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A Point is Normal for Almost All Maps $\beta x + \alpha \mod 1$ or Generalized β-Transformations.

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Abstract. We consider the map $T_{\alpha,\beta}(x) := \beta x + \alpha \mod 1$, which admits a unique probability measure of maximal entropy $\mu_{\alpha,\beta}$. For $x \in [0,1]$, we show that the orbit of x is $\mu_{\alpha,\beta}$-normal for almost all $(\alpha,\beta) \in [0,1) \times (1,\infty)$ (Lebesgue measure). Nevertheless we construct analytic curves in $[0,1) \times (1,\infty)$ along them the orbit of $x = 0$ is at most at one point $\mu_{\alpha,\beta}$-normal. These curves are disjoint and they fill the set $[0,1) \times (1,\infty)$. We also study the generalized β-transformations (in particular the tent map). We show that the critical orbit $x = 1$ is normal with respect to the measure of maximal entropy for almost all β.

1. *Introduction*

In this paper, we consider a dynamical system (X, d, T) where (X, d) is a compact metric space endowed with its Borel σ-algebra $\mathcal{B}$ and $T : X \to X$ is a measurable map. Let $C(X)$ denote the set of all continuous functions from X into $\mathbf{R}$. The set $M(X)$ of all Borel probability measures is equipped with the weak*-topology. $M(X,T) \subset M(X)$ is the subset of all T-invariant probability measures. For $\mu \in M(X,T)$, let $h(\mu)$ denote the measure-theoretic entropy of μ. For all $x \in X$ and $n \geq 1$, the empirical measure of order n at x is

$$\mathcal{E}_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} \delta_x \circ T^{-i} \in M(X), \tag{1}$$

where δ_x is the Dirac mass at x. Let $V_T(x) \subset M(X,T)$ denote the set of all cluster points of $\{\mathcal{E}_n(x)\}_{n\geq 1}$ in the weak*-topology.

DEFINITION 1. *Let $\mu \in M(X,T)$ be an ergodic measure and $x \in X$. The orbit of x under T is μ-**normal**, if $V_T(x) = \{\mu\}$, ie for all continuous $f \in C(X)$, we have*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \int f d\mu.$$

By the Birkhoff Ergodic Theorem, μ-almost all points are μ-normal, however it is difficult to identify a μ-normal point. This paper is devoted to the study of the normality of orbits for piecewise monotone continuous maps of the interval. We consider a family $\{T_\kappa\}_{\kappa\in K}$ of piecewise monotone continuous maps parameterized by a parameter $\kappa \in K$, such that for all $\kappa \in K$ there is a unique measure of maximal entropy μ_κ. In our case K is a subset of $\mathbf{R}$ or $\mathbf{R}^2$. For a given $x \in X$, we estimate the Lebesgue measure of the subset of K such that the orbit of x under T_κ is μ_κ-normal.

For example, let $T_{\alpha,\beta} : [0,1] \to [0,1]$ be the piecewise monotone continuous map defined by $T_{\alpha,\beta}(x) = \beta x + \alpha \mod 1$; here $\kappa = (\alpha,\beta) \in [0,1) \times (1,\infty)$. In [**13**], Parry constructed a $T_{\alpha,\beta}$-invariant probability measure $\mu_{\alpha,\beta}$ absolutely continuous with respect to Lebesgue measure, which is the unique measure of maximal entropy. The main result of section 3 is Theorem 3, which shows that for all $x \in [0,1]$ the set

$$\mathcal{N}(x) := \{(\alpha,\beta) \in [0,1) \times (1,\infty) : \text{the orbit of } x \text{ under } T_{\alpha,\beta} \text{ is } \mu_{\alpha,\beta}\text{-normal}\}$$

has full 2-dimensional Lebesgue measure. This is a generalization of a theorem of Schmeling in [**17**], where the case $\alpha = 0$ and $x = 1$ is studied. For the β-transformations, the orbit of 1 plays a particular role, so the restriction to $x = 1$ considered by Schmeling is natural. Similarly for $T_{\alpha,\beta}$, the orbits of 0 and 1 are very important. In Theorem 4, we show that there exist curves in the plane (α,β) defined by $\alpha = \alpha(\beta)$ along which the orbits of 0 or 1 are never $\mu_{\alpha,\beta}$-normal. The curve $\alpha = 0$ is a trivial example of such a curve for the fixed point $x = 0$. In section 4, we study the generalized β-transformations introduced by Góra [**7**]. A generalized β-transformation is similar to a β-transformation, but each lap is replaced by an increasing or decreasing lap of constant slope β according to a sequence of signs. For a given class of generalized β-transformations, there exists β_0 such that for all $\beta > \beta_0$, there is a unique measure of maximal entropy μ_β and the set

$$\{\beta > \beta_0 : \text{the orbit of 1 under } T_\beta \text{ is } \mu_\beta\text{-normal}\}$$

has full Lebesgue measure, denoted below by λ. Since the tent maps are generalized β-transformations, we obtain an alternative proof of results of Bruin in [**3**].

2. *Preliminaries*

Let us define properly the coding for a piecewise monotone continuous map of the interval. The classical papers are [**15**], [**13**] and [**10**]. We consider the piecewise monotone continuous maps of the following type. Let $k \geq 2$ and $0 = a_0 < a_1 < \cdots < a_k = 1$. We set $\mathtt{A} := \{0,\ldots,k-1\}$, $I_0 = [a_0,a_1)$, $I_j = (a_j,a_{j+1})$

for all $j \in 1, \dots, k-2$, $I_{k-1} = (a_{k-1}, a_k]$ and $S_0 = \{a_j : j \in 1, \dots, k-1\}$. For all $j \in \mathtt{A}$, let $f_j : I_j \to [0,1]$ be a strictly monotone continuous map. A piecewise monotone continuous map $T : [0,1] \backslash S_0 \to [0,1]$ is defined by

$$T(x) = f_j(x) \quad \text{if } x \in I_j.$$

We will state later in each specific case how to define T on S_0. We set $X_0 = [0,1]$ and for all $n \geq 1$

$$X_n = X_{n-1} \backslash S_{n-1} \quad \text{and} \quad S_n = \{x \in X_n : T^n(x) \in S_0\}, \tag{2}$$

so that T^n is well defined on X_n. Finally we define $S = \bigcup_{n \geq 0} S_n$ such that $T^n(x)$ is well defined for all $x \in [0,1] \backslash S$ and all $n \geq 0$.

Let $\mathtt{A}$ be endowed with the discrete topology and $\Sigma_k = \mathtt{A}^{\mathbb{Z}_+}$ be the product space. The elements of Σ_k are denoted by $\underline{x} = x_0 x_1 \dots$. A finite string $\underline{w} = w_0 \dots w_{n-1}$ with $w_j \in \mathtt{A}$ is a **word**. The **length** of $\underline{w}$ is $|\underline{w}| = n$. There is a single word of length 0, the **empty word** ε. The set of all words is $\mathtt{A}^*$. For two words $\underline{w}, \underline{z}$, we write $\underline{w}\,\underline{z}$ for the concatenation of the two words. For $\underline{x} \in \Sigma_k$, let $\underline{x}_{[i,j)} = x_i \dots x_{j-1}$ denote the word formed by the coordinates i to $j-1$ of $\underline{x}$. For a word $\underline{w} \in \mathtt{A}^*$ of length n, the **cylinder** $[\underline{w}]$ is the set

$$[\underline{w}] := \{\underline{x} \in \Sigma_k : \underline{x}_{[0,n)} = \underline{w}\}.$$

The family $\{\, [\underline{w}] : \underline{w} \in \mathtt{A}^*\}$ is a base for the topology and a semi-algebra generating the Borel σ-algebra. For all $\beta > 1$, there exists a metric d_β compatible with the topology defined by

$$d_\beta(\underline{x}, \underline{x}') := \begin{cases} 0 & \text{if } \underline{x} = \underline{x}' \\ \beta^{-\min\{n \geq 0:\ \underline{x}_n \neq \underline{x}'_n\}} & \text{otherwise.} \end{cases}$$

The left shift map $\sigma : \Sigma_k \to \Sigma_k$ is defined by

$$\sigma(\underline{x}) = x_1 x_2 \dots.$$

It is a continuous map. We define a total order on Σ_k denoted by $\prec$. We set

$$\delta(j) = \begin{cases} +1 & \text{if } f_j \text{ is increasing} \\ -1 & \text{if } f_j \text{ is decreasing} \end{cases}$$

and for word $\underline{w}$

$$\delta(\underline{w}) = \begin{cases} 1 & \text{if } \underline{w} = \varepsilon \\ \delta(w_0) \cdots \delta(w_{n-1}) & \text{if } \underline{w} \text{ has length } n. \end{cases}$$

Let $\underline{x} \neq \underline{x}' \in \Sigma_k$ and define $n = \min\{j \geq 0 : x_j \neq x'_j\}$, then

$$\underline{x} \prec \underline{x}' \Leftrightarrow \begin{cases} x_n < x'_n & \text{if } \delta(\underline{x}_{[0,n)}) = +1 \\ x_n > x'_n & \text{if } \delta(\underline{x}_{[0,n)}) = -1. \end{cases}$$

When all maps f_j are increasing, this is the lexicographic order.

