ENDO-PERMUTATION MODULES, A GUIDED TOUR

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1. Introduction

This survey paper gives an overview of the recent classification of all endo-permutation kP-modules, where P is a finite p-group and k is a field of characteristic p. It is an expanded version of two talks given in April 2005 during a special workshop on endo-permutation modules, organized within the program of the Bernoulli Centre of EPFL.

The classification of endo-permutation modules was completed in 2004, a quarter of a century after the first decisive results of Dade [Da2] in 1978. The final results are due to the combined efforts of several authors during the years 1998–2004. The first crucial step was the classification of all modules in an important subclass, namely the class of endo-trivial kP-modules. This appears in the work of Carlson and Thévenaz [CaTh1], [CaTh2], [CaTh3]. The classification of all endo-permutation kP-modules when P is extraspecial (or almost extraspecial), due to Bouc and Mazza [BoMa], was obtained shortly afterwards. The final completion of the classification in all cases is due to Bouc [Bo7], based on the above mentioned papers and on several aspects of his previous work [Bo2], [Bo4], [Bo5]. The important role of relative syzygies had been discovered a few years before by Alperin [Al3].

This survey will not follow the chronological order of the various publications, but rather what appears to be a logical development of the subject, at least in our opinion.

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2. Endo-permutation modules

Throughout this paper, P denotes a finite p-group and k a field of characteristic p. The whole classification is independent of the choice of k, except for some exceptional behaviour concerning the quaternion group of order 8. We first recall the basic definitions and constructions. We refer to any standard book on modular representation theory for all unexplained facts, e.g. [Be], [Fe] or [Th].

We are interested in modules over the group algebra kP and we assume throughout this paper that every kP-module is finitely generated, or in other words finite-dimensional over k. Recall that the trivial kP-module k is the unique simple kP-module up to isomorphism and that the free module kP is the unique indecomposable projective kP-module up to isomorphism (projective modules

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are free). However, there are infinitely many indecomposable kP-modules up to isomorphism, except in the very special case where P is cyclic. In his seminal paper [Da2], Dade started with the comments that 'there are just too many modules over p-groups' and that the family of endo-permutation kP-modules is 'small enough to be classified and large enough to be useful'. This remark was correct, but 25 years were necessary to complete the classification, which is far from being trivial.

Concerning the usefulness of endo-permutation modules, let us just mention that they play a crucial role in representation theory. They appear as sources of simple modules for p-solvable groups (see Section 30 in [Th]), in Puig's description of the source algebra of a nilpotent block (see [Pu1] or Section 50 in [Th]), and also in the local analysis of Morita or derived equivalences between blocks [Pu4]. The subclass of endo-trivial modules appears in connection with self-equivalences of the stable category of a block (see [CaRo]). So endo-permutation modules are fundamental objects which need to be fully understood.

A kP-module M is called a *permutation module* if it has a basis X which is invariant under the action of P. In that case we write $M = kX$. The finite P-set X decomposes as a disjoint union of orbits and each orbit is isomorphic to a set of cosets P/Q for some subgroup Q of P. Thus kX decomposes accordingly as a direct sum of submodules of the form $k[P/Q]$. Every such module $k[P/Q]$ is indecomposable, because its socle is k (generated by the sum of the basis elements, which is the only fixed point up to a scalar multiple). Therefore, the indecomposable permutation kP-modules are parametrized by the subgroups of P up to conjugation. In particular, there are finitely many of them. Since $k[P/Q] \cong \mathrm{Ind}_Q^P(k)$, its vertex is Q, and therefore the only indecomposable permutation module with vertex P is the trivial module k. In view of this discussion, we note that any direct summand of a permutation kP-module is again a permutation module. This very special property of p-groups plays an important role in what follows.

Now we come to the definition, due to Dade [Da2] (but this definition will be changed slightly below). A kP-module M is called an *endo-permutation module (in the weak sense)* if $\mathrm{End}_k(M)$ is a permutation module, where $\mathrm{End}_k(M)$ is endowed with its natural kP-module structure coming from the action of P by conjugation: if $g \in P$ and $\phi \in \mathrm{End}_k(M)$, then ${}^g\phi(m) = g{\cdot}\phi(g^{-1}{\cdot}m)$ for all $m \in M$. Recall that $\mathrm{End}_k(M) \cong M \otimes M^*$ as a kP-module, where $M^* = \mathrm{Hom}(M, k)$ is the dual module and where the tensor product is over k (with diagonal action of P). Thus M is an endo-permutation module if and only if $M \otimes M^*$ has a P-invariant basis.

A kP-module M is called *endo-trivial* if there exists a projective kP-module F such that $\mathrm{End}_k(M) \cong k \oplus F$ as a kP-module. Since every projective kP-module is free and since a free kP-module is a direct sum of copies of kP, which has an obvious P-invariant basis, it is clear that any endo-trivial module is an endo-permutation module. This notion is also due to Dade [Da2], and independently to Alperin, who called them *invertible* modules in [Al1]. More recently, another characterization of endo-trivial modules in terms of stable homomorphisms was obtained by Carlson [Ca3]. We shall see in Section 4 several examples of endo-trivial and endo-permutation modules. Note that some authors write 'endopermutation' and 'endotrivial'.

The first basic properties are the following.

Proposition 2.1. *The class of endo-permutation modules (in the weak sense) contains the permutation modules. Moreover, it is closed under taking direct summands, duals, tensor products, Heller translates, restrictions to a subgroup, and tensor inductions to an overgroup.*

Proof. All these properties except the last one are proved in § 2 of [Da2] or in § 28 of [Th]. For tensor induction, a detailed treatement appears in [BoTh1], but in a more general setting. Since the argument is not difficult, we give here a sketch. Let M be an endo-permutation kQ-module, where Q is a subgroup of P, and let $\mathrm{Ten}_Q^P(M)$ denote the tensor induced module (as defined for instance in § 3.15 of [Be]). Notice that $\mathrm{End}_k(M\otimes N) \cong \mathrm{End}_k(M)\otimes\mathrm{End}_k(N)$ and apply this property to M and all its conjugates under the action of P to obtain $\mathrm{Ten}_Q^P(\mathrm{End}_k(M)) \cong \mathrm{End}_k(\mathrm{Ten}_Q^P(M))$. This isomorphism is proved in detail as Lemma 2.1 of [BoTh1]. Then the tensor product of the P-conjugates of a Q-invariant basis of $\mathrm{End}_k(M)$ yields a P-invariant basis of $\mathrm{Ten}_Q^P(\mathrm{End}_k(M))$. □

Note that the class of endo-permutation modules is not closed under taking direct sums and induction to overgroups (but, by § 2 of [Da2], one knows precisely the conditions under which this holds). The natural operation to be used is tensor product instead of direct sum, and similarly tensor induction instead of induction. It is Puig who first observed, in some unpublished notes [Pu2], that tensor induction preserves the class of endo-permutation modules, and he used it to prove remarkable results for p-solvable groups.

We are actually mainly interested in indecomposable endo-permutation kP-modules with maximal vertex P, because they are the ones that appear naturally in representation theory. But in fact, all endo-permutation modules can be described from the knowledge of the indecomposable ones having maximal vertex. This is proved in § 6 of [Da2] and is based on the fact that any indecomposable endo-permutation module is absolutely indecomposable and that Green's theorem can be applied (i.e. indecomposable endo-permutation modules remain indecomposable under induction).

Since indecomposable modules with maximal vertex may not remain indecomposable under restriction or tensor product, we need a more general definition. An endo-permutation kP-module M is said to be *capped* if it has at least one indecomposable direct summand with vertex P. In particular, if M is indecomposable, then M is capped if and only if it has vertex P, or equivalently, M is not induced from a proper subgroup.

Recall that the Brauer quotient is defined by

$$\big(\mathrm{End}_k(M)\big)[P] = \mathrm{End}_{kP}(M)\Big/\sum_{Q<P} tr_Q^P(\mathrm{End}_{kQ}(M)),$$

where tr_Q^P denotes the relative trace map. If X is a P-invariant basis of $\mathrm{End}_k(M)$, then the subset X^P of P-fixed points is a basis of $\big(\mathrm{End}_k(M)\big)[P]$. By using Higman's criterion, one can prove the following.

Lemma 2.2. *Let M be an endo-permutation kP-module. The following conditions are equivalent :*

(a) *M is capped.*
(b) *The Brauer quotient $\big(\mathrm{End}_k(M)\big)[P]$ is nonzero.*
(c) *There is a fixed point in a P-invariant basis of* $\mathrm{End}_k(M)$*.*
(d) $\mathrm{End}_k(M) \cong k \oplus kY$ *as a kP-module, for some P-set Y.*
(e) $\mathrm{End}_k(M) = k{\cdot}\mathrm{id} \oplus kY$ *as a kP-module, for some P-set Y.*

In view of the general description of endo-permutation modules in terms of capped endo-permutation modules (Theorem 6.6 in [Da2]), we are reduced to the classification of the latter modules. For this reason, we change the terminology and define an *endo-permutation module (in the strong sense)* to be an endo-permutation module (in the weak sense) which is capped. In other words, we assume from now on that *all endo-permutation modules are capped.* This amounts to the additional assumption that $M \otimes M^* \cong k \oplus kY$ as a kP-module, for some P-set Y, by condition (d) above.

Note that Proposition 2.1 remains correct with this new definition, except the first statement, which is replaced by the assertion that a permutation module is an endo-permutation module (in the strong sense) if and only if it has (at least) one trivial direct summand. In particular the only indecomposable permutation module which is an endo-permutation module (in the strong sense) is the trivial module k. Note also that any endo-trivial module is an endo-permutation module in the strong sense.

3. The Dade group

The following basic result of Dade is fundamental for understanding endo-permutation modules. It appears as Theorem 3.8 in [Da2] or as Corollary 28.9 in [Th].

Proposition 3.1. *If M is an endo-permutation kP-module, then any two indecomposable direct summands of M with vertex P are isomorphic.*

Thus M has, up to isomorphism, a unique indecomposable summand with vertex P, called the *cap* of M and written M_0. Note that the cap may appear with some multiplicity as a direct summand. The proposition allows for the definition of an equivalence relation in the class of endo-permutation modules. Two endo-permutation kP-modules M and N are said to be *equivalent* if their caps are isomorphic. We then write $M \sim N$, so that

$$M \sim N \quad \Leftrightarrow \quad M_0 \cong N_0 \ .$$

It is equivalent to require that $M \oplus N$ is again an endo-permutation module (Corollary 6.12 in [Da2]), but we shall not need this characterization of the relation.

Let $D(P)$ be the set of equivalence classes of endo-permutation kP-modules, with respect to the relation $\sim$. In every equivalence class, there is precisely one indecomposable module (up to isomorphism), namely the cap of any member of the class. Therefore, if $\widehat{D}(P)$ is the set of isomorphism classes of indecomposable endo-permutation kP-modules (hence with vertex P in view of our new definition), then the natural map $\widehat{D}(P) \to D(P)$ is a bijection. We are actually interested in $\widehat{D}(P)$, but it is much more convenient to work with $D(P)$ in order to be able to use restriction and tensor product in a straightforward way. We shall always write $[M]$ for the equivalence class of the endo-permutation module M.

