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Subgroups of Type A_1 Containing Semiregular Unipotent Elements

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1. INTRODUCTION

Let G be a simple algebraic group of adjoint type over an algebraically closed field K of characteristic $p > 0$. If p is assumed to be a good prime, then it is shown in [14] that elements of order p are always contained in connected subgroups of G of type A_1. For certain purposes, in particular the study of the subgroup structure of finite groups of Lie type, a variation of this result is important. Namely, it is desirable to show that simple subgroups isomorphic to $PSL_2(q)$ lie in connected subgroups of type A_1.

It can be seen by examples that this sort of result is not always possible, but we show in this article that it can be achieved in some important cases, where the unipotent elements of the finite group are of a particular sort. The results of this paper have been applied in [8] to yield a general theorem on the subgroup structure of finite groups of Lie type.

A unipotent element of G is said to be *semiregular* if its centralizer is a unipotent group. Such elements are fundamental to the analysis of all unipotent elements of G, as all others can be obtained by taking a semiregular unipotent element within the derived group of the centralizer of a semisimple element.

A finite quasisimple group of type A_1 containing a semiregular unipotent element must be simple, which is why we consider simple groups of type $PSL_2(q)$ rather than groups of type $SL_2(q)$.

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We consider two situations. We first show that almost simple subgroups of G isomorphic to $PGL_2(p)$ containing semiregular unipotent elements lie in connected groups of type A_1, provided G is of exceptional type. It is shown, by way of examples, that no general result of this nature holds for classical groups, even if the subgroup contains regular unipotent elements. Next we show that groups of type $PSL_2(q)$, with $q > p$, containing semiregular unipotent elements can always be extended to connected groups of type A_1. This result is established for both classical and exceptional groups.

More precisely, the main results are as follows.

THEOREM 1.1. *Let $PSL_2(p) \cong X < G$, where G is of exceptional type, and assume X contains a semiregular unipotent element of G. If $p \geq 5$ and $N_G(X) \geq PGL_2(p)$, then $N_G(X) \cong PGL_2(p)$ and $N_G(X)$ is contained in a connected group of type A_1.*

THEOREM 1.2. *Let $G > X \cong PSL_2(q)$ with $q > p$ and q a power of p. Assume X contains a semiregular unipotent element of G. Then there exists a connected subgroup $\overline{X} < G$ of such that $X < \tilde{X} \cong PSL_2(K)$. Moreover, except for the case $p = 2$ with G of type $B_2 = C_2$, we have $N_G(X) = N_{\overline{X}}(X) \cong PGL_2(q)$.*

The $p = 2$ and G of type $B_2 = C_2$ case is indeed an exception to the normalizer assertion in Theorem 1.2. It will be shown in the proof of Lemma 2.1 that in this case there are subgroups $X = PSL_2(q) < Sp_4(q)$ having centralizer a (unipotent) root subgroup. This can even happen when the action of X on the natural symplectic module is completely reducible.

There are some strong connections between the above results and the results in [8]. Theorem 1 of [8] makes use of the first theorem of this paper. On the other hand, the second theorem in this paper uses Theorems 2 and 9 from [8], results that are proved independently of Theorem 1 of [8].

2. PRELIMINARIES ON SEMIREGULAR ELEMENTS

In this section we establish some fundamental results regarding semiregular unipotent elements and their embeddings in groups of type A_1. We fix an algebraically closed field K of characteristic p. We will be interested in semiregular unipotent elements of order p.

In order to present uniform notation as in [4] and to apply the results of [14] we must first show that we can make p to be a good prime for G. This is the content of the first lemma. To aid exposition, we defer the proof of the lemma to the end of this section.

LEMMA 2.1. *Suppose $PSL_2(q) = X < G$, where X is simple and contains a semiregular unipotent element of G. Then either p is a good prime or*

$(G, p) = (G_2, 3)$, $(B_2, 2)$, *or* $(C_2, 2)$. *In each of the exceptional cases*, $q > p$ *and Theorem* 1.2 *holds*.

In view of this lemma we assume from now on that p is a good prime for G.

Let L be a simple complex Lie algebra, L_Z a Z-form of L, given by a Chevalley basis, and $L_{Z(p)} = Z(p) \otimes L_Z$, where $Z(p)$ denotes the localization of Z at the prime p. Set $V = K \otimes L_{Z(p)}$. We take $G = G(K)$ to be the corresponding adjoint Chevalley group with root system Σ and base Π.

Let P be a standard parabolic subgroup corresponding to a subset J of Π and let Q be the unipotent radical of P. Recall that P is said to be *distinguished* if $\dim(P/Q) = \dim(Q/Q')$, where Q' denotes the derived group of Q.

The distinguished parabolic subgroups correspond to certain Dynkin diagrams labelled with 0s and 2s, which in turn correspond to conjugacy classes of distinguished unipotent elements. We refer the reader to Section 5 of [4] for a more complete discussion. Under the correspondence, a labelled diagram is sent to a standard parabolic whose Levi factor corresponds to the 0s in the labeled diagram. The corresponding unipotent class is represented by an element of the dense orbit of P on $R_u(P) = Q$. Certain of these unipotent elements are semiregular and all semiregular unipotent elements occur in this way.

