\\1}}$		
	8	0	$e_{\substack{1111110\\1}} - e_{\substack{1121100\\1}} - 2e_{\substack{0121110\\1}} - e_{\substack{1222100\\1}}$		
	10	0	$e_{\substack{1221110\\1}} + e_{\substack{1232100\\2}}$		
	10	0	$e_{\substack{1221110\\1}} + e_{\substack{1122110\\1}} - e_{\substack{0122210\\1}}$		
3	9	λ_1	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
	11	λ_1	$e_{\substack{2464321\\3}}$		
	14	0	$e_{\substack{1232210\\1}} + e_{\substack{1233210\\2}}$		
2	10	0	$e_{\substack{1121110\\1}} - e_{\substack{1232100\\1}}$		
	15	λ_1	$e_{\substack{2465431\\3}}$		
1	2	0	e	z_2	z_2
	14	0	$e_{\substack{1232210\\2}} + e_{\substack{1233210\\1}}$	z_{14}	z_{14}
	16	0	$e_{\substack{1243210\\2}}$	z_{16}	
	18	0	$e_{\substack{2343210\\2}}$	z_{18}	z_{18}

E_8, orbit 54: $E_8(b_6)$

L :  τ : $\substack{0\ 0\ 2\ 0\ 0\ 0\ 2\\0}$ $\Delta = \substack{0\ 0\ 2\ 0\ 0\ 0\ 2\\0}$

$e = e_{\substack{0011000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{1111100\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0111110\\1}} - e_{\substack{1111110\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1110000\\1}}$

$C^\circ = 1 \qquad C/C^\circ = \langle c_1C^\circ, c_2C^\circ\rangle \cong S_3$

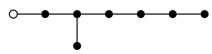
$c_1 = h_1(\omega)h_2(\omega)h_5(\omega^2)$,
$c_2 = n_{\substack{1000000\\0}}\, n_{\substack{0000000\\1}}\, n_{\substack{0001000\\0}}\, h_2(-1)h_3(-1)h_4(-1)h_5(-1)h_8(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
8	2	–	$2e_{\substack{0011000\\0}} + 2e_{\substack{1110000\\0}} - e_{\substack{0000111\\0}} + e_{\substack{0011100\\1}} - e_{\substack{0111110\\0}} - 3e_{\substack{0111100\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011110\\1}} - 4e_{\substack{1111000\\1}}$		
	2	–	$2e_{\substack{0010000\\1}} + 2e_{\substack{0111000\\1}} + e_{\substack{0001111\\0}} + e_{\substack{0011100\\0}} + e_{\substack{1111110\\1}} - 3e_{\substack{1111100\\1}} + e_{\substack{1111000\\0}} + e_{\substack{0011110\\0}} - 4e_{\substack{0110000\\0}}$		
	2	–	$e_{\substack{1110000\\1}} + e_{\substack{0111000\\0}}$		
7	4	–	$e_{\substack{0111111\\1}} + e_{\substack{1111111\\0}} + e_{\substack{1122110\\1}} + e_{\substack{0121110\\1}} - 3e_{\substack{0121100\\1}} + 3e_{\substack{1122100\\1}} - 2e_{\substack{1222210\\1}}$		
	4	–	$e_{\substack{0111111\\0}} - 2e_{\substack{1121000\\1}} + e_{\substack{0122110\\1}} + 3e_{\substack{1221100\\1}} + e_{\substack{1221110\\1}} - e_{\substack{0122100\\1}}$		
	4	–	$e_{\substack{1111111\\1}} + 2e_{\substack{0121000\\1}} + e_{\substack{1121110\\1}} + 3e_{\substack{1222100\\1}} - e_{\substack{1222110\\1}} + e_{\substack{1121100\\1}}$		
	4	–	$e_{\substack{1221000\\1}}$		
6	6	–	$e_{\substack{0122221\\1}} + e_{\substack{1121111\\1}} + e_{\substack{1222111\\1}} + e_{\substack{0122211\\1}} - 2e_{\substack{1232100\\1}} + 2e_{\substack{1232210\\2}}$		
	6	–	$e_{\substack{1122221\\1}} + e_{\substack{0122111\\1}} - e_{\substack{1221111\\1}} - e_{\substack{1122211\\1}} - 2e_{\substack{1232100\\2}} - 2e_{\substack{1233210\\1}}$		
	6	–	$e_{\substack{0121111\\1}} - e_{\substack{1122111\\1}} + e_{\substack{1222221\\1}} - e_{\substack{1232210\\1}} - e_{\substack{1233210\\2}}$		
	6	–	$e_{\substack{1221111\\1}} + e_{\substack{1232110\\2}} - e_{\substack{1232100\\2}}$		
	6	–	$e_{\substack{1222111\\1}} + e_{\substack{1232110\\1}} + e_{\substack{1232100\\1}}$		
5	8	–	$e_{\substack{1233211\\2}} - e_{\substack{1232211\\1}} + e_{\substack{1233221\\2}} + e_{\substack{1232221\\1}} + 2e_{\substack{1243210\\2}}$		
	8	–	$e_{\substack{1232111\\1}} - e_{\substack{1232211\\2}} - e_{\substack{1232221\\2}} - 2e_{\substack{1343210\\2}}$		
	8	–	$e_{\substack{1232111\\2}} - e_{\substack{1233211\\1}} + e_{\substack{1233221\\1}} + 2e_{\substack{2343210\\2}}$		
4	10	–	$e_{\substack{1244321\\2}} - e_{\substack{1343211\\2}} + e_{\substack{2343321\\2}}$		
	10	–	$e_{\substack{1243321\\2}} + e_{\substack{2343211\\2}} + e_{\substack{1344321\\2}}$		
	10	–	$e_{\substack{1243221\\2}} + e_{\substack{1343321\\2}} + e_{\substack{2344321\\2}}$		
	10	–	$e_{\substack{1243211\\2}} + e_{\substack{1343321\\2}} - e_{\substack{2344321\\2}}$		
	10	–	$e_{\substack{2343211\\2}} - e_{\substack{2343221\\2}}$		
	10	–	$e_{\substack{1343211\\2}} + e_{\substack{1343221\\2}}$		

(table continues on next page)