The set $D(P)$ is an abelian group for the following operation

$$[M] + [N] = [M \otimes N]\,,$$

which makes sense because $M \otimes N$ is an endo-permutation module whenever M and N are. The zero element of $D(P)$ is the class $[k]$ of the trivial module, while the opposite of $[M]$ is the class $[M^*]$ of the dual module, because $M \otimes M^* \cong k \oplus kY$ as a kP-module for some P-set Y, so that $[M \otimes M^*] = [k]$. We see here the role of our definition of endo-permutation modules (in the strong sense).

The group $D(P)$ is called *the Dade group* of P. From this definition, the group $D(P)$ depends on the choice of the base field k, but we shall see later that it does in fact not depend on k, except when the quaternion group Q_8 is involved. Classifying all endo-permutation modules comes down to the same thing as finding the detailed structure of the group $D(P)$.

There are several maps induced by change of group. First if Q is a subgroup of P, there is an obvious restriction map

$$\mathrm{Res}^P_Q : D(P) \longrightarrow D(Q)\,,$$

and also a map in the other direction, induced by tensor induction

$$\mathrm{Ten}^P_Q : D(Q) \longrightarrow D(P)\,.$$

Now if R is a normal subgroup of P, there is an obvious inflation map

$$\mathrm{Inf}^P_{P/R} : D(P/R) \longrightarrow D(P)\,,$$

and again a map in the other direction, which we called *deflation*,

$$\mathrm{Def}^P_{P/R} : D(P) \longrightarrow D(P/R)\,,$$

defined in the following way. If M is an endo-permutation kP-module, let $A = \mathrm{End}_k(M)$, which is a k-algebra endowed with an action of P by conjugation. We consider the subalgebra of R-fixed points $A^R = \mathrm{End}_{kR}(M)$ and the Brauer quotient

$$A[R] = A^R \Big/ \sum_{Q<R} tr^P_Q(A^Q)\,.$$

Both A^R and $A[R]$ are endowed with an action of the quotient group P/R. It was first proved by Dade that $A[R] \cong \mathrm{End}_k(M_R)$ for some $k[P/R]$-module M_R, which is uniquely defined up to isomorphism and which is again an endo-permutation module (see Theorem 4.15 in [Da2] or Corollary 28.7 in [Th]). This procedure defines the deflation map

$$\mathrm{Def}^P_{P/R} : [M] \mapsto [M_R] .$$

Special notation is useful for the composition of some of the above maps. If Q is a subgroup of P and if R is a normal subgroup of Q, then Q/R is called a *section* of P and we define

$$\mathrm{Defres}^P_{Q/R} = \mathrm{Def}^Q_{Q/R}\,\mathrm{Res}^P_Q : D(P) \longrightarrow D(Q/R)$$

and similarly

$$\mathrm{Teninf}^P_{Q/R} = \mathrm{Ten}^P_Q\,\mathrm{Inf}^Q_{Q/R} : D(Q/R) \longrightarrow D(P) .$$

The map $\mathrm{Teninf}^P_{Q/R}$ is useful for constructing elements in $D(P)$ from given elements in the Dade group of the smaller group Q/R. In the other direction, the deflation–restriction maps are used in the detection theorem, which asserts that the product of the deflation–restriction maps to a suitable family of sections of P is injective (see Section 9).

There is an explicit formula expressing the composition $\mathrm{Defres}^P_{S/T}\,\mathrm{Teninf}^P_{Q/R}$ where S/T is another section of P. For instance tensor induction followed by restriction can be expressed by the Mackey formula, but the formula is more complicated whenever deflation appears, because it involves isomorphisms of sections and Galois isomorphisms. This is discussed in Section 3 of [BoTh1] (see in particular Proposition 3.10). There is a uniform way of expressing all this in terms of functors, explained in Section 10 below. This functorial approach plays a crucial role in the final classification of endo-permutation modules.

We now explain how to construct an abelian group $T(P)$ with endo-trivial modules. Any endo-trivial kP-module M can be written in a unique way (up to isomorphism) $M = M_0 \oplus F$, where M_0 is an indecomposable endo-trivial kP-module and F is a free kP-module. Note that M is a capped endo-permutation module and that M_0 is its cap. Two endo-trivial modules M and N are equivalent if $M_0 \cong N_0$. Any equivalence class consists of an indecomposable endo-trivial module L and all modules of the form $L \oplus (\text{free})$. The set $T(P)$ of equivalence classes of endo-trivial modules is endowed with an abelian group structure induced by tensor product, in the same way as $D(P)$. The group $T(P)$ is simply called the *group of endo-trivial modules* (or also *endo-trivial group*).

Note that if L is an indecomposable endo-trivial module, then its equivalence class in $T(P)$ is smaller than its equivalence class in $D(P)$, because only free modules can be used in the definition of $T(P)$, while more general permutation modules are allowed in $D(P)$. In other words, a non indecomposable endo-permutation module M can have its cap M_0 which is endo-trivial, but M itself may not be endo-trivial. There is a canonical injective homomorphism

$$i : T(P) \longrightarrow D(P) ,$$

mapping the class of an endo-trivial module M to its class $[M]$ in $D(P)$.

The following important characterization of endo-trivial modules is due to Puig (see Statement 2.1.2 in [Pu3]).

Proposition 3.2. *Let M be an indecomposable endo-pemutation kP-module. Then M is endo-trivial if and only if the class $[M]$ in $D(P)$ belongs to the subgroup*

$$\widetilde{T}(P) = \bigcap_{1<Q\leq P} \mathrm{Ker}(\mathrm{Defres}^P_{N_P(Q)/Q})\,.$$

In other words, the image of $T(P)$ by the canonical map $i : T(P) \longrightarrow D(P)$ is equal to the subgroup $\widetilde{T}(P)$.

A subgroup similar to $\widetilde{T}(P)$ is considered by Dade [Da2] by using only the deflation maps to P/Q where Q runs over normal subgroups. Dade then applies this when P is abelian, in which case the subgroup coincides with $\widetilde{T}(P)$ above.

For simplicity, we shall from now on identify $T(P)$ with its image $\widetilde{T}(P)$ and therefore view $T(P)$ as a subgroup of $D(P)$.

4. Examples

The first examples are the Heller translates $\Omega^n_P(k)$ of the trivial module k (also called syzygies of k). Recall that $\Omega^1_P(k)$ is the kernel of a projective cover of k (in other words, the augmentation ideal of the group algebra kP) and, more generally, $\Omega^{n+1}_P(k)$ is the kernel of the n-th boudary map in a minimal projective resolution of k. By dualizing, we get $\Omega^n_P(k)^* = \Omega^{-n}_P(k)$ and by standard properties of tensor products and Heller translates, we have

$$\Omega^m_P(k) \otimes \Omega^n_P(k) \cong \Omega^{m+n}_P(k) \oplus (\text{projective}) \qquad \text{for all } m,n \in \mathbb{Z}\,.$$

In particular

$$\mathrm{End}_k(\Omega^n_P(k)) \cong \Omega^n_P(k) \otimes \Omega^n_P(k)^* \cong k \oplus (\text{free})$$

(because projective modules are free) and therefore $\Omega^n_P(k)$ is endo-trivial. Moreover, we have

$$[\Omega^m_P(k)] + [\Omega^n_P(k)] = [\Omega^{m+n}_P(k)]$$

in the Dade group $D(P)$ and it follows that all Heller translates of the trivial module build a cyclic subgroup of $T(P)$, generated by $\Omega_P := [\Omega^1_P(k)]$.

Theorem 4.1. *The subgroup of $T(P)$ generated by Ω_P is:*

(a) *trivial if P has order 1 or 2,*
(b) *cyclic of order 2 if P is cyclic of order ≥ 3,*
(c) *cyclic of order 4 if P is generalized quaternion,*
(d) *infinite cyclic otherwise.*

Proof. A detailed treatment appears in Proposition 12.2 in [Da2]. The result essentially follows from the fact that there is a projective resolution of the trivial module which is periodic (or equivalently periodic group cohomology) if and only if P is either cyclic or quaternion. See §XII.7 in [CE]. □

An important generalization, due to Alperin [Al3], consists of introducing *relative syzygies* of the trivial module. Let Q be a subgroup of P, let $k[P/Q]$ be the corresponding permutation module, and let $\Omega^1_{P/Q}(k)$ be the kernel of the 'augmentation' map $k[P/Q] \to k$ (mapping every basis element in P/Q to 1).

Theorem 4.2. *(Alperin* [Al3]*)* $\Omega^1_{P/Q}(k)$ *is an endo-permutation module.*

The proof is not very hard and it is a remarkable fact that the result remained unnoticed for more than 20 years after Dade's original paper. Not only did Alperin dig out this fact, but he used it to find the torsion-free rank of the group $T(P)$ of endo-trivial modules. We can also define higher relative syzygies $\Omega^n_{P/Q}(k)$ using relative projective covers and resolutions, but this is not necessary because we need instead only to consider the subgroup of $D(P)$ generated by

$$\Omega_{P/Q} := [\Omega^1_{P/Q}(k)]\,.$$

More generally, for any P-set X, we denote by Ω_X the class in $D(P)$ of the endo-permutation module $\Omega^1_X(k)$, the kernel of the augmentation map $kX \to k$.

When Q is a normal subgroup of P, then $\Omega^1_{P/Q}(k)$ is just the inflation from the quotient group P/Q of the ordinary syzygy of k for the group P/Q. It is an endo-trivial module for the group P/Q and an endo-permutation module for the group P. When Q is not normal in P, then there is no similar description of the relative syzygy $\Omega^1_{P/Q}(k)$ in terms of ordinary syzygies.

Let $D^\Omega(P)$ be the subgroup of $D(P)$ generated by all the relative syzygies Ω_X, where X runs over all non-empty finite P-sets. This subgroup has been first studied by Bouc [Bo2]. Using Lemma 5.2.1 in [Bo2] and induction on the size of subgroups, it is not hard to prove that $D^\Omega(P)$ is actually generated by all $\Omega_{P/Q}$, where Q runs over the subgroups of P. There is even an explicit way of expressing every Ω_X in terms of all the $\Omega_{P/Q}$ (see Lemma 5.2.3 in [Bo2]). The behaviour of relative syzygies under the natural maps (restriction, tensor induction, inflation, deflation) can be described explicitly [Bo2], but the formula for tensor induction is rather involved and not easy to prove. The subgroup $D^\Omega(P)$ plays a crucial role in the classification of endo-permutation modules. In fact, one of the main results of the classification asserts that $D^\Omega(P) = D(P)$ when p is odd.