We define the coding map $\mathtt{i} : [0,1]\backslash S \to \Sigma_k$ by

$$\mathtt{i}(x) := \mathtt{i}_0(x)\mathtt{i}_1(x)\dots \quad \text{with } \mathtt{i}_n(x) = j \Leftrightarrow T^n(x) \in I_j.$$

The coding map $\mathtt{i}$ is left undefined on S. Henceforth we suppose that T is such that $\mathtt{i}$ is injective. A sufficient condition for the injectivity of the coding is the existence of $c > 1$ such that $|f_j'(x)| \geq c$ for all $x \in I_j$ and all $j \in \mathtt{A}$, see [**13**]. This condition is satisfied in all cases considered in the paper. The coding map is order preserving, ie for all $x, x' \in [0,1]\backslash S$

$$x < x' \Rightarrow \mathtt{i}(x) \prec \mathtt{i}(x').$$

Define $\Sigma_T := \overline{\mathtt{i}([0,1]\backslash S)}$. We introduce now the φ-expansion as defined by Parry. For all $j \in \mathtt{A}$, let $\varphi^j : [j, j+1] \to [a_j, a_{j+1}]$ be the unique monotone extension of $f_j^{-1} : (c,d) \to (a_j, a_{j+1})$ where $(c,d) := f_j\big((a_j, a_{j+1})\big)$. The map $\varphi : \Sigma_k \to [0,1]$ is defined by

$$\varphi(\underline{x}) = \lim_{n\to\infty} \varphi^{x_0}\Big(x_0 + \varphi^{x_1}\big(x_1 + \cdots + \varphi^{x_n}(x_n)\big)\Big).$$

Parry proved that this limit exists if $\mathtt{i}$ is injective. The map φ is order preserving. Moreover $\varphi|_{\mathtt{i}([0,1]\backslash S)} = \mathtt{i}^{-1}$ and for all $n \geq 0$ and all $x \in [0,1]\backslash S$

$$T^n(x) = \varphi \circ \sigma^n \circ \mathtt{i}(x). \tag{3}$$

If the coding map is injective, one can show that the map φ is continuous (see Theorem 2.3 in [**6**]). Using the continuity and the monotonicity of φ, we have $\varphi(\Sigma_T) = [0,1]$. Remark that there is in general no extension of $\mathtt{i}$ on $[0,1]$ such that equation (3) is valid on $[0,1]$. For all $j \in \mathtt{A}$, define

$$\underline{u}^j := \lim_{x\downarrow a_j} \mathtt{i}(x) \quad \text{and} \quad \underline{v}^j := \lim_{x\uparrow a_{j+1}} \mathtt{i}(x) \qquad \text{with } x \in [0,1]\backslash S.$$

The strings $\underline{u}^j$ and $\underline{v}^j$ are called critical orbits and (see for instance [**10**])

$$\Sigma_T = \{\underline{x} \in \Sigma_k : \underline{u}^{x_n} \preceq \sigma^n \underline{x} \preceq \underline{v}^{x_n}\ \forall n \geq 0\}. \tag{4}$$

Moreover the critical orbits $\underline{u}^j, \underline{v}^j$ satisfy for all $j \in \mathtt{A}$

$$\begin{cases} \underline{u}^{u_n^j} \preceq \sigma^n \underline{u}^j \preceq \underline{v}^{u_n^j} \\ \underline{u}^{v_n^j} \preceq \sigma^n \underline{v}^j \preceq \underline{v}^{v_n^j} \end{cases} \qquad \forall n \geq 0. \tag{5}$$

Let us recall the construction of the Hausdorff dimension. Let (X,d) be a metric space and $E \subset X$. Let $\mathcal{D}_\varepsilon(E)$ be the set of all finite or countable cover of E with sets of diameter smaller then ε. For all $s \geq 0$, define

$$H_\varepsilon(E,s) := \inf\{\sum_{B\in\mathcal{C}} (\operatorname{diam} B)^s : \mathcal{C} \in \mathcal{D}_\varepsilon(E)\}$$

and the s-Hausdorff measure of E, $H(E,s) := \lim_{\varepsilon\to 0} H_\varepsilon(E,s)$. The Hausdorff dimension of E is

$$\dim_H E := \inf\{s \geq 0 : H(E,s) = 0\}.$$

In [**1**], Bowen introduced a definition of the topological entropy of non compact set for a continuous dynamical system on a metric space. We recall this definition. Let (X, d) be a metric space, $T : X \to X$ a continuous map. For $n \geq 1$, $\varepsilon > 0$ and $x \in X$, let

$$B_n(x, \varepsilon) = \{y \in X : d(T^j(x), T^j(y)) < \varepsilon \ \forall j = 0, \dots, n-1\}.$$

For $E \subset X$, such that $T(E) \subset E$, let $\mathcal{G}_n(E, \varepsilon)$ be the set of all finite or countable covers of E with Bowen's balls $B_m(x, \varepsilon)$ for $m \geq n$. For all $s \geq 0$, define

$$C_n(E, \varepsilon, s) := \inf\{\sum_{B_m(x,\varepsilon)\in\mathcal{C}} e^{-ms} : \mathcal{C} \in \mathcal{G}_n(x, \varepsilon)\}$$

and $C(E, \varepsilon, s) := \lim_{n\to\infty} C_n(E, \varepsilon, s)$. Now, let

$$h_{\text{top}}(E, \varepsilon) := \inf\{s \geq 0 : C(E, \varepsilon, s) = 0\}$$

and finally $h_{\text{top}}(E) = \lim_{\varepsilon\to 0} h_{\text{top}}(E, \varepsilon)$ (this last limit increase to $h_{\text{top}}(E)$). There is an evident similarity of this definition with the Hausdorff dimension. This similarity is the key of the next lemma.

LEMMA 1. *For $\beta > 1$, consider the dynamical system $(\Sigma_k, d_\beta, \sigma)$. Let $E \subset \Sigma_k$ be such that $\sigma(E) \subset E$, then*

$$\dim_H E \leq \frac{h_{\text{top}}(E)}{\log \beta}.$$

Proof: Let $\varepsilon \in (0, 1), s \geq 0, n \geq 0$ and $\mathcal{C} \in \mathcal{G}_n(E, \varepsilon)$. Since $\operatorname{diam} B_m(x, \varepsilon) \leq \varepsilon\beta^{-m+1} \leq \varepsilon\beta^{-n+1}$ for all $B_m(x, \varepsilon) \in \mathcal{C}$, $\mathcal{C}$ is a cover of E with sets of diameter smaller than $\varepsilon\beta^{-n+1}$. Moreover

$$\sum_{B_m(x,\varepsilon)\in\mathcal{C}} \operatorname{diam}(B_m(x, \varepsilon))^{\frac{s}{\log\beta}} \leq (\varepsilon\beta)^{\frac{s}{\log\beta}} \sum_{B_m(x,\varepsilon)\in\mathcal{C}} e^{-ms}.$$

Thus $H_\delta(E, \frac{s}{\log\beta}) \leq (\varepsilon\beta)^{\frac{s}{\log\beta}} C_n(E, \varepsilon, s)$ with $\delta = \varepsilon\beta^{-n+1}$. Taking the limit $n \to \infty$, we obtain

$$H(E, \frac{s}{\log\beta}) \leq (\varepsilon\beta)^{\frac{s}{\log\beta}} C(E, \varepsilon, s).$$

If $s > h_{\text{top}}(E, \varepsilon)$, then $H(E, \frac{s}{\log\beta}) = 0$ and $\frac{s}{\log\beta} \geq \dim_H E$. This is true for all $s > h_{\text{top}}(E, \varepsilon)$, thus

$$\dim_H E \leq \frac{h_{\text{top}}(E, \varepsilon)}{\log\beta} \leq \frac{h_{\text{top}}(E)}{\log\beta}. \quad \square$$

The next lemma is a classical result about the Hausdorff dimension, it is Proposition 2.3 in [**4**].

LEMMA 2. *Let $(X, d), (X', d')$ be two metric spaces and $\rho : X \to X'$ be an α-Hölder continuous map with $\alpha \in (0, 1]$. Let $E \in X$, then*

$$\dim_H \rho(E) \leq \frac{\dim_H E}{\alpha}.$$

Finally we report Theorem 4.1 from [**14**]. This theorem is used to estimate the topological entropy of sets we are interested in.

Theorem 1. *Let (X,d,T) be a continuous dynamical system and $F \subset M(X,T)$ be a closed subset. Define*

$$G := \{x \in X : V_T(x) \cap F \neq \emptyset\}.$$

Then

$$h_{\mathrm{top}}(G) \leq \sup_{\nu \in F} h(\nu).$$

3. *Normality for the maps* $\beta x + \alpha \mod 1$

In this section, we study the piecewise monotone continuous maps $T_{\alpha,\beta}$ defined by $T_{\alpha,\beta}(x) = \beta x + \alpha \mod 1$ with $\beta > 1$ and $\alpha \in [0,1)$. These maps were studied by Parry in [**13**] as a generalization of the β-transformations. In his paper Parry constructed a $T_{\alpha,\beta}$-invariant probability measure $\mu_{\alpha,\beta}$, which is absolutely continuous with respect to Lebesgue measure. Its density is

$$h_{\alpha,\beta}(x) := \frac{d\mu_{\alpha,\beta}}{d\lambda}(x) = \frac{1}{N_{\alpha,\beta}} \frac{\sum_{n\geq 0} 1_{x<T^n_{\alpha,\beta}(1)} - \sum_{n \geq 0} 1_{x<T^n_{\alpha,\beta}(0)}}{\beta^{n+1}}, \tag{6}$$

with $N_{\alpha,\beta}$ the normalization factor. In [**8**], Halfin proved that $h_{\alpha,\beta}(x)$ is nonnegative for all $x \in [0,1]$. Let $\mathtt{i}^{\alpha,\beta}$ denote the coding map under $T_{\alpha,\beta}$, $\varphi^{\alpha,\beta}$ the corresponding φ-expansion, $\Sigma_{\alpha,\beta} := \Sigma_{T_{\alpha,\beta}} \subset \Sigma_k$ with $k := \lceil \alpha + \beta \rceil$, $\underline{u}^{\alpha,\beta} := \lim_{x \downarrow 0} \mathtt{i}^{\alpha,\beta}(x)$ and $\underline{v}^{\alpha,\beta} := \lim_{x\uparrow 1} \mathtt{i}^{\alpha,\beta}(x)$. We specify how $T_{\alpha,\beta}$ is defined at the discontinuity points. We choose to define $T_{\alpha,\beta}$ by right-continuity at $a_j \in S_0$. Doing this we can also extend the definition of the coding map $\mathtt{i}^{\alpha,\beta}$ using the disjoint intervals $[a_j, a_{j+1})$ for $j \in \mathtt{A}$, so that $\mathtt{i}^{\alpha,\beta}$ is now defined for all $x \in [0,1)$ †. We can show that $\underline{u}^{\alpha,\beta} = \mathtt{i}^{\alpha,\beta}(0)$ and

$$\mathtt{i}([0,1)) = \{\underline{x} \in \Sigma_k : \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{x} \prec \underline{v}^{\alpha,\beta} \quad \forall n \geq 0\}$$

and equation (3) is true for all $x \in [0,1)$. It is easy to check that formula (4) becomes

$$\Sigma_{\alpha,\beta} = \{\underline{x} \in \Sigma_k : \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{x} \preceq \underline{v}^{\alpha,\beta} \quad \forall n \geq 0\} \tag{7}$$

and inequalities (5) become

$$\begin{cases} \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{u}^{\alpha,\beta} \preceq \underline{v}^{\alpha,\beta} \\ \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{v}^{\alpha,\beta} \preceq \underline{v}^{\alpha,\beta} \end{cases} \quad \forall n \geq 0. \tag{8}$$