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	12	–	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$		
	12	–	$e_{\substack{1354321\\2}} + e_{\substack{2454321\\3}}$		
2	14	–	$e_{\substack{2464321\\3}}$		
	14	–	$e_{\substack{2465321\\3}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{2465421\\3}}$	z_{14}	z_{14}
	16	–	$e_{\substack{2465432\\3}}$	z_{16}	

E_8, orbit 55: $D_7(a_1)$

L :  τ : $\begin{smallmatrix} -16 & 2 & 0 & 2 & 2 & 2 & 2 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 0 & 0 & 2 & 0 & 0 & 2 \\ & & 0 & & & & \end{smallmatrix}$

$e = e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$

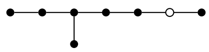
$C^\circ = T_1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$T_1 = \{h_1(\mu^4)h_2(\mu^5)h_3(\mu^7)h_4(\mu^{10})h_5(\mu^8)h_6(\mu^6)h_7(\mu^4)h_8(\mu^2) : \mu \in k^*\}$

$c = n_{\substack{2354321\\3}} n_{\substack{2454321\\2}} h_2(-1)h_4(-1)h_6(-1)h_8(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	–	$f_{\substack{2354321\\2}}$		
	2	–	$e_{\substack{2454321\\3}}$		
	2	–	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$		
	4	–	$f_{\substack{1111111\\1}} - f_{\substack{1221110\\1}} + 2f_{\substack{1122110\\1}} + f_{\substack{1121111\\1}} - f_{\substack{1222100\\1}} + 3f_{\substack{1232100\\1}}$		
	4	–	$e_{\substack{1233210\\2}} - e_{\substack{1233211\\1}} + 2e_{\substack{1232211\\2}} + e_{\substack{1243210\\2}} + e_{\substack{1232221\\1}} - 3e_{\substack{1222221\\1}}$		
4	6	–	$f_{\substack{1111111\\0}} - f_{\substack{1122100\\1}} - f_{\substack{1121110\\1}}$		
	6	–	$e_{\substack{1232221\\2}} - e_{\substack{1233211\\2}} + e_{\substack{1343210\\2}}$		
	8	–	$2e_{\substack{0111100\\1}} + e_{\substack{0011110\\1}} + e_{\substack{0111110\\0}} + e_{\substack{0011111\\0}}$		
3	6	–	$e_{\substack{0001110\\0}} + e_{\substack{0000111\\0}} + e_{\substack{0011100\\1}} - e_{\substack{0111100\\0}} + 2e_{\substack{0121000\\1}}$		
	8	–	$f_{\substack{1111100\\1}} - f_{\substack{1121100\\1}} + 2f_{\substack{1111110\\0}} - f_{\substack{1221000\\1}}$		
	8	–	$e_{\substack{1233221\\2}} - e_{\substack{1233321\\1}} + 2e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}}$		
	10	–	$f_{\substack{2343210\\2}}$		
	10	–	$e_{\substack{2465432\\3}}$		
	10	–	$e_{\substack{0111110\\1}} - e_{\substack{0121110\\1}} + e_{\substack{0111111\\0}} - e_{\substack{0122100\\1}}$		
	10	–	$2e_{\substack{0111110\\1}} + e_{\substack{0011111\\1}} + e_{\substack{0111111\\0}}$		
2	10	–	$f_{\substack{1111100\\0}} - f_{\substack{1121000\\1}}$		
	10	–	$e_{\substack{1233321\\2}} + e_{\substack{1343221\\2}}$		
	12	–	$e_{\substack{0111111\\1}}$		
	14	–	$f_{\substack{1100000\\0}} + f_{\substack{1110000\\0}}$		
	14	–	$e_{\substack{1344321\\2}} - e_{\substack{1354321\\2}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{0122111\\1}} + e_{\substack{0122210\\1}}$	z_{14}	z_{14}
	16	–	$f_{\substack{1000000\\0}}$		
	16	–	$e_{\substack{1354321\\3}}$		
	18	–	$e_{\substack{0122221\\1}}$	z_{18}	z_{18}

E_8, orbit 56: E_6A_1

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & 2 & 2 & -17 & 2 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 1 & 0 & 1 & 0 & 1 & 2 & 2 \\ & & 0 & & & & \end{smallmatrix}$

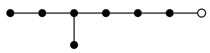
$e = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{1244321\\2}}(t) x_{\pm \substack{1343321\\2}}(-t) x_{\pm \substack{2343221\\2}}(t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	1	$3\lambda_1$	$e_{\substack{2465432\\3}}$		
	7	λ_1	$e_{\substack{1233210\\1}} - e_{\substack{1232210\\2}} - 2e_{\substack{1232111\\2}} + e_{\substack{1232211\\1}}$		
4	2	0	$e_{\substack{0000001\\0}}$		
	8	$2\lambda_1$	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$		
3	9	λ_1	$e_{\substack{1232211\\2}} - e_{\substack{1233211\\1}}$		
	15	λ_1	$e_{\substack{2343210\\2}} + e_{\substack{1343211\\2}}$		
2	10	0	$e_{\substack{1111000\\1}} + 2e_{\substack{1111100\\0}} - e_{\substack{0121000\\1}} - e_{\substack{0111100\\1}}$		
	16	$2\lambda_1$	$e_{\substack{2465421\\3}}$		
1	2	0	e	z_2	z_2
	14	0	$e_{\substack{1221000\\1}} + e_{\substack{1121100\\1}} - e_{\substack{0122100\\1}}$	z_{14}	z_{14}
	17	λ_1	$e_{\substack{2343211\\2}}$		
	22	0	$e_{\substack{1232100\\2}}$	z_{22}	z_{22}

E_8, orbit 57: $E_7(a_2)$

L :  τ : $\begin{smallmatrix}2&2&0&2&0&2&-17\\&&2&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}0&1&0&1&0&2&2\\&&1&&&&\end{smallmatrix}$