In the following lemma we indicate the labelled diagrams corresponding to semiregular classes by indicating those nodes labelled by a 0. For example, in the case of regular unipotent elements this will be the empty set and correspondingly P is a Borel subgroup. Orders of the semiregular unipotent elements are given in (2.3) and (3.3) of [14]. Such elements always exist, but in order for the element to have order p, the prime p must be suitably large. We indicate this prime restriction for each labelled diagram in the following lemma.

LEMMA 2.2. *Assume p is a good prime*. *Labelled diagrams corresponding to semiregular unipotent elements are indicated below together with the prime restriction necessary for the corresponding unipotent element to have order p*. *For each type the nodes indicated are those labelled by a* 0.

A_k, B_k, C_k: ϕ, ($p \geq k + 1, 2k + 1, 2k$, *respectively*).

D_k: ϕ, $\{i, i + 2, i + 4, \ldots, k - 2\}$, $(i > 1)$, ($p \geq 2k - 1, k + i - 2$, *respectively*).

E_6: ϕ, $\{4\}$, ($p \geq 13, 11$, *respectively*).

E_7: ϕ, $\{4\}$, $\{4, 6\}$, ($p \geq 19, 17, 13$, *respectively*).

E_8: ϕ, $\{4\}$, $\{4, 6\}$, ($p \geq 31, 29, 23$, *respectively*).

F_4: ϕ, ($p \geq 13$).

G_2: ϕ, ($p \geq 7$).

As mentioned above, we assume throughout that p is a good prime. Since we will always be working with semiregular elements of prime order, Lemma 2.2 provides a further restriction on p.

We next discuss the exponentiation process developed in [14], connecting semiregular unipotent elements in exceptional groups and corresponding nilpotent elements of $L(G)$. Fix $e = \sum d_i e_{\beta_i}$, a nilpotent element in $L(Q)$, with integral coefficients. The element e is initially taken as an element of $L_{Z(p)}$, but we identify it with the element $1 \otimes e \in V$. Testerman [14] shows that for suitable choices of e it is possible to carry out an exponentiation process, even through we are working over a field of characteristic p.

In these cases, for each $0 \neq c \in K$ we obtain a unipotent element $U(c) = \exp(\mathrm{ad}(ce))$ in G. These elements lie in Q and are constructed by reduction modulo p, starting from a matrix group over $Z(p)[t]$. For exceptional groups, representatives of each class of semiregular elements occur in this way (although for the regular class in E_6 the unipotent element occurs within F_4 and the exponentiation occurs within this subgroup). Let $U = \{U(c): c \in K\}$. Then U is a 1-dimensional unipotent subgroup of the matrix group $G \leq SL(V)$.

Subject to the above prime restrictions, Testerman also shows that a semiregular unipotent element in the exceptional group G is contained in a connected group $\overline{X} = PSL_2(K)$, which can be taken to contain the subgroup U above. Similar results for classical groups are obtained using representation theory. Namely, $SL_2(K)$ has a restricted irreducible representation of degree r for each $r \leq p$. This representation is symplectic or orthogonal, according to whether r is even or odd. In all cases the unipotent elements are regular, having a single Jordan block on the corresponding module. This produces the elements and subgroups for groups of type A_k, B_k, and C_k. The semiregular elements and corresponding overgroups of type A_1 for groups of type D_k occur within subgroups of the form $\overline{X} < B_j B_{k-j-1}$.

In certain cases $\overline{X}$ is properly contained in a proper connected subgroup of G. For the regular case in D_k, $\overline{X} < B_{k-1}$. For the nonregular classes in D_k, $\overline{X} < B_j B_{k-j-1}$, where the connection between j and the integer i above is given by $j = (k + i - 2)/2$. Observe that in this case $j > k - j - 1$. For E_6, we have $\overline{X} < F_4$ or C_4, respectively. Finally, for E_7, the last class satisfies $\overline{X} < A_1 F_4$. These assertions for exceptional groups are established in [6]. In all these cases the corresponding semiregular element is a regular element of the subgroup and we may occasionally refer to these unipotent elements as elements of type $B_j B_{k-j-1}$, F_4, C_4, or $A_1 F_4$.

LEMMA 2.3. *Assume G is an exceptional group, p is a good prime, and $U(1)$ is semiregular, constructed as above Then*

(i) $C_G(U(1)) \leq Q$ *and* $\dim(C_G(U(1))) = \dim(P/Q)$

(ii) $C_G(e)^0 \leq Q$.

Proof. It is shown in [14] that $U(1)$ is in the dense orbit of P on Q. Then 5.2.2 of [4] shows that $C_G(U(1))^0 < P$. Now P is a distinguished parabolic subgroup and $U(1)Q'$ is in the dense orbit of P/Q on Q/Q'. It follows from dimension considerations that $C_G(U(1))Q/Q$ is finite and hence $C_G(U(1))^0 \leq Q$. Since $U(1)$ is semiregular there are no semisimple elements centralizing $U(1)$ and by (3.15) of [12] all unipotent elements of $C_G(U(1))$ are contained in $C_G(U(1))^0$. We conclude that $C_G(U(1)) \leq Q$. The P-conjugacy class of $U(1)$ has dimension equal to $\dim(Q)$, so $\dim C_G(U(1)) = \dim P - \dim Q = \dim(P/Q)$, which establishes (i).