It is known that the dynamical system $(\Sigma_{\alpha,\beta}, \sigma)$ has topological entropy $\log \beta$. Moreover, Hofbauer showed in [**11**] that it has a unique measure of maximal entropy $\hat{\mu}_{\alpha,\beta}$, $\mu_{\alpha,\beta} = \hat{\mu}_{\alpha,\beta} \circ (\varphi^{\alpha,\beta})^{-1}$ and $\mu_{\alpha,\beta}$ is the unique measure of maximal entropy for $T_{\alpha,\beta}$. In view of (7) and (8), for a couple $(\underline{u}, \underline{v}) \in \Sigma_k^2$ satisfying

$$\begin{cases} \underline{u} \preceq \sigma^n \underline{u} \preceq \underline{v} \\ \underline{u} \preceq \sigma^n \underline{v} \preceq \underline{v} \end{cases} \quad \forall n \geq 0, \tag{9}$$

† This convention differs from that made in the previous section; however it is the most convenient choice when all f_j are increasing.

we define the shift space

$$\Sigma_{\underline{u},\underline{v}} := \{\underline{x} \in \Sigma_k : \underline{u} \preceq \sigma^n \underline{x} \preceq \underline{v} \quad \forall n \geq 0\}. \tag{10}$$

We give now a lemma and a proposition which are the keys of the main theorem of this section. In the lemma, we show that for given x and α, there is exponential separation between the orbits of x under the two different dynamical systems T_{α,β_1} and T_{α,β_2}. The proposition asserts that the topological entropy of $\Sigma_{\underline{u},\underline{v}}$ is upper semi-continuous with respect to the critical orbits $\underline{u}$ and $\underline{v}$.

LEMMA 3. *Let $x \in [0,1)$, $\alpha \in [0,1)$ and $1 < \beta_1 \leq \beta_2$. Define $l = \min\{n \geq 0 : \mathtt{i}^1_n(x) \neq \mathtt{i}^2_n(x)\}$ with $\mathtt{i}^j(x) = \mathtt{i}^{\alpha,\beta_j}$ for $j = 1, 2$. If $x \neq 0$, then*

$$\beta_2 - \beta_1 \leq \frac{\beta_2}{x}\beta_2^{-l}.$$

If $x = 0$ and $\alpha \neq 0$, then

$$\beta_2 - \beta_1 \leq \frac{\beta_2^2}{\alpha}\beta_2^{-l}.$$

Proof: Let $\delta := \beta_2 - \beta_1 \geq 0$. We prove by induction that for all $m \geq 1$, $\mathtt{i}^1_{[0,m)}(x) = \mathtt{i}^2_{[0,m)}(x)$ implies

$$T_2^m(x) - T_1^m(x) \geq \beta_2^{m-1}\delta x,$$

where $T_i = T_{\alpha,\beta_i}$. For $m = 1$,

$$T_2(x) - T_1(x) = \beta_2 x + \alpha - \mathtt{i}^2_0(x) - (\beta_1 x + \alpha - \mathtt{i}^1_0(x)) = \delta x.$$

Suppose that this is true for m, then $\mathtt{i}^1_{[0,m+1)} = \mathtt{i}^2_{[0,m+1)}$ implies

$$\begin{aligned} T_2^{m+1}(x) - T_1^{m+1}(x) &= \beta_2 T_2^m(x) + \alpha - \mathtt{i}^2_m(x) - (\beta_1 T_1^m(x) + \alpha - \mathtt{i}^1_m(x)) \\ &= \beta_2(T_2^m(x) - T_1^m(x)) + \delta T_1^m(x) \geq \beta_2^m \delta x. \end{aligned}$$

On the other hand, $1 \geq T_2^m(x) - T_1^m(x) \geq \beta_2^{m-1}\delta x$. Thus $\delta \leq \frac{\beta_2^{-m+1}}{x}$ for all m such that $\mathtt{i}^1_{[0,m)} = \mathtt{i}^2_{[0,m)}$. If $x = 0$, then $T_1(x) = T_2(x) = \alpha$ and we can apply the first statement to $y = \alpha > 0$. □

PROPOSITION 1. *Let the pair $(\underline{u}, \underline{v}) \in \Sigma_k^2$ satisfy (9). For all $\delta > 0$, there exists $L(\delta, \underline{u}, \underline{v})$ such that for all $L \geq L(\delta, \underline{u}, \underline{v})$, the following claim is true: let the pair $(\underline{u}', \underline{v}') \in \Sigma_k^2$ satisfy (9); suppose further that $\underline{u}$, $\underline{u}'$ have a common prefix of length L and $\underline{v}$, $\underline{v}'$ have a common prefix of length L, then*

$$h_{\text{top}}(\Sigma_{\underline{u}',\underline{v}'}) \leq h_{\text{top}}(\Sigma_{\underline{u},\underline{v}}) + \delta\,.$$

To prove Proposition 1 one associates to the subshift $\Sigma_{\underline{u},\underline{v}}$ a graph $\mathcal{G}(\underline{u}, \underline{v})$, called the Markov diagram [**11**]. One then proves an equivalent proposition to Proposition 1 for these graphs, see section 5.

We now state our first theorem and his corollary about the normality of orbits under $T_{\alpha,\beta}$. The proof of the theorem is inspired by the proof of Theorem C in [**17**], where the case $x = 1$ and $\alpha = 0$ is considered.

THEOREM 2. *Let $x \in [0,1)$ and $\alpha \in [0,1)$ except for $(x,\alpha) = (0,0)$. Then the set*

$$\{\beta > 1 : \text{the orbit of } \mathtt{i}^{\alpha,\beta}(x) \text{ under } \sigma \text{ is } \hat{\mu}_{\alpha,\beta}\text{-normal}\}$$

has full λ-measure.

COROLLARY 1. *Let $x \in [0,1)$ and $\alpha \in [0,1)$ except for $(x,\alpha) = (0,0)$. Then the set*

$$\{\beta > 1 : \text{the orbit of } x \text{ under } T_{\alpha,\beta} \text{ is } \mu_{\alpha,\beta}\text{-normal}\}$$

has full λ-measure.

Remark that the theorem and its corollary may also be formulated for $x \in (0,1]$ using a left-continuous extension of $T_{\alpha,\beta}$ on $(0,1]$ and a coding $\mathtt{i}^{\alpha,\beta}$ defined using intervals $(a_j, a_{j+1}]$ for all $j \in \mathtt{A}$.

Proof of the theorem: We briefly sketch the proof. It is sufficient to consider a finite interval $[\underline{\beta}, \overline{\beta}]$. We use the uniqueness of the measure of maximal entropy $\hat{\mu}_{\alpha,\beta}$: for $\underline{x} \in \Sigma_{\alpha,\beta}$ not $\hat{\mu}_{\alpha,\beta}$-normal, there exists $\nu \in V_\sigma(\underline{x})$ such that $h(\nu) < h(\hat{\mu}_{\alpha,\beta}) = \log\beta$. Therefore we cover the set of abnormal β in $[\underline{\beta}, \overline{\beta}]$ by sets Ω_N, $N \in \mathbb{N}$,

$$\Omega_N := \{\beta \in [\underline{\beta}, \overline{\beta}] : \{\mathcal{E}_n(\mathtt{i}^{\alpha,\beta}(x))\}_n \text{ clusters on } \nu \text{ with } h(\nu) < (1-1/N)\log\beta\}\,.$$

We consider each Ω_N separately and cover them by appropriate intervals, which we generically denote by $[\beta_1, \beta_2]$. The main idea is to imbed $\{\mathtt{i}^{\alpha,\beta}(x) : \beta \in [\beta_1, \beta_2]\}$ in a shift space $\Sigma^* := \Sigma_{\underline{u}^*, \underline{v}^*}$ with $\underline{u}^*$ and $\underline{v}^*$ well chosen. Writing $D^* \subset \Sigma^*$ for the range of the imbedding, we estimate the Hausdorff dimension of the subset of D^* corresponding to points $\mathtt{i}^{\alpha,\beta}(x)$ which are not $\hat{\mu}_{\alpha,\beta}$-normal. Then we estimate the coefficient of Hölder continuity of the map ρ_* defined as the inverse of the imbedding. This gives us an estimate of the Hausdorff dimension of the non $\hat{\mu}_{\alpha,\beta}$-normal points in the interval $[\beta_1, \beta_2]$.

To obtain uniform estimates, we restrict our proof to the interval $[\underline{\beta}, \overline{\beta}]$ with $1 < \underline{\beta} < \overline{\beta} < \infty$. All shift spaces below are subshifts of Σ_k with $k = \lceil \alpha + \overline{\beta} \rceil$. Let $\Omega := \{\beta \in [\underline{\beta}, \overline{\beta}] : \mathtt{i}^{\alpha,\beta}(x) \text{ is not } \hat{\mu}_{\alpha,\beta}\text{-normal}\}$. For $\beta \in \Omega$, we have $V_\sigma(\mathtt{i}^{\alpha,\beta}(x)) \neq \{\hat{\mu}_{\alpha,\beta}\}$. Since $\hat{\mu}_{\alpha,\beta}$ is the unique $T_{\alpha,\beta}$-invariant measure of maximal entropy $\log\beta$, there exist $N \in \mathbb{N}$ and $\nu \in V_\sigma(\mathtt{i}^{\alpha,\beta}(x))$ such that $h(\nu) < (1-1/N)\log\beta$. Setting

$$\Omega_N := \{\beta \in [\underline{\beta}, \overline{\beta}] : \exists \nu \in V_\sigma(\mathtt{i}^{\alpha,\beta}(x)) \text{ s.t. } h(\nu) < (1-1/N)\log\beta\},$$

we have $\Omega = \bigcup_{N\geq 1} \Omega_N$. We will prove that $\dim_H \Omega_N < 1$, so that $\lambda(\Omega_N) = 0$ for all $N \geq 1$.