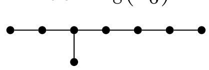
$e = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0000110\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$\beta_1 = {\substack{2465432\\3}}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	0	$e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$		
	6	0	$2e_{\substack{1110000\\1}} + 3e_{\substack{1111100\\0}} - e_{\substack{1111000\\0}} + 3e_{\substack{0121100\\1}} + e_{\substack{0121000\\1}} - e_{\substack{0111100\\1}}$ $+ e_{\substack{0111000\\1}} - e_{\substack{0111110\\0}} + 2e_{\substack{0011110\\1}}$		
4	3	λ_1	$e_{\substack{1233221\\2}}$		
	7	λ_1	$e_{\substack{2343321\\2}} - e_{\substack{2343221\\2}} - e_{\substack{1354321\\2}} + e_{\substack{1344321\\2}}$		
	8	0	$e_{\substack{1111100\\1}} - e_{\substack{1111000\\1}} - e_{\substack{1111110\\0}} + e_{\substack{0122100\\1}} - e_{\substack{0121110\\1}}$		
3	9	λ_1	$e_{\substack{2344321\\2}} - e_{\substack{1354321\\3}}$		
	10	0	$e_{\substack{1111110\\1}} + e_{\substack{0122110\\1}}$		
	14	0	$e_{\substack{1232100\\2}} + e_{\substack{1232210\\1}}$		
2	10	0	$e_{\substack{1111110\\1}} + e_{\substack{0122110\\1}} + e_{\substack{1221000\\1}} + e_{\substack{1122100\\1}} + e_{\substack{1121110\\1}} + 2e_{\substack{0122210\\1}}$		
	15	λ_1	$e_{\substack{2465421\\3}}$		
	16	0	$e_{\substack{1232210\\2}} - e_{\substack{1232110\\2}} + e_{\substack{1233210\\1}}$		
1	2	0	e	z_2	z_2
	14	0	$e_{\substack{1232100\\2}} + e_{\substack{1232110\\1}} + e_{\substack{1222210\\1}} - e_{\substack{1222110\\1}}$	z_{14}	z_{14}
	17	λ_1	$e_{\substack{2465431\\3}}$		
	18	0	$e_{\substack{1233210\\2}}$	z_{18}	z_{18}
	22	0	$e_{\substack{2343210\\2}}$	z_{22}	z_{22}

E_8, orbit 58: $E_8(a_6)$

L :  τ : $\begin{smallmatrix}0&0&2&0&0&2&0\\&&0&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}0&0&2&0&0&2&0\\&&0&&&&\end{smallmatrix}$

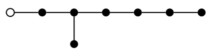
$e = e_{\substack{0011000\\1}} + e_{\substack{1111100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001111\\0}} + e_{\substack{1110000\\1}} + e_{\substack{0111000\\0}}$

$C^\circ = 1 \qquad C/C^\circ = \langle c_1C^\circ, c_2C^\circ\rangle \cong S_3$

$c_1 = h_2(\omega)h_3(\omega)h_5(\omega)h_6(\omega)h_8(\omega^2)$,
$c_2 = n_{\substack{0000000\\1}}\, n_{\substack{0100000\\0}}\, n_{\substack{0000001\\0}}\, n_{\substack{0001100\\0}}\, h_2(-1)h_3(-1)h_4(-1)h_5(-1)h_7(-1)h_8(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
9	2	–	$2e_{\substack{0000110\\0}} + 2e_{\substack{0011100\\1}} - 4e_{\substack{0011000\\0}} - e_{\substack{0110000\\1}} - e_{\substack{1110000\\0}} + 2e_{\substack{0111100\\0}}$		
			$+ 5e_{\substack{1111000\\1}} - e_{\substack{0000011\\0}}$		
	2	–	$2e_{\substack{0000111\\0}} + 2e_{\substack{0110000\\0}} - 4e_{\substack{0111000\\1}} - e_{\substack{0011100\\0}} - e_{\substack{1111100\\1}} + 2e_{\substack{0010000\\1}}$		
			$+ 5e_{\substack{1111000\\0}} - e_{\substack{0001110\\0}}$		
8	4	–	$e_{\substack{0111110\\1}} - e_{\substack{0011111\\0}} - e_{\substack{1111110\\0}} + e_{\substack{1111111\\1}} - e_{\substack{1221000\\1}} - e_{\substack{1122100\\1}}$		
			$+ e_{\substack{0121100\\1}}$		
7	6	–	$e_{\substack{0121110\\1}} + e_{\substack{0122211\\1}} - e_{\substack{1121111\\1}} + e_{\substack{1222210\\1}} + e_{\substack{1122110\\1}} - e_{\substack{1222111\\1}}$		
	6	–	$e_{\substack{1121110\\1}} + e_{\substack{1232100\\1}} - e_{\substack{0122210\\1}} + e_{\substack{1122211\\1}} + 2e_{\substack{1222110\\1}}$		
	6	–	$e_{\substack{1222211\\1}} + e_{\substack{1232100\\2}} + e_{\substack{0121111\\1}} + e_{\substack{1221110\\1}} + 2e_{\substack{1122111\\1}}$		
	6	–	$e_{\substack{1232100\\1}} + e_{\substack{1221111\\1}} - e_{\substack{0122111\\1}} + e_{\substack{1222110\\1}}$		
	6	–	$e_{\substack{1232100\\2}} + e_{\substack{1122210\\1}} + e_{\substack{0122110\\1}} + e_{\substack{1122111\\1}}$		
6	8	–	$e_{\substack{1232210\\1}} + e_{\substack{1232211\\2}} + e_{\substack{1233210\\2}} + e_{\substack{1232111\\1}} - e_{\substack{0122221\\1}}$		
	8	–	$e_{\substack{1122221\\1}} + e_{\substack{1232210\\2}} + e_{\substack{1232110\\1}} - 2e_{\substack{1232111\\2}} - 2e_{\substack{1233211\\1}}$		
	8	–	$e_{\substack{1222221\\1}} - e_{\substack{1232211\\1}} - e_{\substack{1233211\\2}} + 2e_{\substack{1233210\\1}} + 2e_{\substack{1232110\\2}}$		
5	10	–	$e_{\substack{1232221\\2}} - e_{\substack{1233321\\1}} - e_{\substack{1343211\\2}} - e_{\substack{1243210\\2}}$		
	10	–	$e_{\substack{1233221\\1}} + e_{\substack{1343210\\2}} + e_{\substack{2343211\\2}}$		
	10	–	$e_{\substack{1233221\\2}} - e_{\substack{1243211\\2}} - e_{\substack{2343210\\2}}$		
4	12	–	$e_{\substack{1244321\\2}} + e_{\substack{1343221\\2}} + e_{\substack{2343321\\2}}$		
	12	–	$e_{\substack{1243321\\2}} + e_{\substack{1344321\\2}} + e_{\substack{2343221\\2}}$		
	12	–	$e_{\substack{1343321\\2}} + e_{\substack{1243221\\2}} + e_{\substack{2344321\\2}}$		
3	14	–	$e_{\substack{1354321\\3}} - e_{\substack{2354321\\2}}$		
	14	–	$e_{\substack{1354321\\2}} + e_{\substack{2454321\\3}}$		
2	16	–	$e_{\substack{2465321\\3}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$	z_{14}	z_{14}
	18	–	$e_{\substack{2465431\\3}}$	${z_{18}}^1$	
	18	–	$e_{\substack{2465432\\3}}$	${z_{18}}^2$	