Viewing Q/Q' as a module for the Levi factor of P, there is an isomorphism from Q/Q' to $L(Q)/L(Q')$ which sends $U_\gamma(c)Q'$ to $ce_\gamma + L(Q')$ for each root γ of level 1 and each $c \in K$. Lemma 1.2 of [14] shows that $U(1)Q'$ corresponds to $e + L(Q')$ under the isomorphism. Hence, $e + L(Q')$ is in the open dense orbit of P on $L(Q)/L(Q')$.

The proof of (4.5) of [10] shows that if l is in the open dense orbit of P on $L(Q)$, then all elements in the coset $l + L(Q')$ are Q-conjugate. Since this coset is also in the dense orbit of P on $L(Q)/L(Q')$, we conclude that $\langle e \rangle$ and $\langle l \rangle$ are P-conjugate. Now, Corollary 5.2.4 of [4] implies $C_G(e)^0 = C_P(e)^0$. As in the first paragraph, this group is contained in Q.

In the following lemma we will describe composition factors of $\bar{X}$ on $L(G)$ when G is of exceptional type. Let $\bar{T}$ be a maximal torus of $\bar{X}$ normalizing U. To describe composition factors we use the following notation: $(a^r, b^s, \ldots)$ will denote a module for $\bar{X}$ which has composition factors equal to those of r copies of the Weyl module of high weight a, s copies of the Weyl module of high weight b, etc.

LEMMA 2.4. *Let G be an exceptional group and p a good prime for G.*

(i) *There is a maximal torus of G containing $\bar{T}$ such that the weights of $\bar{T}$ on a suitable base of the root system are given by the labelled Dynkin diagram corresponding to the semiregular class of unipotent elements contained in $\bar{X}$.*

(ii) *P is the corresponding parabolic subgroup, $\bar{T}$ is in the center of a Levi subgroup of P, and $u = U(1)$ is in the dense orbit of P on Q.*

(iii) *With the above notation, the composition factors of* $\bar{X}$ *on* $L(G)$ *are given as follows*:

E_6	ϕ	$(22, 16, 14, 10, 8, 2)$
	$\{4\}$	$(16, 14, 10^2, 8, 6, 4, 2)$
E_7	ϕ	$(34, 26, 22, 18, 14, 10, 2)$
	$\{4\}$	$(26, 22, 18, 16, 14, 10^2, 6, 2)$
	$\{4, 6\}$	$(22, 18, 16, 14^2, 10^2, 8, 6, 2^2)$
E_8	ϕ	$(58, 46, 38, 34, 26, 22, 14, 2)$
	$\{4\}$	$(46, 38, 34, 28, 26, 22, 18, 14, 10, 2)$
	$\{4, 6\}$	$(38, 34, 28, 26, 22^2, 18, 16, 14, 10, 6, 2)$
F_4	ϕ	$(22, 14, 10, 2)$
G_2	ϕ	$(10, 2)$

Proof. (i) This follows from Lemma 4.1 of [6].

(ii) Let P_0 be the parabolic subgroup corresponding to the above weights of $\bar{T}$. Then P_0 is a conjugate of the parabolic subgroup, P, mentioned above. Lemma 2.4 of [10] shows that $U \le Q_0 = R_u(P_0)$. If u is not in the dense orbit of P_0 on Q_0, then its orbit has dimension less than $\dim(Q_0) = \dim(Q)$, so its centralizer has dimension greater than $\dim(P_0/Q_0) = \dim(P/Q)$, contradicting 2.2.

We now have u in the dense orbit of P on Q as well as the dense orbit of P_0 on Q_0. Say $P^x = P_0$. Then u, u^x are both in the dense orbit of P_0 on Q_0. Hence, there is an element $y \in P_0$ such that $u^{xy} = u$. Hence, $xy \in C_G(u) \le Q$ and it follows that $P = P^{xy} = P_0^y = P_0$.

To complete (ii), note that since $\bar{T}$ acts by weight 2 on all roots of level 1, it induces scalars on Q_0/Q_0' and hence is in the center of a Levi subgroup.

(iii) The composition factors of $\bar{X}$ on $L(G)$ are determined by the weights of $\bar{T}$, and these are determined by the labelled diagram. The composition factors are listed on p. 65 and p. 195 of [10].

The next lemma relates the unipotent subgroup U mentioned above with the nilpotent element involved in its construction. Recall that $e = \sum d_i e_{\beta_i}$

LEMMA 2.5. *Assume G is of exceptional type, p is a good prime for G, and U is constructed as above. Then* $L(U) = \langle \mathrm{ad}(e) \rangle$.

Proof. We will use the fact that, except for the case of regular elements of E_6, the elements of U are given by exponentiation as described above.

In the exceptional case, we obtain the result by working with $U < F_4 < E_6$, which gives a similar containment for Lie algebras.