The next examples are the exotic modules for the quaternion groups, discovered by Dade [Da1] (a few years before his own definition of endo-permutation modules !).

Proposition 4.3. *Let P be a generalized quaternion* 2*-group. If $|P| = 8$, assume moreover that k contains a cubic root of unity (or equivalently k contains the field $\mathbb{F}_4$). There exists a nontrivial kP-module M with the following properties:*

(a) *M is endo-trivial and indecomposable.*
(b) $\dim(M) = |P|/2 + 1$.
(c) *$M^* \cong M$, or in other words $2[M] = 0$ in $D(P)$.*
(d) *If $|P| = 8$, the module M is not defined over the prime field $\mathbb{F}_2$.*

(e) *For each of the three maximal subgroups Q of P, we have either* $\mathrm{Res}^P_Q(M) \cong k \oplus kQ$ *or* $\mathrm{Res}^P_Q(M) \cong \Omega^2_Q(k)$ *(both cases occur if $|P| \geq 16$ while only the first one occurs if $|P| = 8$ because any such Q is cyclic).*

Dade's original approach [Da1] is not stated in terms of endo-trivial modules, but the result can be essentially found there with some extra work. For a direct and detailed approach, see Section 6 of [CaTh1]. There are actually two possible choices for the module M, because $\Omega^2_P(M)$ has the same properties. Note that $[\Omega^2_P(M)] = [\Omega^2_P(k) \otimes M] = 2\Omega_P + [M]$ and this has order 2 again because $4\Omega_P = 0$ by Theorem 4.1.

From the classification, it turns out that all endo-permutation modules are defined over the prime field $\mathbb{F}_2$, except when the quaternion group of order 8 is involved (actually as a section of the form $N_P(R)/R$ for some subgroup R). Moreover, the two exotic modules are not invariant under Galois automorphisms:

Lemma 4.4. *Assume that k has characteristic 2 and contains a cubic root of unity, and let σ be the field homomorphism $\sigma(x) = x^2$. Let P be the quaternion group of order 8 and let M be an exotic module as in Proposition 4.3. Then the conjugate module ${}^\sigma M$ under σ satisfies ${}^\sigma M \cong \Omega^2_P(M) \not\cong M$.*

Proof. This does not seem to appear in print, except in Remark 3.4 of [BoTh1]. The proof follows from the direct definition of M given in [CaTh1] and from explicit computations. □

5. The abelian case

In his fundamental paper [Da2], Dade also classified all endo-permutation modules in the case where P is an abelian p-group. This result is an essential step for the final classification. Moreover, in the course of his proof, Dade proved a crucial lemma which plays an important role in modular representation theory.

The main theorem is the following.

Theorem 5.1. *(Dade* [Da2]*) Let P be an abelian p-group. Then the group $T(P)$ of endo-trivial modules is cyclic, generated by Ω_P.*

In view of Theorem 4.1, we deduce that $T(P)$ is trivial if $|P| \leq 2$, cyclic of order 2 if P is cyclic of order ≥ 3, and infinite cyclic otherwise.

It is for the proof in the elementary abelian case that Dade introduced cyclic shifted subgroups, which played later a crucial role in Carlson's theory of rank varieties. A key result is what is today known as Dade's lemma, which asserts that, for an elementary abelian p-group P, a kP-module is projective if and only if it is projective on restriction to all cyclic shifted subgroups of P.

The proof of Theorem 5.1 is easy when P is cyclic (and the statement is actually included in Proposition 6.1 of the next section). The most difficult case occurs when P is elementary abelian of order p^2 and the proof is based on Dade's lemma. The argument then proceeds by induction. It should be noted that the theorem was proved independently by Carlson [Ca1], who only published his proof in the case when P is elementary abelian of order p^2. Carlson's proof is different from Dade's and does not use cyclic shifted subgroups.

The knowledge of the subgroup $T(P)$ often gives enough information to obtain by induction the complete structure of the Dade group $D(P)$. This is the case when P is abelian and yields the following complete classification.

Theorem 5.2. *(Dade [Da2]) Let P be an abelian p-group. Then the Dade group $D(P)$ decomposes as follows:*

$$D(P) = \bigoplus_{Q<P} T(P/Q) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^c ,$$

where each factor $T(P/Q)$ is identified with a subgroup of $D(P)$ by inflation from P/Q, and where r is the number of non-cyclic quotients P/Q and c is the number of cyclic quotients P/Q of order ≥ 3.

Proof. The proof is not difficult and can be sketched as follows. For any $Q < P$, the composition $\mathrm{Def}^P_{P/Q}\,\mathrm{Inf}^P_{P/Q}$ is the identity of $D(P/Q)$ and the composition $\mathrm{Inf}^P_{P/Q}\,\mathrm{Def}^P_{P/Q}$ is an idempotent endomorphism of $D(P)$. One can modify any element $a \in D(P)$ by $a' = a - \mathrm{Inf}^P_{P/Q}\,\mathrm{Def}^P_{P/Q}(a)$ and obtain $\mathrm{Def}^P_{P/Q}(a') = 0$. By a series of modifications the goal is to obtain an element in the kernel of all deflation maps. Working by descending induction on the size of Q, one can assume that $\mathrm{Def}^P_{P/R}(a) = 0$ for every subgroup R containing Q properly and this implies that $\mathrm{Def}^P_{P/Q}(a)$ belongs to $T(P/Q)$ by Proposition 3.2. Thus a is modified by an element in $T(P/Q)$ (inflated to P) and the new element a' is then killed by one more deflation map. By descending induction, we finally obtain an element in the kernel of all deflation maps, that is, an element in $T(P)$. This shows how an arbitrary element of $D(P)$ is decomposed as a sum of elements in $T(P/Q)$ where Q runs over the subgroups of P.

The proof that the sum $\bigoplus_{Q<P} T(P/Q)$ is direct also uses an argument by descending induction. If $\sum_Q u_Q = 0$ where $u_Q \in T(P/Q)$ and if $u_R = 0$ for every subgroup R containing Q properly, then we can apply $\mathrm{Def}^P_{P/Q}$ and obtain $u_Q = 0$. □

This theorem was published in 1978, but no real progress was made on the classification for other groups until recently (except for the groups of small rank discussed in the next section). The recent contributions are the classification for metacyclic groups (Mazza [Ma1]), for extraspecial groups (Bouc-Mazza [BoMa]), and finally for all groups (Bouc [Bo7]).

6. Some small groups

Apart from abelian groups, there are a few p-groups for which the classification of endo-permutation modules was known at the end of the 70's, at least to specialists. These are the groups of p-rank 1 (cyclic and quaternion) and more generally of normal p-rank 1 (dihedral and semi-dihedral). The explicit description of $D(P)$ for all these groups appeared in print much later [CaTh1].

The presence of an exotic endo-trivial module has of course an effect on the structure of the endo-trivial group $T(P)$ whenever P is quaternion, but also when P is semi-dihedral (because such a group has a quaternion subgroup of index 2).

The complete structure is described in the following result, which also contains the case of cyclic groups for convenience. The proof is an exercise in the cyclic case and is not very hard in the quaternion and semi-dihedral cases (see Carlson–Thévenaz [CaTh1]).

Proposition 6.1. *The group $T(P)$ contains torsion whenever P is cyclic, quaternion, or semi-dihedral. More precisely, the complete structure of $T(P)$ is as follows.*

(a) *Let C_q be a cyclic group of order $q = p^n$. Then*

$$T(C_q) \cong \begin{cases} 0 & \text{if } q \leq 2, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } q \geq 3, \end{cases}$$

generated by Ω_{C_q}.

(b) *Let Q_8 be the quaternion group of order 8. Then*

$$T(Q_8) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } k \text{ does not contain } \mathbb{F}_4, \\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ contains } \mathbb{F}_4, \end{cases}$$

where the first factor is generated by Ω_{Q_8} and the second by one of the two exotic modules of Proposition 4.3.

(c) *Let Q_{2^n} be a quaternion group of order 2^n, with $n \geq 4$. Then*

$$T(Q_{2^n}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\,,$$

where the first factor is generated by $\Omega_{Q_{2^n}}$ and the second by one of the two exotic modules of Proposition 4.3.

(d) *Let SD_{2^n} be a semi-dihedral group of order 2^n, with $n \geq 4$. Then*

$$T(SD_{2^n}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\,,$$

where the first factor is generated by $\Omega_{SD_{2^n}}$ and the second by a suitably constructed relative syzygy.

The case of dihedral 2-groups also appears in [CaTh1].

Proposition 6.2. *Let D_{2^n} be a dihedral group of order 2^n, with $n \geq 3$. Then*

$$T(D_{2^n}) \cong \mathbb{Z} \oplus \mathbb{Z}\,,$$

where the first factor is generated by $\Omega_{D_{2^n}}$ and the second by the relative syzygy $\Omega_{D_{2^n}/C}$ where C is some noncentral subgroup of order 2.

Passing now to the Dade group, we have the following result (see [CaTh1]), which is an easy consequence of Proposition 3.2.

Lemma 6.3. *Let P be a cyclic p-group or a quaternion, dihedral, or semi-dihedral 2-group. Then*

$$D(P) = T(P) \oplus D(P/Z)\,,$$

where Z is the unique central subgroup of order p.

Since P/Z is dihedral or $C_2 \times C_2$ whenever P is a quaternion, dihedral, or semi-dihedral 2-group, we deduce the following result by induction. The starting point of induction is the group $C_2 \times C_2$, for which we have $D(C_2 \times C_2) = T(C_2 \times C_2) = \mathbb{Z}$, by Theorems 5.1 and 5.2.

Proposition 6.4. (a) *Let C_q be a cyclic group of order $q = p^n$. Then*

$$D(C_q) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{n-1} & \text{if } p = 2, \\ (\mathbb{Z}/2\mathbb{Z})^n & \text{if } p \geq 3. \end{cases}$$

(b) *Let Q_8 be the quaternion group of order 8. Then*

$$D(Q_8) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z} & \text{if } k \text{ does not contain } \mathbb{F}_4, \\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} & \text{if } k \text{ contains } \mathbb{F}_4, \end{cases}$$

(c) *Let Q_{2^n} be a quaternion group of order 2^n, with $n \geq 4$. Then*

$$D(Q_{2^n}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{2n-5} .$$

(d) *Let D_{2^n} be dihedral group of order 2^n, with $n \geq 3$. Then*

$$D(D_{2^n}) \cong \mathbb{Z}^{2n-3} .$$

(e) *Let SD_{2^n} be a semi-dihedral group of order 2^n, with $n \geq 4$. Then*

$$D(SD_{2^n}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{2n-4} .$$

The explicit structure of $T(P)$ and $D(P)$ for these small groups plays a role in the final classification. Note that $D(P) = D^{\Omega}(P)$ in all cases except when P is quaternion. This will turn out to be the general pattern, in the sense that $D(P) = D^{\Omega}(P)$ unless some quaternion group appears as a section of P.