For $N \in \mathbb{N}$ fixed, define $\varepsilon := \frac{\underline{\beta}\log\underline{\beta}}{2N-1} > 0$ and $\delta := \log\big(1 + \varepsilon/\overline{\beta}\big)$. Let $\beta \in [\underline{\beta}, \overline{\beta}]$ and define $L_\beta = L(\delta/2, \underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta})$ as in Proposition 1. Choose q_β in $\mathbf{Q}$ such that $\log\beta - \delta/2 \leq \log q_\beta \leq \log\beta$. Let

$$J(\beta, L_\beta, q_\beta) := \{\beta' \in [q_\beta, \overline{\beta}] : \underline{u}^{\alpha,\beta'}_{[0,L_\beta)} = \underline{u}^{\alpha,\beta}_{[0,L_\beta)}, \underline{v}^{\alpha,\beta'}_{[0,L_\beta)} = \underline{v}^{\alpha,\beta}_{[0,L_\beta)}\}\,.$$

This set is an interval; if $\beta' \in J(\beta, L_\beta, q_\beta)$, $\beta' < \beta'' \in J(\beta, L_\beta, q_\beta)$ then $[\beta', \beta''] \subset J(\beta, L_\beta, q_\beta)$ since the maps $\beta' \mapsto \underline{u}^{\alpha,\beta'}$ and $\beta' \mapsto \underline{v}^{\alpha,\beta'}$ are both monotone increasing. Moreover $\beta \in J(\beta, L_\beta, q_\beta)$. Notice also that the family

$\{J(\beta,L_\beta,q_\beta):\beta\in[\underline{\beta},\overline{\beta}]\}$ is countable. Indeed the interval $J(\beta,L_\beta,q_\beta)$ is entirely characterized by $\underline{u}^{\alpha,\beta}_{[0,L_\beta)}$, $\underline{v}^{\alpha,\beta}_{[0,L_\beta)}$ and q_β. But there are only countably many triples in $\mathtt{A}^*\times\mathtt{A}^*\times\mathbf{Q}$. Thus $\{J(\beta,L_\beta,q_\beta):\beta\in[\underline{\beta},\overline{\beta}]\}$ is a countable cover of $[\underline{\beta},\overline{\beta}]$. To prove that $\lambda(\Omega_N)=0$, it is sufficient to prove that $\lambda(\Omega_N\cap J(\beta,L_\beta,q_\beta))=0$ for all $\beta\in[\underline{\beta},\overline{\beta}]$. The interval $J(\beta,L_\beta,q_\beta)$ may be open, closed or neither open nor closed. We need to work on a closed interval, thus we prove an equivalent result: for any closed interval $[\beta_1,\beta_2]\subset J(\beta,L_\beta,q_\beta)$, we have $\lambda(\Omega_N\cap[\beta_1,\beta_2])=0$.

Let $\underline{u}^j=\underline{u}^{\alpha,\beta_j}$ and $\underline{v}^j=\underline{v}^{\alpha,\beta_j}$. Using (8) and the monotonicity of $\beta\mapsto\underline{u}^{\alpha,\beta}$ and $\beta\mapsto\underline{v}^{\alpha,\beta}$, we have

$$\begin{array}{l}\underline{u}^1\preceq\sigma^n\underline{u}^1\preceq\underline{v}^1\preceq\underline{v}^2\\ \underline{u}^1\preceq\underline{u}^2\preceq\sigma^n\underline{v}^2\preceq\underline{v}^2\end{array}\qquad\forall n\geq 0.$$

Hence the couple $(\underline{u}^1,\underline{v}^2)$ satisfies (9) and we set $\Sigma^*:=\Sigma_{\underline{u}^1,\underline{v}^2}$ and

$$D^*:=\{\underline{z}\in\Sigma^*:\exists\beta\in[\beta_1,\beta_2]\text{ s.t. }\underline{z}=\mathtt{i}^{\alpha,\beta}(x)\}.$$

We define an map $\rho_*:D^*\to[\beta_1,\beta_2]$ by $\rho_*(\underline{z})=\beta\Leftrightarrow\mathtt{i}^{\alpha,\beta}(x)=\underline{z}$. This map is well defined: by definition of D^*, for all $\underline{z}\in D^*$ there exists a β such that $\underline{z}=\mathtt{i}^{\alpha,\beta}(x)$; moreover this β is unique, since by Lemma 3, $\beta\mapsto\mathtt{i}^{\alpha,\beta}(x)$ is strictly increasing. On the other hand, for all $\beta\in[\beta_1,\beta_2]$, we have from (7)

$$\underline{u}^1\preceq\underline{u}^{\alpha,\beta}\preceq\sigma^n\mathtt{i}^{\alpha,\beta}(x)\preceq\underline{v}^{\alpha,\beta}\preceq\underline{v}^2\quad\forall n\geq 0,$$

whence $\mathtt{i}^{\alpha,\beta}(x)\in\Sigma^*$ and $\rho_*:D^*\to[\beta_1,\beta_2]$ is surjective. Let $\log\beta_*:=h_{\mathrm{top}}(\Sigma^*)$; then by Proposition 1

$$\log\beta^*=h_{\mathrm{top}}(\Sigma^*)\leq h_{\mathrm{top}}(\Sigma_{\alpha,\beta})+\delta/2=\log\beta+\delta/2\,.$$

By definition of q_β, we have $\log\beta-\delta/2\leq\log q_\beta\leq\log\beta_1$, thus $\log\beta^*\leq\log\beta_1+\delta$ and

$$\beta_*-\beta_1\leq\beta_1\big(\mathrm{e}^\delta-1\big)\leq\varepsilon\,. \tag{11}$$

Let us compute the coefficient of Hölder continuity of $\rho_*:(D^*,d_{\beta_*})\to[\beta_1,\beta_2]$. Let $\underline{z}\neq\underline{z}'\in D^*$ and $n=\min\{l\geq 0:z_l\neq z'_l\}$, then $d_{\beta_*}(\underline{z},\underline{z}')=\beta_*^{-n}$. By Lemma 3, there exists C such that

$$|\rho_*(\underline{z})-\rho_*(\underline{z}')|\leq C\rho_*(\underline{z})^{-n}\leq C\beta_1^{-n}=C(d_{\beta_*}(\underline{z},\underline{z}'))^{\frac{\log\beta_1}{\log\beta_*}}\,,$$

where

$$C=\max\left\{\frac{\overline{\beta}}{x},\frac{\overline{\beta}^2}{\alpha}\right\}.$$

By equation (11) and the choice of ε, we have

$$\begin{aligned}\beta_*-\beta_1\leq\frac{\underline{\beta}\log\underline{\beta}}{2N-1}\quad&\Rightarrow\quad\beta_*-\beta_1\leq\frac{\beta_1\log\beta_1}{2N-1}\\ &\Leftrightarrow\quad 1+\frac{\beta_*-\beta_1}{\beta_1\log\beta_1}\leq 1+\frac{1}{2N-1}\\ &\Leftrightarrow\quad\frac{\log\beta_1+\frac{\beta_*-\beta_1}{\beta_1}}{\log\beta_1}\leq\frac{2N}{2N-1}\\ &\Rightarrow\quad\frac{\log\beta_1}{\log\beta_*}\geq\frac{\log\beta_1}{\log\beta_1+\frac{\beta_*-\beta_1}{\beta_1}}\geq 1-\frac{1}{2N}.\end{aligned}$$

In the last line, we use the concavity of the logarithm, so the first order Taylor development is an upper estimate. Thus ρ_* has Hölder-exponent $1 - \frac{1}{2N}$.

Define

$$G_N^* := \{\underline{z} \in \Sigma^* : \exists \nu \in V_\sigma(\underline{z}) \text{ s.t. } h(\nu) < (1 - 1/N) \log \beta_*\}.$$

Let $\beta \in \Omega_N \cap [\beta_1, \beta_2]$. Then there exists $\nu \in V_\sigma(\mathtt{i}^{\alpha,\beta}(x))$ such that

$$h(\nu) < (1 - 1/N) \log \beta \leq (1 - 1/N) \log \beta_*.$$

Since $\mathtt{i}^{\alpha,\beta}(x) \in D^* \subset \Sigma^*$, we have $\mathtt{i}^{\alpha,\beta}(x) \in G_N^*$. Using the surjectivity of ρ_*, we obtain $\Omega_N \cap [\beta_1, \beta_2] \subset \rho_*(G_N^* \cap D^*)$. We claim that $h_{\text{top}}(G_N^*) \leq (1 - 1/N) \log \beta_*$. This implies, using Lemmas 2 and 1,

$$\begin{aligned}
\dim_H(\Omega_N \cap [\beta_1, \beta_2]) &\leq \dim_H \rho_*(G_N^* \cap D^*) \\
&\leq \frac{\dim_H G_N^*}{1 - \frac{1}{2N}} \leq \frac{h_{\text{top}}(G_N^*)}{(1 - \frac{1}{2N}) \log \beta_*} \leq \frac{1 - \frac{1}{N}}{1 - \frac{1}{2N}} < 1.
\end{aligned}$$

Thus $\lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0$.

