TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

COMPLETELY REDUCIBLE SL(2)-HOMOMORPHISMS

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ABSTRACT. Let K be any field, and let G be a semisimple group over K. Suppose the characteristic of K is positive and is very good for G. We describe all group scheme homomorphisms $\phi : \mathrm{SL}_2 \to G$ whose image is geometrically G-completely reducible – or G-cr – in the sense of Serre; the description resembles that of irreducible modules given by Steinberg's tensor product theorem. In case K is algebraically closed and G is simple, the result proved here was previously obtained by Liebeck and Seitz using different methods. A recent result shows the Lie algebra of the image of ϕ to be geometrically G-cr; this plays an important role in our proof.

1. INTRODUCTION

Let K be an arbitrary field of characteristic $p > 0$. By a scheme we mean a separated K-scheme of finite type. An algebraic group will mean a smooth and affine K-group scheme; a subgroup will mean a K-subgroup scheme, and a homomorphism will mean a K-homomorphism. A smooth group scheme G is said to be reductive if $G_{/K^{\mathrm{alg}}}$ is reductive in the usual sense – i.e. it has trivial unipotent radical – where K^{alg} is an algebraic closure of K. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ may be regarded as a scheme over K; we permit ourselves to write $\mathfrak{g}$ for the set of K-points $\mathfrak{g}(K)$.

For G a reductive group, a subgroup $H \subset G$ is said to be geometrically G-completely reducible – or G-cr – if whenever k is an algebraically closed field containing K and $H_{/k}$ is contained in a parabolic k-subgroup P of $G_{/k}$, then $H_{/k} \subset L$ for some Levi k-subgroup L of P; see §2.3 for more details. The notion of G-cr was introduced by J-P. Serre; see e.g. [Ser 05] for more on this notion. It is our goal here to describe all homomorphisms $\phi : \mathrm{SL}_2 \to G$ whose image is geometrically G-cr; this we achieve under some assumptions on G which are described in §2.4. For the purposes of this introduction, let us suppose that G is semisimple. Then our assumption is: the characteristic of K is *very good* for G (again see §2.4 for the precise definition of a very good prime).

Let $F : \mathrm{SL}_2 \to \mathrm{SL}_2$ be the Frobenius endomorphism obtained by base change from the Frobenius endomorphism of $\mathrm{SL}_{2/\mathbf{F}_p}$; cf. §2.8 below. We say that a collection of homomorphisms $\phi_0, \phi_1, \dots, \phi_r : \mathrm{SL}_2 \to G$ is *commuting* if

$$\mathrm{im}\,\phi_i \subset C_G(\mathrm{im}\,\phi_j) \quad \text{for all } 0 \le i \ne j \le r.$$

Let $\vec{\phi} = (\phi_0, \dots, \phi_r)$ where the ϕ_i are commuting homomorphisms $\mathrm{SL}_2 \to G$, and let $\vec{n} = (n_0 < \dots < n_r)$ where the n_i are non-negative integers. Then the data $(\vec{\phi}, \vec{n})$ determines

Received by the editors October 18, 2005.
2000 *Mathematics Subject Classification.* 20G15.
Research of McNinch supported in part by the US National Science Foundation through DMS-0437482.
Research of Testerman supported in part by the Swiss National Science Foundation grant PP002-68710.

a homomorphism $\Phi_{\vec{\phi},\vec{n}} : \mathrm{SL}_2 \to G$ given for every commutative K-algebra Λ and every $g \in \mathrm{SL}_2(\Lambda)$ by the rule

$$g \mapsto \phi_0(F^{n_0}(g)) \cdot \phi_1(F^{n_1}(g)) \cdots \phi_r(F^{n_r}(g)).$$

We say that $\Phi = \Phi_{\vec{\phi},\vec{n}}$ is the *twisted-product homomorphism* determined by $(\vec{\phi}, \vec{n})$.

A notion of *optimal* homomorphisms $\mathrm{SL}_2 \to G$ was introduced in [Mc 05]; see §2.7 for the precise definition. When G is a K-form of $\mathrm{GL}(V)$ or $\mathrm{SL}(V)$, a homomorphism $f : \mathrm{SL}_2 \to G$ is optimal just in case the representation $(f_{/K^{\mathrm{sep}}}, V)$ is restricted and semisimple, where K^{sep} is a separable closure of K; see Remark 18. We will say that the list of commuting homomorphisms $\vec{\phi} = (\phi_0, \phi_1, \dots, \phi_r)$ is *optimal* if each ϕ_i is an optimal homomorphism.

Theorem 1. *Let G be a semisimple group for which the characteristic is very good, and let $\Phi : \mathrm{SL}_2 \to G$ be a homomorphism. If the image of Φ is geometrically G-cr, then there are commuting optimal homomorphisms $\vec{\phi} = (\phi_0, \dots, \phi_r)$ and non-negative integers $\vec{n} = (n_0 < n_1 < \cdots < n_d)$ such that Φ is the twisted-product homomorphism determined by $(\vec{\phi}, \vec{n})$. Moreover, $\vec{\phi}$ and $\vec{n}$ are uniquely determined by Φ.*

We actually prove the theorem for the strongly standard reductive groups described below in 2.4; see Theorem 39.

In case K is algebraically closed and G is simple, this theorem was obtained by Liebeck and Seitz [LS 03, Theorem 1]; cf. Remark 17 to see that the notion of restricted – or good – A_1-subgroup used in [LS 03] is "the same" as the notion of optimal homomorphism used here.

Note that Liebeck and Seitz prove a version of Theorem 1 where SL_2 is replaced by any quasisimple group H. If G is a split classical group over K in good characteristic, the more general form of Theorem 1 found in [LS 03] is a consequence of Steinberg's tensor product theorem [Jan 87, Cor. II.3.17]; cf. [LS 03, Lemma 4.1]. The proof given by Liebeck and Seitz of Theorem 1 for a quasisimple group G of exceptional type relies instead on detailed knowledge of the subgroup structure – in particular, of the maximal subgroups – of G; see e.g. [LS 03, Theorem 2.1, Proposition 2.2, and §4.1] for the case $H = \mathrm{SL}_2$. In contrast, when $p > 2$, our proof uses in an essential way the complete reducibility of the Lie algebra of a G-cr subgroup of G [Mc 05a]; cf. the proofs of Lemma 24, Proposition 25, and Lemma 29 [when $p = 2$, we have essentially just used the proof of Liebeck and Seitz].

We obtain also the converse to Theorem 1, though we do so only under a restriction on p. Write $h(G)$ for the maximum value of the Coxeter number of a simple k-quotient of $G_{/k}$, where k is an algebraically closed field containing K.

Theorem 2. *Let G be semisimple in very good characteristic, and suppose that $p > 2h(G)-2$, let $\vec{\phi} = (\phi_0, \dots, \phi_d)$ be commuting optimal homomorphisms $\mathrm{SL}_2 \to G$, and let $\vec{n} = (n_0 < n_1 < \cdots < n_d)$ be non-negative integers. Then the image of the twisted-product homomorphism $\Phi : \mathrm{SL}_2 \to G$ determined by $(\vec{\phi}, \vec{n})$ is geometrically G-cr.*

Again, this result is proved for a more general class of reductive groups; see Theorem 43.

The assumption on p made in the last theorem is unnecessary if G is a classical group – or a group of type G_2 – in good characteristic; see Remark 44. However, it is not clear to the authors how to eliminate the prime restriction in general.

The first named author would like to acknowledge the hospitality of the Centre Interfacultaire Bernoulli at the École Polytechnique Fédérale de Lausanne during a visit in June 2005; this visit permitted much of the collaboration which led to the present manuscript.

2. Preliminaries

2.1. **Reduced subgroups.** Let k be a perfect field – in the application we take k to be algebraically closed. Let B be a group scheme of finite type over k.

Lemma 3. *There is a unique smooth subgroup $B_{\mathrm{red}} \subset B$ which has the same underlying topological space as B. If A is any smooth group scheme over k and $f : A \to B$ is a k-morphism, then f factors in a unique way into a k-morphism $A \to B_{\mathrm{red}}$ followed by the inclusion $B_{\mathrm{red}} \to B$.*

Proof. Use [Li 02, Prop. 2.4.2] to find the reduced k-scheme B_{red} with the same underlying topological space as B; the result just quoted then yields the uniqueness of B_{red}. It is clear that B_{red} is a k-group scheme, and the assertion about A and f follows from *loc. cit.* Prop 2.4.2(d). Since k is perfect, apply [KMRT, Prop. 21.9] to see that a k-group is smooth if and only if it is geometrically reduced if and only if it is reduced. Thus B_{red} is indeed smooth . □

We are going to consider later some group schemes which we do not *a priori* know to be smooth, and we want to choose maximal tori in these group schemes. The following example explains why in those cases we first extend scalars to an algebraically closed field (see e.g. §3.2 below).

Example 4. If B is a group scheme over an imperfect field K, and if k is a perfect field containing K, then a maximal torus of $B_{/k,\mathrm{red}}$ need not arise by base-change from a K-subgroup of B. Let us give an example.

Let $A = \mathbf{G}_m \ltimes \mathbf{G}_a$ where $\mathbf{G}_m$ acts on $\mathbf{G}_a$ "with weight one"; i.e. $K[A] = K[T, T^{-1}, U]$ where the comultiplication μ^* is given by

$$\mu^*(T^{\pm 1}) = (T \otimes T)^{\pm 1} \quad \text{and} \quad \mu^*(U) = U \otimes T + 1 \otimes U.$$

Suppose that K is not perfect, and let $L = K(\beta)$ where $\beta^p = \alpha \in K$ but $\beta \notin K$. Consider the subgroup scheme $B \subset A$ defined by the ideal $I = (\alpha T^p - U^p - \alpha) \lhd K[A]$.

If k is a perfect field containing K, notice that the image $\bar{f} \in k[B]$ of $f = \beta T - U - \beta \in k[A]$ satisfies $\bar{f}^p = 0$ but $\bar{f} \neq 0$; thus $B_{/k}$ is not reduced. The subgroup $B_{/k,\mathrm{red}} \subset A_{/k}$ is defined by $J = (\beta T - U - \beta)$, so that $B_{/k,\mathrm{red}} \simeq \mathbf{G}_{m/k}$ is a torus. The group of k-points $B_{/k,\mathrm{red}}(k) \subset A(k)$ may be described as:

$$B_{/k,\mathrm{red}}(k) = \{(t, \beta t - \beta) \in \mathbf{G}_m(k) \ltimes \mathbf{G}_a(k) \mid t \in k^\times\}.$$

Note that $B_{/k,\mathrm{red}}$ does not arise by base change from a K-subgroup of A, e.g. since the intersection $B_{/k,\mathrm{red}}(k) \cap A(K)$ consists only in the identity element [where the intersection takes place in the group $A(k)$].