There is a basis $v_1, \ldots, v_n$ of V consisting of elements e_δ for δ a root together with the usual elements h_α corresponding to elements $\alpha \in \Pi$. Arrange the basis so that the ordering is consistent with height, with the highest roots first in the ordering and the elements h_α of height 0. Then U is represented by a group of upper triangular matrices. Fix a basis element e_δ (the argument is essentially the same for h_α). Then for $c \in K$,

$$\mathrm{ad}(ce)(e_\delta) = c \sum d_i N_{\beta_i, \delta} e_{\beta_i + \delta}.$$

This expression is at the level of V and agrees with reduction modulo p of the corresponding action on $L_{Z(p)}$. The coefficients $N_{\beta_i, \delta}$ are the usual structure constants for L_Z. There are similar expressions for

$$\left((\mathrm{ad}(ce))^2/2!\right)(e_\delta),$$

up to the divided $p-1$ power. Higher powers are first considered as matrices with entries in $Z(p)[t]$ and then reduced modulo p, sending t to c. For each of the higher powers there is a corresponding larger power of c which precedes the sum. One adds the expressions to obtain the action of $U(c)$.

Regard U as a subgroup of $SL(V)$, with coordinate ring $K[x]$ (a quotient of the coordinate ring of $SL(V)$). Let $L(U) = \langle \gamma \rangle$, with notation chosen such that $\gamma(x) = 1$. Regard γ as an element of $sl(V)$, in the usual way.

Suppose $e_\delta = v_i$. Consider $\gamma(x_{ij})$, where x_{ij} is a coordinate function. Let ϕ denote the inclusion map of U into $SL(V)$. Then $d\phi(\gamma)(x_{ij}) = \gamma(\phi^*(x_{ij}))$. If $i > j$, then $\phi^*(x_{ij}) = 0$. If the jth basis vector is of the same height as e_δ, then again $\phi^*(x_{ij}) = 0$. If v_j has height one less than v_i then $\phi^*(x_{ij}) = dx$, where d is the (i, j) entry in $\mathrm{ad}(e)$. Here, $d\phi(\gamma)(x_{ij}) = \gamma(\phi^*(x_{ij})) = d$. Finally, if v_j has height at least two more than v_i, then $\phi^*(x_{ij}) = kx^s$, for $k \in K$ and $s \geq 2$. Hence, $d\phi(\gamma)(x_{ij}) = 0$.

It follows from the above remarks that $d\phi(\gamma) = ad(e)$. However, $d\phi(\gamma)$ is just the identification of γ with an element of $sl(V)$. So this establishes the result.

Let $\phi\colon \hat{G} \to G$ be the natural surjection from the simply connected covering group of G. In our situation $L(\hat{G})$ is simple. Indeed, since p is a good prime this is automatic except possibly when G is of type A_n, in which case it follows from the prime restriction given earlier. Hence, $d\phi$ is an isomorphism in all cases. Let $\hat{e}$ be the preimage of e.

The following lemma is presumably well known.

LEMMA 2.6. (i) $d\phi(\mathrm{Ad}(\hat{g})(\hat{e})) = \mathrm{Ad}(\phi(\hat{g}))(d\phi(\hat{e}))$, *for all* $\hat{g} \in \hat{G}$.

(ii) $C_G(e) = \phi(C_{\hat{G}}(\hat{e}))$.

Proof. (i) Fix $f \in K[G]$. Regarding $L(G)$ as left invariant derivations of $K[G]$, we have

$$d\phi(\mathrm{Ad}(\hat{g})(\hat{e}))(f) = (\mathrm{Ad}(\hat{g})(\hat{e}))(f \circ \phi) = \hat{e}(f \circ \phi \circ \mathrm{int}(\hat{g})).$$

On the other hand.

$$\mathrm{Ad}(\phi(\hat{g})(d\phi(\hat{e}))(f) = d\phi(\hat{e}))(f \circ \mathrm{int}(\phi(\hat{g}))) = \hat{e}(f \circ \mathrm{int}(\phi(\hat{g})) \circ \phi).$$

So it suffices to show $\phi \circ \mathrm{int}(\hat{g}) = \mathrm{int}(\phi(\hat{g})) \circ \phi$, which is clear.

(ii) As ϕ is surjective we have $C_G(e) = \{\phi(\hat{g})\colon \mathrm{Ad}(\phi(\hat{g}))(e) = e\}$. Using (i) and the equality $e = d\phi(\hat{e})$ yields

$$C_G(e) = \{\phi(\hat{g}) : d\phi(\mathrm{Ad}(\hat{g})(\hat{e})) = d\phi(\hat{e})\} = \{\phi(\hat{g}) : \mathrm{Ad}(\hat{g})(\hat{e}) = \hat{e}\},$$

the last equality holding as $d\phi$ is an isomorphism. It follows that $C_G(e) = \phi(C_{\hat{G}}(\hat{e}))$.

The next lemma is a curious result regarding the unipotent radical, Q, of the parabolic subgroup P. In this lemma we use the notation $Q^{(2)} = [Q, Q']$. Let $u = U(1)$ be a semiregular unipotent element as constructed above.

LEMMA 2.7. $\dim(Q/Q') = \dim(Q'/Q^{(2)}) + \delta$, *where* $\delta = 1$, 2, *or* 3. *More precisely*

(i) $\delta = 3$ *if u is of type* $B_j B_{j-1}$ *in* $G = D_{2j}$.

(ii) $\delta = 2$ *if u is of type* $B_j B_{k-j-1}$ *with* $j > k - j > 1$ *or if u is of type* $A_1 F_4$ *for G of type* E_7.