E_8, orbit 59: D_7

L :  τ : $\begin{smallmatrix}-21 & 2 & 2 & 2 & 2 & 2 & 2\\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix}2 & 1 & 0 & 1 & 1 & 0 & 1\\ & & 1 & & & & \end{smallmatrix}$

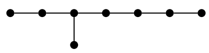
$e = e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$x_{\pm\beta_1}(t) = x_{\pm \substack{2354321\\3}}(t)x_{\pm \substack{2454321\\2}}(-t)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
7	3	λ_1	$e_{\substack{1232210\\2}} - e_{\substack{1233210\\1}} - e_{\substack{1232211\\1}} + 2e_{\substack{1232111\\2}} + e_{\substack{1222221\\1}}$		
6	6	0	$2e_{\substack{0110000\\1}} + e_{\substack{0111000\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}} + e_{\substack{0000111\\0}}$		
5	9	λ_1	$e_{\substack{1343210\\2}} + e_{\substack{1243211\\2}} - e_{\substack{1233221\\2}} + e_{\substack{1233321\\1}}$		
4	12	$2\lambda_1$	$e_{\substack{2465432\\3}}$		
3	15	λ_1	$e_{\substack{1343321\\2}} - e_{\substack{1244321\\2}}$		
2	10	0	$2e_{\substack{0121000\\1}} + 2e_{\substack{0111100\\1}} + e_{\substack{0111110\\0}} + e_{\substack{0011110\\1}} + e_{\substack{0011111\\0}}$		
	11	λ_1	$e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}} + e_{\substack{1233321\\2}}$		
	18	0	$e_{\substack{0122111\\1}} + e_{\substack{0122210\\1}}$		
1	2	0	e	z_2	z_2
	14	0	$e_{\substack{0122100\\1}} + e_{\substack{0121110\\1}} + e_{\substack{0111111\\1}}$	z_{14}	z_{14}
	21	λ_1	$e_{\substack{1354321\\3}}$		
	22	0	$e_{\substack{0122221\\1}}$	z_{22}	z_{22}

E_8, orbit 60: $E_8(b_5)$

L :  τ : $\begin{smallmatrix}0&0&2&0&0&2&2\\&&0&&&&\end{smallmatrix}$ Δ = $\begin{smallmatrix}0&0&2&0&0&2&2\\&&0&&&&\end{smallmatrix}$

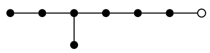
$e = e_{\substack{0011000\\1}} + e_{\substack{0000001\\0}} + e_{\substack{1111100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1110000\\1}}$

$C^\circ = 1 \qquad C/C^\circ = \langle c_1C^\circ, c_2C^\circ\rangle \cong S_3$

$c_1 = h_1(\omega)h_2(\omega)h_5(\omega^2)$,
$c_2 = n_{\substack{1000000\\0}}\, n_{\substack{0000000\\1}}\, n_{\substack{0001000\\0}}\, h_1(-1)h_3(-1)h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	–	$e_{\substack{1110000\\0}} - e_{\substack{1111000\\1}} - e_{\substack{0110000\\1}}$		
	2	–	$e_{\substack{0111000\\1}} - e_{\substack{0110000\\0}} - e_{\substack{1111000\\0}}$		
	2	–	$e_{\substack{1110000\\1}} + e_{\substack{0111000\\0}}$		
	6	–	$e_{\substack{0011111\\0}} - 2e_{\substack{1111111\\1}} - e_{\substack{1121110\\1}} + e_{\substack{0122210\\1}} + 3e_{\substack{1232100\\1}} + 2e_{\substack{1222110\\1}}$		
	6	–	$e_{\substack{0011111\\1}} - 2e_{\substack{0111111\\0}} - e_{\substack{0122110\\1}} - e_{\substack{1122210\\1}} - 3e_{\substack{1232100\\2}} - 2e_{\substack{1221110\\1}}$		
4	4	–	$e_{\substack{1221000\\1}}$		
	8	–	$e_{\substack{1121111\\1}} + 2e_{\substack{1222111\\1}} + e_{\substack{1232110\\1}} + e_{\substack{1232210\\2}}$		
	8	–	$e_{\substack{0122111\\1}} - 2e_{\substack{1221111\\1}} - e_{\substack{1232110\\2}} - e_{\substack{1233210\\1}}$		
	8	–	$e_{\substack{1232210\\1}} + e_{\substack{1233210\\2}} - e_{\substack{0121111\\1}} + e_{\substack{1122111\\1}}$		
3	10	–	$e_{\substack{1222221\\1}} - e_{\substack{1232211\\1}} + e_{\substack{1233211\\2}} + 2e_{\substack{1243210\\2}}$		
	10	–	$e_{\substack{1232111\\1}} - e_{\substack{1343210\\2}}$		
	10	–	$e_{\substack{1232111\\2}} + e_{\substack{2343210\\2}}$		
	14	–	$e_{\substack{1244321\\2}} + e_{\substack{1343221\\2}} + e_{\substack{2343321\\2}}$		
	14	–	$e_{\substack{1243321\\2}} + e_{\substack{2343221\\2}} + e_{\substack{1344321\\2}}$		
2	16	–	$e_{\substack{2354321\\2}} - e_{\substack{1354321\\3}}$		
	16	–	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$		
	16	–	$e_{\substack{1354321\\2}} + e_{\substack{2454321\\3}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{1243221\\2}} + e_{\substack{1343321\\2}} + e_{\substack{2344321\\2}}$	z_{14}	z_{14}
	18	–	$e_{\substack{2464321\\3}}$	${z_{18}}^1$	
	18	–	$e_{\substack{2465321\\3}}$	${z_{18}}^2$	
	22	–	$e_{\substack{2465432\\3}}$	z_{22}	z_{22}