2.2. **Cocharacters and parabolic subgroups.** A *cocharacter* of an algebraic group A is a homomorphism $\gamma : \mathbf{G}_m \to A$. We write $X_*(A)$ for the set of cocharacters of A.

A linear representation (ρ, V) of A yields a linear representation $(\rho \circ \gamma, V)$ of $\mathbf{G}_m$ which in turn is determined by the morphism

$$(\rho \circ \gamma)^* : V \to K[\mathbf{G}_m] \otimes_K V = K[t, t^{-1}] \otimes_K V.$$

Then V is the direct sum of the weight spaces

$$V(\gamma; i) = \{v \in V \mid (\rho \circ \gamma)^* v = t^i \otimes v\} \tag{2.2.1}$$

for $i \in \mathbf{Z}$.

Consider now the reductive group G. If $\gamma \in X_*(G)$, then

$$P_G(\gamma) = P(\gamma) = \{x \in G \mid \lim_{t \to 0} \gamma(t) x \gamma(t^{-1}) \text{ exists}\}$$

is a parabolic subgroup of G whose Lie algebra is $\mathfrak{p}(\gamma) = \sum_{i \geq 0} \mathfrak{g}(\gamma; i)$; see e.g. [Spr 98, §3.2] for the notion of limit used here. Moreover, each parabolic subgroup of G has the form $P(\gamma)$ for some cocharacter γ; for all this cf. [Spr 98, 3.2.15 and 8.4.5].

We note that γ "exhibits" a Levi decomposition of $P = P(\gamma)$. Indeed, $P(\gamma)$ is the semi-direct product $C_G(\gamma) \cdot U(\gamma)$, where $U(\gamma) = \{x \in P \mid \lim_{t \to 0} \gamma(t) x \gamma(t^{-1}) = 1\}$ is the unipotent radical of $P(\gamma)$, and the reductive subgroup $C_G(\gamma) = C_G(\gamma(\mathbf{G}_m))$ is a Levi factor in $P(\gamma)$; cf. [Spr 98, 13.4.2].

2.3. **Complete reducibility, Lie algebras.** Let G be a reductive group, and write $\mathfrak{g}$ for its Lie algebra.

A smooth subgroup $H \subset G$ is geometrically G-cr if whenever k is an algebraically closed field containing K and $H_{/k} \subset P$ for a parabolic k-subgroup $P \subset G_{/k}$, then $H_{/k} \subset L$ for some Levi k-subgroup $L \subset P$.

Similarly, if $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, we say that $\mathfrak{h}$ is geometrically G-cr if whenever k is an algebraically closed field containing K and $P \subset G_{/k}$ is a parabolic k-subgroup with $\mathfrak{h}_{/k} = \mathfrak{h} \otimes_K k \subset \mathrm{Lie}(P)$, then $\mathfrak{h}_{/k} \subset \mathrm{Lie}(L)$ for some Levi k-subgroup $L \subset P$.

Lemma 5. *Let X and Y be schemes of finite type over K, and let $f : X \to Y$ be a (K-) morphism. The following are equivalent:*

i) *f is surjective,*
ii) *$f_{/k} : X(k) \to Y(k)$ is surjective for all algebraically closed fields k containing K, and*
iii) *$f_{/k} : X(k) \to Y(k)$ is surjective for some algebraically closed field k containing K.*

Proof. This follows from [DG70, I §3.6.10] □

Lemma 6. *Fix an algebraically closed field k containing K. Let G be a reductive group, let $J \subset G$ be a smooth subgroup, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then*

(1) *J is geometrically G-cr if and only if $J_{/k}$ is $G_{/k}$-cr.*
(2) *$\mathfrak{h}$ is geometrically G-cr if and only if $\mathfrak{h}_{/k}$ is $G_{/k}$-cr.*

Proof. We prove (1); the proof of (2) is essentially the same. We are going to apply the previous Lemma.

First let $\mathcal{P}$ be the scheme of all parabolic subgroups of G, and let $Y = \mathcal{P}^J$ be the fixed point scheme for the action of J; thus Y is the closed subscheme of those parabolic subgroups containing J. [1]

Let also $\mathcal{PL}$ be the scheme such that for each commutative K-algebra Λ, the Λ-points $\mathcal{PL}(\Lambda)$ are the pairs $P \supset L$ where P is a parabolic of $G_{/\Lambda}$ and L is a Levi subgroup of P; cf. [SGA3, Exp. XXVI, §3.15]. Let $X = (\mathcal{PL})^J$ be the scheme of those pairs $P \supset L$ where L contains J.

There is an evident morphism $\mathcal{PL} \to \mathcal{P}$ given by $(P \supset L) \mapsto P$; cf. [SGA3, Exp. XXVI, §3.15]. By restriction one gets a morphism $f : X \to Y$. Then f is surjective if and only if J is G-cr, and (1) follows from the preceding Lemma. □

Proposition 7. *Let G be reductive, and let $M \subset G$ be a Levi subgroup. Suppose that $J \subset M$ is a smooth subgroup, and that $\mathfrak{h} \subset \mathrm{Lie}(M)$ is a Lie subalgebra. Then J is geometrically G-cr if and only if J is geometrically M-cr and $\mathfrak{h}$ is geometrically G-cr if and only if $\mathfrak{h}$ is geometrically M-cr.*

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[1]For assertion (2), one should instead regard $\mathcal{P}$ as the scheme of parabolic subalgebras of $\mathfrak{g}$, which may be regarded as a closed subscheme of a product of Grassman schemes $\mathrm{Gr}_d(\mathfrak{g})$ for various d. Now the subscheme $X \subset \mathcal{P}$ of parabolic subalgebras containing $\mathfrak{h}$ is the intersection of $\mathcal{P}$ with the subscheme Z of the product of Grassman schemes consisting of those subspaces containing $\mathfrak{h}$. Since Z is closed in the product, Y is closed in $\mathcal{P}$. Similar remarks apply to the definition of the subscheme $Y \subset \mathcal{PL}$ to be given in the next paragraph.

Proof. For the proof, it is enough to suppose that K is algebraically closed. The proof for J is found in [BMR 05, Theorem 3.10]. The proof for $\mathfrak{h}$ is deduced from [Ser 05, 2.1.8]; see [Mc 05a, Lemma 2] for the argument. ☐

The following theorem was proved in [Mc 05a].

Theorem 8. *Let $H \subset G$ be a smooth subgroup which is geometrically G-cr. Then $\mathfrak{h} = \mathrm{Lie}(H)$ is geometrically G-cr.*

We recall a similar result of B. Martin [Ser 05, Théorème 3.6].

Theorem 9 (Martin)**.** *Let $H \subset G$ be a smooth subgroup which is geometrically G-cr, and let $H' \lhd H$ be a smooth normal subgroup. Then H' is geometrically G-cr as well.*

Finally, we note:

Lemma 10. *Let $\pi : G \to G_1$ be a central isogeny where G_1 is a second reductive group, let $J \subset G$ be a smooth subgroup, and let $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}(G)$ be a Lie subalgebra. Then*

(1) *J is geometrically G-cr if and only if $\pi(J)$ is geometrically G_1-cr, and*
(2) *$\mathfrak{h}$ is geometrically G-cr if and only if $d\pi(\mathfrak{h})$ is geometrically G_1-cr.*

Proof. We may and will suppose that K is algebraically closed for the proof. It is clear that J is contained in a parabolic subgroup P of G if and only if $\pi(J)$ is contained in the parabolic subgroup $\pi(P)$ of G_1, and similarly $\mathfrak{h}$ is contained in $\mathrm{Lie}(P)$ if and only if $d\pi(\mathfrak{h})$ is contained in $d\pi(\mathrm{Lie}(P)) = \mathrm{Lie}(\pi(P))$, the result follows since $P \mapsto \pi(P)$ determines a bijection between the parabolic subgroups of G and those of G_1. ☐

2.4. **Strongly standard reductive groups.** If G is geometrically quasisimple with absolute root system R [2], the characteristic p of K is said to be a bad prime for R in the following circumstances: $p = 2$ is bad whenever $R \neq A_r$, $p = 3$ is bad if $R = G_2, F_4, E_r$, and $p = 5$ is bad if $R = E_8$. Otherwise, p is good. [Here is a more intrinsic definition of good prime: p is good just in case it divides no coefficient of the highest root in R].

If p is good, then p is said to be very good provided that either R is not of type A_r, or that $R = A_r$ and $r \not\equiv -1 \pmod p$.

There is a possibly inseparable central isogeny [3]

$$\prod_{i=1}^{r} G_i \times T \to G \tag{2.4.1}$$

for some torus T and some $r \geq 1$, where for $1 \leq i \leq r$ there is an isomorphism $G_i \simeq R_{L_i/K} H_i$ for a finite separable field extension L_i/K and a geometrically simple, simply connected L_i-group scheme H_i; here, $R_{L_i/K} H_i$ denotes the "Weil restriction" of H_i to K.

Then p is good, respectively very good, for G if and only if that is so for H_i for every $1 \leq i \leq r$. Since the H_i are uniquely determined by G up to central isogeny, the notions of good and very good primes depend only on the central isogeny class of the derived group (G, G). Moreover, these notions are geometric in the sense that they depend only on the group $G_{/k}$ for an algebraically closed field k containing K.

[2]The absolute root system of G is the root system of $G_{/K^{\mathrm{sep}}}$ where K^{sep} is a separable closure of K.

[3]Indeed, the center of the reductive group G is a smooth subgroup scheme; this follows e.g. from [SGA3, II Exp. XII Théorème 4.1] since for reductive G, the center is the same as the "centre réductif". The radical $R(G)$ is the maximal torus of the center of G, so $R(G)$ is a smooth torus, and we take $T = R(G)$ in (2.4.1). Now, multiplication gives a central isogeny $G' \times R(G) \to G$ where G' is the derived group of G. So (2.4.1) follows from the corresponding result for semisimple groups; see e.g. [KMRT, Theorems 26.7 and 26.8] or [TW 02, Appendix (42.2.7)].

One says that a smooth K-group D is of multiplicative type if $D_{/K'}$ is diagonalizable for some algebraic extension $K \subset K'$; i.e. that $D_{/K'} \simeq \mathrm{Diag}(\Gamma)$ for some commutative group Γ. See [Jan 87, I.2.5] for the definition of $\mathrm{Diag}(\Gamma)$ – it is implicitly defined in [Spr 98, Corollary 3.2.4] as well. A torus is of multiplicative type, as is any finite, smooth, commutative subgroup all of whose geometric points are semisimple.

If G is reductive and if $D \subset G$ is a subgroup of multiplicative type, then $C_G(D)$ is a reductive subgroup containing a maximal torus of G – use [SGA3, II Exp. XI, Cor 5.3] to see that $C_G(D)$ is smooth, use [Spr 98, Theorem 3.2.3] to see that $D_{K'}$ lies in a maximal torus of $G_{K'}$, and finally use [SS 70, II. §4.1] to see that $C_G(D)$ is reductive.