(iii) $\delta = 1$ *otherwise*.

Proof. We use the labelled diagrams corresponding to the semiregular element as discussed above. Then by Lemma 4 of [3], $\dim(Q/Q')$ is the number of positive roots of level 1, while $\dim(Q'/Q^{(2)})$ is the number of roots of level 2. Writing a positive root as a sum of fundamental roots, the first number is just the number of positive roots involving a single fundamental root of label 2 and this root has coefficient 1. Similarly for the second number, where one counts positive roots involving two of the roots of label 2, each with coefficient one, or one root with coefficient 2.

In the case where u is regular, all labels are 2 and it is clear that $\dim(Q/Q') = \mathrm{rank}(G)$. It is easy to check that $\delta = 1$ in this case.

The nonregular, semiregular case is slightly more complicated. We illustrate with the most difficult case, $G = D_n$. In this case the labelled diagram begins with a string of $i - 1$ 2s, followed by a string $0202 \cdots 020$, with j 2s, ending on the triality note. The remaining end nodes are each 2. One can now simply count the number of roots of levels 1 and 2. In this case we find that there are $4 + i + 4j$ roots of level 1 and $2 + i + 4j$ roots of level 2, except when $i = 2$, where there are $1 + i + 4j$ roots of level 2. In this way we establish the result.

Let δ be as in Lemma 2.7. In the cases where $\delta > 1$, $\bar{X}$ is diagonally embedded in a subgroup of type A_1F_4 or B_jB_{k-j-1}, according to whether $G = E_7$ or D_k, where we have $j > k - j - 1$. In these cases, let D_1, D_2 be the projections of $\bar{X}$ to the factors, U_1, U_2 the projections of U, and T_1, T_2 the projections of $\bar{T}$.

Then for $i = 1, 2$, T_i is a 1-dimensional torus of a regular A_1 in D_i.

LEMMA 2.8. (i) *If* $\delta = 1$, *then* $\bar{T} < P$.
(ii) *If* $\delta > 1$, *then* $T_1T_2 < P$.

Proof. (i) follows from (2.4, ii). For (ii) let $u = U(1)$ and let $t \in T_1T_2$. We have, $U_1U_2 \le C_G(u) \le Q$, so $v = u^t \in Q$. Now $\dim C_G(v) \ge \dim C_P(v) \ge \dim C_P(u) = \dim C_G(u) = \dim C_G(v)$, where the second inequality follows as u is in the dense orbit of P on Q. So the inequalities are equalities, showing that v is also in the dense orbit. Thus $v = u^r$, for some $r \in P$, and so $rt^{-1} \in C_G(u) \le Q$. Therefore $t \in P$, as required.

The following result makes use of 2.7 and plays a key role in our proofs of the main theorems.

LEMMA 2.9. *Assume u is not of type B_jB_{j-1} in D_{2j} (i.e., $\delta < 3$). Then $C_G(u)Q'/Q'$ has dimension δ. Moreover,*

(i) *If* $\delta = 1$, *then* $C_G(u)Q'/Q' = UQ'/Q'$.

(ii) *If* $\delta = 2$, *then* $C_G(u)Q'/Q' = U_1U_2Q'/Q'$.

Proof. We first claim that $C_G(u) \le Q$ and $C_G(u)Q'/Q'$ is a subgroup of Q/Q' of dimension at most δ. We know that u is in the dense orbit of P acting on Q and Lemma 2.3 shows that $C_G(u) \le Q$. As P is distinguished, it is shown in (4.5) of [10] that uQ' is fused under the conjugation action of Q. Thus, the map $q \to [u, q]$ is a surjective map from Q to Q'. As $Q/Q^{(2)}$ is nilpotent of class 2 (as before, $Q^{(2)} = [Q, Q']$), the map induces a surjective homomorphism $Q/Q' \to Q'/Q^{(2)}$. The kernel of the homomorphism has dimension precisely δ and contains $C_G(u)Q'/Q'$. So this establishes the claim.

First assume $\delta = 1$. Then $U \le C_G(u)$ and all nonidentity elements of U are conjugate to u under the action of $\bar{T}$. Now by [3], Q/Q' has the structure of a vector space with $\bar{T}$ inducing scalar action. As U is $\bar{T}$-invariant we conclude that UQ'/Q' is a 1-dimensional subspace. So (i) follows from the claim.

Now assume $\delta = 2$. Let $V = U_1U_2 \cap Q'$ and suppose this group is nontrivial. Then V is T_1T_2-invariant. Also, V cannot contain any G-conjugates of u, for such an element would have centralizer in P of dimension at least $\dim(P) - \dim(Q')$, which is strictly bigger than $\dim(C_G(u)) = \dim(P) - \dim(Q)$, a contradiction. Since all elements of $U_1U_2 - (U_1 \cup U_2)$ are conjugate under $T_1T_2 < P$ (by 2.8), we conclude that $V = U_i$, for $i = 1$ or 2. But then by (4.5) of [10] all elements of uU_i are conjugate to u. However, this coset contains elements of U_j, where $\{i, j\} = \{1, 2\}$ and elements of U_j are centralized by D_i, a contradiction. So V is trivial and (ii) follows from the claim.