7. Detection of endo-trivial modules

As often happens in representation theory, the key for understanding the objects under study is to restrict them to a suitable family of small groups and prove that one does not loose information in the sense that this restriction map is injective. For the group $T(P)$ of endo-trivial modules one uses ordinary restriction, but for the Dade group $D(P)$ we shall use deflation–restriction maps in Section 9.

Let $\mathcal{X}$ be a class of p-groups, closed under isomorphisms and under taking subgroups. We consider the restriction map

$$\mathrm{Res}_{\mathcal{X}} : T(P) \longrightarrow \prod_{\substack{Q<P \\ Q\in\mathcal{X}}} T(Q) .$$

The class $\mathcal{X}$ is called a *detecting family* (for $T(P)$) if the map $\mathrm{Res}_{\mathcal{X}}$ is injective.

The first tentative choice for $\mathcal{X}$ is the class $\mathcal{E}$ of elementary abelian p-groups, in view of Dade's theorem 5.1 and in view of the crucial role played by elementary abelian groups in group cohomology. This choice was made by Puig, who proved in 1980 that the kernel of $\mathrm{Res}_{\mathcal{E}}$ is finite (but the result only appeared 10 years later [Pu3]).

Unfortunately, the class $\mathcal{E}$ of elementary abelian p-groups is not a detecting family in general because of cyclic 2-groups and quaternion 2-groups. In that case, there is a unique elementary abelian 2-subgroup Z of order 2 and $T(Z) = 0$, so that $\mathrm{Res}_{\mathcal{E}} = \mathrm{Res}^P_Z$ is the zero map. However, by Theorem 5.1, $\mathrm{Res}_{\mathcal{E}}$ is injective if P is abelian (and noncyclic when $p = 2$) and it was still hoped that the detecting family should not be far from $\mathcal{E}$. This problem remained open for more than 20 years.

The final answer shows that Puig's choice turns out to be the right one when p is odd, and almost the right one when $p = 2$. The following theorem gives an even more precise answer.

Theorem 7.1. *(Carlson–Thévenaz* [CaTh2]*) Let P be a p-group. Then the endo-trivial group $T(P)$ is detected on restriction to the class $\mathcal{X}$ consisting of elementary abelian p-groups of rank 2, cyclic groups of order p for odd p, cyclic groups of order 4, and quaternion groups of order 8.*

It is in fact only for the cases where P is cyclic, quaternion, or semi-dihedral that one needs to include cyclic groups of order p or 4 and quaternion groups of order 8 in the detecting family. For all the other cases, there is the following more specific result.

Theorem 7.2. *(Carlson–Thévenaz* [CaTh2]*) Suppose that P is a finite p-group which is not cyclic, quaternion, or semi-dihedral. Then the class of elementary abelian p-groups of rank 2 is a detecting family for $T(P)$.*

In view of Dade's theorem 5.1, we have $T(Q) \cong \mathbb{Z}$ whenever Q is elementary abelian of rank 2 and therefore, in the situation of Theorem 7.2, $T(P)$ embeds by restriction into a product of copies of $\mathbb{Z}$. This proves the following corollary.

Corollary 7.3. *Suppose that P is a finite p-group which is not cyclic, quaternion, or semi-dihedral. Then the torsion subgroup of $T(P)$ is trivial and $T(P)$ is a finitely generated free abelian group.*

Note that torsion occurs in the excluded cases by Proposition 6.1. We also deduce in general the following result, originally due to Puig [Pu3].

Corollary 7.4. *The group $T(P)$ is a finitely generated abelian group.*

The proof of Theorem 7.1 (or its variant Theorem 7.2) is rather long and involves many different aspects. We now sketch some of the main ideas. Excluding known cases and working by induction, we are easily reduced to the proof of the following statement.

Theorem 7.5. *Let P be a non-abelian p-group and assume that P is not quaternion of order 8. Then the restriction map*

$$\mathrm{Res} : T(P) \longrightarrow \prod_{Q} T(Q)$$

is injective, where Q runs over all maximal subgroups of P.

An endo-trivial module M whose class is in the kernel of the restriction to all maximal subgroups is called a *critical* module. This amounts to the condition that $\mathrm{Res}^P_Q(M) = k \oplus (\text{free})$ for every maximal subgroup Q. We have to prove that every critical module is in the class of the trivial module, that is, isomorphic to $k \oplus (\text{free})$. Note that the exotic modules for Q_8 are nontrivial critical modules (see Proposition 4.3), so the assumption in Theorem 7.5 is necessary.

Let Z be a central subgroup of P of order p contained in the Frattini subgroup of P and let $\overline{P} = P/Z$. If M is a critical kP-module, we define

$$\overline{M} = M/\{m \in M \mid (z-1)^{p-1}m = 0\}\,, \qquad \text{where } z \text{ is a generator of } Z.$$

Since $(z-1)\overline{M} = 0$, the module $\overline{M}$ can be viewed as a $k\overline{P}$-module. The key property of $\overline{M}$ is the following.

Lemma 7.6. *Let M be a critical module. Then $M \cong k \oplus (\text{free})$ if and only if $\overline{M}$ is free as a $k\overline{P}$-module.*

The proof is not difficult (see Lemma 3.3 in [CaTh1] or Lemma 5.3 in [CaTh2]) and is based on the fact (for p-groups) that a nonzero free summand exists in a module if and only if the action of the sum of all group elements is nonzero.

Applying this lemma to both M and $\mathrm{Res}^P_Q(M)$, we see that we have to prove that $\overline{M}$ is free under the assumption that $\mathrm{Res}^{\overline{P}}_{\overline{Q}}(\overline{M})$ is free for every maximal subgroup Q of P (where $\overline{Q} = Q/Z$). By Chouinard's theorem, we are done whenever $\overline{P}$ is not elementary abelian. However, if $\overline{P}$ is elementary abelian, we may very well have a non-free module whose restriction to every maximal subgroup is free. This shows that we need more input in the latter case, which turns out to be the really hard case.

This case occurs when every quotient of P is elementary abelian. This implies that P is either extraspecial or almost extraspecial (that is, a central product of an extraspecial group and a cyclic group of order p^2). There is a unique central subgroup Z of order p, with elementary abelian quotient $\overline{P} = P/Z$. The critical $k\overline{P}$-module $\overline{M}$ is free on restriction to every maximal subgroup and is therefore in particular a periodic module. One of the important ingredients used in the proof is the theory of rank varieties (see [Be] or [Ca6]), applied here to the elementary abelian group $\overline{P}$. The fact that $\overline{M}$ is periodic implies that the variety $V_{\overline{P}}(\overline{M})$ is a union of lines.

We have to prove Theorem 7.5 whenever P is either extraspecial or almost extraspecial. The argument has two very distinct parts:

1. Assuming by contradiction that a nontrivial critical module exists, construct a critical module of very large dimension.
2. Using group cohomology, find an explicit upper bound for the dimension of a critical module.

The final contradiction comes from the fact that the dimension found in part 1 exceeds the upper bound obtained in part 2.

Part 1. If M is a nontrivial critical kP-module, then $\overline{M}$ is not free and the variety $V_{\overline{P}}(\overline{M})$ is a union of lines. By cutting the module into suitable pieces, one can reduce to the case where it is a single line ℓ. Now we apply the automorphism group of P and get other critical modules with corresponding lines $\sigma(\ell)$, where $\sigma \in \mathrm{Aut}(P)$. The tensor product of these modules yields a new critical module N whose corresponding variety $V_{\overline{P}}(\overline{N})$ is the union of the lines $\sigma(\ell)$, where σ runs over $\mathrm{Aut}(P)/B$ and B is the stabilizer of ℓ. This critical module N turns out to have a very large dimension, because the group $\mathrm{Aut}(P)$ is large and the stabilizer

B is quite small. More precisely, $\mathrm{Aut}(P)$ is essentially a symplectic group when p is odd and an orthogonal group when $p = 2$ (the automorphism group of an extraspecial group is well-known), while B is just a cyclic subgroup of order prime to p. An explicit estimate of the cardinality $|\,\mathrm{Aut}(P)/B|$ is necessary in each case.

Part 2. It has been known for more than 20 years that the dimension of a critical module M is bounded. This was one of the crucial arguments used by Puig for his result on the kernel of restriction to elementary abelian subgroups. The bound is obtained by applying a theorem of Carlson [Ca2] which produces a bound for $\dim(M)$ in terms of the dimensions of the projective-free part of $\mathrm{Res}^P_E(M)$, where E runs over elementary abelian subgroups of P. When M is critical, the projective-free part of $\mathrm{Res}^P_E(M)$ is always the trivial module and so the bound is independent of the choice of M.

The new aspect is that we have today a much better control on the bound. The main ingredient is Serre's theorem on the vanishing of products of Bocksteins in group cohomology and we have now explicit values on the minimal number of terms in such a product whenever P is extraspecial or almost extraspecial (by work of Yalçin [Ya], based on [BeCa]). Moreover, Serre's theorem has now a module-theoretic counterpart, due to Carlson [Ca4], which gives a filtration of $k \oplus \Omega^s_P(k) \oplus F$ (for a suitable s and a suitable free module F) by modules induced from maximal subgroups (see Section 5 of [Ca6] for a general exposition). When tensored with a critical module M, this filtration gives an upper bound for $\dim(M)$ in terms of cohomological data for the maximal subgroups of P. Since P is extraspecial or almost extraspecial, there is enough information about the cohomology of the maximal subgroups to obtain numerical upper bounds.

We have sketched the arguments which appear in [CaTh2], but they are actually not used for all cases. When p is odd and P is either extraspecial of exponent p^2 or almost extraspecial, then a different cohomological argument allows for a more direct proof of Theorem 7.5. This argument ressembles Chouinard's theorem and appears in Section 4 of [CaTh1]. Nevertheless, the proof in [CaTh2] could most probably be adapted to cover also the case where p is odd and P is either extraspecial of exponent p^2 or almost extraspecial.

8. Classification of endo-trivial modules

Let P be a finite p-group. The classification of all endo-trivial kP-modules is equivalent to the complete description of the structure of the group $T(P)$ of equivalence classes of endo-trivial kP-modules. Whenever P is either cyclic, quaternion, or semi-dihedral, then torsion occurs and the detailed structure of $T(P)$ appears in Proposition 6.1. In all other cases, $T(P)$ is free abelian by Corollary 7.3.

It was believed for some time after Dade's fundamental paper [Da2] that $T(P)$ was likely to be infinite cyclic generated by the class of Ω_P (except for the special cases of Section 6 and a few other cases known to some specialists). This is true in most cases but not always: Alperin proved in 2001 [Al3] that the torsion-free rank of $T(P)$ can be 2 or more if P has maximal elementary abelian subgroups of rank 2.

But let us state first the most general case, which appears in [CaTh2]. The result follows from an observation due to Alperin (in his computation of the rank of $T(P)$) together with the fact that elementary abelian subgroups of rank 2 form a detecting family by Theorem 7.2.