Consider reductive groups of the form

$$(*) \quad H = H_1 \times S$$

where H_1 is a semisimple group for which the characteristic of K is very good, and where S is a torus. We say that G is *strongly standard* if there is a group H as in $(*)$, a subgroup of multiplicative type $D \subset H$, and a separable isogeny between G and the reductive subgroup $C_H(D)$ of H.

Remark 11. This definition of *strongly standard* is more general than that given e.g. in [Mc 05]. It follows from Proposition 12 below that the main result of *loc. cit.* in fact applies to a strongly standard group in this stronger sense.

Proposition 12. *Let G be strongly standard.*

(1) *If $D \subset G$ is a subgroup of multiplicative type, then the reductive group $C_G(D)$ is strongly standard.*
(2) *The characteristic of K is good for the derived group of G, and there is a non-degenerate, G-invariant bilinear form on* $\mathrm{Lie}(G)$.
(3) *Each conjugacy class and each adjoint orbit is separable. In particular, if $g \in G(K)$ and $X \in \mathfrak{g}(K)$, then $C_G(g)$ and $C_G(X)$ are smooth.* [4]

Remark 13. The centralizers considered in (3) – and elsewhere in this paper – are the *scheme-theoretic* centralizers. Thus e.g. $C_G(X)$ is the group scheme with Λ-points $C_G(X)(\Lambda) = \{g \in G(\Lambda) \mid \mathrm{Ad}(g)X = X\}$ for each commutative K-algebra Λ.

Proof of Proposition 12. For the proofs of (1) and (2), we may replace G by a separably isogenous group and suppose G to be the centralizer of a subgroup of multiplicative type $D_1 \subset H$ where H has the form $(*)$.

For (1), note that since D_1 centralizes D, the group $D_2 = D \cdot D_1 \subset H$ is of multiplicative type and (1) is immediate.

To prove (2), note first that the characteristic of K is good for the derived group of G by [Mc 05, Lemma 1](2). Now, there is a non-degenerate H-invariant bilinear form β on a group H of the form $(*)$ by [Mc 05, Lemma 1](1). Moreover, it suffices to see that the restriction of β to $\mathrm{Lie}(G)$ is nondegenerate after making a field extension; thus, we may suppose that $D_1 \simeq \mathrm{Diag}(\Gamma)$. We have $\mathrm{Lie}(H) = \bigoplus_{\gamma\in\Gamma} \mathrm{Lie}(H)_\gamma$ where D_1 acts on $\mathrm{Lie}(H)_\gamma$ through γ; see e.g. [Jan 87, §I.2.11]. The subspaces $\mathrm{Lie}(H)_\gamma$ and $\mathrm{Lie}(H)_\tau$ are evidently orthogonal unless $\gamma \cdot \tau = 1$ in Γ [5]. Since $\mathrm{Lie}(G) = \mathrm{Lie}(H)^{D_1} = \mathrm{Lie}(H)_1$, the restriction of β to $\mathrm{Lie}(G)$ must remain non-degenerate.

In view of (2), the proof of [Mc 05, Prop. 5] yields (3). □

[4] In older language, these centralizers are defined over K.

[5] We are writing Γ multiplicatively

2.5. **Nilpotent elements and associated cocharacters.** Let G be a reductive group, and let $X \in \mathfrak{g} = \mathrm{Lie}(G)$ be nilpotent. A cocharacter $\Psi \in X_*(G)$ is said to be associated with X if the following conditions hold:

(A1) $X \in \mathfrak{g}(\Psi; 2)$, and
(A2) there is a maximal torus S of $C_G(X)$ such that $\Psi \in X_*(L_1)$ where $L = C_G(S)$ and $L_1 = (L, L)$ is its derived group.

Assume now that G is strongly standard.

Proposition 14. *Let $X \in \mathfrak{g}$ be nilpotent.*

(1) *There is a cocharacter Ψ associated with X.*
(2) *If Ψ is associated to X and $P = P(\Psi)$ is the parabolic determined by Ψ, then $C_G(X) \subset P$. In particular, $\mathfrak{c}_{\mathfrak{g}}(X) \subset \mathrm{Lie}(P)$.*
(3) *If $\Psi, \Phi \in X_*(G)$ are associated with X, then $\Psi = \mathrm{Int}(x) \circ \Phi$ for some $x \in C_G(X)(K)$.*
(4) *The parabolic subgroups $P(\Psi)$ for cocharacters Ψ associated with X all coincide.*

Proof. (1) is [Mc 04, Theorem 26], (2) is [Ja 04, Prop. 5.9]. (3) follows from [Mc 05, Prop/Defn 21(4)], and (4) is [Mc 05, Prop/Defn 21(5)]. □

Let Ψ be a cocharacter associated with X as in the previous Proposition. Then the parabolic subgroup $P(X) = P(\Psi)$ of (4) is known as the the instability parabolic of X.

Let $X \in \mathfrak{g}$, and let $[X] \in \mathbf{P}(\mathfrak{g})(K)$ be the K-point which "is" the line determined by X in the corresponding projective space.

Proposition 15. *Write $N_G(X) = \mathrm{Stab}_G([X])$.*

(1) *$N_G(X)$ is a smooth subgroup of G.*
(2) *For each maximal torus T of $N_G(X)$, there is a unique cocharacter $\lambda \in X_*(T)$ associated to X.*

Proof. Recall that $N_G(X)$ is the *scheme-theoretic* stabilizer of the point $[X] \in \mathbf{P}(\mathfrak{g})(K)$. (1) follows from [Mc 04, Lemma 23], and in view of Proposition/Definition 14(1), assertion (2) follows from [Mc 04, Lemma 25]. □

2.6. **Notation for** SL_2**.** We fix here some convenient notation for SL_2. We first choose the "standard" basis for the Lie algebra $\mathfrak{sl}_2$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad [E, F] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now consider the homomorphisms $e, f : \mathbf{G}_a \to \mathrm{SL}_2$ given for each commutative K-algebra Λ and each $t \in \mathbf{G}_a(\Lambda) = \Lambda$ by the rules

$$e(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Finally, write $\mathcal{T}$ for the "diagonal" maximal torus of SL_2; we fix the cocharacter

$$(t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) : \mathbf{G}_m \to \mathcal{T}$$

and use this cocharacter to identify $\mathcal{T}$ with $\mathbf{G}_m$.

2.7. **Optimal homomorphisms.** We will use without mention the notation of §2.6. Let G be a reductive group. We say that a homomorphism $\phi : \mathrm{SL}_2 \to G$ is optimal for $X = d\phi(E)$, or simply that ϕ is an optimal SL_2-homomorphism, if $\lambda = \phi_{|\mathcal{T}}$ is a cocharacter associated with X.

Theorem 16 ([Mc 05]). *Suppose that G is strongly standard. Let $X \in \mathfrak{g}$ satisfy $X^{[p]} = 0$, and let $\lambda \in X_*(G)$ be associated with X. There is a unique homomorphism*

$$\phi : \mathrm{SL}_2 \to G$$

such that $d\phi(E) = X$ and $\phi_{|\mathcal{T}} = \lambda$. Moreover, the image of ϕ is geometrically G-cr.

Proof. In view of Proposition 12(3), this follows from [Mc 05, Theorem 47 and Prop. 52]. □

Remark 17. Seitz has introduced a notion of "good A_1-subgroup" of a quasisimple group in [Sei 00]; in [LS 03], these subgroups are called "restricted". Refer to [Mc 05, §8.5] to see that a subgroup of type A_1 of a quasisimple group G is restricted if and only if it is the image of an optimal homomorphism $\mathrm{SL}_2 \to G$. It is not hard to see that the image of an optimal homomorphism is restricted; cf. [Mc 05, Prop. 30]. The proof that a restricted A_1-subgroup is the image of an optimal homomorphism is more involved.

Remark 18. If V is a finite dimensional vector space, a homomorphism $\phi : \mathrm{SL}_2 \to \mathrm{SL}(V)$ is optimal if and only if V is a restricted semisimple SL_2-module. Indeed, if V is restricted and semisimple, one sees at once that $\phi_{|\mathcal{T}}$ is associated with $d\phi(E)$ so that ϕ is indeed optimal. On the other hand, if $\lambda = \phi_{|\mathcal{T}}$ is associated to $X = d\phi(E)$, then the the character of the SL_2-module V is determined by the cocharacter λ; it follows that the composition factors of V as SL_2-module are restricted. If $0 \le n < p$, write $L(n)$ for the restricted simple SL_2-module of highest weight n [Jan 87, §II.2]. The linkage principle [Jan 87, Corollary II.6.17] implies that $\mathrm{Ext}^1_{\mathrm{SL}_2}(L(n), L(m)) = 0$ whenever $0 \le n, m < p$. Thus, V is semisimple as well.

Proposition 19. *Let S be the image of the optimal SL_2-homomorphism ϕ, and let $\lambda = \phi_{|\mathcal{T}} \in X_*(G)$. Write $X = d\phi(E)$ and $Y = d\phi(F)$. Then:*

1. *$C_G(\mathrm{im}\, d\phi) = C_G(S)$.*
2. *The scheme-theoretic intersection $C_G(S) = C_G(X) \cap C_G(\lambda)$ is a smooth subgroup of G.*

Proof. For (1), recall that if $\varepsilon : \mathbf{G}_a \to G$ is given by $\varepsilon = \phi \circ e$, then $C_G(X) = C_G(\mathrm{im}\, \varepsilon)$ by [Mc 05, Prop. 35]. Similarly, $C_G(Y) = C_G(\mathrm{im}\, \phi \circ f)$. Since $\mathrm{im}\, d\phi$ is spanned by X and Y and since S is generated as a group scheme by the image of $\phi \circ e$ and the image of $\phi \circ f$, we have

$$C_G(\mathrm{im}\, d\phi) = C_G(X) \cap C_G(Y) = C_G(\mathrm{im}\, \phi \circ e) \cap C_G(\mathrm{im}\, \phi \circ f) = C_G(S).$$

For (2), the inclusion $C_G(S) \subset C_G(X) \cap C_G(\lambda)$ is clear. To prove the other inclusion, let Λ be a commutative K-algebra, and let $g \in G(\Lambda)$ be such that $\mathrm{Ad}(g)X = X$ and $\mathrm{Int}(g) \circ \lambda = \lambda$. By (1), it is enough to show that g centralizes Y. Since $Y \in \mathfrak{g}(\lambda; -2)(K)$ and since $\mathrm{Int}(g) \circ \lambda = \lambda$, we have $\mathrm{Ad}(g)Y \in \mathfrak{g}(\lambda; -2)(\Lambda)$. Notice that

$$[X, \mathrm{Ad}(g)Y] = [\mathrm{Ad}(g^{-1})X, Y] = [X, Y].$$

Thus $[X, Y - \mathrm{Ad}(g)Y] = 0$ so that $Y - \mathrm{Ad}(g)Y \in \mathfrak{c}_{\mathfrak{g}}(X)(\lambda; -2)(\Lambda)$. Since $\mathfrak{c}_{\mathfrak{g}}(X) \subset \mathrm{Lie}(P(\lambda))$ by Proposition 14, we have $\mathfrak{c}_{\mathfrak{g}}(X)(\lambda; -2) = 0$ so that $Y = \mathrm{Ad}(g)Y$ as required. Now, $C_G(X)$ is smooth by Proposition 12, hence $C_G(X) \cap C_G(\lambda)$ is smooth by [SGA3, II Exp. XI, Cor 5.3].