E_8, orbit 61: $E_7(a_1)$

L :  τ : $\substack{2\ 2\ 0\ 2\ 2\ 2\ {-21}\\ 2}$ $\Delta = \substack{2\ 1\ 0\ 1\ 0\ 2\ 2\\ 1}$

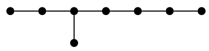
$e = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$\beta_1 = \substack{2465432\\3}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	5	λ_1	$e_{\substack{1233321\\2}} - e_{\substack{1343221\\2}} + e_{\substack{2343211\\2}}$		
	6	0	$2e_{\substack{1110000\\1}} - e_{\substack{1111000\\0}} - e_{\substack{0121000\\1}} + 2e_{\substack{0111000\\1}} - e_{\substack{0111100\\0}} + 2e_{\substack{0011100\\1}} - 3e_{\substack{0011110\\0}} + 2e_{\substack{0001110\\0}}$		
4	10	0	$2e_{\substack{1111110\\0}} + e_{\substack{0121110\\1}} + e_{\substack{0122100\\1}} - e_{\substack{1221000\\1}} - e_{\substack{1121100\\1}}$		
	10	0	$e_{\substack{1111100\\1}} + e_{\substack{0111110\\1}} + e_{\substack{1221000\\1}}$		
	11	λ_1	$e_{\substack{2354321\\2}} - e_{\substack{2344321\\2}} + e_{\substack{1354321\\3}}$		
3	15	λ_1	$e_{\substack{2454321\\3}}$		
	16	0	$e_{\substack{1232100\\2}} - e_{\substack{1232110\\1}} + e_{\substack{1222110\\1}} - e_{\substack{1122210\\1}}$		
2	21	λ_1	$e_{\substack{2465431\\3}}$		
1	2	0	e	z_2	z_2
	14	0	$e_{\substack{1232100\\1}} + e_{\substack{1221110\\1}} - e_{\substack{1122110\\1}} + e_{\substack{0122210\\1}}$	z_{14}	z_{14}
	18	0	$e_{\substack{1232110\\2}} - e_{\substack{1222210\\1}}$	z_{18}	z_{18}
	22	0	$e_{\substack{1243210\\2}} - e_{\substack{1233210\\2}}$	z_{22}	z_{22}
	26	0	$e_{\substack{2343210\\2}}$	z_{26}	z_{26}

E_8, orbit 62: $E_8(a_5)$

L :  τ : $\substack{2\ 0\ 2\ 0\ 0\ 2\ 0\\0}$ $\Delta = \substack{2\ 0\ 2\ 0\ 0\ 2\ 0\\0}$

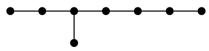
$e = e_{\substack{0011000\\1}} + e_{\substack{0000111\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}} + e_{\substack{0010000\\0}}$
$+ e_{\substack{0000010\\0}}$

$C^\circ = 1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$c = h_4(-1)h_7(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
11	2	–	$e_{\substack{1100000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0111000\\1}} + 2e_{\substack{0000110\\0}} + 4e_{\substack{0010000\\1}} - 7e_{\substack{0000011\\0}}$		
			$- 6e_{\substack{0011000\\0}} + e_{\substack{0111100\\0}} - 2e_{\substack{0011100\\1}} + e_{\substack{0001111\\0}}$		
	2	–	$e_{\substack{0010000\\0}} + e_{\substack{0000010\\0}}$		
10	4	–	$e_{\substack{1110000\\0}} - e_{\substack{0121000\\1}} - 2e_{\substack{0011110\\1}} + e_{\substack{0111110\\0}} - e_{\substack{0011111\\0}}$		
9	6	–	$e_{\substack{1111110\\1}} + e_{\substack{1121100\\1}} - e_{\substack{1221000\\1}} + e_{\substack{1111111\\0}} - 2e_{\substack{0122210\\1}} - e_{\substack{0122111\\1}}$		
8	8	–	$e_{\substack{1221110\\1}} + 2e_{\substack{1122110\\1}} + e_{\substack{1121111\\1}} + e_{\substack{1222210\\1}} - e_{\substack{1232100\\2}} + e_{\substack{1222111\\1}}$		
			$- 4e_{\substack{0122221\\1}} + e_{\substack{1122211\\1}}$		
7	10	–	$2e_{\substack{1232210\\1}} + e_{\substack{1232110\\2}} - e_{\substack{1233211\\1}} + e_{\substack{1233210\\2}} - e_{\substack{1232111\\1}} - e_{\substack{1232211\\2}}$		
	10	–	$2e_{\substack{1232210\\2}} + e_{\substack{1222221\\1}} - e_{\substack{1232211\\1}} + 2e_{\substack{1233210\\1}} - e_{\substack{1232111\\2}}$		
	10	–	$e_{\substack{1232210\\1}} + e_{\substack{1232110\\2}} + e_{\substack{1122221\\1}} + e_{\substack{1232111\\1}}$		
	10	–	$2e_{\substack{1232110\\1}} - e_{\substack{1222221\\1}} - e_{\substack{1232211\\1}} - e_{\substack{1233211\\2}}$		
6	12	–	$e_{\substack{1243210\\2}} - e_{\substack{1232221\\2}} - e_{\substack{1233221\\1}}$		
	12	–	$2e_{\substack{1232221\\1}} + e_{\substack{1233221\\2}} + e_{\substack{1243211\\2}}$		
5	14	–	$2e_{\substack{1243321\\2}} + e_{\substack{1343221\\2}} + e_{\substack{1344321\\2}} - e_{\substack{2343211\\2}}$		
	14	–	$e_{\substack{1243221\\2}}$		
4	16	–	$e_{\substack{2343221\\2}} - e_{\substack{1354321\\2}}$		
3	18	–	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$		
2	20	–	$e_{\substack{2464321\\3}} - e_{\substack{2465421\\3}}$		
1	2	–	e	z_2	z_2
	14	–	$7e_{\substack{1243221\\2}} - e_{\substack{1244321\\2}} - e_{\substack{1343321\\2}} - e_{\substack{2343210\\2}}$	z_{14}	z_{14}
	22	–	$e_{\substack{2465431\\3}}$	${z_{22}}^1$	
	22	–	$e_{\substack{2465432\\3}}$	${z_{22}}^2$	${z_{22}}^2$