Theorem 8.1. *Let P be a p-group such that every maximal elementary abelian subgroup of P has rank at least 3. Then $T(P) \cong \mathbb{Z}$, generated by Ω_P.*

Proof. The restriction map to all elementary abelian subgroups of rank at least 2 is injective by Theorem 7.2. The partially ordered set of all elementary abelian subgroups of rank at least 2 is connected, in view of the assumption and using a well-known result of the theory of p-groups. For any such subgroup H, the restriction map $T(H) \longrightarrow T(E) \cong \mathbb{Z}$ to an elementary abelian subgroup of rank 2 is an isomorphism. It follows that all restrictions to such rank 2 subgroups E are equal. □

In the remaining cases, there are maximal elementary abelian subgroups of rank 2 and we let c be the number of conjugacy classes of such subgroups. Define $r = c$ if the rank of P is 2 and $r = c + 1$ if the rank of P is ≥ 3.

Theorem 8.2. *Assume that P has at least one maximal elementary abelian subgroup of rank 2 and that P is not semi-dihedral. Let r be the integer defined above. Then $T(P)$ is free abelian on r explicit generators.*

The fact that the dimension of $\mathbb{Q} \otimes T(P)$ is equal to r is due to Alperin [Al3] and is based on a construction using relative syzygies. To control $T(P)$ integrally is more delicate and there are two proofs of Theorem 8.2 which provide generators in a quite different way (but the generators turn out to be the same). One proof is due to Carlson–Thévenaz [CaTh3] and is based on Alperin's construction. The other proof is due to Carlson [Ca5] and uses cohomology (see also Section 8 of [Ca6]). Actually both proofs make use of cohomology and support varieties, but the construction of the generators in [CaTh3] is independent of any cohomological argument. Both proofs use the restriction map to elementary abelian subgroups, which was already used by Alperin.

Alperin proved that the torsion-free rank of $T(P)$ is equal to r by using the restriction map

$$\mathrm{Res}_{\mathcal{E}} : \mathbb{Q} \otimes T(P) \longrightarrow \prod_{\substack{E<P \\ E\in\mathcal{E}}} \mathbb{Q} \otimes T(E) \cong \prod_{\substack{E<P \\ E\in\mathcal{E}}} \mathbb{Q}\,,$$

where $\mathcal{E}$ is a set of representatives of conjugacy classes of elementary abelian subgroups of rank 2. This restriction map was known to be injective (Puig [Pu3]), but the partially ordered set of all elementary abelian subgroups of rank at least 2 is now disconnected, contrary to the situation of the previous theorem. Alperin constructed explicit elements in $\mathbb{Q} \otimes T(P)$ (namely classes of suitable relative syzygies of the trivial module) and proved that they map to 0 in all components of the direct product on the right hand side above, except one. For any maximal elementary abelian subgroup of rank 2, we may have such a nonzero component and zero elsewhere. From this follows the fact that the dimension of $\mathbb{Q}\otimes T(P)$ is equal to r. Details appear in [Al3] and again in [CaTh3] and [Ca5].

The second part of the proof of Theorem 8.2 is the final step in the classification of all endo-trivial kP-modules. It consists of the proof that the relative syzygies found by Alperin generate all of $T(P)$ and not just a subgroup of finite index. This appears in [CaTh3]. In the other proof ([Ca5], [Ca6]), the generators are constructed in a different way, but they turn out to be the same because they have the same image under the injective map $\mathrm{Res}_{\mathcal{E}}$. However, it is interesting to note that there is no known way of proving directly that the two constructions yield isomorphic modules.

The method used in both proofs ressembles the one used in [CaTh2] which was sketched in the previous section. There is a reduction to the case of small (almost) extraspecial groups and then, for a given endo-trivial kP-module M, the main argument is about the module

$$\overline{M} = M/\{m \in M \mid (z-1)^{p-1}m = 0\}\,,$$

where z is a generator of the centre of P. If M is nontrivial but is trivial on restriction to some maximal elementary abelian subgroup of rank 2, then $\overline{M}$ is not free and the variety $V_{\overline{P}}(\overline{M})$ is a union of lines. Again, the arguments involve cutting the module into suitable pieces (in order to reduce to the case where the variety is a single line ℓ), applying automorphisms (in order to obtain other endo-trivial modules), tensoring (in order to construct large endo-trivial modules), and finally using cohomological information in order to deduce the possible dimensions of the ordinary syzygies involved in the argument.

9. Detection of endo-permutation modules

The structural results obtained for the group $T(P)$ of endo-trivial modules have important consequences for the Dade group $D(P)$ of all endo-permutation modules. The reason is Proposition 3.2, which allows for induction arguments. The first main consequence is concerned with detection. Unlike the case of endo-trivial modules where restriction maps were the basic tools, we need now to use deflation–restriction maps $\mathrm{Defres}^P_{K/H}$ to sections K/H of P. More precisely, the detection theorem for the group $T(P)$ (Theorem 7.1) implies the following detection theorem for the Dade group $D(P)$.

Theorem 9.1. *(Carlson–Thévenaz* [CaTh2]*) Let P be a p-group. The product of all deflation–restriction maps*

$$\prod_{K/H} \mathrm{Defres}^P_{K/H} \;:\; D(P) \quad\longrightarrow\quad \prod_{K/H} D(K/H)$$

is injective, where K/H runs through the set of all sections of P which are elementary abelian of rank 2, cyclic of order p with p odd, cyclic of order 4, or quaternion of order 8.

The proof is an easy consequence of Theorem 7.1, using a straightforward induction argument and Proposition 3.2 to reduce to the group $T(P)$. Theorem 9.1 is one of the main ingredients used by Bouc in his final classification of all endo-ermutation modules (see Section 13).

For the torsion subgroup $D_t(P)$ of the group $D(P)$, we have the following more specific result.

Theorem 9.2. *(Carlson–Thévenaz* [CaTh2]*) Let $D_t(P)$ be the torsion subgroup of the group $D(P)$.*

(a) *The product of all deflation–restriction maps*

$$\prod_{K/H} \mathrm{Defres}^P_{K/H} \ : \ D_t(P) \quad \longrightarrow \quad \prod_{K/H} D_t(K/H)$$

is injective, where K/H runs through the set of all sections of P which are cyclic of order p if p is odd, quaternion of order 8 or cyclic of order 4 if $p = 2$.

(b) *If p is odd, any nontrivial torsion element in $D(P)$ has order 2. If $p = 2$, any nontrivial torsion element in $D(P)$ has order 2 or 4.*

Proof. The proof of (a) is similar to the proof of the previous result. For (b), it suffices to apply (a) and the fact that the nontrivial elements of $D_t(C_p)$ and $D_t(C_4)$ have order 2, while those of $D_t(Q_8)$ have order 2 or 4. □

There is also another detection theorem for the torsion subgroup $D_t(P)$ which plays a role in the final classification. It asserts that $D_t(P)$ is detected by deflation–restriction to all sections of the form $N_P(Q)/Q$, where Q runs over all subgroups such that $N_P(Q)/Q$ is cyclic of order ≥ 4, quaternion of order ≥ 8, or semi-dihedral of order ≥ 16 (see [CaTh2] for details).

10. Functorial approach

We have already seen in Section 3 that there are several maps between Dade groups, namely restriction, tensor induction, inflation and deflation. All these maps can be unified and viewed as morphisms in a suitable category, to the effect that the Dade group becomes simply a functor on this category with values in the category of abelian groups (except that this is not quite correct if $p = 2$). This functorial approach is an essential ingredient for the classification of endo-permutation modules. Considering only restrictions and tensor inductions, one would obtain a structure of Mackey functor for the Dade groups, but we need more morphisms, namely inflations and deflations, and this yields the definition of a Bouc functor which we now need to explain. Bouc functors were introduced by Bouc [Bo1] and then further studied in [BoTh1] as a tool to analyze the Dade group.

In order to describe morphisms from a p-group P to a p-group Q, we are going to use bisets as follows. A (Q, P)-biset U is a finite set endowed with a left action of Q and a right action of P which 'commute', or 'associate', in the sense that

$(a \cdot u) \cdot b = a \cdot (u \cdot b)$ for all $a \in Q$, $u \in U$, $b \in P$. Any biset U can be decomposed as a disjoint union of transitive bisets, so we need to understand the latter.

The main examples are the following:

• *Restriction.* If Q is a subgroup of P, then P is a (Q, P)-biset by left and right multiplication. This will correspond to restriction from P to Q.

• *Induction.* If Q is a subgroup of P, then P is a (P, Q)-biset by left and right multiplication. This will correspond to induction (or tensor induction) from Q to P.

• *Inflation.* If R is a normal subgroup of P, then P/R is a $(P, P/R)$-biset by left and right multiplication. This will correspond to inflation from P/R to P.

• *Deflation.* If R is a normal subgroup of P, then P/R is a $(P/R, P)$-biset by left and right multiplication. This will correspond to deflation from P to P/R.

• *Isomorphism.* If $\alpha : P \to Q$ is an isomorphism, then P is a (Q, P)-biset by left and right multiplication (using α^{-1} for the left action).

Let $\mathcal{C}_p$ be the category whose objects are the finite p-groups and morphisms $\mathrm{Hom}_{\mathcal{C}_p}(P, Q) = \Gamma(Q, P)$, where $\Gamma(Q, P)$ denotes the Grothendieck group of all (Q, P)-bisets (which is free abelian on transitive bisets). The composition of morphisms is induced by the product of bisets defined as follows. If V is a (R, Q)-biset and U is (Q, P)-biset, their product is the (R, P)-biset

$$V \times_Q U = (V \times U)/\sim\ , \quad \text{where } (v{\cdot}q, u) \sim (v, q{\cdot}u)\ \forall v \in V, u \in U, q \in Q\ .$$

It can be shown (see Lemma 7.4 in [BoTh1]) that any transitive biset is a product of bisets of the five types above (more precisely a restriction, a deflation, an isomorphism, an inflation, and an induction, in this order from right to left). Thus the category $\mathcal{C}_p$ captures in fact exactly the five types of morphisms described above and their composites. There is a similar category $\mathcal{C}$ whose objects are all finite groups (see [Bo1]), but we only consider p-groups here.

We define a *Bouc functor* to be a functor from the category $\mathcal{C}_p$ to the category of abelian groups (and again there is a more general definition involving all finite groups and the category $\mathcal{C}$). The Burnside rings and the ordinary representation rings provide natural examples of Bouc functors (see below), but we first wish to view the Dade group as a functor, if possible. For every object P in $\mathcal{C}_p$, the abelian group $D(P)$ is defined to be the Dade group of P. For any (Q, P)-biset U, it is possible to define a homomorphism

$$D(U) : D(P) \to D(Q)$$

in such a way that whenever the biset U is a restriction, an inflation, a deflation or an isomorphism as above, then $D(U)$ is the usual restriction, inflation, deflation or isomorphism between the corresponding Dade groups. If U is an induction as above, then $D(U)$ is the tensor induction between the corresponding Dade groups. This passes to the Grothendieck group $\mathrm{Hom}_{\mathcal{C}_p}(P, Q)$ and therefore, for every morphism α in the category $\mathcal{C}_p$, we have defined a homomorphism $D(\alpha)$ between corresponding Dade groups.