□

Remark 20. In the notation of the previous Proposition, we have $\operatorname{im} d\phi = \operatorname{Lie} S$ whenever $p > 2$, since the adjoint representation of SL_2 is irreducible for $p > 2$.

Proposition 21. *Let G be a strongly standard reductive group.*

(1) *Let $L \subset G$ be a Levi subgroup, and assume that $\phi : \mathrm{SL}_2 \to L$ is a homomorphism. Then ϕ is an optimal homomorphism in G if and only if it is an optimal homomorphism in L.*
(2) *Let $\pi : G_1 \to G$ be a central isogeny, let $f : \mathrm{SL}_2 \to G$ be a homomorphism, and suppose that $\widetilde{f} : \mathrm{SL}_2 \to G_1$ satisfies $\pi \circ \widetilde{f} = f$. Then f is optimal if and only if $\widetilde{f}$ is optimal.*

Proof. We first prove (1). In view of Theorem 16, it suffices to prove the following: Let $X \in \operatorname{Lie}(L)$ be nilpotent and let $\lambda \in X_*(L)$ be a cocharacter with $X \in \operatorname{Lie}(L)(\lambda; 2)$. Then λ is associated to X in L if and only if λ is associated to X in G.

Note that $\lambda \in X_*(N_L(X))$, so the image of λ normalizes $C_L(X)$. In particular, we may choose a maximal torus S_0 of $C_L(X)$ centralized by the image of λ, and we may choose a maximal torus S of $C_G(X)$ with $S_0 \subset S$. Notice that S_0 – and hence also S – contains the center of L. Since L is the centralizer in G of the connected center of L, we have $S \subset L$ so that $S = S_0$. Moreover, since $C_G(S) \subset L$, it is clear that $C_G(S) = C_L(S) = M$. Moreover, it is clear that $\lambda \in X_*(M)$, and the Proposition follows since the condition that λ be associated to X is just that $\lambda \in X_*((M, M))$; this condition is the same for L and for G.

We now prove (2). Note first that it suffices to prove (2) in case K is algebraically closed. Let $X = df(E)$ and $\widetilde{X} = d\widetilde{f}(E)$. Then π induces a surjective morphism $N_{G_1}(\widetilde{X})_{\mathrm{red}} \to N_G(X)$. [6] We may thus choose a maximal torus $\widetilde{T}$ of $C_{G_1}(\widetilde{X})_{\mathrm{red}}$ centralized by $\operatorname{im} \widetilde{f}_{|\mathcal{T}}$ and a maximal torus T of $C_G(X)$ centralized by $\operatorname{im} f_{|T}$ such that $\pi(\widetilde{T}) = T$.

Now, $\widetilde{X}$ is distinguished in the Levi subgroup $L_1 = C_{G_1}(\widetilde{T})$ and X is distinguished in the Levi subgroup $L = C_G(T)$. Since the maximal tori in $C_{G_1}(\widetilde{X})$ are all conjugate, one sees that $\widetilde{f}_{|\mathcal{T}}$ is associated with $\widetilde{X}$ if and only if $\operatorname{im} \widetilde{f}_{|\mathcal{T}}$ lies in the derived group of L_1; since the maximal tori in $C_G(X)$ are all conjugate, $f_{|\mathcal{T}}$ is associated with X if and only if $\operatorname{im} f_{\mathcal{T}}$ lies in the derived group of L. Since π induces a central isogeny $L_1 \to L$, it follows that $\widetilde{f}_{|T}$ is associated with $\widetilde{X}$ if and only if $f_{|T}$ is associated with X; (2) is an immediate consequence. $\square$

2.8. **Frobenius endomorphisms.** Let H be a connected, split, quasi-simple algebraic group; recall that H arises by base change from a corresponding group scheme $H_{/\mathbf{F}_p}$ over the prime field $\mathbf{F}_p$. There is a Frobenius endomorphism $F : H \to H$ which arises by base change from the corresponding Frobenius endomorphism of $H_{/\mathbf{F}_p}$.

Proposition 22. *Let G be an algebraic group, and let $\phi : H \to G$ be a homomorphism. The following are equivalent:*

(1) $d\phi = 0$
(2) *there is a unique integer $t \geq 1$ and a unique homomorphism $\psi : H \to G$ such that $d\psi \neq 0$ and $\phi = \psi \circ F^t$.*

Proof. (1) $\Longrightarrow$ (2) is a consequence of [Mc 05, Cor. 20]. (2) $\Longrightarrow$ (1) is straightforward. $\square$

Of course, the above result holds in particular when H is the group SL_2.

[6] Note that G_1 need not be strongly standard.

3. The tangent map of a G-completely reducible SL_2-homomorphism

3.1. **The set-up.** Now fix a homomorphism $\phi : \mathrm{SL}_2 \to G$ whose image is geometrically G-cr. Assume that $d\phi \neq 0$, and write

$$X = d\phi(E), \quad Y = d\phi(F), \quad H = d\phi([E,F]) \in \mathfrak{g},$$

Also put $\mathfrak{s} = \operatorname{im} d\phi$, and write $\lambda = \phi_{|\mathcal{T}}$. Consider the smooth subgroups $N_G(X), N_G(Y) \subset G$ which are the stabilizers of the points $[X],[Y] \in \mathbf{P}(\mathfrak{g})(K)$. Then λ is evidently a cocharacter of $N_G(X) \cap N_G(Y)$.

Consider the group schemes $C(X,Y) = (C_G(X) \cap C_G(Y))$ and $N(X,Y) = (N_G(X) \cap N_G(Y))$. We observe the following:

Lemma 23. *$C(X,Y)$ is a normal subgroup scheme of $N(X,Y)$.*

In particular, the image of λ normalizes $C(X,Y)$.

3.2. **Working geometrically.** Fix an algebraically closed field k containing K and consider $G_{/k}$, $\mathfrak{s}_{/k} = \mathfrak{s} \otimes_K k$ etc. In this section, we are forced to consider the reduced subgroups corresponding to various subgroup schemes; recall the results of 2.1. Thus, *for the remainder of §3.2, we replace K by k and so suppose that K is algebraically closed.*

According to §2.1, the image of λ normalizes $C(X,Y)$ and hence also $C(X,Y)_{\mathrm{red}}$. Thus, we may choose a maximal torus $T \subset C(X,Y)_{\mathrm{red}}$ centralized by the image of λ.

Consider now the Levi subgroup $M = C_G(T)$ of G; M is a strongly standard reductive group by Proposition 12(1). Since X and Y are centralized by T, and since $\mathfrak{s}$ is generated as a Lie algebra by X and Y, we have $\mathfrak{s} \subset \mathrm{Lie}(M)$. Of course, the image of the homomorphism ϕ need not lie in M.

Lemma 24. *$\mathfrak{s}$ is not contained in $\mathrm{Lie}(P)$ for any proper parabolic subgroup $P \subset M$.*

Proof. Any torus $T_1 \subset M$ centralizing $\mathfrak{s}$ of course centralizes X and Y; thus T lies in a maximal torus of $C(X,Y)_{\mathrm{red}}$. Since T is central in M, T_1 centralizes T. Since T is a maximal torus of $C(X,Y)_{\mathrm{red}}$, we find that $T_1 \subset T$ hence T_1 is central in M. Since $\mathfrak{s}$ is the Lie algebra of a G-cr subgroup of G, the Lie algebra $\mathfrak{s}$ is itself G-cr by Theorem 8. Hence $\mathfrak{s}$ is also M-cr by Proposition 7 . If $\mathfrak{s}$ is contained in $\mathrm{Lie}(P)$ for a parabolic subgroup $P \subset M$, then $\mathfrak{s}$ is contained in $\mathrm{Lie}(L)$ for some Levi subgroup L of P. But then any central torus of L is central in M, so that $P = M$. $\square$

Proposition 25. *Let T_1 be a maximal torus of $N_M(X)$ with $\lambda \in X_*(T_1)$, and let $\lambda_0 \in X_*(T_1)$ be the unique cocharacter of T_1 associated to X [see Proposition 15(2)]. Let $\phi_0 : \mathrm{SL}_2 \to M$ be the optimal homomorphism determined by X and λ_0 [Theorem 16]. Write $\mu = \lambda_0 - \lambda$ for the cocharacter*

$$t \mapsto \lambda_0(t) \cdot \lambda(t^{-1})$$

of T_1. Then the image of μ is central in M.

Proof. We have $H \in \mathfrak{m}(\lambda_0; 0) \cap \mathfrak{m}(\lambda; 0)$ by the choice of T_1; thus $H \in \mathfrak{m}(\mu; 0)$. We have also $X \in \mathfrak{m}(\lambda_0; 2) \cap \mathfrak{m}(\lambda; 2)$ so that $X \in \mathfrak{m}(\mu; 0)$ as well.

Write $Y = \sum_{j \in \mathbf{Z}} Y^j$ with $Y^j \in \mathfrak{m}(\lambda_0; j)$. Since $[X,Y] = H \in \mathfrak{m}(\lambda_0; 0)$, we have $Y - Y^{-2} \in \mathfrak{c}_{\mathfrak{m}}(X)$, so that

$$Y = Y^{-2} + \sum_{j \geq 0} Y^j.$$

Since the images of λ and λ_0 commute, and since $Y \in \mathfrak{m}(\lambda; -2)$, we have $Y^j \in \mathfrak{m}(\lambda; -2)$ for all j. Thus, $Y^j \in \mathfrak{m}(\lambda_0 - \lambda; j+2) = \mathfrak{m}(\mu; j+2)$ for all j, hence $Y \in \sum_{\ell \geq 0} \mathfrak{m}(\mu; \ell) = \mathrm{Lie}\, P_M(\mu)$.

