

Irreducible subgroups of algebraic groups

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1 Introduction

Let G be a semisimple algebraic group over an algebraically closed field K of characteristic $p \geq 0$. Following Serre, we define a subgroup Γ of G to be G -irreducible if Γ is closed, and lies in no proper parabolic subgroup of G . When $G = SL(V)$, this definition coincides with the usual notion of irreducibility on V . The definition follows the philosophy, developed over the years by Serre, Tits and others, of generalizing standard notions of representation theory (morphisms $\Gamma \rightarrow SL(V)$) to situations where the target group is an arbitrary semisimple algebraic group. For an exposition, see for example Part II of the lecture notes [8].

In this paper we study the collection of connected G -irreducible subgroups of semisimple algebraic groups G . Our first theorem is a finiteness result, showing that connected G -irreducible subgroups are “nearly maximal”.

Theorem 1 *Let G be a connected semisimple algebraic group, and let A be a connected G -irreducible subgroup of G . Then A is contained in only finitely many subgroups of G .*

Since connected G -irreducible subgroups are necessarily semisimple (see Lemma 2.1), the smallest possibility for such a subgroup is A_1 . The next result shows that G -irreducible A_1 's usually exist. In large characteristic this is hardly surprising, as maximal A_1 's usually exist; but in low characteristic maximal A_1 's do not exist (see [4]), and the result provides a supply of “nearly maximal” A_1 's.

Theorem 2 *Let G be a simple algebraic group over K . If $G = A_n$, assume that $p > n$ or $p = 0$. Then G has a G -irreducible subgroup of type A_1 .*

In the excluded case $G = A_n$, $0 < p \leq n$, it is easy to see that an irreducible subgroup A_1 exists if and only if all prime factors of $n + 1$ are at most p .

In a subsequent paper [6] we shall use the G -irreducible A_1 's constructed in the proof of Theorem 2 to exhibit examples of *epimorphic* subgroups of minimal

dimension in simple algebraic groups, as defined in [2]. (A closed subgroup H of the connected algebraic group G is said to be epimorphic if any morphism of G into an algebraic group is determined by its restriction to H . Theorem 1 of [2] has a number of equivalent formulations of this definition: for example, H is epimorphic if and only if, whenever V is a rational G -module and $V \downarrow H = X \oplus Y$, then X, Y are G -invariant.)

Our final theorem concerns the description of conjugacy classes of connected G -irreducible subgroups of semisimple algebraic groups G . When G is simple, it has only finitely many classes of maximal connected subgroups (see [4, Corollary 3]). This is in general not the case for connected G -irreducible subgroups (see for example Corollary 4.5 below). However, Theorem 3 below shows that there is a finite collection of conjugacy classes of closed connected subgroups such that every G -irreducible subgroup is embedded in a specified way in a member of one of these classes. For the precise statement we require the following definition.

Definition Let X, Y be connected linear algebraic groups over K .

(i) Suppose X is simple. We say X is a *twisted diagonal* subgroup of Y if $Y = Y_1 \dots Y_t$, a commuting product of simple groups Y_i of the same type as X , and each projection $X \rightarrow Y_i/Z(Y_i)$ is nontrivial and involves a different Frobenius twist.

(ii) More generally, if X is semisimple, say $X = X_1 \dots X_r$ with each X_i simple, we say X is a *twisted diagonal* subgroup of Y if $Y = Z_1 \dots Z_r$, a commuting product of semisimple subgroups Z_i , and, writing $\bar{X} = X/Z(X) = \bar{X}_1 \dots \bar{X}_r$ and $\bar{Y} = Y/Z(Y) = \bar{Z}_1 \dots \bar{Z}_r$, each \bar{X}_i is a twisted diagonal subgroup of \bar{Z}_i .