E_8, orbit 63: $E_8(b_4)$

L :  τ : $\substack{2\ 0\ 2\ 0\ 0\ 2\ 2\\0}$ $\Delta = \substack{2\ 0\ 2\ 0\ 0\ 2\ 2\\0}$

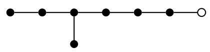
$e = e_{\substack{0000001\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0010000\\0}}$

$C^\circ = 1$ $C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
7	2	–	$e_{\substack{0010000\\0}}$		
	4	–	$e_{\substack{0011110\\0}} + e_{\substack{1111000\\1}} - e_{\substack{0121000\\1}} + e_{\substack{0001111\\0}} - 3e_{\substack{1110000\\0}} + e_{\substack{0111110\\1}} + 3e_{\substack{0121100\\1}}$		
	6	–	$e_{\substack{1111110\\0}} + e_{\substack{0111111\\0}} + 2e_{\substack{1221000\\1}} - 2e_{\substack{0011111\\1}} - 2e_{\substack{0122110\\1}} - 2e_{\substack{1122100\\1}} - 3e_{\substack{1221100\\1}} - e_{\substack{0122210\\1}}$		
6	6	–	$e_{\substack{0011111\\0}} + e_{\substack{1121000\\1}} + e_{\substack{0121110\\1}}$		
	10	–	$2e_{\substack{1121111\\1}} + e_{\substack{1222111\\1}} + e_{\substack{1232210\\1}} - e_{\substack{1232110\\1}} + e_{\substack{1222211\\1}} - e_{\substack{1233210\\2}}$		
5	10	–	$e_{\substack{0122221\\1}} - e_{\substack{1122211\\1}} + 2e_{\substack{1221111\\1}} - e_{\substack{1232210\\2}} - e_{\substack{1233210\\1}}$		
	10	–	$e_{\substack{1122111\\1}} + e_{\substack{0122221\\1}} - e_{\substack{1232110\\2}}$		
	12	–	$e_{\substack{1232111\\1}} + e_{\substack{1232211\\1}} - e_{\substack{1243210\\2}}$		
4	14	–	$e_{\substack{1233221\\2}} + e_{\substack{1232221\\1}} - e_{\substack{1243211\\2}}$		
	16	–	$e_{\substack{1244321\\2}} + e_{\substack{1343221\\2}} + e_{\substack{1343321\\2}} - e_{\substack{2343211\\2}}$		
3	16	–	$e_{\substack{1243221\\2}}$		
	18	–	$e_{\substack{1354321\\3}} - e_{\substack{2343221\\2}}$		
	20	–	$e_{\substack{2354321\\2}} - e_{\substack{2454321\\3}}$		
2	22	–	$e_{\substack{2464321\\3}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{1232221\\2}} - e_{\substack{1233321\\1}} - e_{\substack{1343211\\2}} - e_{\substack{2343210\\2}}$	z_{14}	z_{14}
	22	–	$e_{\substack{2465321\\3}} + e_{\substack{2465421\\3}}$	z_{22}	z_{22}
	26	–	$e_{\substack{2465432\\3}}$	z_{26}	z_{26}

E_8, orbit 64: E_7

L :  τ : $\begin{smallmatrix} 2 & 2 & 2 & 2 & 2 & 2 & -27 \\ & & 2 & & & & \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2 & 1 & 0 & 1 & 2 & 2 & 2 \\ & & 1 & & & & \end{smallmatrix}$

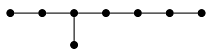
$e = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$

$C^\circ = A_1 \qquad C/C^\circ = 1$

$\beta_1 = \substack{2465432\\3}$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	9	λ_1	$e_{\substack{1244321\\2}} - e_{\substack{1343321\\2}} + e_{\substack{2343221\\2}}$		
2	10	0	$e_{\substack{1111000\\1}} + 2e_{\substack{1111100\\0}} - e_{\substack{0111100\\1}} - 3e_{\substack{0011110\\1}} + 2e_{\substack{0111110\\0}} - e_{\substack{0121000\\1}}$		
	17	λ_1	$e_{\substack{2454321\\2}} - e_{\substack{2354321\\3}}$		
	18	0	$e_{\substack{1221110\\1}} - e_{\substack{1122110\\1}} + e_{\substack{0122210\\1}}$		
1	2	0	e	z_2	z_2
	14	0	$e_{\substack{1221000\\1}} + e_{\substack{1121100\\1}} + 2e_{\substack{1111110\\1}} - e_{\substack{0121110\\1}} - e_{\substack{0122100\\1}}$	z_{14}	z_{14}
	22	0	$e_{\substack{1232100\\2}} + e_{\substack{1232110\\1}} - e_{\substack{1222210\\1}}$	z_{22}	z_{22}
	26	0	$e_{\substack{1232210\\2}} - e_{\substack{1233210\\1}}$	z_{26}	z_{26}
	27	λ_1	$e_{\substack{2465431\\3}}$		
	34	0	$e_{\substack{2343210\\2}}$	z_{34}	z_{34}

E_8, orbit 65: $E_8(a_4)$

L :  τ : $\begin{smallmatrix}2&0&2&0&2&0&2\\&&0&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}2&0&2&0&2&0&2\\&&0&&&&\end{smallmatrix}$

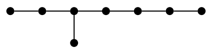
$e = e_{\substack{0011000\\1}} + e_{\substack{0000100\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0001110\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000001\\0}}$