Since $X, Y, H \in \operatorname{Lie} P_M(\mu)$, we have proved that $\mathfrak{s} = \operatorname{im} d\phi$ lies in $\operatorname{Lie} P_M(\mu)$. Thus by Lemma 24, we have $P_M(\mu) = M$; we conclude that the image of μ is central in M. □

Proposition 26. *Let T_1 be a maximal torus of $N_M(X)$, and write $\phi_0 : \mathrm{SL}_2 \to M$ for the optimal homomorphism determined by the cocharacter $\lambda_0 \in X_*(T_1)$ associated with X as in Proposition 25. Then $d\phi = d\phi_0$.*

Proof. Recall that T is a fixed maximal torus of $C(X, Y)_{\mathrm{red}}$, and $M = C_G(T)$. Using (2.4.1), one finds a (possibly inseparable) central isogeny

$$\pi : G_{\mathrm{sc}} \to G,$$

where the derived group of G_{sc} is simply connected. There is a torus $\widetilde{T} \subset G_{\mathrm{sc}}$ with $\pi(\widetilde{T}) = T$; then the Levi subgroup $M_{\mathrm{sc}} = C_{G_{\mathrm{sc}}}(\widetilde{T})$ has simply connected derived group, and π restricts to a central isogeny $\pi : M_{\mathrm{sc}} \to M$.

Since SL_2 is simply connected, there are homomorphisms

$$\widetilde{\phi} : \mathrm{SL}_2 \to G_{\mathrm{sc}} \quad \text{and} \quad \widetilde{\phi}_0 : \mathrm{SL}_2 \to M_{\mathrm{sc}}$$

such that $\phi = \pi \circ \widetilde{\phi}$ and $\phi_0 = \pi \circ \widetilde{\phi}_0$. We write $\widetilde{\lambda}$ and $\widetilde{\lambda}_0$ for the cocharacters obtained by restricting these homomorphisms to the maximal torus $\mathcal{T}$ of SL_2. Moreover, write

$$\mu = \lambda_0 - \lambda_1 \quad \text{and} \quad \widetilde{\mu} = \widetilde{\lambda}_0 - \widetilde{\lambda}_1.$$

Since $\lambda, \lambda_0 \in X_*(T_1)$, we know that $\widetilde{\lambda}, \widetilde{\lambda}_0$ are cocharacters of M_{sc}. Since $\pi \circ \widetilde{\mu} = \mu$, it follows from Proposition 25 that the image of $\widetilde{\mu}$ is central in M_{sc}.

Since T centralizes $\operatorname{im} \phi_0$ and $\operatorname{im} d\phi$, it is clear that the images of $d\widetilde{\phi}_0$ and $d\widetilde{\phi}$ lie in $\operatorname{Lie}(M_{\mathrm{sc}})$. Write $H_0 = d\phi_0([E, F])$. We claim that the Proposition will follow if we show that $H = H_0$. Indeed, by Proposition 25, the image of $\mu = \lambda_0 - \lambda$ centralizes Y; thus, we have $Y \in \mathfrak{m}(\lambda_0; -2)$ and $Y - Y_0 \in \mathfrak{m}(\lambda_0; -2)$. If $H = H_0$, then $[X, Y - Y_0] = H - H_0 = 0$ so that $Y - Y_0 \in \mathfrak{c}_{\mathfrak{m}}(X)$. Since λ_0 is associated to X, we have $\mathfrak{c}_{\mathfrak{m}}(X) \subset \operatorname{Lie} P(\lambda_0) = \sum_{i \geq 0} \mathfrak{m}(\lambda_0; i)$ by Proposition/Definition 14. We may thus conclude that $Y = Y_0$ so that $d\phi$ and $d\phi_0$ are indeed equal.

It remains now to show that $H = H_0$. Let $\widetilde{H} = d\widetilde{\phi}([E, F])$ and $\widetilde{H}_0 = d\widetilde{\phi}_0([E, F])$. It is clearly enough to show that $\widetilde{H} = \widetilde{H}_0$.

Now, since the derived group M'_{sc} of M_{sc} is simply connected, one knows that M_{sc} is a direct product

$$M_{\mathrm{sc}} = Z^o(M_{\mathrm{sc}}) \times M'_{\mathrm{sc}}$$

where $Z^o(M_{\mathrm{sc}})$ is the connected component of the center of M_{sc} (it is a torus). Thus also

$$\operatorname{Lie}(M_{\mathrm{sc}}) = \operatorname{Lie}(Z^o(M_{\mathrm{sc}})) \oplus \operatorname{Lie}(M'_{\mathrm{sc}}).$$

The derived subalgebra $[\operatorname{Lie}(M_{\mathrm{sc}}), \operatorname{Lie}(M_{\mathrm{sc}})]$ is contained in $\operatorname{Lie}(M'_{\mathrm{sc}})$; since $\mathfrak{sl}_2 = [\mathfrak{sl}_2, \mathfrak{sl}_2]$, we have

$$\operatorname{im} d\widetilde{\phi} \subset \operatorname{Lie}(M'_{\mathrm{sc}}) \quad \text{and} \quad \operatorname{im} d\widetilde{\phi}_0 \subset \operatorname{Lie}(M'_{\mathrm{sc}}).$$

In particular,

$$\widetilde{H}_0 - \widetilde{H} \in \operatorname{Lie}(M'_{\mathrm{sc}}). \tag{3.2.1}$$

On the other hand, $\operatorname{im} \widetilde{\mu}$ lies in $Z^o(M_{\mathrm{sc}})$, and so

$$\widetilde{H}_0 - \widetilde{H} \in \operatorname{im} d\widetilde{\mu} \subset \operatorname{Lie}(Z^o(M_{\mathrm{sc}})). \tag{3.2.2}$$

Since $\operatorname{Lie}(Z^o(M_{\mathrm{sc}})) \cap \operatorname{Lie}(M'_{\mathrm{sc}}) = 0$, we deduce that $\widetilde{H}_0 = \widetilde{H}$ by applying (3.2.1) and (3.2.2). This completes the proof. □

3.3. **The tangent map over any field.** We now suppose that K is an arbitrary field of characteristic $p > 0$. As in the previous section, we fix a homomorphism $\phi : \mathrm{SL}_2 \to G$ whose image is geometrically G-cr, and we assume that $d\phi \neq 0$.

We also fix an algebraically closed field k containing K.

Corollary 27. *The K-subgroup $C(X,Y) = C_G(X) \cap C_G(Y)$ is smooth.*

Proof. Working over the algebraically closed field $k \supset K$, let $\phi_0 : \mathrm{SL}_{2/k} \to G_{/k}$ be any optimal k-homomorphism as in Proposition 26. Write $S_0 \subset G_{/k}$ for the image of ϕ_0, and recall that $\mathfrak{s}_{/k} = \operatorname{im} d\phi_{/k} = \operatorname{im} d\phi_0$. Then we know that

$$C_{G_{/k}}(S_0) = C_{G_{/k}}(\mathfrak{s}_{/k}) = C_{G_{/k}}(X) \cap C_{G_{/k}}(Y)$$

by Proposition 19, hence $C_{G_{/k}}(X) \cap C_{G_{/k}}(Y) = (C_G(X) \cap C_G(Y))_{/k}$ is smooth. But then $C_G(X) \cap C_G(Y)$ is smooth, since that is so after extension of the ground field. □

Corollary 28. *There is a cocharacter λ_0 of G associated to X such that if $\phi_0 : \mathrm{SL}_2 \to G$ is the optimal homomorphism determined by X and λ_0, then $d\phi = d\phi_0$. Moreover, ϕ_0 is uniquely determined by ϕ.*

Proof. Since $C_G(X) \cap C_G(Y)$ is smooth by the previous corollary, we can find a maximal torus T of $C_G(X) \cap C_G(Y)$ centralized by the image of the torus $\lambda = \phi_{|\mathcal{T}}$. Then the Levi subgroup $M = C_G(T)$ is a strongly standard reductive K-subgroup. As in Proposition 26(1), we may find maximal tori [now over K] of $C_M(X)$ centralized by the image of λ; Proposition 26 then gives the first assertion of the Corollary. According to Theorem 16, an optimal homomorphism is uniquely determined by its tangent mapping; the uniqueness assertion follows at once. □

4. Proof of the main theorem

4.1. **A general setting.** Let H be a connected and simple algebraic group. For each strongly standard reductive group G, suppose that one is given a set $\mathcal{C}_G$ of homomorphisms $H \to G$ with the properties to be enumerated below.

Let G be strongly standard and let $f_0 \in \mathcal{C}_G$ be arbitrary; write S_0 for the image of f_0. We assume the following hold for each f_0:

(C1) S_0 is geometrically G-cr.
(C2) $C_G(S_0)$ is a smooth subgroup of G, and $C_G(S_0) = C_G(\mathrm{Lie}(S_0))$.
(C3) $\mathrm{Lie}(S_0) = \operatorname{im} df_0$.

We also suppose:

(C4) Given any homomorphism $f : H \to G$ for which $df \neq 0$ and for which $\operatorname{im} f$ is geometrically G-cr, there is a unique $f_0 \in \mathcal{C}_G$ such that $df = df_0$.
(C5) If $f : H \to G$ is a homomorphism and if $L \subset G$ is a Levi subgroup with $\operatorname{im} f \subset L$, then $f \in \mathcal{C}_G$ if and only if $f \in \mathcal{C}_L$.

The following Lemma gives a useful application of (C1) and (C2).

Lemma 29. *Let G be a reductive group and let $S \subset G$ be a subgroup with the property $C_G(S) = C_G(\mathrm{Lie}(S))$. Suppose that S is geometrically G-cr. If $K \subset k$ is any field extension and $P \subset G_{/k}$ is a k-parabolic subgroup, then $S_{/k} \subset P$ if and only if $\mathrm{Lie}(S)_{/k} \subset \mathrm{Lie}(P)$,*

Proof. Since the Lemma follows once it is proved for algebraically closed extensions k, it suffices to suppose that K itself is algebraically closed and to prove the conclusion of the Lemma for a (K-)parabolic subgroup $P \subset G$. First notice that if $S \subset P$, then clearly $\mathrm{Lie}(S) \subset \mathrm{Lie}(P)$.

Now suppose that $\mathfrak{s} = \mathrm{Lie}(S) \subset \mathrm{Lie}(P)$. Since S is G-cr, Theorem 8 shows that $\mathfrak{s}$ is also G-cr. Thus, we may find a Levi subgroup $L \subset P$ with $\mathfrak{s} \subset \mathrm{Lie}(L)$. Then $L = C_G(T)$ where $T = Z(L)$, and we see that

$$T \subset C_G(\mathfrak{s}) = C_G(S);$$

thus T centralizes S, so that $S \subset C_G(T) = L \subset P$, as required. □

We now observe:

Proposition 30. *Let $p > 2$, let $H = \mathrm{SL}_2$, and for each strongly standard reductive group G, let $\mathcal{C}_G$ be the set of optimal homomorphisms $\mathrm{SL}_2 \to G$. Then conditions (C1) – (C5) of §4.1 hold for the sets $\mathcal{C}_G$.*

Proof. (C1) follows from Theorem 16, (C2) is Proposition 19, (C4) is Corollary 28, and (C5) is Proposition 21(1).