Our final piece of business in this section is to furnish the proof of the first lemma.

Proof of Lemma 2.1. We are assuming the nonidentity unipotent elements of X are semiregular in G.

For $p = 2$, centralizers of involutions are determined in [2]. Although this article concerns the finite groups of Lie type over fields of characteristic 2, the arguments also cover algebraic groups. Applying these results we see that semiregular involutions occur only when $G = A_1$, B_2, or C_2. In the first case $p = 2$ is a good prime. The latter two are accounted for in the lemma.

It will suffice to consider the case of $X < C_2$. We carry the analysis a bit further than necessary in order to verify the remark made after the statement of Theorem 1.2. First note that the simplicity of X forces $q > p$. From [1] we see that either X acts completely reducibly on the 4-dimensional symplectic module or (up to field twists) is indecomposable of type $1/2/1$, where there are an invariant 1-space and hyperplane.

Consider the completely reducible case. In order for involutions in X to be semiregular we must have X preserving a decomposition of the symplectic space into irreducible nondegenerate 2-spaces (otherwise X is centralized by a torus). The full connected stabilizer of the decomposition is $Sp_2 \times Sp_2$. It is then clear that X is contained in a connected group of type A_1. Moreover, if the summands afford inequivalent representations for X then the normalizer preserves the decomposition. The homogeneous case is more subtle and we will return to this in a minute.

In the indecomposable case argue as follows. The embedding $B_1 < A_1A_1 = SO_4 < Sp_4$ produces a connected group of type A_1 with the same indecomposable action and we take such a group stabilizing the 1-space that X stabilizes. Let P_1 be the full stabilizer of this 1-space, a parabolic subgroup of G, and set $Q_1 = R_u(P_1)$. Let $L = SL_2(K)$ be the derived group of the Levi subgroup. There is a 1-dimensional normal subgroup $Q_0 < Q$ such that Q_0 is a root group for a long root and Q/Q_0 affords the usual module for L, scalar action being induced by a torus of P centralizing L. Using the information on H^1 contained in [1] we conclude that, modulo Q_0, there are two P-classes of complements to Q in XQ. One class is represented by a subgroup of L, the other by a subgroup of B_1. For both classes, the extension over Q_0 splits, from which we conclude that X is conjugate to a subgroup of B_1, as required.

Note that $N_G(X)$ leaves the fixed 1-space invariant and hence is contained in P. It is now easy to check that $N_G(X) = X \times Q_0$.

Now G admits a morphism interchanging long and short root groups. The image of SO_4 under this morphism is of the form $Sp_2 \times Sp_2$, so the corresponding image of X acts on the symplectic module as the orthogonal sum of two natural modules, as discussed earlier. Here, the centralizer is a root group for a short root.

Suppose p is odd. For classical groups, p is then a good prime, and there is nothing to prove. So assume G is an exceptional group. As indicated in [4], the parametrization of unipotent elements is the same as for good primes, with two exceptions, where $p = 3$ and $G = G_2, E_8$. In each case there is one additional unipotent class.

For semiregular elements parametrized as for good primes, we see from Proposition (2.2) of [14] (a result not assuming that p is good) that a semiregular element of order p forces p to be a good prime So this leaves the two exceptional classes. It follows from [9] that the extra class in E_8 is not semiregular. The final case is where $p = 3$ and $G = G_2$.

Consider the action of X on an orthogonal module, V, of dimension 7. It is shown in [7] that G is transitive on nonsingular 1-spaces of V. So if X fixes a nonsingular 1-space, it is contained in a connected group of type A_2. Working within this group and using the fact that $Z(X) = 1$, we embed X in a connected subgroup of type A_1. Further, we see from the embedding that X fixes a unique 1-space, so $N_G(X) = N_{A_2}(X) = PGL_2(q)$, as required.

Now suppose X does not stabilize a nonsingular 1-space. Since $X \cong PSL_2(q)$ composition factors of X on V have dimensions 1, 3, and 4. By [1], only a 4-dimensional irreducible module can extend a trivial module. Using these remarks and the above paragraph we see that if X fixes a 1-space, then it must have three trivial composition factors and thus act trivially on a 2-space, which by our supposition is singular. It follows that there is a unique nontrivial composition factor of dimension 3, which does not extend the trivial module. We may then write V as the orthogonal sum of a 3-space and a 4-space, the latter affording a trivial module for X. But then X fixes a nondegenerate 1-space, a contradiction.

We are left with the possibility that V decomposes as the orthogonal sum of a 3-space and a 4-space. To settle this case, consider $X < G_2 < D_4$, where we take the D_4 to be simply connected so that D_4 admits a triality automorphism centralizing the G_2 and acts on an 8-dimensional orthogonal module. We take this orthogonal space as the orthogonal sum of a 1-space and V. It follows that X acts irreducibly on a unique 4-space. Hence, X is contained in a unique connected subgroup of D_4 having type

$A_1A_1A_1A_1$. This subgroup is invariant under triality, which necessarily normalizes one of the A_1 factors. The involution in the center of this factor is fixed by triality, so G_2 contains an involution centralizing X. This contradicts the fact that X contains a semiregular unipotent element and completes the proof.