Theorem 3 *Let G be a connected semisimple algebraic group of rank l . Then there is a finite set \mathcal{C} of conjugacy classes of connected semisimple subgroups of G , of size depending only on l , with the following property. If X is any connected G -irreducible subgroup of G , then there is a subgroup $Y \in \bigcup \mathcal{C}$ such that X is a twisted diagonal subgroup of Y .*

The above results concern G -irreducible connected subgroups. Examples of non-connected G -irreducible subgroups X such that X^0 is not G -irreducible are easy to come by: for instance, $X = N_G(T)$, the normalizer of a maximal torus T is such an example, and there are many others for which $C_G(X^0)$ contains a nontrivial torus. However we have not found any examples for which $C_G(X^0)$ contains no nontrivial torus. It may be the case that if X is a non-connected G -irreducible subgroup such that X^0 is not G -irreducible, then $C_G(X^0)$ necessarily contains a nontrivial torus; this is easily seen to be true when $G = A_n$.

Notation For G a simple algebraic group over K and λ a dominant weight, we denote by $V_G(\lambda)$ (or just λ) the rational irreducible KG -module of high weight λ . When $p > 0$, the irreducible module λ twisted by a p^r -power field morphism of G is denoted by $\lambda^{(p^r)}$. Finally, if V_1, \dots, V_k are X -modules then $V_1/\dots/V_k$ denotes a G -module having the same composition factors as $V_1 \oplus \dots \oplus V_k$.

2 Preliminaries

As above, let G be a semisimple connected algebraic group over the algebraically closed field K of characteristic p . We begin with two elementary results concerning G -irreducible subgroups.

Lemma 2.1 *If X is a connected G -irreducible subgroup of G , then X is semisimple, and $C_G(X)$ is finite.*

Proof Suppose $C = C_G(X)^0 \neq 1$. If C contains a nontrivial torus T , then $X \leq C_G(T)$, which lies in a parabolic; otherwise, C is unipotent, so $X \leq N_G(C)$ which lies in a parabolic by [3]. In either case we have a contradiction, and so $C_G(X)^0 = 1$, giving the result. ■

Lemma 2.2 *Suppose G is classical, with natural module $V = V_G(\lambda_1)$. Let X be a semisimple connected closed subgroup of G . If X is G -irreducible then one of the following holds:*

- (i) $G = A_n$ and X is irreducible on V ;
- (ii) $G = B_n, C_n$ or D_n and $V \downarrow X = V_1 \perp \dots \perp V_k$ with the V_i all non-degenerate, irreducible and inequivalent as X -modules;
- (iii) $G = D_n$, $p = 2$, X fixes a nonsingular vector $v \in V$, and X is a G_v -irreducible subgroup of $G_v = B_{n-1}$.

Proof Part (i) is clear, so assume $G = Sp(V)$ or $SO(V)$. Let W be a minimal nonzero X -invariant subspace of V . Then W is either non-degenerate or totally isotropic. In the first case induction gives a non-degenerate decomposition as in (ii); note that no two of the V_i are equivalent as X -modules, since otherwise, if say $V_1 \downarrow X \cong V_2 \downarrow X$ via an isometry $\phi : V_1 \rightarrow V_2$, then X fixes the diagonal totally singular subspace $\{v + i\phi(v) : v \in V_1\}$ of $V_1 + V_2$ (where $i^2 = -1$), hence lies in a parabolic. Finally, if W is totally isotropic it can have no nonzero singular vectors (as X does not lie in a parabolic), so we must have $G = SO(V)$ with $p = 2$ and $W = \langle v \rangle$ nonsingular, yielding (iii). ■

The next result is fairly elementary for classical groups G , but rests on the full weight of the memoirs [7, 4] for exceptional groups.