It remains to prove $h_{\text{top}}(G_N^*) \leq (1 - 1/N) \log \beta_*$. Recall that $h(\nu) = \lim_n \frac{1}{n} H_n(\nu)$, where $H_n(\nu)$ is the entropy of ν with respect to the algebra $\mathcal{A}_n$ of cylinder sets of length n,

$$H_n(\nu) = - \sum_{[\underline{w}] \in \mathcal{A}_n} \nu([\underline{w}]) \log \nu([\underline{w}]) \,.$$

Since the cylinders are both open and closed, $\nu \mapsto H_n(\nu)$ is continuous in the weak*-topology. Moreover $\frac{1}{n} H_n(\nu)$ is decreasing in n. For all $m \geq 1$, we set

$$\begin{aligned}
F_N^*(m) &:= \{\nu \in M(\Sigma^*, \sigma) : \frac{1}{m} H_m(\nu) \leq (1 - 1/N) \log \beta_*\} \\
G_N^*(m) &:= \{\underline{z} \in \Sigma^* : V_\sigma(\underline{z}) \cap F_N^*(m) \neq \emptyset\}.
\end{aligned}$$

Let $\underline{z} \in G_N^*$, then there exists $\nu \in V_\sigma(\underline{z})$ such that $h(\nu) < (1 - \frac{1}{N}) \log \beta_*$. Since $\frac{1}{m} H_m(\nu) \downarrow h(\nu)$, there exists $m \geq 1$ such that $\frac{1}{m} H_m(\nu) \leq (1 - 1/N) \log \beta_*$, whence $\nu \in F_n^*(m)$ and $\underline{z} \in G_N^*(m)$. This implies $G_N^* \subset \bigcup_{m \geq 1} G_N^*(m)$. Since $H_m(\cdot)$ is continuous, $F_N^*(m)$ is closed for all $m \geq 1$. Finally we obtain using Theorem 1

$$\begin{aligned}
h_{\text{top}}(G_N^*) = \sup_m h_{\text{top}}(G_N^*(m)) &\leq \sup_m \sup_{\nu \in F_N^*(m)} h(\nu) \\
&\leq \sup_m \sup_{\nu \in F_N^*(m)} \frac{1}{m} H_m(\nu) \leq (1 - 1/N) \log \beta_* \,. \quad \square
\end{aligned}$$

Proof of the Corollary: Let $\beta > 1$ be such that the orbit of $\mathtt{i}^{\alpha,\beta}(x)$ under σ is $\hat{\mu}_{\alpha,\beta}$-normal. Let $f \in C([0,1])$, then $\hat{f} : \Sigma_{\alpha,\beta} \to \mathbf{R}$ defined by $\hat{f} := f \circ \varphi^{\alpha,\beta}$ is continuous, since $\varphi^{\alpha,\beta}$ is continuous. Using $\mu_{\alpha,\beta} := \hat{\mu}_{\alpha,\beta} \circ (\varphi^{\alpha,\beta})^{-1}$, we have

$$\begin{aligned}
\int_{[0,1]} f d\mu_{\alpha,\beta} &= \int_{\Sigma_{\alpha,\beta}} \hat{f} d\hat{\mu}_{\alpha,\beta} = \lim_{n \to \infty} \sum_{i=0}^{n-1} \hat{f}(\sigma^i \mathtt{i}^{\alpha,\beta}(x)) \\
&= \lim_{n \to \infty} \sum_{i=0}^{n-1} f(\varphi^{\alpha,\beta}(\sigma^i \mathtt{i}^{\alpha,\beta}(x))) = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(T_{\alpha,\beta}^i(x)).
\end{aligned}$$

The second equality comes from the $\hat{\mu}_{\alpha,\beta}$-normality of the orbit of $\mathtt{i}^{\alpha,\beta}(x)$ under σ, the last one is (3) which is true for all $x \in [0,1)$ with our convention for the extension of $T_{\alpha,\beta}$ and $\mathtt{i}^{\alpha,\beta}$ on $[0,1)$. □

The next step is to consider the question of $\mu_{\alpha,\beta}$-normality in the whole plane (α,β) instead of working with α fixed. Define $\mathcal{R} := [0,1) \times (1,\infty)$.

THEOREM 3. *For all $x \in [0,1)$, the set*

$$\mathcal{N}(x) := \{(\alpha,\beta) \in \mathcal{R} : \textit{the orbit of } x \textit{ under } T_{\alpha,\beta} \textit{ is } \mu_{\alpha,\beta}\textit{-normal}\}$$

has full 2-dimensional Lebesgue measure.

Proof: We have only to prove that $\mathcal{N}(x)$ is measurable and to apply Fubini's Theorem and Corollary 1. The first step is to prove that for all $x \in [0,1)$ and all $n \geq 0$, the maps $(\alpha,\beta) \mapsto \mathtt{i}^{\alpha,\beta}(x)$ and $(\alpha,\beta) \mapsto T^n_{\alpha,\beta}(x)$ are measurable. First remark that for all $n \geq 1$

$$T^n_{\alpha,\beta}(x) = \beta^n x + \alpha \frac{\beta^n - 1}{\beta - 1} - \sum_{j=0}^{n-1} \mathtt{i}^{\alpha,\beta}_j(x)\ \beta^{n-j-1}. \tag{12}$$

The proof by induction is immediate. To prove that $(\alpha,\beta) \mapsto \mathtt{i}^{\alpha,\beta}(x)$ is measurable, it is enough to prove that for all $n \geq 0$ and for all words $\underline{w} \in \mathtt{A}^*$ of length n

$$\{(\alpha,\beta) \in \mathcal{R} : \mathtt{i}^{\alpha,\beta}_{[0,n)}(x) = \underline{w}\}$$

is measurable, since the σ-algebra on Σ_k is generated by the cylinders. This set is the subset of $\mathbf{R}^2$ such that

$$\begin{cases} \beta > 1 \\ 0 \leq \alpha < 1 \\ w_j < \beta T^j_{\alpha,\beta}(x) + \alpha \leq w_j + 1 \quad \forall 0 \leq j < n \end{cases}$$

Using (12), this system of inequalities can be rewritten

$$\begin{cases} \beta > 1 \\ 0 \leq \alpha < 1 \\ \alpha > \frac{\beta-1}{\beta^{j+1}-1} \left(\sum_{i=0}^{j} w_i \beta^{j-i} - \beta^{j+1} x \right) \qquad \forall 0 \leq j < n \\ \alpha \leq \frac{\beta-1}{\beta^{j+1}-1} \left(1 + \sum_{i=0}^{j} w_i \beta^{j-i} - \beta^{j+1} x \right) \quad \forall 0 \leq j < n \end{cases}$$

From this, the measurability of $\mathtt{i}^{\alpha,\beta}$ follows. If $(\alpha,\beta) \mapsto \mathtt{i}^{\alpha,\beta}(x)$ is measurable, then by formula (12), $(\alpha,\beta) \mapsto T^n_{\alpha,\beta}(x)$ is clearly measurable for all $n \geq 0$. Then for all $f \in C([0,1])$ and all $n \geq 1$, the map $(\alpha,\beta) \mapsto S_n(f) := \frac{1}{n} \sum_{i=0}^{n-1} f(T^i_{\alpha,\beta}(x))$ is measurable and consequently

$$\{(\alpha,\beta) : \lim_{n\to\infty} S_n(f) \text{ exists}\}$$

is a measurable set.

On the other hand, if $f \in C([0,1])$, then $(\alpha,\beta) \mapsto \int f d\mu_{\alpha,\beta}$ is measurable. Indeed

$$\int f d\mu_{\alpha,\beta} = \int f h_{\alpha,\beta} d\lambda$$

and in view of equation (6) and the measurability of $(\alpha,\beta) \mapsto T_{\alpha,\beta}(x)$, the map $(\alpha,\beta) \mapsto h_{\alpha,\beta}$ is clearly measurable. Therefore

$$\{(\alpha,\beta) : \lim_{n\to\infty} S_n(f) = \int f d\mu_{\alpha,\beta}\}$$

is measurable for all $f \in C([0,1])$. Let $\{f_m\}_{m\in\mathbb{N}} \subset C([0,1])$ be countable subset which is dense with respect to the uniform convergence. Then setting

$$D_m := \{(\alpha,\beta) \in \mathcal{R} : \lim_{n\to\infty} S_n(f_m) = \int f_m d\mu_{\alpha,\beta}\},$$

we have $\mathcal{N}(x) = \bigcap_{m\in\mathbb{N}} D_m$, whence it is a measurable set. □

We have shown that for a given $x \in [0,1)$, the orbit of x under $T_{\alpha,\beta}$ is $\mu_{\alpha,\beta}$-normal for almost all (α,β). The orbits of 0 and 1 are of particular interest (see equation (6)). Now we show that through any point (α_0,β_0), there passes a curve defined by $\alpha = \alpha(\beta)$ such that the orbit of 0 under $T_{\alpha(\beta),\beta}$ is $\mu_{\alpha(\beta),\beta}$-normal for at most one β. A trivial example of such a curve is $\alpha = 0$, since $x = 0$ is a fixed point. The idea is to consider curves along which the coding of 0 is constant, ie to define $\alpha(\beta)$ such that $\underline{u}^{\alpha(\beta),\beta}$ is constant. The results below depend on reference [**6**], where we solve the following inverse problem: given $\underline{u}$ and $\underline{v}$ verifying (9), can we find α,β such that $\underline{u} = \underline{u}^{\alpha,\beta}$ and $\underline{v} = \underline{v}^{\alpha,\beta}$?

Let

$$\mathcal{U} := \{\underline{u} : \exists\ (\alpha,\beta) \in \mathcal{R} \text{ s.t. } \underline{u} = \underline{u}^{\alpha,\beta}\}.$$

We define an equivalence relation in $\mathcal{R}$ by

$$(\alpha,\beta) \sim (\alpha',\beta') \iff \underline{u}^{\alpha,\beta} = \underline{u}^{\alpha',\beta'}.$$

An equivalence class is denoted by $[\underline{u}]$. The next lemma describes $[\underline{u}]$.