3. GROUPS OF TYPE $PGL_2(p)$ IN EXCEPTIONAL GROUPS

In this section we establish Theorem 1.1. Let G be a simple adjoint algebraic group of exceptional type. Let $PSL(2, p) = X < G$, be such that the unipotent elements of X are semiregular elements of G. Suppose t is an element of $N_G(X)$ such that $X\langle t\rangle \cong PGL(2, p)$.

We will show that X is contained in a connected subgroup of type A_1. In view of Lemma 2.1 we assume p is a good prime. Then Lemma 2.2 provides additional restrictions on p which are used throughout. For technical reasons, it will be convenient to first settle the case of $G = G_2$.

LEMMA 3.1. *Theorem* 1.1 *holds if* $G = G_2$.

Proof. Consider $X < G < E$, with $E = D_4(K)$, simply connected, and let R be an 8-dimensional orthogonal module for E. Write $G = C_E(\tau)'$, where τ is a triality automorphism of E.

Working with root elements it is clear that regular unipotent elements of G are also regular in E, hence unipotent elements of X act on R as the sum of a fixed point and a Jordan block of size 7. Write $R = R_1 \oplus R_7$, accordingly. Both subspaces can be taken to be invariant under the action of G.

Consider the action of X on R_7, a nondegenerate space. If this action is reducible, then using [1], the information on Jordan blocks, and the fact that $p \geq 7$, we see that X must be indecomposable, fixing a 1-space, say $\langle r\rangle$, for r a singular vector, and acting irreducibly on $\langle r\rangle^{\perp}/\langle r\rangle$. It follows from [7] that G is transitive on singular 1-spaces of R_7, with point stabilizer a parabolic subgroup, say P. But then X is contained in a conjugate of P, whereas $P'/R_u(P) \cong SL_2(K)$, a contradiction. (We note, however, that in Proposition 3.2, we shall show that there does exist such a subgroup of SO_7, when $p = 7$).

We now have X irreducible on R_7 and we can embed this group in a connected group A of type A_1 fixing the orthogonal form, unique up to conjugacy in E. By [14], A is contained in a conjugate of G. Thus, there is an element $g \in E$, such that $A < C_E(g\tau)$. But then $X < C_G(g)$, whereas $C_E(X) = Z(E)$. Therefore, $X < A < G$, giving the required containment. The information on normalizers is easy, since $N_A(X) \cong PGL_2(p)$.

In view of the above lemma we assume $G \neq G_2$, so the prime restrictions of 2.2 indicate $p \geq 11$.

We may take t to have order $p-1$, the image of an element of $SL_2(p^2)$ of order $2(p-1)$. Indeed, let $h(\gamma)$ denote a diagonal matrix with entries γ, γ^{-1}, where γ is an element of order $2(p-1)$. We may regard t as the image of $h(\gamma)$, for suitable γ.

Let u be a nonidentity unipotent element of X, such that t normalizes $\langle u \rangle$. We take $u \in Q \leq P$, where P is a distinguished parabolic subgroup, as discussed in Section 2. By [14], there is a closed connected subgroup $\overline{X}$ of G, with $u \in \overline{X}$ and $\overline{X}$ of type A_1. Let $\overline{U}$ be the unipotent subgroup of $\overline{X}$ containing u and $\overline{T}$ a 1-dimensional torus of $\overline{X}$ normalizing $\overline{U}$. Embed $\overline{T}$ in a maximal torus of G and choose a system of root subgroups for this maximal torus so that Lemma 2.4 holds. Since $L(G)$ is simple we identify $L(G)$ with its image under the adjoint representation. Replacing u by a conjugate, if necessary, we may assume $u = U(1)$ is as described earlier. Then Lemma 2.5 gives $L(\overline{U}) = \langle e \rangle$, where e is the nilpotent element giving rise to the construction of u.

LEMMA 3.2. *Replacing $\overline{X}$ by a conjugate, we may assume $t \in \overline{T}$.*

Proof. Choose an element $t' \in \overline{T}$ such that $u^t = u^{t'}$. Then $t(t')^{-1} \in C_G(u) = C$, where C is a nilpotent group. So $t \in t'C$. As t normalizes $C_G(u) = C_G(\langle u \rangle)$, the argument in Claim 5 of [14] shows that all semisimple elements of $t'C$ are conjugate in C. So t and t' are conjugate by an element in C. Conjugating $\overline{X}$ by this element we have the result.

Notice that $\overline{X} = PSL(2, K)$. This follows either from the observation that all weights of $\overline{T}$ are even (see 2.4) or from noting that otherwise the involution in $\langle t \rangle$ would lie in $Z(\overline{X})$, whence it would centralize u. However, in $X\langle t \rangle$ this is not the case.

In considering composition factors of $\overline{X}$ on $L(G)$ and the corresponding weights, the weights involved are weights for $SL_2(K)$, where t acts as an element of order $2(p-1)$. Here we regard t as the image of $h(\beta) \in SL_2(K)$. From the action of t on $\langle u \rangle$ we see that $\beta = \pm\gamma$.

Let s be an involution in X such that s inverts t.