All this is very satisfactory, except that D is unfortunately not a Bouc functor in general because there is a problem with the composition of morphisms. If V

is a (R,Q)-biset and U a (Q,P)-biset, we may have $D(V\times_Q U)\neq D(V)\circ D(U)$ (this occurs for the composition of a deflation with a tensor induction). We have instead $D(V\times_Q U)=\gamma\circ D(V)\circ D(U)$ where γ is an automorphism induced by a Galois automorphism of the field k. This more complicated behaviour is due to the fact that we work with tensor induction instead of ordinary induction. Details about this Galois twist can be found in Section 3 of [BoTh1].

Fortunately, this problem does not occur in the following important cases.

Theorem 10.1. *Let $D^\Omega(P)$ denote the subgroup of $D(P)$ generated by relative syzygies.*

(a) *D^Ω is a Bouc functor.*
(b) *If p is odd, D is a Bouc functor.*
(c) *$\mathbb{Q}\otimes_{\mathbb{Z}} D$ is a Bouc functor.*

Proof. The proof requires a number of technical verifications (see Proposition 7.6 in [BoTh1] for the main ideas). The reason why no Galois twist occurs can be sketched as follows. Relative syzygies of the trivial module are invariant under Galois automorphisms because they are defined over the ground field $\mathbb{F}_p$. Therefore every Galois twist is the identity whenever we work with D^Ω and it follows that D^Ω is a Bouc functor.

In particular, no Galois twist occurs for abelian p-groups, since the Dade group is generated by all $\Omega_{P/Q}$ with $Q<P$ (see Theorems 5.1 and 5.2). Now if p is odd, $D(P)$ embeds by deflation–restriction into Dade groups of abelian groups $D(K/H)$ by Theorem 9.1. Since a Galois automorphism γ has to commute with restriction and deflation, we have a commutative diagram

$$\begin{array}{ccc} D(P) & \longrightarrow & \prod\limits_{K/H} D(K/H) \\ \gamma\big\downarrow & & \big\downarrow \gamma=\mathrm{id} \\ D(P) & \longrightarrow & \prod\limits_{K/H} D(K/H) \end{array}$$

with injective horizontal maps. Therefore the Galois automorphisms are the identity on $D(P)$ and (b) follows.

When everything is tensored with $\mathbb{Q}$, the torsion subgroup disappears and we don't need cyclic and quaternion groups in the detecting family. Therefore $\mathbb{Q}\otimes_{\mathbb{Z}} D(P)$ embeds by deflation–restriction into Dade groups of abelian groups $\mathbb{Q}\otimes_{\mathbb{Z}} D(K/H)$ by Theorem 9.1. The previous argument holds again and (c) follows. □

The problem of Galois twists really occurs when the quaternion group Q_8 appears as a section of our group. The exotic endo-trivial modules for Q_8 are not invariant under the non-identity Galois automorphism of the field $\mathbb{F}_4$ (see Lemma 4.4) and this means that extra care is needed when $p=2$ and Q_8 is involved. In practice, several arguments which are straightforward when D is a functor have to be adapted in order to include extra Galois automorphisms.

Other important constructions turn out to be Bouc functors. This is the case in particular for the Burnside ring $B(P)$ and the rational representation ring $R_{\mathbb{Q}}(P)$, which turn out to have strong connections with the Dade group.

Recall that $B(P)$ is the Grothendieck group of finite P-sets, with $\mathbb{Z}$-basis consisting of all transitive P-sets P/Q, where Q runs over all subgroups of P up to conjugation. For every (Q,P)-biset U, we have a map $B(U) : B(P) \to B(Q)$ defined by $X \mapsto U \times_P X$ for every P-set X. This provides a structure of Bouc functor on B, called simply the Burnside functor. Note that if the morphism U is a restriction, induction, inflation, or deflation, then the morphism $B(U)$ is indeed the corresponding morphism between Burnside rings.

On the other hand $R_{\mathbb{Q}}(P)$ is the Grothendieck group of all $\mathbb{Q}P$-modules (that is, the rational representations of P) and the irreducible rational representations form a $\mathbb{Z}$-basis of $R_{\mathbb{Q}}(P)$. For every (Q,P)-biset U, we have a map $R_{\mathbb{Q}}(P) \to R_{\mathbb{Q}}(Q)$ defined by $V \mapsto \mathbb{Q}U \otimes_{\mathbb{Q}P} V$ for every $\mathbb{Q}P$-module V (where $\mathbb{Q}U$ is the permutation $(\mathbb{Q}Q, \mathbb{Q}P)$-bimodule with $\mathbb{Q}$-basis U). Again we recover the usual morphisms of restriction, induction, inflation, or deflation for representation rings.

Both B and $R_{\mathbb{Q}}$ have very intimate connections with the Dade group. We will come back to this in the next section, but here we present a result which is not really needed for the classification of endo-permutation modules and which has to do with the rational Dade group $\mathbb{Q}D(P) = \mathbb{Q} \otimes_{\mathbb{Z}} D(P)$. This captures the torsion-free part of the Dade group and it is connected with the rational Burnside ring $\mathbb{Q}B(P)$ and the rational version of the rational representation ring $\mathbb{Q}R_{\mathbb{Q}}(P)$. Note that the natural basis of $B(P)$ is a $\mathbb{Q}$-basis of $\mathbb{Q}B(P)$ and that the irreducible rational representations form a $\mathbb{Q}$-basis of $\mathbb{Q}R_{\mathbb{Q}}(P)$.

Theorem 10.2. *(Bouc–Thévenaz* [BoTh1]*) Consider $\mathbb{Q}D$, $\mathbb{Q}B$, and $\mathbb{Q}R_{\mathbb{Q}}$ as Bouc functors (that is, functors from the category $\mathcal{C}_p$ to the category of $\mathbb{Q}$-vector spaces).*

- (a) *The functor $\mathbb{Q}D$ is a simple functor. The unique minimal group on which $\mathbb{Q}D$ does not vanish is the elementary abelian p-group E of rank 2 and $\mathbb{Q}D(E) = \mathbb{Q}$.*
- (b) *There is an exact sequence of functors*

$$0 \to \mathbb{Q}D \to \mathbb{Q}B \to \mathbb{Q}R_{\mathbb{Q}} \to 0\,,$$

  where, for every p-group P, the surjection $B(P) \to R_{\mathbb{Q}}(P)$ is the natural homomorphism mapping a P-set X to the corresponding permutation representation $\mathbb{Q}X$.
- (c) *The dimension of $\mathbb{Q}D(P)$ is the number of conjugacy classes of noncyclic subgroups of P.*

Note that a simple functor as in (a) above is characterized by the property of the second sentence, so $\mathbb{Q}D \cong S_{E,\mathbb{Q}}$, where $S_{E,\mathbb{Q}}$ is the standard notation for such a simple functor (see Proposition 7.10 in [BoTh1]). Note also that the injective map $\mathbb{Q}D \to \mathbb{Q}B$ is not explicitly described in [BoTh1] and that a change of point of view is necessary in order to understand how it can be defined (see the next section). Finally note that (c) follows from (b) because $\dim \mathbb{Q}B(P)$ is the number of conjugacy classes of subgroups of P and $\dim \mathbb{Q}R_{\mathbb{Q}}(P)$ is the number of conjugacy classes of cyclic subgroups of P. A direct proof of (c) appears in Theorem 4.1 in [BoTh1] and a proof based on different arguments follows from Corollaries 6.4.4 and 7.4.10 in [Bo2].

11. The dual Burnside ring

Trying to understand the deep meaning of the exact sequence of Theorem 10.2, Bouc came to the conclusion that the sequence should be dualized. It is in fact the dual of $\mathbb{Q}D$ which appears in the kernel of the exact sequence, but this remained first unnoticed because the simple functor $\mathbb{Q}D \cong S_{E,\mathbb{Q}}$ is actually self-dual. The advantage of the dual version below is that it holds integrally. It plays a crucial role in the final classification of endo-permutation modules.

The dual Burnside ring of a p-group P is by definition $B^*(P) = \mathrm{Hom}_{\mathbb{Z}}(B(P), \mathbb{Z})$. This is free abelian with a basis $\{\delta_{P/Q}\}$, dual to the canonical basis of $B(P)$ (here Q runs over subgroups of P up to conjugation). But another basis turns out to be useful:

$$\omega_{P/R} = \sum_{Q \leq_P R} \delta_{P/Q} \,,$$

where the sum means that Q runs over subgroups of R up to P-conjugation. More generally, for any P-set X, an element $\omega_X \in B^*(P)$ can be defined by $\omega_X(P/Q) = 1$ if the set of fixed points X^Q is nonempty, and $\omega_X(P/Q) = 0$ otherwise. It is easy to see that we recover the previous definition when $X = P/R$.

It is not hard to prove that one can dualize Bouc functors. Dualizing is contravariant, but by swapping the role of induction and restriction, and also inflation and deflation, we recover a covariant functor. It follows that B^* is a Bouc functor, because so is B. Similarly, the dual $R^*_{\mathbb{Q}}$ of the functor of rational representations is again a Bouc functor.

The following theorem provides the connection between $B^*(P)$ and the Dade group $D(P)$.

Theorem 11.1. *(Bouc* [Bo4]*)*

(a) *There is a group homomorphism $\Theta_P : B^*(P) \to D(P)$ such that*

$$\Theta_P(\omega_X) = \Omega_X$$

for every P-set X.

(b) *The family of maps Θ_P defines a natural transformation $\Theta : B^* \to D^\Omega$ between the Bouc functors B^* and D^Ω.*

In order to understand the remarkable aspects of this result, notice first that $B(P)$, hence also $B^*(P)$, is an abelian group for an addition induced by adding P-sets, whereas the abelian group law on $D(P)$ is induced by tensor product. So the simple fact that Θ_P is a group homomorphism is not obvious and actually depends on a nontrivial lemma (Lemma 5.2.3 in [Bo2]).

Also, the fact that Θ commutes with all the natural maps (restriction, inflation, deflation) becomes particularly striking for induction, because we have an ordinary induction for B^* and a tensor induction for D^Ω. In fact, the proof that

Θ is natural with respect to induction is based on an explicit formula for tensor induction of relative syzygies, which is one of the main nontrivial results of [Bo2].