Since $p > 2$, the adjoint representation of SL_2 is irreducible; since any optimal homomorphism $f : \mathrm{SL}_2 \to G$ has $df(E) \neq 0$, the map df must be injective and so (C3) is immediate. □

4.2. **Some results about twisted-product homomorphisms.** Let H be a reductive group and let $\mathcal{C}_G$ be a collection of homomorphisms $H \to G$ for each strongly standard group G which satisfies (C1)–(C5) of §4.1.

We are going to prove several technical results about twisted product homomorphisms; to avoid repetition in the statements, we fix the following notation:

Let $\vec{h} = (h_0, h_1, \ldots, h_r)$ with $h_i \in \mathcal{C}_G$ be commuting homomorphisms [as in the introduction], and let $\vec{n} = (n_0 < n_1 < \cdots < n_r)$ be non-negative integers; the data $(\vec{h}, \vec{n})$ determines a twisted-product homomorphism $\Phi = \Phi_{\vec{h},\vec{n}} : H \to G$ given for each commutative K-algebra Λ and each $g \in H(\Lambda)$ by the rule

$$g \mapsto h_0(F^{n_0} g) \cdot h_1(F^{n_1} g) \cdots h_r(F^{n_r} g). \tag{4.2.1}$$

Several of the results proved in this section hold only assuming a subset of the conditions (C1)–(C5); for simplicity of exposition, we assume all five conditions hold – we don't bother to identify the subset.

Lemma 31. *Let G be strongly standard, let $\vec{h}$, $\vec{n}$, and $\Phi = \Phi_{\vec{h},\vec{n}}$ be as in the beginning of §4.2. Then $d\Phi = 0$ if and only if $n_0 > 0$. If $d\Phi = 0$ let Ψ be the twisted-product homomorphism determined by $(\vec{h}, (0, n_1 - n_0, \ldots, n_r - n_0))$. Then $\Phi = \Psi \circ F^{n_0}$ and $d\Psi \neq 0$. Moreover, $\mathrm{im}\, \Phi = \mathrm{im}\, \Psi$.*

Proof. Straightforward and left to the reader. □

Proposition 32. *Let G be strongly standard, let (f_1, f_2) be commuting homomorphisms $H \to G$ with $f_1 \in \mathcal{C}_G$. Let $(n_1 < n_2)$ be non-negative integers, and let f be the twisted-product homomorphism determined by (f_1, f_2) and (n_1, n_2). Write S_i for the image of f_i, $i = 1, 2$, and write S for the image of f. Then:*

1. *for each field extension $K \subset k$ and each parabolic subgroup $P \subset G_{/k}$, we have $S_{/k} \subset P$ if and only if $S_{1/k} \subset P$ and $S_{2/k} \subset P$.*
2. *$S_1 \cdot S_2$ is geometrically G-cr if and only if S is geometrically G-cr.*

Proof. Note that (1) will follow once it is proved for algebraically closed extension fields k of K. Thus, we suppose that $k = K$ is algebraically closed, and prove the conclusion of (1) for parabolic subgroups $P \subset G$.

If the parabolic subgroup $P \subset G$ contains S_1 and S_2, it is clear by the definition of a twisted-product homomorphism that P contains S. Suppose now that P contains S; we show it contains also S_1 and S_2.

Applying Lemma 31, one knows that if $g : H \to G$ is the twisted-product homomorphism determined by (f_1, f_2) and $(0, n_2 - n_1)$, then $\operatorname{im} g = S$ as well. We may thus suppose that $n_1 = 0$, so that $df \neq 0$.

It is clear that $df = df_1$. Since $\operatorname{im} df_1 = \operatorname{Lie}(S_1)$, it follows that $\operatorname{Lie}(S) = \operatorname{Lie}(S_1)$. Since $S \subset P$, we have $\operatorname{Lie}(S_1) = \operatorname{Lie}(S) \subset \operatorname{Lie}(P)$; since (C1) and (C2) hold, we may apply Lemma 29 and conclude that $S_1 \subset P$. Since f_2 is given by the rule

$$g \mapsto f_1(g)^{-1} f(g)$$

it is then clear that $S_2 \subset P$ as well. This proves (1).

Since (2) is a geometric statement, we may again suppose that K is an algebraically closed field. Write X for the building of G; cf. [Ser 05, §2 and §3.1]. Then X is a simplicial complex whose simplices are in bijection with the parabolic subgroups of G. We have shown the equality of fixed-point sets: $X^S = (X^{S_2})^{S_1} = X^{S_1 \cdot S_2}$.

According to [Ser 05, Théorème 2.1], the group S is G-cr if and only if $X^S = X^{S_1 S_2}$ is contractible if and only if $S_1 \cdot S_2$ is G-cr. This proves (2). □

Corollary 33. *Let G be strongly standard, and let $\vec{h}$, $\vec{n}$, $\Phi = \Phi_{\vec{h},\vec{n}}$ be as in the beginning of §4.2. Write S for the image of h and S_i for the image of h_i. If $P \subset G_{/k}$ is a k-parabolic subgroup for an extension field $K \subset k$, then $S_{/k} \subset P$ if and only if $S_{i/k} \subset P$ for $i = 1, \dots, r$.*

Proof. It is enough to give the proof assuming that $K = k$ is algebraically closed. If $S_i \subset P$ for each i, it is clear by construction that $S \subset P$.

Now suppose that $S \subset P$. To prove that each $S_i \subset P$, we proceed by induction on r. If $r = 1$, the result is immediate. Suppose that $r > 1$, and let $\Psi : H \to G$ be the twisted-product homomorphism determined by $(h_2, \dots, h_r)$ and $(n_2 - 1, \dots, n_r - 1)$. Then f may be regarded as the twisted-product homomorphism determined by (h_1, Ψ) and $(0, 1)$. Thus we may apply Proposition 32(1) to see that $S_1 \subset P$ and $\operatorname{im} \Psi \subset P$. Now apply the induction hypothesis to Ψ to learn that $S_i \subset P$ for $2 \leq i \leq r$. This completes the proof. □

Proposition 34. *Let G be strongly standard, and let $\vec{h}$, $\vec{n}$, $\Phi = \Phi_{\vec{h},\vec{n}}$ be as in the beginning of §4.2. If S denotes the image of Φ and S_i the image of h_i, then $C_G(S) \subset C_G(S_i)$ for $1 \leq i \leq r$.*

Proof. If $r = 1$, the result is immediate. Suppose $r > 1$, write Ψ for the homomorphism determined by $(h_2, \dots, h_r)$ and $(n_2, \dots, n_r)$ and write $T = \operatorname{im} \Psi$. It suffices by induction on r to show that $C_G(S) \subset C_G(S_1)$ and $C_G(S) \subset C_G(T)$, since then for $2 \leq i \leq r$ we have

$$C_G(S) \subset C_G(T) \subset C_G(S_i)$$

by the induction hypothesis.

Applying Lemma 31, we may assume that $n_1 = 0$ and $dh \neq 0$ without changing S. Thus $\operatorname{Lie}(S) = \operatorname{Lie}(S_1)$. By (C2), we have

$$C_G(S) \subset C_G(\operatorname{Lie}(S)) = C_G(\operatorname{Lie}(S_1)) = C_G(S_1).$$

Finally, it remains to check that $C_G(S) \subset C_G(T)$. Write $\Psi^*, h_1^*, \Phi^* : K[G] \to K[H]$ for the comorphisms of Ψ, h_1, and Φ. Then by construction, Ψ^* is given by the composition

$$K[G] \xrightarrow{\mu} K[G] \otimes_K K[G] \xrightarrow{\iota \otimes id} K[G] \otimes_K K[G] \xrightarrow{h_1^* \otimes \Phi^*} K[H] \otimes_K K[H] \xrightarrow{\Delta} K[H]$$

where the map μ defines the multiplication in G, the map ι defines the inversion in G, and Δ is given by multiplication in $K[H]$.

Let $g \in C_G(S)(\Lambda)$ for some commutative K-algebra Λ. To show that $g \in C_G(T)(\Lambda)$, it is enough to argue that the inner automorphism $\mathrm{Int}(g)$ of G induces the identity on the subgroup scheme $T_{/\Lambda}$. Since T is defined by the ideal $\ker \Psi^* \lhd K[G]$, it is enough to require that $\Psi^*(\mathrm{Int}(g)^* f) = \Psi^*(f)$ for each $f \in \Lambda[G]$. [Note: we write Ψ^* rather than $\Psi^*_{/\Lambda}$ for simplicity.]

Since $g \in C_G(S)(\Lambda)$ and $g \in C_G(S_1)(\Lambda)$, we know for each $f \in \Lambda[G]$ that

$$h_1^*(\mathrm{Int}(g)^* f) = h_1^*(f) \quad \text{and} \quad \Phi^*(\mathrm{Int}(g)^* f) = \Phi^*(f).$$

If $f_1 \otimes f_2 \in \Lambda[G] \otimes_\Lambda \Lambda[G]$, then

$$\begin{aligned}(h_1^* \otimes \Phi^*)((\mathrm{Int}(g)^* \otimes \mathrm{Int}(g)^*)(f_1 \otimes f_2)) &= h_1^*(\mathrm{Int}(g)^* f_1) \otimes \Phi^*(\mathrm{Int}(g)^* f_2) \\ &= h_1^*(f_1) \otimes \Phi^*(f_2) \\ &= (h_1^* \otimes \Phi^*)(f_1 \otimes f_2).\end{aligned}$$

It follows for any $f_1 \in \Lambda[G] \otimes_\Lambda \Lambda[G]$ that

$$(h_1^* \otimes \Phi^*)((\mathrm{Int}(g)^* \otimes \mathrm{Int}(g)^*) f_1) = (h_1^* \otimes \Phi^*) f_1 \tag{4.2.2}$$

Since $\mathrm{Int}(g)$ is an automorphism of G, we have for $f \in \Lambda[G]$ that

$$((\iota \otimes id) \circ \mu)(\mathrm{Int}(g)^* f) = (\mathrm{Int}(g)^* \otimes \mathrm{Int}(g)^*)((\iota \otimes id) \circ \mu)(f). \tag{4.2.3}$$

Combining (4.2.2) and (4.2.3), we see that $\Psi^*(\mathrm{Int}(g)^* f) = \Psi^*(f)$ for each $f \in \Lambda[G]$, as required. This completes the proof. $\square$

Remark 35. With notation as before, one can even show that

$$C_G(S) = \bigcap_{i=1}^{r} C_G(S_i).$$

The inclusion $C_G(S) \subset \bigcap_i C_G(S_i)$ follows from the previous Proposition, and the reverse inclusion may be proved by showing for each commutative K-algebra Λ that if $g \in C_G(S_i)(\Lambda)$ for each i, then $\Phi^*(\mathrm{Int}(g)^* f) = \Phi^*(f)$ for each $f \in \Lambda[G]$; the proof is like that used for the Proposition.