Lemma 4. *Let $\underline{u} \in \mathcal{U}$ and set*

$$\alpha(\beta) = (\beta - 1)\sum_{j\geq 0} \frac{u_j}{\beta^{j+1}}.$$

Then there exists $\beta_{\underline{u}} \geq 1$ such that

$$[\underline{u}] = \{(\alpha(\beta),\beta) : \beta \in I_{\underline{u}}\}$$

with $I_{\underline{u}} = (\beta_{\underline{u}},\infty)$ or $I_{\underline{u}} = [\beta_{\underline{u}},\infty)$.

Proof: If $\underline{u} = 000\ldots$, then the statement is trivially true with $\alpha(\beta) \equiv 0$ and $\beta_{\underline{u}} = 1$. Suppose $\underline{u} \neq 000\ldots$. First we prove that

$$(\alpha,\beta) \sim (\alpha',\beta) \implies \alpha = \alpha'$$

then

$$(\alpha,\beta) \in [\underline{u}] \implies (\alpha(\beta'),\beta') \in [\underline{u}] \quad \forall \beta' \geq \beta.$$

Let $(\alpha,\beta) \in [\underline{u}]$. Using (3), we have $\varphi^{\alpha,\beta}(\sigma\underline{u}) = T_{\alpha,\beta}(0) = \alpha$. Since the map $\alpha \mapsto \varphi^{\alpha,\beta}(\sigma\underline{u}) - \alpha$ is continuous and strictly decreasing (Lemmas 3.5 and 3.6 in [**6**]), the first statement is true. Let $\beta' > \beta$. By Corollary 3.1 in [**6**], we have that $\varphi^{\alpha,\beta}(\sigma\underline{u}) > \varphi^{\alpha,\beta'}(\sigma\underline{u})$. Therefore there exists a unique $\alpha' < \alpha$ such that $\varphi^{\alpha',\beta'}(\sigma\underline{u}) = \alpha'$. We prove that $\underline{u}^{\alpha',\beta'} = \underline{u}$. By point 1 of Proposition 2.5 in [**6**], we have $\underline{u} \preceq \underline{u}^{\alpha',\beta'}$. By Proposition 3.3 in [**6**], we have

$$h_{\mathrm{top}}(\Sigma_{\underline{u},\underline{v}^{\alpha',\beta'}}) = h_{\mathrm{top}}(\Sigma_{\alpha',\beta'}) = \log\beta'.$$

Since $\Sigma_{\alpha,\beta} = \Sigma_{\underline{u},\underline{v}^{\alpha,\beta}}$ and $\beta' > \beta$, we must have $\underline{v}^{\alpha,\beta} \prec \underline{v}^{\alpha',\beta'}$. Therefore

$$\begin{cases} \underline{u} \preceq \sigma^n\underline{u} \prec \underline{v}^{\alpha,\beta} \prec \underline{v}^{\alpha',\beta'} \\ \underline{u} \preceq \underline{u}^{\alpha',\beta'} \prec \sigma^n\underline{v}^{\alpha',\beta'} \preceq \underline{v}^{\alpha',\beta'} \end{cases} \qquad \forall n \geq 0,$$

are the inequalities (4.1) in [**6**] for the pair $(\underline{u}, \underline{v}^{\alpha',\beta'})$. We can apply Proposition 3.2 and Theorem 4.1 in [**6**] to this pair and get $\underline{u} = \underline{u}^{\alpha',\beta'}$. It remains to show that $\alpha' = \alpha(\beta')$. Following the definition of the φ-expansion of Rényi, we have for all $x \in [0,1)$ and all $n \geq 0$

$$x = \sum_{j=0}^{n-1} \frac{\mathtt{i}_j^{\alpha,\beta}(x) - \alpha}{\beta^{j+1}} + \frac{T^n_{\alpha,\beta}(x)}{\beta^n}.$$

Since $T^n_{\alpha,\beta}(x) \in [0,1)$, for all $\beta > 1$ we find an explicit expression for $\varphi^{\alpha,\beta}$ on $\Sigma_{\alpha,\beta}$

$$x = \sum_{j\geq 0} \frac{\mathtt{i}_j^{\alpha,\beta}(x) - \alpha}{\beta^{j+1}}.$$

In particular, applying this equation to $x = 0$, we have for all $(\alpha,\beta) \in \mathcal{R}$

$$\alpha = (\beta - 1)\sum_{j\geq 0} \frac{u_j^{\alpha,\beta}}{\beta^{j+1}}.$$

Since for all $\beta > \beta_{\underline{u}}$, we have $\underline{u} \in \Sigma_{\alpha,\beta}$, this completes the proof. □

For each $\underline{u} \in \mathcal{U}$, the equivalence class $[\underline{u}]$ defines an analytic curve in $\mathcal{R}$, which is strictly monotone decreasing (except for $\underline{u} = 000\ldots$),

$$[\underline{u}] = \{(\alpha,\beta) : \alpha = (\beta - 1)\sum_{j\geq 0} \frac{u_j}{\beta^{j+1}},\ \beta \in I_{\underline{u}}\}.$$

These curves are disjoint two by two and their union is $\mathcal{R}$.

THEOREM 4. *Let $(\alpha,\beta) \in \mathcal{R}$, $\underline{u} = \underline{u}^{\alpha,\beta}$ and define $\alpha(\beta)$ and $\beta_{\underline{u}}$ as in Lemma 4. Then for all $\beta > \beta_{\underline{u}}$, the orbit of $x = 0$ under $T_{\alpha(\beta),\beta}$ is not $\mu_{\alpha(\beta),\beta}$-normal.*

Proof: Let $\hat{\nu} \in M(\Sigma_k, \sigma)$ (with k large enough) be a cluster point of $\{\mathcal{E}_n(\underline{u})\}_{n\geq 1}$ (see (1)). By Lemma 4, $\underline{u}^{\alpha(\beta),\beta} = \underline{u}$ for any $\beta > \beta_{\underline{u}}$. Therefore

$$h(\hat{\nu}) \leq h_{\mathrm{top}}(\Sigma_{\alpha(\beta),\beta}) = \log\beta \qquad \forall \beta > \beta_{\underline{u}}$$

and $\hat{\nu}$ is not a measure of maximal entropy, as well as $\nu_\beta := \hat{\nu} \circ (\varphi^{\alpha(\beta),\beta})^{-1}$ for all $\beta > \beta_{\underline{u}}$ (see [**10**]). □

Recall that

$$\mathcal{N}(0) = \{(\alpha,\beta) \in \mathcal{R} : \text{the orbit of } 0 \text{ under } T_{\alpha,\beta} \text{ is } \mu_{\alpha,\beta}\text{-normal}\}.$$

By Theorem 3, $\mathcal{N}(0)$ has full Lebesgue measure. On the other hand, by Theorem 4, we can decompose $\mathcal{R}$ into a family of disjoint analytic curves such that each curve meets $\mathcal{N}(0)$ in at most one point. This situation is very similar to the one presented in [**12**] by Milnor following an idea of Katok.

4. *Normality in generalized β-transformations*

In this section, we consider another class of piecewise monotone continuous maps, the generalized β-transformations. Introduced by Góra in [**7**], they have only one critical orbit like β-transformations, but they admit increasing and decreasing laps. A family $\{T_\beta\}_{\beta>1}$ of generalized β-transformations is defined by $k \geq 2$ and a sequence $s = (s_n)_{0\leq n<k}$ with $s_i \in \{-1,1\}$. For any $\beta \in (k-1,k]$, let $a_j = j/\beta$ for $j = 0,\ldots,k-1$ and $a_k = 1$. Then for all $j = 0,\ldots,k-1$, the map $f_j = I_j \to [0,1]$ is defined by

$$f_j(x) := \begin{cases} \beta x \mod 1 & \text{if } s_j = +1 \\ 1 - (\beta x \mod 1) & \text{if } s_j = -1. \end{cases}$$

In particular when $s = (1,-1)$, then T_β is a tent map. Here we left the map undefined on a_j for $j = 1,\ldots,k-1$.

Góra constructed the unique measure μ_β absolutely continuous with respect to Lebesgue measure (Theorem 6 and Proposition 8 in [**7**]). Using the same argument as Hofbauer in [**9**], we deduce that a measure of maximal entropy is always absolutely continuous with respect to Lebesgue measure, hence the measure μ_β is the unique measure of maximal entropy. Let $k = \lceil\beta\rceil$ and let us denote $\mathtt{i}^\beta$ for the coding map under T_β, $\varphi^\beta := (\mathtt{i}^\beta)^{-1}$ for the inverse of the coding map, $\Sigma_\beta := \Sigma_{T_\beta}$ and $\underline{\eta}^\beta := \lim_{x\uparrow 1} \mathtt{i}^\beta(x)$. Now it is easy to check that formula (4) becomes

$$\Sigma_\beta = \{\underline{x} \in \Sigma_k : \sigma^n \underline{x} \preceq \underline{\eta}^\beta \quad \forall n \geq 0\} \tag{13}$$

and inequalities (5) become

$$\sigma^n \underline{\eta}^\beta \preceq \underline{\eta}^\beta \quad \forall n \geq 0. \tag{14}$$

It is known, in all cases treated below, that the dynamical system (Σ_β, σ) has topological entropy $\log\beta$ and, by general theory of Hofbauer in [12], it has a unique measure of maximal entropy $\hat{\mu}_\beta$ such that $\mu_\beta = \hat{\mu}_\beta \circ (\varphi^\beta)^{-1}$ (see [**5**]).

As in the previous section, we state two lemmas which we need for the proof of the main theorem of this section. We study the normality only of $x = 1$, so these lemmas are formulated only for $x = 1$. Let $S_n(\beta) \equiv S_n$ and $S(\beta) \equiv S$ be defined by (2).