Proposition 2.3 ([4, Corollary 3]) *If G is a simple algebraic group then G has only finitely many conjugacy classes of maximal closed subgroups of positive dimension. The number of conjugacy classes is bounded in terms of the rank of G .*

We shall also require a description of the maximal closed connected subgroups of semisimple algebraic groups. Let G be a semisimple algebraic group, and write $G = G_1 \cdots G_r$, a commuting product of simple factors G_i . Define $\mathcal{M}(G)$ to be the following set of connected subgroups of G :

- (1) for $j \in \{1, \dots, r\}$, subgroups $(\prod_{i \neq j} G_i) \cdot M_j$, with M_j a maximal connected proper subgroup of G_j , and
(2) for $r \geq 2$ and distinct $j, k \in \{1, \dots, r\}$ such that there is a surjective morphism $\phi : G_j \rightarrow G_k$, subgroups of the form

$$G_{j,k}(\phi) = (\prod_{i \neq j,k} G_i) \cdot D_{j,k},$$

where $D_{j,k} = \{(g, \phi(g)) : g \in G_j\}$, a closed connected diagonal subgroup of $G_j G_k$.

Lemma 2.4 *The collection $\mathcal{M}(G)$ comprises all the maximal closed connected subgroups of the semisimple group G .*

Proof It is clear that the members of $\mathcal{M}(G)$ are maximal closed connected subgroups of G . Conversely, suppose that M is a maximal closed connected subgroup of G . Factoring out $Z(G)$, we may assume that $Z(G) = 1$. Let π_i be the projection map $M \rightarrow G_i$. If some π_i is not surjective, then M lies in $(\prod_{j \neq i} G_j) \cdot \pi_i(M)$, which is contained in a member of $\mathcal{M}(G)$ under (1) of the definition above. Otherwise, all π_i are surjective and we easily see that M lies in a member of $\mathcal{M}(G)$ under (2) above. \blacksquare

By Proposition 2.3, there are only finitely many G -classes of subgroups in $\mathcal{M}(G)$ under (1) in the definition above. If the collection of subgroups under (2) is non-empty, then it consists of finitely many G -classes if $p = 0$, and infinitely many classes if $p > 0$, since in this case we can adjust the morphism ϕ by an arbitrary field twist.

Write $\mathcal{M}_1(G)$ for the collection of subgroups of G under (1), so that $\mathcal{M}_1(G)$ consists of finitely many G -classes of subgroups.

If H is a proper connected G -irreducible subgroup of G , then there is a sequence of subgroups

$$H = H_0 < H_1 < \dots < H_s = G$$

such that for each i , H_i is semisimple and $H_i \in \mathcal{M}(H_{i+1})$. Write $\mathcal{M}_0(G)$ for the collection of G -irreducible subgroups H for which there is such a sequence with $H_i \in \mathcal{M}_1(H_{i+1})$ for all i . By Proposition 2.3 again, there are only finitely many G -classes of subgroups in $\mathcal{M}_0(G)$.

3 Proof of Theorem 1

Let G be a connected semisimple algebraic group, and let A be a connected G -irreducible subgroup of G . We prove that A is contained in only finitely many subgroups of G .

The proof proceeds by induction on $\dim G$. The base case $\dim G = 3$ is obvious. Clearly we may assume without loss that $Z(G) = 1$. Write $G = G_1 \cdots G_r$, a direct product of simple groups G_i , and let $\pi_i : G \rightarrow G_i$ be the i^{th} projection map.

Lemma 3.1 *If H is a subgroup of G containing A , then H is closed and H^0 is semisimple.*

Proof Observe that $A^H = \langle A^h : h \in H \rangle$ is closed and connected, and hence $N_{\bar{H}}(A^H)$ is also closed. This normalizer contains H , hence contains \bar{H} . Thus $A^{\bar{H}} \triangleleft \bar{H}^0$. By Lemma 2.1, \bar{H}^0 is semisimple and $C_G(A)^0 = 1$. It follows that $A^H = \bar{H}^0$. Thus $\bar{H}^0 \leq H \leq \bar{H}$. This means that H is a union of finitely many cosets of \bar{H}^0 , hence is closed, as required. ■

In view of this lemma, it suffices to show that the number of closed connected overgroups of A in G is finite. Suppose this is false, so that A is contained in infinitely many connected subgroups of G . We shall obtain a contradiction in a series of lemmas.