4.3. **Finding the twisted factors of a homomorphism with G-cr image.** Let H be a reductive group and let $\mathcal{C}_G$ be a collection of homomorphisms $H \to G$ for each strongly standard group G which satisfies (C1)–(C5) of §4.1. In this section, we are going to give the proof of Theorem 1.

We first have the following:

Proposition 36. *Fix a strongly standard reductive group G, and let the homomorphism $f : H \to G$ have geometrically G-cr image S. Assume that $df \neq 0$ and let $f_0 \in \mathcal{C}_G$ be the unique map – as in (C4) – such that $df = df_0$. Then:*

1. *the map $f_1 : H \to G$ given by the rule $g \mapsto f_0(g^{-1}) \cdot f(g)$ is a group homomorphism.*
2. *$S_1 = \mathrm{im}\, f_1 \subset C_G(\mathrm{im}\, f_0)$.*
3. *$df_1 = 0$.*
4. *S_1 is geometrically G-cr.*

Proof. Write $S_0 = \mathrm{im}\, f_0$, and write $f_1 : H \to G$ for the morphism defined by the rule in (1). Let Λ be an arbitrary commutative K-algebra, let $g \in H(\Lambda)$, and let $X \in \mathrm{Lie}(H)(\Lambda)$. Since $df = df_0$, we know that

$$\mathrm{Ad}(f(g)) df_0(X) = \mathrm{Ad}(f(g)) df(X) = df(\mathrm{Ad}(g)X) = df_0(\mathrm{Ad}(g)X) = \mathrm{Ad}(f_0(g)) df_0(X).$$

It follows that $\mathrm{Ad}(f_1(g)) = \mathrm{Ad}(f_0(g^{-1})f(g))$ centralizes $df_0(X)$ for each $X \in \mathrm{Lie}(H)(\Lambda)$. Since $\mathrm{im}\, df_0 = \mathrm{Lie}(S_0)$ by (C3), it follows that the image of f_1 lies in $C_G(\mathrm{Lie}(S_0))$. If now $g, h \in H(\Lambda)$, then we see that

$$f_1(gh) = f_0(h^{-1})f_0(g^{-1})f(g)f(h) = f_0(h^{-1})f_1(g)f(h) = f_1(g)f_0(h^{-1})f(h) = f_1(g)f_1(h).$$

Thus f_1 is a homomorphism, so that (1) and (2) are proved. By construction, the tangent map of f_1 is $df - df_0 = 0$; this proves (3).

Note that (2) implies that $S_0 \cdot S_1$ is a subgroup. Since S is geometrically G-cr, we may apply Proposition 32 to see that $S_0 \cdot S_1$ is geometrically G-cr. Since $S_1 \lhd S_0 \cdot S_1$ is a normal subgroup, it follows from the result of B. Martin (Theorem 9) that S_1 is G-cr; this proves (4). □

Corollary 37. *Let H be quasisimple and suppose that the homomorphism $f : H \to G$ has geometrically G-cr image. Then there are uniquely determined commuting $\mathcal{C}_G$-homomorphisms $h_0, h_1, \dots, h_r$ and uniquely determined non-negative integers $n_0 < n_1 \cdots < n_r$ such that f is the twisted-product homomorphism determined by $(\vec{h}, \vec{n})$.*

Proof. We may use Proposition 22 to find a homomorphism $h : H \to G$ and an integer $t \geq 0$ such that $f = h \circ F^t$ where F is the Frobenius endomorphism of H. Moreover, $dh \neq 0$. If the conclusion of the Theorem holds for h, we claim that it holds for f as well. Indeed, if h is the twisted-product homomorphism determined by the commuting $\mathcal{C}_G$-homomorphisms $\vec{h} = (h_0, h_1, \dots, h_r)$ and the non-negative integers $\vec{n} = (0 = n_0 < \cdots < n_r)$, then f is the commuting-product homomorphism determined by $\vec{h}$ and the non-negative integers $\vec{m} = (t < n_1 + t < \cdots < n_r + t)$. If f had a second representation as a commuting-product homomorphism determined by $\vec{h}' = (h_0', \dots, h_t')$ and $\vec{m}' = (m_0' < \cdots < m_t')$ then using Lemma 31 one deduces that $m_0' = t$ and one finds a representation of h as the twisted product homomorphism determined by $\vec{h}'$ and $(0 < m_1' - t < \cdots < m_t')$. Thus $\vec{h} = \vec{h}'$ and $\vec{m} = \vec{m}'$; this proves the claim. So we may and will suppose that $df \neq 0$.

Let us first prove the uniqueness assertion; namely, suppose that $\vec{h} = (h_0, h_1, \dots, h_t)$ and $\vec{h}' = (h_0', h_1', \dots, h_s)$ are commuting homomorphisms with $h_i, h_j' \in \mathcal{C}_G$, and suppose that $\vec{n} = (n_0 < \cdots < n_t)$ and $\vec{n}' = (n_0' < \cdots < n_s')$ are non-negative integers with $0 = n_0 = n_0'$, and suppose that $f = \Phi_{\vec{h},\vec{n}} = \Phi_{\vec{h}',\vec{n}'}$. We must argue that $s = t$, $\vec{h} = \vec{h}'$ and $\vec{n} = \vec{n}'$. We know that $df = dh_0 = dh_0'$. Since $h_0 \in \mathcal{C}_G$ is the unique mapping with $df = dh_0$ by (C4), we have $h_0 = h_0'$. It then follows that

$$\Phi_{(h_1,\dots,h_t),(n_1<\cdots<n_t)} = \Phi_{(h_1',\dots,h_t'),(n_1'<\cdots<n_t')}$$

so by induction on $\min(s,t)$, we find that $s = t$, $h_i = h_i'$ and $n_i = n_i'$ for $1 \leq i \leq t$; this completes the proof of uniqueness.

For the existence, we choose by (C4) the unique map $f_0 \in \mathcal{C}_G$ such that $df = df_0$. We now write $f_1 : H \to G$ for the homomorphism of Proposition 36(1). Thus f is given by the rule

$$g \mapsto f_0(g) \cdot f_1(g) \tag{4.3.1}$$

Write S for the image of f, and write S_0 and S_1 for the respective images of f_0 and f_1.

We proceed by induction on the semisimple rank r of G. If r is smaller than the rank of the simple group H, there are no homomorphisms $H \to G$. If the semisimple rank of G is the same as the rank of H, then apply Lemma 38 to $S_0 \subset G'$, where G' is the derived group of G. One deduces that $C_{G'}(S_0)$ has no non-trivial torus, hence that any torus in $C_G(S_0)$ is central in G. Since SL_2 is its own derived group, $\mathrm{im}\, f_1$ lies in G'; thus $\mathrm{im}\, f_1$ is contained in $C_{G'}(S_0)$

by Proposition 36(2) and it follows that the map f_1 is trivial. We conclude in this case that $f = f_0 \in \mathcal{C}_G$.

We now suppose that the semisimple rank of G is strictly greater than the rank of H. Since $S_1 \subset C_G(S_0)$, a maximal torus of S_0 centralizes S_1. Thus, the image of the G-cr homomorphism f_1 lies in some proper Levi subgroup L. Since the semisimple rank of L is smaller than that of G, we may apply induction; we find commuting homomorphisms $h_1, \dots, h_r \in \mathcal{C}_L$, and non-negative integers $n_1 < n_2 < \cdots < n_r$ such that f_1 is the twisted-product map determined by $(\vec{h}, \vec{n})$. Since $df_1 = 0$, we have $0 < n_1$. It follows from (C5) that $h_1, \dots, h_r \in \mathcal{C}_G$.

Since $\operatorname{im} f_0 = S_0 \subset C_G(S_1)$, it follows from Proposition 34 applied to f_1 that $S_0 \subset C_G(\operatorname{im} h_i)$ for $1 \le i \le r$. Thus the homomorphisms $(f_0, h_1, \dots, h_r)$ are commuting. In view of (4.3.1), f is the twisted-product homomorphism determined by $(f_0, h_1, \dots, h_r)$ and $(0 < n_1 < \cdots < n_r)$. □

Lemma 38. *Let X and Y be semisimple groups of the same rank, and suppose that $X \subset Y$. Then $C_Y(X)$ contains no non-trivial torus.*

Proof. Let $S \subset Y$ be any torus centralizing X, and let T be a maximal torus of X. Since T is centralized by S and is also maximal in Y, we have $S \subset T$ so that $S \subset X$. Thus S is a central torus in X. Since X is semisimple, S is trivial as required. □

We can now prove the following; note that Theorem 1 is a special case.

Theorem 39. *Let G be a strongly standard reductive group, and let $\Phi : \mathrm{SL}_2 \to G$ be a homomorphism. If the image of Φ is geometrically G-cr, then there are commuting optimal homomorphisms $\vec{\phi} = (\phi_0, \dots, \phi_r)$ and non-negative integers $\vec{n} = (n_0 < n_1 < \cdots < n_d)$ such that Φ is the twisted-product homomorphism determined by $(\vec{\phi}, \vec{n})$. Moreover, $\vec{\phi}$ and $\vec{n}$ are uniquely determined by Φ.*

Proof. For a strongly standard reductive group G, write $\mathcal{C}_G$ for the set of optimal homomorphisms $\mathrm{SL}_2 \to G$. Suppose first that $p > 2$. Then Theorem 1 is a consequence of Proposition 30 together with Corollary 37.

Now suppose that $p = 2$. Use (2.4.1) to find a central isogeny $\pi : G_{\mathrm{sc}} \to G$ where the derived group of G_{sc} is simply connected. Since SL_2 is simply connected, there is a homomorphism $\widetilde{\Phi} : \mathrm{SL}_2 \to G_{\mathrm{sc}}$ with $\Phi = \pi \circ \widetilde{\Phi}$. It follows from Lemma 10 that $\widetilde{\Phi}$ has geometrically G-cr image. Proposition 21(2) shows that a homomorphism $f : \mathrm{SL}_2 \to G_{\mathrm{sc}}$ is optimal if and only if $\pi \circ f : \mathrm{SL}_2 \to G$ is optimal.

If $\widetilde{\Phi}$ is the twisted product homomorphism determined by the optimal homomorphisms $\vec{\phi} = (\phi_0, \dots, \phi_r)$ and the non-negative integers $\vec{n} = (n_0 < n_1 < \cdots < n_r)$, it is then clear that Φ is the twisted product homomorphism determined by the optimal homomorphisms $\vec{\phi}' = (\pi \circ \phi_0, \dots, \pi \circ \phi_r)$ and $\vec{n}$. Moreover, the uniqueness of $\vec{\phi}$ implies the uniqueness of $\vec{\phi}'$; thus it suffices to prove the theorem after replacing G by G_{sc}. So we now assume that the derived group of G is simply connected.