Now the rational representation ring comes into play by means of the natural homomorphism $\pi_P : B(P) \to R_{\mathbb{Q}}(P)$ mapping a P-set X to the permutation $\mathbb{Q}P$-module $\mathbb{Q}X$ with basis X. This is surjective, by a theorem of Ritter and Segal, and defines a natural transformation of Bouc functors $\pi : B \to R_{\mathbb{Q}}$. By duality, we have an injective natural transformation of Bouc functors $\pi^* : R_{\mathbb{Q}}^* \to B^*$. Together with the transformation of the previous theorem, this is used for the following result.

Theorem 11.2. *(Bouc* [Bo4]*) Let $D_t^\Omega(P)$ be the torsion subgroup of $D^\Omega(P)$.*

(a) *There is an exact sequence*

$$0 \longrightarrow R_{\mathbb{Q}}^* \xrightarrow{\pi^*} B^* \xrightarrow{\overline{\Theta}} D^\Omega / D_t^\Omega \longrightarrow 0$$

where $\overline{\Theta}$ is induced by the natural transformation Θ of Theorem 11.1.

(b) *There is an exact sequence*

$$0 \longrightarrow L \xrightarrow{\pi^*} B^* \xrightarrow{\Theta} D^\Omega \longrightarrow 0$$

where L is a suitable subfunctor of $R_{\mathbb{Q}}^*$ such that $R_{\mathbb{Q}}^*/L \cong D_t^\Omega$.

Note that $L(P)$ can be described explicitly, but this does not appear in print. In fact L only appears in the proof of the main classification theorem of Bouc (Theorem 9.5 in [Bo7]).

Note also that $D^\Omega = D$ when p is odd, by the main classification theorem (see Section 13). Therefore Theorem 11.2 provides a presentation of D as a quotient of B^*. The situation is more complicated when $p = 2$.

12. Rational representations and an induction theorem

It is surprising that the structure of the Dade group $D(P)$ depends heavily on results concerning the rational representations of P. There is a parametrization of simple $\mathbb{Q}P$-modules in terms of certain special subgroups called genetic subgroups. It turns out that genetic subgroups appear in several arguments concerned with Bouc functors and also in the statement of the final classification theorem. Secondly, there is a crucial induction theorem which is concerned with relations arising from permutation representations over $\mathbb{Q}$. All the machinery which is used in this part is due to Bouc and is based on several recent papers concerned with rational representations [Bo3], [Bo5], [Bo6].

A subgroup S of a p-group P is called a *genetic* subgroup if the following two conditions hold:

(a) $N_P(S)/S$ is cyclic, generalized quaternion, dihedral of order at least 16, or semi-dihedral (in other words $N_P(S)/S$ has normal p-rank one).

(b) For every $x \in P$ such that $S^x \cap Z \leq S$, we have $x \in N_P(S)$, where Z denotes the subgroup of $N_P(S)$ defined by $Z/S = Z(N_P(S)/S)$.

For any such subgroup S, there is a unique faithful $\mathbb{Q}[N_P(S)/S]$-module Φ_S and the rational representation

$$V_S = \operatorname{Ind}_{N_P(S)}^{P} \operatorname{Inf}_{N_P(S)/S}^{N_P(S)}(\Phi_S)$$

is a simple $\mathbb{Q}P$-module.

Moreover there is an equivalence relation on the set of genetic subgroups (defined in purely group theoretic terms) to the effect that equivalent genetic subgroups S and T define isomorphic modules $V_S \cong V_T$ and we obtain a parametrization of all the simple $\mathbb{Q}P$-modules V_S when S runs over genetic subgroups up to equivalence (see Theorem 3.11 in [Bo5]). Note that this parametrization is stated in terms of genetic *sections* in [Bo5], but every genetic section is actually uniquely determined by a genetic subgroup (Proposition 4.4 in [Bo6]) and therefore everything can in fact be stated in terms of genetic subgroups rather than genetic sections. This simplification appears in the final classification [Bo7].

Now a *genetic basis* $\mathcal{S}$ is a set of representatives of genetic subgroups for the equivalence relation bewteen genetic subgroups. By the result above, the cardinality of such a set $\mathcal{S}$ is the number of simple $\mathbb{Q}P$-modules, which is well known to be the number of conjugacy classes of cyclic subgroups of P. We will see in the next section that genetic bases are also used in a crucial way for the description of the torsion subgroup $D_t(P)$.

Now we turn to an induction theorem concerned with rational representations. By Theorem 11.2, D^Ω/D_t^Ω is isomorphic to the cokernel of the natural map $\pi^* : R_\mathbb{Q}^* \to B^*$. By duality, we see that the kernel of $\pi : B \to R_\mathbb{Q}$ is an important functor to consider. Define

$$K = \operatorname{Ker}\big(\pi : B \longrightarrow R_\mathbb{Q}\big)\,,$$

so that K is the $\mathbb{Z}$-dual of D^Ω/D_t^Ω. The group $K(P)$ is generated by all differences of P-sets $X - Y$ such that $\mathbb{Q}X \cong \mathbb{Q}Y$. This is clearly related to the old problem of finding conditions under which two non-isomorphic P-sets define isomorphic permutation representations.

One of the key ingredients for the classification of all endo-permutation kP-modules is an induction theorem for the Bouc functor K. This appears in Section 6 of [Bo7] and is also a result of independent interest. For simplicity, we only state the result for odd p.

Theorem 12.1. *(Bouc* [Bo7]*) Let $K = \operatorname{Ker}\big(\pi : B \longrightarrow R_\mathbb{Q}\big)$ and assume that p is odd. Let X_{p^3} denote the extraspecial p-group of order p^3 and exponent p.*

(a) *There is an (explicit) element $\delta \in B(X_{p^3})$ such that K is generated by δ, in the sense that, for any p-group P, any element of $K(P)$ has the form $\sum_{i=1}^m \pm\psi_i(\delta)$ for some m, where $\psi_i : X_{p^3} \to P$ is a morphism in the category $\mathcal{C}_p$ induced by a (P, X_{p^3})-biset.*

(b) *For any p-group P, we have*

$$K(P) = \sum_{T/S} \operatorname{Ind}_T^P \operatorname{Inf}_{T/S}^T K(T/S)\,,$$

where T/S runs over all sections of P which are isomorphic to either X_{p^3} or $C_p \times C_p$.

Note that (b) is essentially a restatement of (a). It is easy to see that $K(C_p \times C_p)$ is free of rank 1 generated by some element ε (see Corollary 6.5 in [Bo7]). Then any element of $K(P)$ is obtained from ε and δ by isomorphisms, inflations and inductions.

The reader can consult [Bo7] for the case $p = 2$. Let us only mention that the group X_{p^3} has to be replaced by all dihedral 2-groups, which play a special role analogous to X_{p^3}.

Note finally that the proof of the theorem uses again in an essential way the genetic bases defined above.

13. Classification of endo-permutation modules

The final classification of all endo-permutation kP-modules for a finite p-group P is due to Bouc [Bo7]. It is based on the detection theorem of Section 9 and uses many other ingredients, in particular the induction theorem of the previous section. By Theorem 10.2, we know that we have a good understanding of $\mathbb{Q}D(P)$, so the real problem has to do with questions of torsion, in two different ways. We have to control the torsion subgroup $D_t(P)$ and also the quotient $D(P)/D^{\Omega}(P)$, which is a finite group. We shall discuss briefly both issues.

For the torsion subgroup $D_t(P)$, there is a direct approach obtained in [BoTh1], but which works only when p is odd. The main tool is tensor induction, which works well because the prime p is odd and hence does not annihilate the elements of $D_t(P)$, which all have order 2 by Theorem 9.2. For every nontrivial cyclic subgroup C of P, we define

$$M_C = \mathrm{Ten}_C^P(\Omega_{C/\Phi(C)}) \,, \quad \text{where } \Phi(C) \text{ is the unique maximal subgroup of } C.$$

This depends only on the conjugacy class of C.

Theorem 13.1. *(Bouc–Thévenaz* [BoTh1]*) If p is odd and P is a finite p-group, the torsion subgroup $D_t(P)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$, where n is the number of conjugacy classes of nontrivial cyclic subgroups of P. The set of all classes $[M_C]$ of the elements M_C above forms a $\mathbb{Z}/2\mathbb{Z}$-basis of $D_t(P)$.*

The main idea consists of applying the restriction–deflation map

$$\sum_B \mathrm{Defres}^P_{B/\Phi(B)} : D_t(P) \longrightarrow \bigoplus_B D(B/\Phi(B)) \cong (\mathbb{Z}/2\mathbb{Z})^s \,,$$

where B runs over all nontrivial cyclic subgroups up to conjugation, and proving that it is an isomorphism. It is injective by the detection theorem and surjective because $\mathrm{Defres}^P_{B/\Phi(B)}(M_C)$ is zero whenever B is not conjugate to C and is $\Omega_{C/\Phi(C)}$ if $B = C$. Note that the proof requires the detection theorem (Theorem 9.2) which was not proved when [BoTh1] was written, but Corollary 6.3 in [BoTh1] states precisely that the theorem holds provided the detection theorem holds. Thus the first complete proof of the theorem depends on the main result of [CaTh2] and appears in that paper.

There is another approach for the torsion subgroup which has the advantage of being independent of p but requires more machinery. This appears in Bouc's classification paper [Bo7] and is based on the notion of genetic basis. The result was first conjectured in [Bo6]. Let $\mathcal{S}$ be a genetic basis of P, as defined in the previous section. For every $S \in \mathcal{S}$, the group $N_P(S)/S$ has normal p-rank 1 and therefore we know $D(N_P(S)/S)$ by the results of Section 6 and in particular the torsion subgroup $T_t(N_P(S)/S)$ by Proposition 6.1. The main result is the following.

Theorem 13.2. *(Bouc* [Bo7]*) Let $\mathcal{S}$ be a genetic basis of P.*

(a) *The map*

$$\sum_{S\in\mathcal{S}} \mathrm{Teninf}^P_{N_P(S)/S} : \bigoplus_{S\in\mathcal{S}} T_t(N_P(S)/S) \longrightarrow D_t(P)$$

is an isomorphism.