By Lemma 2.1, $C_G(A)$ and $N_G(A)/A$ are finite. Recall the definitions in Section 2 of the collections $\mathcal{M}(G)$ and $\mathcal{M}_1(G)$ of maximal connected subgroups of G .

Lemma 3.2 *There exists $M \in \mathcal{M}(G)$ such that A lies in infinitely many G -conjugates of M .*

Proof First, if $A \leq M \in \mathcal{M}(G)$, then M is semisimple by Lemma 2.1, and by induction A has only finitely many overgroups in M . It follows that A lies in infinitely many members of $\mathcal{M}(G)$.

We next claim that the overgroups of A in $\mathcal{M}(G)$ represent only finitely many G -conjugacy classes of subgroups. For if not, there must exist j, l such that A lies in subgroups $G_{j,l}(\phi)$ for morphisms ϕ involving infinitely many different field twists. Since the high weights of composition factors of $L(G_l) \downarrow A$ are ϕ -twists of those of $L(G_j) \downarrow A$ this implies that the highest weight of A on $L(G)$ is arbitrarily large, a contradiction. This proves the claim, and the lemma follows. ■

From now on, let M be the subgroup provided by Lemma 3.2.

Lemma 3.3 *M contains infinitely many G -conjugates of A , no two of which are M -conjugate.*

Proof By the previous lemma, A lies in infinitely many conjugates of M ; say A lies in distinct conjugates M^g for $g \in C$, where C is an infinite subset of G . Let $g, h \in C$, so $A^{g^{-1}}$ and $A^{h^{-1}}$ lie in M ; if these subgroups are M -conjugate, say $A^{g^{-1}} = A^{h^{-1}m}$ with $m \in M$, then $h^{-1}mg \in N_G(A)$. Letting n_1, \dots, n_t be coset representatives for A in $N_G(A)$, we have $h^{-1}mg = an_i$ for some $a \in A$ and some i . Thus $M^g = M^{han_i}$, so as $a \in M^h$, we have $M^g = M^{hn_i}$.

To summarise: fix $g \in C$; then if $h \in C$ is such that $A^{g^{-1}}$ and $A^{h^{-1}}$ are M -conjugate, we have $M^h = M^{gn_i^{-1}}$ for some i , so there are only finitely many such h . The lemma follows. ■

Lemma 3.4 *We have $M \in \mathcal{M}_1(G)$.*

Proof Suppose not. Then there exist distinct $j, k \in \{1, \dots, r\}$ and a surjective morphism $\phi : G_j \rightarrow G_k$, such that

$$M = G_{j,k}(\phi) = G_0 \cdot D_{j,k},$$

where $G_0 = \prod_{i \neq j,k} G_i$ and $D_{j,k} = \{g \cdot \phi(g) : g \in G_j\}$.

We may take it that $A \leq M$, so that each element of A is of the form $a = a_0 \cdot a_j \cdot \phi(a_j)$, where $a_0 \in G_0, a_j \in G_j$. Since M contains infinitely many G -conjugates of A , no two of them M -conjugate, it follows that M contains infinitely many conjugates of the form A^{g_k} ($g_k \in G_k$). If $a \in A$ is as above, then $a^{g_k} = a_0 \cdot a_j \cdot \phi(a_j)^{g_k}$, so it follows that $\phi(a_j)^{g_k} = \phi(a_j)$ for all $a_j \in \pi_j(A)$. But this means that $g_k \in C_{G_k}(\pi_k(A))$, which is finite, a contradiction. ■

Lemma 3.5 *There exists $M_1 \in \mathcal{M}_1(M)$ such that M_1 contains infinitely many G -conjugates of A , no two of which are M -conjugate.*

Proof By Lemma 3.3, M contains infinitely many G -conjugates of A , no two of which are M -conjugate. Call these conjugates A^{g_λ} ($\lambda \in \Lambda$) where Λ is an infinite index set. For each $\lambda \in \Lambda$, there exists $M_\lambda \in \mathcal{M}(M)$ containing A^{g_λ} . Then infinitely many M_λ are in $\mathcal{M}_1(M)$, since otherwise there exist j, k such that $A^{g_\lambda} \leq M_{j,k}(\phi)$ for morphisms ϕ involving infinitely many different field twists, which is impossible as in the proof of Lemma 3.2.