$C^\circ = 1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$c = h_4(-1)h_8(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	4	$-$	$3e_{\substack{1111000\\1}} - e_{\substack{1110000\\0}} - e_{\substack{0000111\\0}} - e_{\substack{0111110\\0}} + 3e_{\substack{0111100\\1}} - 5e_{\substack{0011100\\0}} - 2e_{\substack{0121000\\1}} - e_{\substack{0011110\\1}}$		
	6	$-$	$e_{\substack{1111110\\1}} + e_{\substack{0111111\\1}} + e_{\substack{1111100\\0}} - e_{\substack{1221000\\1}} - 2e_{\substack{0011111\\0}} - e_{\substack{0122100\\1}} - e_{\substack{0121110\\1}}$		
	8	$-$	$2e_{\substack{0122210\\1}} - 3e_{\substack{1222100\\1}} + e_{\substack{1122110\\1}} + e_{\substack{1221110\\1}} - e_{\substack{1111111\\0}} + 2e_{\substack{0122111\\1}} - 3e_{\substack{1121100\\1}}$		
	10	$-$	$e_{\substack{1222210\\1}} + 2e_{\substack{1121111\\1}} - e_{\substack{1232110\\1}} - 2e_{\substack{0122221\\1}} - e_{\substack{1232100\\2}} + e_{\substack{1222111\\1}}$		
4	10	$-$	$2e_{\substack{0122211\\1}} + e_{\substack{1122210\\1}} + e_{\substack{1232100\\1}} - e_{\substack{1122111\\1}} + e_{\substack{1232110\\2}}$		
	14	$-$	$e_{\substack{1232211\\2}} + e_{\substack{1232221\\1}} + e_{\substack{1233211\\1}} + e_{\substack{1233221\\2}}$		
3	14	$-$	$e_{\substack{1243210\\2}} + e_{\substack{1233211\\2}}$		
	16	$-$	$e_{\substack{2343210\\2}} - e_{\substack{1243221\\2}} + e_{\substack{1233321\\2}} + e_{\substack{1343211\\2}}$		
	18	$-$	$e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}} - e_{\substack{1244321\\2}}$		
2	18	$-$	$e_{\substack{2343211\\2}} - e_{\substack{1344321\\2}} - e_{\substack{1243321\\2}}$		
	22	$-$	$e_{\substack{2354321\\2}} - e_{\substack{2454321\\3}}$		
1	2	$-$	e	z_2	z_2
	14	$-$	$4e_{\substack{1243210\\2}} + 3e_{\substack{1233211\\2}} - e_{\substack{1232211\\1}} - e_{\substack{1233221\\1}} - e_{\substack{1232221\\2}}$	z_{14}	z_{14}
	22	$-$	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$	z_{22}	z_{22}
	26	$-$	$e_{\substack{2465421\\3}}$	z_{26}	z_{26}
	28	$-$	$e_{\substack{2465432\\3}}$	z_{28}	

E_8, orbit 66: $E_8(a_3)$

L :  τ : $\substack{2\ 0\ 2\ 0\ 2\ 2\ 2\\ 0}$ $\Delta = \substack{2\ 0\ 2\ 0\ 2\ 2\ 2\\ 0}$

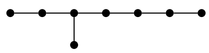
$e = e_{\substack{0000001\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0000010\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0111000\\0}} + e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0010000\\0}}$

$C^\circ = 1 \qquad C/C^\circ = \langle cC^\circ \rangle \cong S_2$

$c = h_4(-1)$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	2	–	$e_{\substack{0010000\\0}}$		
	8	–	$e_{\substack{1111110\\1}} + e_{\substack{0111111\\1}} - 3e_{\substack{0011111\\0}} - e_{\substack{1121100\\1}} - 2e_{\substack{0121110\\1}} - e_{\substack{1222100\\1}}$		
4	10	–	$e_{\substack{1232100\\1}} - e_{\substack{0121111\\1}} - e_{\substack{1121110\\1}}$		
3	10	–	$e_{\substack{0122210\\1}} - e_{\substack{1122110\\1}} + 2e_{\substack{1221110\\1}} - 2e_{\substack{1111111\\0}} + e_{\substack{0122111\\1}} + 3e_{\substack{1232100\\2}}$		
	16	–	$e_{\substack{1222221\\1}} - e_{\substack{1232211\\1}} + 2e_{\substack{1243210\\2}} + e_{\substack{1233211\\2}}$		
	18	–	$e_{\substack{2343210\\2}} + e_{\substack{1343211\\2}} - e_{\substack{1232221\\2}}$		
2	18	–	$e_{\substack{1232221\\1}} + e_{\substack{1243211\\2}}$		
	26	–	$e_{\substack{2354321\\2}} - e_{\substack{2454321\\3}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{0122221\\1}} - e_{\substack{1122211\\1}} - e_{\substack{1232210\\2}} - 2e_{\substack{1232111\\2}} - e_{\substack{1233210\\1}}$	z_{14}	z_{14}
	22	–	$e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}} - e_{\substack{1244321\\2}}$	z_{22}	z_{22}
	26	–	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$	z_{26}	z_{26}
	28	–	$e_{\substack{2464321\\3}}$	z_{28}	
	34	–	$e_{\substack{2465432\\3}}$	z_{34}	z_{34}

E_8, orbit 67: $E_8(a_2)$

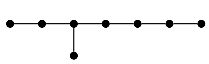
L :  τ : $\begin{smallmatrix}2&2&0&2&0&2&2\\&&2&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}2&2&0&2&0&2&2\\&&2&&&&\end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0000001\\0}}$