Lemma 5. *For any family of generalized β-transformations defined by $(s_n)_{0\leq n<k}$, the set $\{\beta \in (k-1,k] : 1 \in S(\beta)\}$ is countable.*

Proof: For a fixed $n \geq 1$, we study the map $\beta \mapsto T_\beta^n(1)$. This map is well defined everywhere in $(k-1, k]$ except for finitely many points and it is continuous on each interval where it is well defined. Indeed this is true for $n = 1$. Suppose it is true for n, then $T_\beta^{n+1}(1)$ is well defined and continuous wherever $T_\beta^n(1)$ is well defined and continuous, except for $T_\beta^n(1) \in S_0(\beta)$. By the induction hypothesis, there exists a finite family of disjoint open intervals J_i and continuous functions $g_i : J_i \to [0,1]$ such that $(k-1, k]\backslash(\bigcup_i J_i)$ is finite and

$$T_\beta^n(x) = g_i(\beta) \quad \text{if } \beta \in J_i.$$

Then

$$\{\beta \in (k-1, k] : T_\beta^n(1) \text{ is well defined and } T_\beta^n(1) \in S_0(\beta)\} = \bigcup_{i,j} \{\beta \in J_i : g_i(\beta) = \frac{j}{\beta}\}.$$

We claim that $\{\beta \in J_i : g_i(\beta) = \frac{j}{\beta}\}$ has finitely many points. From the form of the map T_β, it follows immediately that each $g_i(\beta)$ is a polynomial of degree n. Since $\beta > 1$,

$$g_i(\beta) = \frac{j}{\beta} \quad \Longleftrightarrow \quad \beta g_i(\beta) - j = 0.$$

This polynomial equation has at most $n + 1$ roots. In fact, using the monotonicity of the map $\beta \mapsto \underline{\eta}^\beta$, we can prove that this set has at most one point. The lemma follows, since $S(\beta) = \bigcup_{n \geq 0} S_n(\beta)$. □

Lemma 6. *Consider a family* $\{T_\beta\}_{\beta > 1}$ *of generalized* β*-transformations defined by a sequence* $s = (s_n)_{0 \leq n < k}$. *Let* $1 < \beta_1 \leq \beta_2$ *and* $\underline{\eta}^j := \underline{\eta}^{\beta_j}$ *for* $j = 1, 2$*; define* $l := \min\{n \geq 0 : \underline{\eta}_n^1 \neq \underline{\eta}_n^2\}$.
If $k \geq 3$*, for all* $\beta_0 > 2$*, there exists* K *such that* $\beta_1 \geq \beta_0$ *implies*

$$\beta_2 - \beta_1 \leq K \beta_2^{-l}.$$

If $s = (+1, +1)$*, then*

$$\beta_2 - \beta_1 \leq \beta_2^{-l+1}.$$

If $s = (+1, -1)$ *or* $(-1, +1)$*, then for all* $\beta_0 > 1$*, there exists* K *such that* $\beta_1 \geq \beta_0$ *implies*

$$\beta_2 - \beta_1 \leq K \beta_2^{-l}.$$

If $s = (-1, -1)$*, then there exists* $\beta_0 > 1$ *and* K *such that* $\beta_1 \geq \beta_0$ *implies*

$$\beta_2 - \beta_1 \leq K \beta_2^{-l}.$$

The proof is very similar to the proof of Brucks and Misiurewicz for Proposition 1 of [**2**], see also Lemma 23 of Sands in [**16**].

Proof: Let $\delta := \beta_2 - \beta_1 \geq 0$ and denote $T_j = T_{\beta_j}$ and $\mathtt{i}^j = \mathtt{i}^{\beta_j}$ for $j = 1, 2$. Let $a_1, a_2 \in [0, 1]$ such that $r := \mathtt{i}_0^1(a_1) = \mathtt{i}_0^2(a_2)$. Considering four cases according to the signs of $a_2 - a_1$ and s_r, we have

$$|T_2(a_2) - T_1(a_1)| \geq \beta_2 |a_2 - a_1| - \delta.$$

Applying this formula n times, we find that $\mathtt{i}^1_{[0,n)}(a_1) = \mathtt{i}^2_{[0,n)}(a_2)$ implies

$$|T_2^n(a_2) - T_1^n(a_1)| \geq \beta_2^n \left(|a_2 - a_1| - \frac{\delta}{\beta_2 - 1} \right).$$

Consider the case $k \geq 3$. Then $a_i = T_i(1)$ for $i = 1, 2$ are such that

$$|a_2 - a_1| = \delta > \frac{\delta}{\beta_0 - 1} \geq \frac{\delta}{\beta_2 - 1}.$$

Using $|T_2^n(a_2) - T_1^n(a_1)| \leq 1$, we conclude that for all $\beta_0 \leq \beta_1 \leq \beta_2$, if $\underline{\eta}^1_{[0,n)} = \underline{\eta}^2_{[0,n)}$ then

$$\delta \leq \frac{\beta_0 - 1}{\beta_0 - 2}\, \beta_2^{-n+1}.$$

For the case $s = (+1, +1)$, we can apply Lemma 3 with $\alpha = 0$ and $x = 1$.
The case $s = (+1, -1)$ or $(-1, +1)$ is considered in Lemma 23 of [**16**].
For the case $s = (-1, -1)$: for a fixed n, we want to find β_0 such that for all $\beta_0 \leq \beta_1 \leq \beta_2$ we have

$$|T_2^n(1) - T_1^n(1)| > \frac{\delta}{\beta_2 - 1}. \tag{15}$$

Then we conclude as in the case $k \geq 3$. Formula (15) is true, if $|\frac{d}{d\beta} T_\beta^n(1)| > \frac{1}{\beta - 1}$ for all $\beta \geq \beta_0$. When n increases, β_0 decreases. With $n = 3$, we have $\beta_0 \approx 1.53$. □

In the tent map case, the separation of orbits is proved for $\beta \in (\sqrt{2}, 2]$ and then extended arbitrarily near $\beta_0 = 1$ using the renormalization. In the case $s = (-1, -1)$, there is no such argument and we are forced to increase n to obtain a lower bound β_0. With the help of a computer, we obtain $\beta_0 \approx 1.27$ for $n = 12$. For more details, see [**5**].

Now we turn to the question of normality for generalized β-transformations. The structure of the proof is very similar to the proof of Theorem 2 and Corollary 1.

Theorem 5. *Consider a family $\{T_\beta\}_{k-1<\beta\leq k}$ of generalized β-transformations defined by a sequence $s = (s_n)_{0\leq n<k}$. Let β_0 be defined as in Lemma 6. Then the set*

$$\{\beta > \beta_0 : \text{the orbit of } \underline{\eta}^\beta \text{ under } \sigma \text{ is } \hat{\mu}_\beta\text{-normal}\}$$

has full λ-measure.

Corollary 2. *Consider a family $\{T_\beta\}_{\beta>1}$ of generalized β-transformations defined by a sequence $s = (s_n)_{n\geq 0}$. Let β_0 be defined as in Lemma 6. Then the set*

$$\{\beta > \beta_0 : \text{the orbit of } 1 \text{ under } T_\beta \text{ is } \mu_\beta\text{-normal}\}$$

has full λ-measure.

Proof of Theorem: Let

$$B_0 := \{\beta \in (\beta_0, \infty) : 1 \notin S(\beta)\}.$$

From Lemma 5, this subset has full Lebesgue measure. To obtain uniform estimates, we restrict our proof to the interval $[\underline{\beta},\overline{\beta}]$ with $\beta_0<\underline{\beta}<\overline{\beta}<\infty$. Let $k:=\lceil\overline{\beta}\rceil$ and $\Omega:=\{\beta\in[\underline{\beta},\overline{\beta}]\cap B_0 : \underline{\eta}^\beta$ is not $\hat{\mu}_\beta$-normal$\}$. As before, setting

$$\Omega_N:=\{\beta\in[\underline{\beta},\overline{\beta}]\cap B_0 : \exists\nu\in V_\sigma(\underline{\eta}^\beta) \text{ s.t. } h(\nu)<(1-1/N)\log\beta\},$$

we have $\Omega=\bigcup_{N\geq 1}\Omega_N$. We prove that $\dim_H\Omega_N<1$. For $N\in\mathbb{N}$ fixed, define $\varepsilon:=\frac{\beta\log\beta}{2N-1}>0$ and L such that $\underline{\eta}^\beta_{[0,L)}=\underline{\eta}^{\beta'}_{[0,L)}$ implies $|\beta-\beta'|\leq\varepsilon$ (see Lemma 6). Consider the family of subsets of $[\underline{\beta},\overline{\beta}]$ of the following type

$$J(\underline{w})=\{\beta\in[\underline{\beta},\overline{\beta}] : \underline{\eta}^\beta_{[0,L)}=\underline{w}\}$$

where $\underline{w}$ is a word of length L. $J(\underline{w})$ is either empty or it is an interval. We cover the non-closed $J(\underline{w})$ with countably many closed intervals if necessary. We prove that $\lambda(\Omega_N\cap[\beta_1,\beta_2])=0$ where $\beta_1<\beta_2$ are such that $\underline{\eta}^{\beta_1}_{[0,L)}=\underline{\eta}^{\beta_2}_{[0,L)}$.

Let $\underline{\eta}^j=\underline{\eta}^{\beta_j}$. Let

$$D^*:=\{\underline{z}\in\Sigma_{\underline{\eta}^2} : \exists\beta\in[\beta_1,\beta_2]\cap B_0 \text{ s.t. } \underline{z}=\underline{\eta}^\beta\}.$$

Define $\rho_*: D^*\to[\beta_1,\beta_2]\cap B_0$ by $\rho_*(\underline{z})=\beta\Leftrightarrow\underline{\eta}^\beta=\underline{z}$. As before, from formula (13) and strict monotonicity of $\beta\mapsto\underline{\eta}^\beta$, we deduce that ρ_* is well defined and surjective. We compute the coefficient of Hölder continuity of $\rho_*:(D^*,d_{\beta_*})\to[\beta_1,\beta_2]$. Let $\underline{z}\neq\underline{z}'\in D^*$ and $n=\min\{l\geq 0 : z_l\neq z'_l\}$, then $d_{\beta_*}(\underline{z},\underline{z}')=\beta_*^{-n}$. By Lemma 6, there exists C such that

$$|\rho_*(\underline{z})-\rho_*(\underline{z}')|\leq C\rho_*(\underline{z})^{-n}\leq C\beta_1^{-n}=C(d_{\beta_*}(\underline{z},\underline{z}'))^{\frac{\log\beta_1}{\log\beta_*}}.$$

By the choice of L and ε, we have

$$\frac{\log\beta_1}{\log\beta_*}\geq 1-\frac{1}{2N},$$

thus ρ_* has Hölder-exponent of continuity $1-\frac{1}{2N}$. Define

$$G_N^*:=\{\underline{z}\in\Sigma^* : \exists\nu\in V_\sigma(\underline{z}) \text{ s.t. } h(\nu)<(1-1/N)\log\beta_*\}.$$

As before, we have $\Omega_N\cap[\beta_1,\beta_2]\subset\rho_*(G_N^*\cap D^*)$ and $h_{\rm top}(G_N^*)\leq(1-1/N)\log\beta_*$. Finally $\dim_H(\Omega_N\cap[\beta_1,\beta_2])<1$ and $\lambda(\Omega_N\cap[\beta_1,\beta_2])=0$. □

Proof of the Corollary: The proof is similar to the proof of Corollary 1. Equation (3) is true, since we work on B_0. □

In particular, when we consider the tent map ($s=(1,-1)$), we recover the main Theorem of Bruin in [**3**]. We do not state this theorem for all $x\in[0,1]$ as for the map $T_{\alpha,\beta}$, because we do not have an equivalent of Lemma 3 for all $x\in[0,1]$. This is the unique missing step of the proof.