Since there are only finitely many M -classes of subgroups in $\mathcal{M}_1(M)$, infinitely many of the M_λ lie in a single M -class of subgroups, with representative say M_1 . Then M_1 contains infinitely many G -conjugates $A^{g_\lambda m_\lambda}$ ($m_\lambda \in M$), no two of which are M -conjugate. ■

Recall the definition of $\mathcal{M}_0(G)$ from Section 2. Choose $N \in \mathcal{M}_0(G)$, minimal subject to containing infinitely many G -conjugates of A , no two of which are N -conjugate.

Lemma 3.6 *There are infinitely many distinct G -conjugates of A lying in $\mathcal{M}(N)$, no two of which are N -conjugate.*

Proof Say A^{g_λ} ($\lambda \in \Lambda$) are infinitely many conjugates of A lying in N , no two of them N -conjugate. If the conclusion of the lemma is false, then for infinitely many λ , there is a subgroup $N_\lambda \in \mathcal{M}(N)$ such that $A^{g_\lambda} \leq N_\lambda$. As in the previous proof, infinitely many of these N_λ are in $\mathcal{M}_1(N)$, of which there are only finitely many N -classes, so infinitely many N_λ are N -conjugate to some $N_1 \in \mathcal{M}_1(N)$. But then N_1 contains infinitely many G -conjugates of A (namely $A^{g_\lambda n_\lambda}$ for some $n_\lambda \in N$), no two of which are N -conjugate, contradicting the minimal choice of N . ■

At this point we can obtain a contradiction. Write $N = N_1 \cdots N_k$, a commuting product of simple factors N_i . By Lemma 3.6, there are infinitely many distinct G -conjugates A^{g_λ} lying in $\mathcal{M}(N)$, no two of which are N -conjugate. As $\mathcal{M}_1(N)$ consists of only finitely many N -classes of subgroups, infinitely many of the A^{g_λ} are in $\mathcal{M}(N) \setminus \mathcal{M}_1(N)$. Hence there exist j, l such that infinitely many A^{g_λ} are of the form $N_{j,l}(\phi_\lambda)$, where ϕ_λ is a surjective morphism $N_j \rightarrow N_l$, and no two of these subgroups are N -conjugate. Then the morphisms ϕ_λ must involve

infinitely many different field twists, which is a contradiction as usual, as it implies that the highest weight of A on $L(G)$ (which is of course the highest weight of each conjugate $A^{g\lambda}$) is arbitrarily large.

This completes the proof of Theorem 1.

4 Proof of Theorem 2

Let G be a simple algebraic group over K in characteristic p , as in Theorem 2 (so that if $G = A_n$ then $p > n$ or $p = 0$). We aim to construct a G -irreducible subgroup $A \cong A_1$.

Lemma 4.1 *The conclusion of Theorem 2 holds if $p = 0$.*

Proof Suppose $p = 0$. First consider the case where G is classical. The irreducible representation of A_1 of high weight r embeds A_1 in Sp_{r+1} if r is odd, and in SO_{r+1} if r is even. Hence SL_n , Sp_{2n} and SO_{2n+1} all have irreducible subgroups A_1 . As for the remaining case $G = SO_{2n}$, an A_1 embedded irreducibly in a subgroup SO_{2n-1} is G -irreducible.

When G is of exceptional type, but not E_6 , it has a maximal subgroup A_1 (see [7]), and this is obviously G -irreducible; and for $G = E_6$, a maximal A_1 in a subgroup F_4 is G -irreducible (its connected centralizer in G is trivial, so it cannot lie in any Levi subgroup). ■

In view of Lemma 4.1, we assume from now on that $p > 0$.