(b) *Assume that the base field k contains cubic roots of unity. Let m be the number of subgroups $S \in \mathcal{S}$ such that $N_P(S)/S$ is quaternion and let n be the number of subgroups $S \in \mathcal{S}$ such that $N_P(S)/S$ is cyclic of order ≥ 3 or semi-dihedral. Then there exists a subgroup $D_t^{ex}(P)$ of $D_t(P)$ such that*

$$D_t(P) = D_t^{\Omega}(P) \oplus D_t^{ex}(P)\,.$$

Moreover

$$D_t^{\Omega}(P) \cong (\mathbb{Z}/4\mathbb{Z})^m \oplus (\mathbb{Z}/2\mathbb{Z})^n \quad \textit{and} \quad D_t^{ex}(P) \cong (\mathbb{Z}/2\mathbb{Z})^m\,.$$

(c) *The summand $D_t^{\Omega}(P)$ is generated by the elements $\mathrm{Teninf}^P_{N_P(S)/S}(\Omega_{N_P(S)/S})$ for $S \in \mathcal{S}$.*

(d) *The summand $D_t^{ex}(P)$ is generated by the elements $\mathrm{Teninf}^P_{N_P(S)/S}([L_S])$ where $S \in \mathcal{S}$ with S quaternion and L_S is an exotic endo-trivial module for $N_P(S)/S$ (as in Proposition 6.1).*

(e) *If the base field k does not contain cubic roots of unity, then a direct summand $\mathbb{Z}/2\mathbb{Z}$ must be omitted from $D_t^{ex}(P)$ whenever the corresponding $S \in \mathcal{S}$ is such that $N_P(S)/S$ is quaternion of order 8.*

Note that (b) and (c) follow easily from (a) and the structure of $T_t(N_P(S)/S)$ given by Proposition 6.1. Note also that the summand $D_t^{ex}(P)$ is not uniquely defined (because the exotic modules for the quaternion groups are not unique, see Proposition 6.1). Finally notice that this theorem is consistent with the previous one when p is odd because the cardinality of $\mathcal{S}$ is the number of conjugacy classes of cyclic subgroups and only one of them is excluded by the condition $|N_P(S)/S| \geq 3$, namely the subgroup $S = P$. However, the generators of $D_t(P)$ appearing in each theorem are quite different.

The proof of part (a) is rather involved. The fact that the map is a split injection is proved in [Bo6] and uses some machinery of Bouc functors (namely specific calculations with bisets) together with the properties of genetic subgroups, which come from the theory of rational representations. The proof that the map is surjective (or rather that the corresponding retraction in the other direction is injective) appears in [Bo7] and is based on a delicate induction argument which reduces to the case of a few small groups of order p^3 or 16 (extraspecial or almost extraspecial). For each of these groups, the complete structure of the Dade group is known by results of Bouc and Mazza [BoMa]. This is one of the two places where the paper [BoMa] is used in the final classification. Another ingredient is a detection theorem proved in [CaTh2] and which is mentioned at the end of Section 9.

The final part of the classification consists in the proof that the finite group $D(P)/D^\Omega(P)$ does not contain any unexpected element, and in particular that it is trivial if p is odd.

Theorem 13.3. *(Bouc* [Bo7]*)*
If p is odd, then $D(P) = D^\Omega(P)$.
If $p = 2$, then $D(P) = D^\Omega(P) \oplus D_t^{ex}(P)$, where $D_t^{ex}(P)$ denotes the 'exotic' part of $D_t(P)$ described in Theorem 13.2.

We now give some indications about the proof, restricting to the case where p is odd for simplicity.

First it should be noted that $D(P)/D^\Omega(P)$ is a finite group, more precisely that it is annihilated by a power of p. The main idea appears in [Bo2] and is the following. Every element $a \in D(P)$ defines by deflation–restriction a family $a_{T,S} \in D(T/S)$ where T/S runs over elementary abelian sections of P. To each family $(a_{T,S})$ obtained in this way, one can associate an element $b \in D^\Omega(P)$ by some formula involving tensor induction and Möbius functions, using the fact that $a_{T,S} \in D^\Omega(T/S)$ (note that $D(T/S) = D^\Omega(T/S)$ since T/S is abelian). Applying again deflation–restriction, we obtain a family $b_{T,S} \in D(T/S)$ and the whole point of the formula defining b is that we obtain $b_{T,S} = |P| \cdot a_{T,S}$. By the detection theorem (Theorem 9.1) and the fact that p is odd, we obtain $b = |P| \cdot a$, hence $|P| \cdot a \in D^\Omega(P)$.

The next step is to consider $\overline{D^\Omega} = D^\Omega/D_t^\Omega$, which is isomorphic to the cokernel of the natural map $\pi^* : R_{\mathbb{Q}}^* \to B^*$ (Theorem 11.2) and is $\mathbb{Z}$-dual to the kernel K of $\pi : B \longrightarrow R_{\mathbb{Q}}$. For any positive integer n, consider $nD \cap D^\Omega$ and its image $\overline{nD \cap D^\Omega}$ in $\overline{D^\Omega}$. This is easily seen to be a subfunctor of $\overline{D^\Omega}$. The main point is to prove that

$$\overline{nD \cap D^\Omega} = \overline{nD^\Omega}\,.$$

By Theorem 12.1, we know that the Bouc functor K is generated by its values at the groups X_{p^3} and $C_p \times C_p$. Using the duality between K and $\overline{D^\Omega}$, it can be shown that it suffices to prove the equality above for the groups X_{p^3} and $C_p \times C_p$. This is where the induction theorem of Section 12 plays a crucial role. Now the complete structure of the Dade group is known for the group $C_p \times C_p$ (Theorem 5.2) and also for the group X_{p^3} by a result of Bouc and Mazza [BoMa] (this is the second place where the paper [BoMa] is used in the final classification). The above equality is trivial if P is one of these small groups because $D^\Omega(P) = D(P)$. It follows that the equality above holds for all groups.

Now the proof of Theorem 13.3 is easy. Let $a \in D(P)$ and let n be the exponent of $D(P)/D^\Omega(P)$. Then $na \in nD(P) \cap D^\Omega(P)$, so $na \equiv nb \pmod{D_t^\Omega(P)}$ for some

$b \in D^\Omega(P)$, by the equality above. It follows that $n(a-b)$ is a torsion element, hence $a-b$ too. But, since p is odd, we know from either Theorem 13.1 or Theorem 13.2 that $D_t(P) = D_t^\Omega(P)$, hence $a = b + (a-b)$ belongs to $D^\Omega(P)$.

The proof when $p = 2$ is similar and has only to be adapted in various places.

Let us finally mention that it is possible to describe explictly $D(P)$ as an abelian group by generators and relations. This appears in Section 9 of [Bo7]. But for many purposes the description of D^Ω as a quotient of the dual Burnside functor B^* is sufficient (see Theorem 11.2). This description gives the whole of $D(P)$ when p is odd, while we only have to add an exotic part $D_t^{ex}(P)$ when $p = 2$ (see Theorem 13.2).

14. Consequences of the classification

As before, P denotes a finite p-group. The first consequence of the classification has to do with torsion and was already mentioned in Theorem 9.2.

Theorem 14.1. *The Dade group $D(P)$ has no torsion of odd order. More precisely :*

(a) *If p is odd, then every nontrivial torsion element of $D(P)$ has order 2. In other words, for any indecomposable endo-permutation kP-module M with vertex P, the class of M is a torsion element if and only if M is self-dual.*

(b) *If $p = 2$, then every nontrivial torsion element of $D(P)$ has order 2 or 4. If no section of P is quaternion, then every nontrivial torsion element of $D(P)$ has order 2.*

Proof. See Theorem 9.2. Moreover, an element of order 2 corresponds to a self-dual module by definition of the group law. □

Note that the theorem does not require the whole classification of endo-permutation modules, but only the detection theorem of Section 9.

This theorem is interesting in view of the fact that many invariants lying in the Dade group (e.g. sources of simple modules) are either known or expected to lie in the torsion subgroup. For instance, all sources of simple modules for p-solvable groups define torsion elements of the Dade group [Pu2]. Also all torsion elements of the Dade group are actually sources of a simple module for a p-nilpotent group [Ma2]. For nilpotent blocks, the source of the unique simple module is an endo-permutation module and it is a torsion element in all known cases. Now when p is odd, all such invariants must have order 2 while it is not at all clear why the corresponding modules should be self-dual. This is a rather intriguing question in block theory.

The next consequence of the classification is concerned with lifting to a p-adic ring. Let $\mathcal{O}$ be a complete discrete valuation ring of characteristic zero with maximal ideal $\mathbf{m}$ and residue field $k = \mathcal{O}/\mathbf{m}$. A large part of representation theory uses the ring $\mathcal{O}$ to pass from characteristic zero to characteristic p, and conversely and actually all blocks are best defined as direct summands of a group algebra over $\mathcal{O}$. It is therefore quite important to be able, if possible, to lift a

kP-module to an $\mathcal{O}P$-lattice (where the word 'lattice' means that the module is free as an $\mathcal{O}$-module).

It was an open question for a long time whether or not endo-permutation modules can be lifted. This question is stated explicitly in the introduction of the second of Dade's original papers [Da2]. More than 20 years later, Alperin [Al2] was able to prove directly that any endo-trivial kP-module can be lifted to an endo-trivial $\mathcal{O}P$-lattice. However, for arbitrary endo-permutation modules, the result depends on the classification.

Theorem 14.2. *Let M be an endo-permutation kP-module. Then there exists an $\mathcal{O}P$-lattice $\widehat{M}$ such that $k \otimes_{\mathcal{O}} \widehat{M} \cong M$.*

Proof. Any relative syzygy lifts in an obvious fashion to a relative syzygy over $\mathcal{O}$. Moreover, one can prove directly that the exotic modules for the quaternion groups can be lifted. From the classification of endo-permutation modules (Theorem 13.3), it follows that any element of $D(P)$ can be lifted, that is, any indecomposable endo-permutation kP-module can be lifted. Since all endo-permutation modules can be described in terms of the indecomposable ones (§ 6 of [Da2]), it follows that all endo-permutation kP-modules lift. □

This result can be used in block theory in the following situation. Let b be a nilpotent block and let S be the source module of the unique simple b-module. It is known that S is an endo-permutation module and therefore it can be lifted to $\mathcal{O}$. This lifting property was known for a long time but the proof requires a rather complicated argument (this is the whole § 51 in [Th]). Now the classification of endo-permutation modules allows for another proof, but unfortunately the classification is so long that this new proof does not really compete with the previous one !

Another consequence of the classification is concerned with kP-modules having an endo-split permutation resolution, in the sense of Rickard (see Section 7 of [Ri]). It is easy to see that any relative syzygy has an endo-split permutation resolution. Moreover, it has been noticed by Rickard (unpublished communication) that the exotic modules for Q_8 cannot have an endo-split permutation resolution. His argument actually works for the exotic modules for Q_{2^n} (as noticed by Mazza in her PhD thesis). The next result follows from these remarks and from the methods in [Ri], using of course the classification (Theorem 13.3).

Theorem 14.3. *Let M be an endo-permutation kP-module. Then M has an endo-split permutation resolution if and only if the class of M lies in $D^{\Omega}(P)$. In particular, if p is odd, every endo-permutation kP-module has an endo-split permutation resolution.*

Finally, the classification also implies that some further work is now possible, particularly because we have a good grasp of the methods needed for the classification. Let us mention some research work which has been carried out after the classification. There is the question of classifying all endo-p-permutation modules for an arbitrary finite group (see [Ur]), and in particular all endo-trivial modules for an arbitrary finite group (see [CaMaNa] and [Ma3]). There is also the problem of 'gluing' endo-permutation modules from a given compatible family of such modules, a question which has some importance in block theory. This is considered in [BoTh2] and yields to new questions about the Dade group [BoTh3].

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