$C^\circ = 1 \qquad C/C^\circ = 1$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
5	6	–	$e_{\substack{1111000\\0}} - 2e_{\substack{1110000\\1}} - e_{\substack{0111000\\1}} + e_{\substack{0111110\\0}} - 3e_{\substack{0001111\\0}} - 2e_{\substack{0011110\\1}}$		
			$- 3e_{\substack{1111100\\0}} + e_{\substack{0111100\\1}} + e_{\substack{0011111\\0}} - e_{\substack{0121000\\1}} - 3e_{\substack{0121100\\1}}$		
	10	–	$3e_{\substack{1111110\\1}} - 2e_{\substack{1221000\\1}} + 6e_{\substack{1111111\\0}} - 2e_{\substack{1121110\\1}} - 3e_{\substack{0111111\\1}}$		
			$+ 3e_{\substack{0122110\\1}} + 4e_{\substack{0121111\\1}} - 2e_{\substack{1122100\\1}} - 4e_{\substack{0122210\\1}}$		
4	16	–	$e_{\substack{1222111\\1}} + e_{\substack{1232110\\2}} + e_{\substack{1122221\\1}} + e_{\substack{1222211\\1}} - e_{\substack{1232210\\2}} - e_{\substack{1233210\\1}}$		
			$- 2e_{\substack{1232211\\1}}$		
3	18	–	$e_{\substack{1232111\\2}} + 2e_{\substack{1233210\\2}} + e_{\substack{1233211\\1}} + e_{\substack{1232211\\2}} + e_{\substack{1232221\\1}}$		
	22	–	$e_{\substack{1233221\\2}}$		
2	28	–	$e_{\substack{2344321\\2}} - e_{\substack{1354321\\3}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{1222110\\1}} + e_{\substack{1122111\\1}} - e_{\substack{1232110\\1}} - e_{\substack{1222210\\1}} - e_{\substack{1221111\\1}} - e_{\substack{1232100\\2}}$	z_{14}	z_{14}
			$+ e_{\substack{0122221\\1}} - 2e_{\substack{1122211\\1}}$		
	22	–	$e_{\substack{2343210\\2}} + e_{\substack{1343211\\2}} - e_{\substack{1243321\\2}} - e_{\substack{1233221\\2}}$	z_{22}	z_{22}
	26	–	$e_{\substack{2343221\\2}} - e_{\substack{1344321\\2}} - e_{\substack{2343321\\2}} + e_{\substack{1354321\\2}}$	z_{26}	z_{26}
	34	–	$e_{\substack{2465421\\3}}$	z_{34}	z_{34}
	38	–	$e_{\substack{2465432\\3}}$	z_{38}	z_{38}

E_8, orbit 68: $E_8(a_1)$

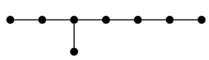
L :  τ : $\begin{smallmatrix}2&2&0&2&2&2&2\\&&2&&&&\end{smallmatrix}$ $\Delta = \begin{smallmatrix}2&2&0&2&2&2&2\\&&2&&&&\end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

$C^\circ = 1 \qquad C/C^\circ = 1$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
3	10	–	$4e_{\substack{1111100\\1}} + 2e_{\substack{0111110\\1}} + 2e_{\substack{1111110\\0}} + 2e_{\substack{0111111\\0}} - e_{\substack{1121100\\1}} + e_{\substack{0121110\\1}} + 4e_{\substack{1221000\\1}} - 3e_{\substack{0011111\\1}} + e_{\substack{0122100\\1}}$		
	18	–	$2e_{\substack{0122221\\1}} + e_{\substack{1222210\\1}} + e_{\substack{1222111\\1}} - 2e_{\substack{1122211\\1}} - e_{\substack{1232110\\2}} - 2e_{\substack{1232111\\1}}$		
2	28	–	$e_{\substack{1343221\\2}} - e_{\substack{1233321\\2}} - e_{\substack{2343211\\2}}$		
1	2	–	e	z_2	z_2
	14	–	$e_{\substack{0122210\\1}} - e_{\substack{1122110\\1}} - e_{\substack{1221110\\1}} - e_{\substack{1222100\\1}} + e_{\substack{0122111\\1}} - 2e_{\substack{1121111\\1}} - e_{\substack{1111111\\1}} + e_{\substack{1232100\\1}}$	z_{14}	z_{14}
	22	–	$2e_{\substack{1222221\\1}} - e_{\substack{1232221\\1}} + e_{\substack{1233211\\1}} - e_{\substack{1232211\\2}} - e_{\substack{1243210\\2}}$	z_{22}	z_{22}
	26	–	$e_{\substack{1233221\\2}} - e_{\substack{1233321\\1}} - e_{\substack{2343210\\2}} - e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}}$	z_{26}	z_{26}
	34	–	$e_{\substack{1354321\\3}} - e_{\substack{2354321\\2}}$	z_{34}	z_{34}
	38	–	$e_{\substack{2454321\\3}}$	z_{38}	z_{38}
	46	–	$e_{\substack{2465432\\3}}$	z_{46}	z_{46}

E_8, orbit 69: E_8

L :  τ : $\begin{smallmatrix} 2&2&2&2&2&2&2\\ &&2&&&& \end{smallmatrix}$ $\Delta = \begin{smallmatrix} 2&2&2&2&2&2&2\\ &&2&&&& \end{smallmatrix}$

$e = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

$C^\circ = 1 \qquad C/C^\circ = 1$

n	m	λ	v	$\mathcal{Z}^\natural$	$\mathcal{Z}$
1	2	–	e	z_2	z_2
	14	–	$2e_{\substack{1111110\\1}} - e_{\substack{0111111\\1}} + e_{\substack{1121100\\1}} - e_{\substack{0121110\\1}} + e_{\substack{1221000\\1}} + 3e_{\substack{1111111\\0}} - e_{\substack{0122100\\1}}$	z_{14}	z_{14}
	22	–	$e_{\substack{1222210\\1}} - e_{\substack{1232110\\1}} - 2e_{\substack{0122221\\1}} - e_{\substack{1232100\\2}} - e_{\substack{1222111\\1}} + 2e_{\substack{1122211\\1}}$	z_{22}	z_{22}
	26	–	$e_{\substack{1232210\\2}} + e_{\substack{1222221\\1}} - e_{\substack{1233210\\1}} - e_{\substack{1232211\\1}} + 2e_{\substack{1232111\\2}}$	z_{26}	z_{26}
	34	–	$e_{\substack{2343210\\2}} + e_{\substack{1343211\\2}} - e_{\substack{1243221\\2}} + e_{\substack{1233321\\2}}$	z_{34}	z_{34}
	38	–	$e_{\substack{2343221\\2}} - e_{\substack{1343321\\2}} + e_{\substack{1244321\\2}}$	z_{38}	z_{38}
	46	–	$e_{\substack{2354321\\3}} - e_{\substack{2454321\\2}}$	z_{46}	z_{46}
	58	–	$e_{\substack{2465432\\3}}$	z_{58}	z_{58}

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