Assume first that K is separably closed. Recall that since $p = 2$ is good for the derived group of G, each of its simple factors has type A_m for some m. Since G is split and simply connected, we find that $G \simeq T \times \prod_{i=1}^{t} G_i$ where T is a central torus, and $G_i \simeq \mathrm{SL}(V_i)$ for a vector space V_i. Write $\pi_i : G \to G_i$ for the i-th projection, and $\Phi_i = \pi_i \circ \Phi$. Steinberg's tensor product theorem [Jan 87, Cor. II.3.17] shows that Φ_i may be written as a twisted-product homomorphism for a unique collection of commuting optimal homomorphisms and a unique increasing list of non-negative integers; see [LS 03, Lemma 4.1]. The same then clearly holds for Φ, so the Theorem is proved in this case.

For general K, the above argument represents the base-changed morphism $\Phi_{/K^{\mathrm{sep}}}$ as the twisted product homomorphism $\Phi_{\vec{\phi},\vec{n}}$ for unique commuting optimal K^{sep}-homomorphisms $\vec{\phi} = (\phi_0, \dots, \phi_t)$ and unique $\vec{n} = (n_0 < n_1 < \cdots < n_t)$. Since $\vec{\phi}$ and $\vec{n}$ are unique, we may apply Galois descent to see that each ϕ_i arises by base change from an optimal K-homomorphism, and the proof is complete. □

5. Proof of a partial converse to the main theorem

In this section, we will prove Theorem 2, which is a geometric statement – it depends only on G and H over an algebraically closed field. Thus we will suppose in this section that K is algebraically closed, and we write "G-cr" rather than "geometrically G-cr".

We begin with a result on G-cr subgroups.

Proposition 40. *Let G be reductive, let $h(G)$ be the maximum Coxeter number of a simple quotient of G, and suppose that $p > 2h(G)-2$. Let $A, B \subset G$ be smooth, connected, and G-cr, and suppose that $B \subset C_G(A)$. Then $A \cdot B$ is G-cr.*

Proof. Under our assumptions on p, it follows from [Ser 05, Corollaire 5.5] that a subgroup $\Gamma \subset G$ is G-cr if and only if the representation of Γ on $\mathrm{Lie}(G)$ is semisimple.

Since a smooth, connected G-cr subgroup is reductive [Ser 05, Prop. 4.1], the proposition is now a consequence of the lemma which follows. □

Lemma 41. *Let $G_1, G_2 \subset \mathrm{GL}(V)$ be connected and reductive, and suppose $G_2 \subset C_{\mathrm{GL}(V)}(G_1)$. Then V is semisimple for $G_1 \cdot G_2$.*

Proof. Write $H = G_1 \cdot G_2$. Since H is a quotient of the reductive group $G_1 \times G_2$ by a central subgroup, H is reductive.

Since G_1 and G_2 commute, G_2 leaves stable the G_1-isotypic components of V. Thus we may write V as a direct sum of H-submodules which are isotypic for both G_1 and G_2. Thus we may as well assume that V itself is isotypic for G_1 and for G_2.

Let $B_i \subset G_i$ be Borel subgroups and let $T_i \subset B_i$ be maximal tori for $i = 1, 2$. Note that the choice of a Borel subgroup determines a system of positive roots in each $X^*(T_i)$; the weights of T_i on $U_i = R_u(B_i)$ are *positive.* Our hypothesis means that there are dominant weights $\lambda_i \in X^*(T_i)$ such that each simple G_i-submodule of V is isomorphic to $L_{G_i}(\lambda_i)$, the simple G_i-module with highest weight λ_i.

Now, $B = B_1 \cdot B_2$ is a Borel subgroup of H, and $T = T_1 \cdot T_2$ is a maximal torus of B. Since $T_1 \cap T_2$ lies in the center of H, one knows that there is a unique character $\lambda \in X^*(T)$ such that $\lambda_{|T_i} = \lambda_i$ for $i = 1, 2$. Moreover, it is clear that λ is dominant. Put $U = U_1 \cdot U_2 = R_u(B)$.

It follows from [Jan 87, II.2.12(1)] that there are no non-trivial self-extensions of simple H-modules; thus the Lemma will follow if we show that all simple H-submodules of V are isomorphic to $L_H(\lambda)$.

Let $L \subset V$ be a simple H-submodule; we claim that $L \simeq L_H(\lambda)$. Since L is simple, the fixed point space of U on L satisfies $\dim_K L^U = 1$ and our claim will follow once we show that $L^U \subset L_{T;\lambda}$ since then $L^U = L_{T;\lambda}$ and $L \simeq L_H(\lambda)$; for all this, see [Jan 87, Prop. II.2.4] [we are writing $L_{T;\lambda}$ for the λ weight space of the torus T on L]. Since L is semisimple and G_1-isotypic, $L^{U_1} = L_{T_1;\lambda_1}$. Since $G_2 \subset C_{\mathrm{GL}(V)}(G_1)$, L^{U_1} is a G_2-submodule. Since L^{U_1} is semisimple and isotypic as G_2-module, we know that $L^U = (L^{U_1})^{U_2} = (L^{U_1})_{T_2;\lambda_2}$. Thus $L^U \subset L_{T_1;\lambda_1} \cap L_{T_2;\lambda_2}$ so indeed $L^U = L_{T;\lambda}$ as required. □

Suppose that H is a simple group, and that for each strongly standard group G, one has a set $\mathcal{C}_G$ of homomorphisms $H \to G$ satisfying (C1)–(C5) of §4.1.

Theorem 42. *Let G be strongly standard and assume that $p > 2h(G) - 2$. Let $h_0, \dots, h_r$ be commuting $\mathcal{C}_G$-homomorphisms, and let $n_0 < n_1 < \cdots < n_r$ be non-negative integers. Then the image of the twisted-product homomorphism h determined by $(\vec{h}, \vec{n})$ is geometrically G-cr.*

Proof. Write S_i for the image of h_i, $0 \le i \le r$. By (C1), S_i is G-cr for $0 \le i \le r$. In view of our assumption on p, it follows from Proposition 40 that the subgroup $A = S_0 \cdot S_1 \cdots S_r$ is G-cr.

Write X for the building of G. If $S = \operatorname{im} h$, Corollary 33 shows that $X^S = X^A$. Since A is G-cr, $X^A = X^S$ is not contractible, so that S is G-cr [Ser 05, Théorème 2.1]. □

Theorem 43. *Let G be a strongly standard reductive group, suppose that $p > 2h(G) - 2$, let $\vec{\phi} = (\phi_0, \dots, \phi_d)$ be commuting optimal homomorphisms $\mathrm{SL}_2 \to G$, and let $\vec{n} = (n_0 < n_1 < \cdots < n_d)$ be non-negative integers. Then the image of the twisted-product homomorphism $\Phi : \mathrm{SL}_2 \to G$ determined by $(\vec{\phi}, \vec{n})$ is geometrically G-cr.*

Proof. As in the proof of Theorem 1, write $\mathcal{C}_G$ for the set of optimal homomorphisms $\mathrm{SL}_2 \to G$ for a strongly standard group G. Note that the condition $p > 2h(G) - 2$ implies that $p > 2$. Then Theorem 2 is a consequence of Proposition 30 and Theorem 42. □

Of course, Theorem 2 is a special case of the previous result.

Remark 44. Let G be one of the following groups: (i) $\mathrm{GL}(V)$, (ii) the symplectic group $\mathrm{Sp}(V)$, (iii) the orthogonal group $\mathrm{SO}(V)$, or (iv) a group of type G_2. In cases (ii), (iii) assume $p > 2$ while in case (iv) assume that $p > 3$; then p is very good for G. In case (iv), write V for the 7 dimensional irreducible module for G; thus in each case V is the "natural" module for G. Then a closed subgroup $H \subset G$ is G-cr if and only if V is semisimple as an H-module; see [Ser 05, 3.2.2]. Thus, the conclusion of Theorem 2 holds for G (with no further prime restrictions). Indeed, in view of Lemma 41, one finds that the conclusion of Proposition 40 is valid with no further assumption on p by using V rather than the adjoint representation of G. Now argue as in the proof of Theorem 42 when $p > 2$, or just use Steinberg's tensor product theorem when $p = 2$ (since we are supposing $G = \mathrm{GL}(V)$ in that case).

References

[BMR 05] M. Bate, B.M.S. Martin, and G. Röhrle, *A Geometric Approach to Complete Reducibility*, Inv. Math. **161** (2005), 177 –218.

[DG70] M. Demazure and P. Gabriel, *Groupes Algébriques*, Masson/North-Holland, Paris/Amsterdam, 1970.

[SGA3] M. Demazure and A. Grothendieck, *Schémas en Groupes (SGA 3)*, Séminaire de Géometrie Algébrique du Bois Marie, 1965.

[Jan 87] Jens Carsten Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003.

[Ja 04] Jens Carsten Jantzen, *Nilpotent orbits in representation theory*, Lie Theory: Lie Algebras and Representations (J -P. Anker and B. Orsted, eds.), Progress in Mathematics, vol. 228, Birkhäuser, Boston, 2004, pp. 1–211.

[Hu 95] James E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Math. Surveys and Monographs, vol. 43, Amer. Math. Soc., 1995.

[KMRT] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, *The book of involutions*, Amer. Math. Soc. Colloq. Publ., vol. 44, Amer. Math. Soc., 1998.

[LS 03] Martin W. Liebeck and Gary M. Seitz, *Variations on a theme of Steinberg*, J. Algebra **260** (2003), 261–297, Special issue celebrating the 80th birthday of Robert Steinberg.

[Li 02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, Oxford University Press, 2002, Translated from the French by Reinie Erné.

[Mc 03] George J. McNinch, *Sub-principal homomorphisms in positive characteristic*, Math. Zeitschrift **244** (2003), 433–455.

[Mc 04] ———, *Nilpotent orbits over ground fields of good characteristic*, Math. Annalen **329** (2004), 49–85, arXiv:math.RT/0209151.

[Mc 05] ———, *Optimal* SL(2)*-homomorphisms*, Comment. Math. Helv. **80** (2005), 391 –426.

[Mc 05a] ———, *Completely reducible Lie subalgebras*, Transformation Groups (to appear), arXiv math.RT/0509590.

[Sei 00] Gary M. Seitz, *Unipotent elements, tilting modules, and saturation*, Invent. Math. **141** (2000), 467–502.

[Ser 05] Jean-Pierre Serre, *Complète Réductibilité*, Astérisque **299** (2005), Exposés 924-937, pp. 195–217, Séminaire Bourbaki 2003/2004.

[Spr 98] Tonny A. Springer, *Linear algebraic groups*, 2nd ed., Progr. in Math., vol. 9, Birkhäuser, Boston, 1998.

[SS 70] Tonny A. Springer and Robert Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Springer, Berlin, 1970, pp. 167–266, Lecture Notes in Mathematics, Vol. 131. MR 42 #3091

[TW 02] Jacques Tits and Richard M. Weiss, *Moufang polygons*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002.

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