Lemma 4.2 *The conclusion of Theorem 2 holds if G is classical.*

Proof Assume G is classical. If $G = A_n = SL_{n+1}$ then $p > n$ by hypothesis, so G has a subgroup A_1 acting irreducibly on the natural $n + 1$ -dimensional G -module (with high weight n); clearly this subgroup does not lie in a parabolic of G .

Next, if $G = C_n = Sp_{2n}$, then G has a subgroup $(Sp_2)^n = (A_1)^n$, and we choose a subgroup $A \cong A_1$ of this via the embedding $1, 1^{(p)}, 1^{(p^2)}, \dots, 1^{(p^{n-1})}$; then A fixes no nonzero totally isotropic subspace of the natural module, hence lies in no parabolic of G . Similarly, if $G = D_{2n} = SO_{4n}$, then G has a subgroup $(SO_4)^n = (A_1)^{2n}$, and we choose $A \cong A_1$ in this via the embedding $1, 1^{(p)}, \dots, 1^{(p^{2n-1})}$.

Now let $G = D_{2n+1} = SO_{4n+2}$. Then G has a subgroup $SO_6 \times (SO_4)^{n-1} \cong A_3 \times (A_1)^{2(n-1)}$, which contains a subgroup $(A_1)^{2n}$ lying in no parabolic of G ; choose $A \cong A_1$ in this $(A_1)^{2n}$ via the embedding $1, 1^{(p)}, \dots, 1^{(p^{2n-1})}$ again.

Finally, for $G = B_{2n} = SO_{4n+1}$, choose $A \cong A_1$ in a subgroup $(SO_4)^n = (A_1)^{2n}$ via the above embedding, while for $G = B_{2n+1} = SO_{4n+3}$ choose A in a subgroup $SO_3 \times (SO_4)^n \cong (A_1)^{2n+1}$. This completes the proof. ■

Assume from now on that G is of exceptional type. We choose our subgroup $A \cong A_1$ as follows. For $G = E_8, E_7, F_4$ or G_2 , there is a maximal rank subgroup $(A_1)^l$ (where $l = 8, 7, 4$ or 2 respectively), and we choose

$$A < (A_1)^l, \text{ via embedding } 1, 1^{(p^2)}, 1^{(p^4)}, \dots, 1^{(p^{2^{l-1}})}.$$

For $G = E_6$ with $p > 2$, there is a maximal rank subgroup $(A_2)^3$, and we choose

$$A < (A_2)^3, \text{ via embedding } 2, 2^{(p^2)}, 2^{(p^4)}.$$

Finally, for $G = E_6$ with $p = 2$, take a subgroup F_4 of G , and a subgroup C_4 of that, generated by short root groups in F_4 ; now take $A < C_4$, embedded via the irreducible symplectic 8-dimensional representation $1 \otimes 1^{(2)} \otimes 1^{(4)}$.

Lemma 4.3 (i) For $G \neq E_6$, $L(G)/L(A_1^l)$ restricts to A as follows:

$G = E_8$: 14 distinct 4-fold tensor factors,

$G = E_7$: 7 distinct 4-fold tensor factors,

$G = F_4$: one 4-fold factor and 6 distinct 2-fold factors,

$G = G_2$: $1 \otimes 3^{(p^2)}$ ($p \neq 2, 3$); $1 \otimes 1^{(9)}/1 \otimes 1^{(27)}$ ($p = 3$); $1 \otimes 1^{(4)} \otimes 1^{(8)}$ ($p = 2$).

Moreover, $L(A_1^l)$ restricts to A as $2/2^{(p^2)}/\dots/2^{(p^{2^{l-1}})}$ if $p \neq 2$, and as $1^{(2)}/1^{(8)}/\dots/1^{(2^{2l-1})}/0^l$ if $p = 2$.

In particular, the nontrivial composition factors of $L(G) \downarrow A$ are all distinct.