5. *Appendix*

Let $\mathcal{G}$ be an oriented labeled right-resolving graph and denote by $\mathtt{V}$ the set of vertices of $\mathcal{G}$. We assume that $\mathcal{G}$ has a root $\mathtt{v}_0\in\mathtt{V}$. Let $\mathtt{v}\in\mathtt{V}$, the **level** of $\mathtt{v}$ is the length of the shortest path on $\mathcal{G}$ from $\mathtt{v}_0$ to $\mathtt{v}$. For $K \in \mathbb{N}$, the graph $\mathcal{G}_K$ is the subgraph of $\mathcal{G}$ whose set of vertices is

$$\mathtt{V}_K := \{\mathtt{v} \in \mathtt{V} : \text{ the level of } \mathtt{v} \text{ is at most } K\}\,.$$

We set

$$\ell(n,\mathcal{G}) := \mathrm{card}\{\text{paths of length } n \text{ in } \mathcal{G} \text{ starting at } \mathtt{v}_0\}\,.$$

Since the graph is right-resolving, a path in $\mathcal{G}$ is uniquely prescribed by the initial vertex of the path and the (ordered) set of labels of its edges. The right-resolving rooted graph $\mathcal{G}$ has the property $\mathcal{P}$, if *for any path starting at* $\mathtt{v}$ *there is a unique path starting at the root* $\mathtt{v}_0$ *with the same set of labels*. If $\mathcal{G}$ has the property $\mathcal{P}$, then

$$\ell(n+m,\mathcal{G}) \leq \ell(n,\mathcal{G})\ell(m,\mathcal{G})\,.$$

It follows that

$$h(\mathcal{G}) := \lim_{n\to\infty} \frac{1}{n}\log \ell(n,\mathcal{G}) = \inf_n \frac{1}{n}\log \ell(n,\mathcal{G})\,. \tag{16}$$

The quantity $h(\mathcal{G})$ is the **entropy** of $\mathcal{G}$.

LEMMA 7. *Let $\mathcal{G}$ be a right-resolving rooted graph which has the property $\mathcal{P}$. For all $\delta > 0$, there exists $L(\mathcal{G},\delta)$ such that for all $L \geq L(\mathcal{G},\delta)$ and for all right-resolving rooted graph $\mathcal{G}'$ satisfying the property $\mathcal{P}$, we have $\mathcal{G}_L = \mathcal{G}'_L$ implies that*

$$h(\mathcal{G}') \leq h(\mathcal{G}) + \delta\,.$$

Proof: Given $\mathcal{G}$ and $\delta > 0$, choose $L(\mathcal{G},\delta)$ such that, for all $L \geq L(\mathcal{G},\delta)$, we have

$$\frac{1}{L}\log \ell(L,\mathcal{G}) \leq h(\mathcal{G}) + \delta\,.$$

Let $\mathcal{G}'$ be a right-resolving rooted graph with the property $\mathcal{P}$ such that $\mathcal{G}'_L = \mathcal{G}_L$. Then using (16) and the fact that a path of length L in $\mathcal{G}$ (or in $\mathcal{G}'$) remains in $\mathcal{G}_L$ (or in $\mathcal{G}'_L$), we get

$$\begin{aligned} h(\mathcal{G}') &\leq \frac{1}{L}\log \ell(L,\mathcal{G}') = \frac{1}{L}\log \ell(L,\mathcal{G}'_L) \\ &= \frac{1}{L}\log \ell(L,\mathcal{G}_L) = \frac{1}{L}\log \ell(L,\mathcal{G}) \leq h(\mathcal{G}) + \delta\,. \quad \square \end{aligned}$$

Let $(\underline{u},\underline{v})$ satisfy (9); we define a labeled graph $\mathcal{G} = \mathcal{G}(\underline{u},\underline{v})$. A vertex $\mathtt{v}$ of the graph is a couple $(p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. We define the out-going labeled edges from $\mathtt{v} = (p,q)$ to $\mathtt{v}' = (p',q')$, the successors of $\mathtt{v}$.

1. If $u_p = v_q$, then there is a unique out-going edge labeled by u_p from $\mathtt{v}$ to $\mathtt{v}' = (p+1,q+1)$.
2. If $u_p < v_q$, then there is an out-going edge labeled by u_p from $\mathtt{v}$ to $\mathtt{v}' = (p+1,0)$, and an out-going edge labeled by v_q from $\mathtt{v}$ to $\mathtt{v}' = (0,q+1)$. Furthermore, if there exists a, $u_p < a < v_q$, then there is an out-going edge labeled by a from $\mathtt{v}$ to $\mathtt{v}' = (0,0)$.

The graph $\mathcal{G}$ is the minimal graph containing $(0,0)$, the root of $\mathcal{G}$, such that if v is a vertex of $\mathcal{G}$, then all successors of v are vertices of $\mathcal{G}$. All vertices of $\mathcal{G}$ are of the form (p,q) with $p\neq q$, except for the root. Furthermore, (p,q) is a vertex of $\mathcal{G}$ with $p>q$ if and only if the longest suffix of $u_0\cdots u_{p-1}$, which is a prefix of $\underline{v}$ has length q. Using the map from the vertices of $\mathcal{G}$ to the subsets of $\Sigma_{\underline{u},\underline{v}}$,

$$(p,q)\mapsto[\sigma^p\underline{u},\sigma^q\underline{v}]:=\{\underline{x}\in\Sigma_{\underline{u},\underline{v}}\colon \sigma^p\underline{u}\preceq\underline{x}\preceq\sigma^q\underline{v}\}\,,$$

and the results of section 3.1 of [**6**], one checks that $\mathcal{G}$ has property $\mathcal{P}$, $h(\mathcal{G})=h_{\rm top}(\Sigma_{\underline{u},\underline{v}})$ and the level of $\mathtt{v}=(p,q)$ is $\max\{p,q\}$. This last result implies that for $(\underline{u}',\underline{v}')$ satisfying (9), if $\underline{u}$ and $\underline{u}'$ have a common prefix of length L and $\underline{v}$ and $\underline{v}'$ have a common prefix of length L, then $\mathcal{G}_L=\mathcal{G}'_L$. Therefore Lemma 7 implies Proposition 1.

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References

[**1**] R. Bowen, *Topological entropy for noncompact sets*, Trans. Amer. Math. Soc., **184** (1973), 125-136.

[**2**] K. Brucks and M. Misiurewicz, *The trajectory of the turning point is dense for almost all tent maps*, Ergod. Th. and Dynam. Sys., **16** (1996), 1173-1183.

[**3**] H. Bruin, *For almost every tent map, the turning point is typical*, Fund. Math., **155** (1998), 215-235.

[**4**] K. Falconer, *Fractal geometry*, Wiley, Chichester, 2003.

[**5**] B. Faller, PhD thesis, EPF-L, 2008.

[**6**] B. Faller and C.-E. Pfister, *Computation of Topological Entropy via φ-expansion, an Inverse Problem for the Dynamical Systems* $\beta x+\alpha \mod 1$, arXiv:0806.0914v1 [mathDS].

[**7**] P. Góra, *Invariant densities for generalized β-maps*, Ergod. Th. and Dynam. Sys., **27** (2007), 1-16.

[**8**] S. Halfin, *Explicit construction of invariant measures for a class of continuous state Markov processes*, The Annals of Probability, **3** (1975), 859-864.

[**9**] F. Hofbauer, *Maximal measures for piecewise monotonically increasing transformations on* $[0,1]$, in Ergodic Theory, vol. 729 of Lecture Notes in Mathematics, 1979, 66-77.

[**10**] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy*, Israel J. Math., **34** (1979), 213-237.

[**11**] F. Hofbauer, *Maximal measures for simple piecewise monotonic transformations*, Z. Wahrschein. Gebiete, **52** (1980), 289-300.

[**12**] J. Milnor, *Fubini foiled: Katok's paradoxical example in measure theory*, The Math. Intelligencer, **19** (1997), 30-32.

[**13**] W. Parry, *Representations for real numbers*, Acta Math. Acad. Sci. Hung., **15** (1964), 95-105.

[**14**] C.-E. Pfister and W. Sullivan, *On the topological entropy of satured sets*, Ergod. Th. and Dynam. Sys., **27** (2007), 929-956.

[**15**] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hung., **8** (1957), 477-493.

[**16**] D. Sands, *Topological conditions for positive Lyapunov exponent in unimodal maps*, PhD thesis, University of Cambridge, St John's College, 1993.

[**17**] J. Schmeling, *Symbolic dynamics for β-shifts and self-normal numbers*, Ergod. Th. and Dynam. Sys., **17** (1997), 675-694.

[**18**] P.C. Shields, The Ergodic Theory of Discrete Sample Paths, Graduate Studies in Mathematics Volume **13**, American Mathematical Society (1996).