(ii) For $G = E_6$ ($p \neq 2$), $L(G)/L(A_2^3)$ restricts to A as $(2 \otimes 2^{(p^2)} \otimes 2^{(p^4)})^2$; and $L(A_2^3)$ restricts to A as $2/2^{(p^2)}/2^{(p^4)}/4/4^{(p^2)}/4^{(p^4)}$ if $p \neq 3$, and as $2/2^{(3^2)}/2^{(3^4)}/1 \otimes 1^{(3)}/1^{(3^2)} \otimes 1^{(3^3)}/1^{(3^4)} \otimes 1^{(3^5)}/0^3$ if $p = 3$.

(iii) For $G = E_6$ ($p = 2$), letting $V_{27} = V_G(\lambda_1)$, we have

$$V_{27} \downarrow A = 1^{(2)} \otimes 1^{(4)}/1^{(2)} \otimes 1^{(8)}/1^{(4)} \otimes 1^{(8)}/1^{(2)}/1^{(2)}/1^{(4)}/1^{(4)}/1^{(8)}/1^{(8)}/0^3.$$

Proof (i) For $G = E_8$, the restriction of $L(G)$ to a subsystem D_4D_4 is given by [5, 2.1]: it is $L(D_4D_4)/\lambda_1 \otimes \lambda_1/\lambda_3 \otimes \lambda_3/\lambda_4 \otimes \lambda_4$. Now consider the restriction further to A_1^8 . This is embedded as $SO_4 \cdot SO_4$ in each D_4 factor, so the factor $\lambda_1 \otimes \lambda_1$ of $L(G) \downarrow D_4D_4$ restricts to A_1^8 as a sum of 4-fold tensor factors, each of dimension 16. The normalizer $N_G(A_1^8)$ acts as the 3-transitive permutation group $AGL_3(2)$ on the 8 factors, and the smallest orbit of this on 4-sets has size 14. It follows that $L(G) \downarrow A_1^8$ has at least 14 distinct 4-fold tensor factors. Since $14 \cdot 16 + \dim A_1^8 = \dim G$, these 14 modules comprise all the composition factors of $L(G)/L(A_1^8)$ restricted to A_1^8 . Part (i) follows for $G = E_8$. The other types are handled similarly.

(ii) The restriction $L(E_6) \downarrow (A_2)^3$ is given by [5, 2.1], and (ii) follows easily.

(iii) We have $V_{27} \downarrow F_4 = V_{F_4}(\lambda_4)/0$, and $V_{F_4}(\lambda_4) \downarrow C_4 = V_{C_4}(\lambda_2)$. Hence $V_{27} \downarrow C_4$ has the same composition factors as the wedge-square of the natural 8-dimensional C_4 -module, minus 1 trivial composition factor. Now calculate the composition factors of the A_1 -module $\wedge^2(1 \otimes 1^{(2)} \otimes 1^{(4)})$ to get the conclusion. ■

Lemma 4.4 The subgroup A is G -irreducible.

Proof First assume $G \neq E_6$. If $A < P = QL$, a parabolic subgroup with unipotent radical Q and Levi subgroup L , then the composition factors of A on $L(Q)$ are the same as those on $L(Q^{opp})$, the Lie algebra of the opposite unipotent radical. By the last sentence of Lemma 4.3(i), it follows that all composition factors of A on $L(Q)$ must be trivial, whence from Lemma 4.3(i) we see that $\dim Q \leq l/2$, which is impossible.

Now assume $G = E_6$ with $p \neq 2$. If $p \neq 3$ then $L(G) \downarrow A$ has no trivial composition factors, so A cannot lie in a parabolic. Now suppose $p = 3$. By Lemma 4.3(ii), $L(G) \downarrow A$ has two isomorphic 27-dimensional composition factors. If $A < QL$ as above, then these factors must occur in $L(Q) + L(Q^{opp})$, and the only other possible composition factors in $L(Q) + L(Q^{opp})$ are trivial. Hence $\dim Q$ must be 27 or 28. There is no such unipotent radical in E_6 .

Finally, assume $G = E_6$ with $p = 2$. Suppose $A < P = QL$, with the parabolic P chosen minimally. By minimality, A must project irreducibly to any A_r factor of L' ; since the irreducible representations of A have dimension a power of 2, it follows that the only possible such factors are A_3 and A_1 . Consequently either $L' = A_3A_1$, or L' lies in a subsystem D_5 . If $L' = A_3A_1$, then A acts on the natural modules for A_3, A_1 as $1 \otimes 1^{(q)}, 1^{(q')}$ respectively, for some powers q, q' of 2. The restriction $V_{27} \downarrow A_3A_1$ is given by [5, 2.3], and it follows that $V_{27} \downarrow A$ has a composition factor $1 \otimes 1^{(q)} \otimes 1^{(q')}$ if $q \neq q'$, and has two composition factors $1 \otimes 1^{(q)}$ if $q = q'$. This conflicts with Lemma 4.3(iii). Therefore $L' \neq A_3A_1$. The remaining possibilities for L' lie in a subsystem D_5 . The irreducible orthogonal A_1 -modules of dimension 10 or less have dimensions 4 and 8, and do not extend the trivial module (see [1, 3.9]). It follows that $L' \leq D_4$. Observe that $V_{27} \downarrow D_4 = \lambda_1/\lambda_3/\lambda_4/0^3$. Hence it is readily checked that no possible embedding of A in D_4 gives composition factors for $V_{27} \downarrow A$ consistent with Lemma 4.3(iii). ■

This completes the proof of Theorem 2.

By varying the field twists involved in the definitions of A above, we obtain the following.

Corollary 4.5 *Let G be a simple algebraic group in characteristic $p > 0$, and assume that $G \neq A_n$. Then G has infinitely many conjugacy classes of G -irreducible subgroups of type A_1 .*

5 Proof of Theorem 3

Let G be a connected semisimple algebraic group of rank l . The proof proceeds by induction on $\dim G$. The base case $\dim G = 3$ is trivial. Let X be a connected G -irreducible subgroup of G . By Lemma 2.1, X is semisimple. Write $G = G_1 \dots G_r$ and $X = X_1 \dots X_s$, commuting products of simple factors G_i and X_i . Without loss we can factor out the finite group $Z(G)$, and hence assume that $Z(G) = 1$.

Suppose first that X projects onto every simple factor G_i of G . Say X_1 projects onto the factors G_1, \dots, G_t . Identifying the direct product $G_1 \dots G_t$ with $G_1 \times \dots \times G_1$ (t factors), and replacing X by a suitable G -conjugate, we

can take

$$X_1 = \{(x^{\tau_1}, \dots, x^{\tau_t}) : x \in G_1\},$$

where each $\tau_i = \gamma_i q_i$ with γ_i a graph automorphism or 1, and q_i a Frobenius morphism or 1. For each k let $S_k = \{i : q_i = q_k\}$, and define a corresponding subgroup $G_{S_k} \leq \prod_{i \in S_k} G_i$ by

$$G_{S_k} = \left\{ \prod_{i \in S_k} x^{\gamma_i} : x \in G_1 \right\}.$$

Then X_1 is a twisted diagonal subgroup of $G_1^+ := \prod_{S_k} G_{S_k}$. Repeating this construction for each simple factor X_i of X , we obtain a subgroup $G_1^+ \dots G_s^+$ of G containing X as a twisted diagonal subgroup. There are only finitely many such subgroups $G_1^+ \dots G_s^+$ in G . Hence if we include the conjugacy classes of these subgroups in our collection \mathcal{C} , we have the conclusion of Theorem 3 in this case.

Now suppose X does not project onto some factor, say G_1 , of G . Then there exists a maximal connected subgroup M_1 of G_1 such that $X \leq M_1 G_2 \dots G_r$. By Proposition 2.3, up to G_1 -conjugacy there are only finitely many possibilities for M_1 . Since $M_1 G_2 \dots G_r$ is a semisimple group of dimension less than $\dim G$, the result now follows by induction.

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