LEMMA 3.3. *s inverts $\overline{T}$. Thus s sends every $\overline{T}$-weight space on $L(G)$ to the weight space for the negative weight.*

Proof. Let $s' \in \overline{X}$ be an involution inverting $\overline{T}$. It will suffice to show that $ss' \in C_G(\overline{T})$. We have $ss' \in C_G(t)$, so it will suffice to show $C_G(t) = C_G(\overline{T})$ and we will do this in all but two exceptional cases.

It follows from 2.4(iii) and the prime restrictions in 2.2 that there does not exist a nonzero $\overline{T}$-weight of $L(G)$ which is congruent to 0 modulo $2(p-1)$. Hence, $C_{L(G)}(t) = C_{L(G)}(\overline{T})$. Thus, t and $\overline{T}$ have centralizers of the same dimension. On the other hand, $C_G(\overline{T}) \leq C_G(t)$. The centralizer of a torus is connected and so $C_G(\overline{T}) = C_G(t)^0$. It will suffice to show $C_G(t)$ is connected. Suppose this is false. By (II, 4.4) in [12] it follows that $G = E_6$ or E_7.

Let $\hat{G}$ be the simply connected covering group of G and $Z(\hat{G}) = \langle z \rangle$. Let $\hat{X}$ be the connected preimage of $\overline{X}$ and let $\hat{T}$ be the preimage in $\hat{X}$ of $\overline{T}$. Finally, let $\hat{t} \in \hat{T}$ be a preimage of t. In the simply connected group centralizers of semisimple elements are connected. So there is an element of $\hat{G}$ centralizing the coset $\hat{t}Z(\hat{G})$ but not centralizing $\hat{t}$. It follows that $z \neq 1$ and $\hat{t}$ is conjugate in $\hat{G}$ to $\hat{t}z$ (by an element in the preimage of an element in $C_G(t) - C_G(t)^0$).

Let V be a restricted module of dimension 27 or 56, according to $G = E_6$ or E_7. Then $\hat{t}$ and $\hat{t}z$ have the same set of eigenvalues on V, including multiplicities. So if z is represented by δ on V, then multiplication of the eigenvalues of $\hat{t}$ by δ permutes the eigenvalues, preserving multiplicities.

The unipotent class in question determines a labelled Dynkin diagram, which in turn determines the weights of $\overline{T}$ (or $\hat{T}$) on roots. The high weight λ of V can be expressed as a rational combination of roots and the remaining weights can be found by subtracting roots from λ. Consequently, it is easy to find all weights of $\hat{T}$ on V. This determines the composition factors of $\hat{X}$.

For $G = E_6, E_7$ there are two (respectively 3) classes of semiregular unipotent elements corresponding to labelled diagrams as in Lemma 2.2 and subject to certain prime restrictions. In the following argument we exclude the cases where u is regular and $(G, p) = (E_6, 13)$ or $(E_7, 19)$. These cases will be settled later.

Suppose $G = E_6$. Then $\hat{X} \cong \overline{X}$, so we regard $\overline{X}$ as a subgroup of $\hat{G}$ and take $t = \hat{t}$. Here $|z| = 3$, so $|t| = |tz|$ implies $p - 1 \equiv 0 \bmod (3)$. Also, $\lambda = \lambda_6 = \frac{1}{3}(234654)$, where the right hand side gives the coefficients of λ as a combination of simple roots. For the regular class one checks that $V|\overline{X}$ has composition factors which are the same as those of the Weyl modules of high weight 16, 8, and 0. For the other semiregular class the appropriate Weyl modules have high weights 12, 8, and 4.

In each case, the fixed point space of $\hat{t} = t$ has dimension 3 and $\hat{t}z$ induces an element of order 3 on this 3-space. Consequently, there must exist a weight space of dimension 3 on which t induces an element of order 3. The only weight spaces of dimension 3 occur for the nonregular class for weights 0, 2, -2, 4, -4, and on none of these does t induce an element of order 3. So this is a contradiction.

Now suppose $G = E_7$. Then V has high weight $\lambda = \frac{1}{2}(2346543)$. Thus, $\hat{T}$ has weight 27, 21, or 17 on a maximal vector, from which it follows that $\hat{X} = SL_2(K)$ in each case.

As before, compute the $\hat{X}$-composition factors on V. They are the same as those of the Weyl modules of high weights 27, 17, 9; 21, 15, 11, 5; or 17, 15, 9, 7, 3, respectively. Now $\hat{t}$ has order $2(p-1)$. The $\hat{T}$-weight 1 occurs with multiplicity 3, 4, or 5, respectively, and one checks that there is no other $\hat{T}$-weight giving the same value on $\hat{t}$. Say $\hat{t}$ is represented by β on this weight space. As z induces -1 on V it follows that there is a weight space of dimension 3, 4, 5, respectively, on which $\hat{t}$ is represented by $-\beta$. This gives a contradiction in all but the regular class with $\rho = 19$, which we have omitted.

What remains are the two previously excluded cases where u is regular and $p = 13, 19$, according to $G = E_6, E_7$. In these cases Q is the maximal unipotent subgroup generated by all root groups corresponding to fundamental roots.

We have seen that $C_G(t)^0 = C_G(\bar{T})$. From (2.4, iii) this centralizer has dimension 6 or 7, respectively, so it is a maximal torus of G. Hence, s and s' both normalize this torus and permute the corresponding root groups. Suppose we are able to show that $s(\Pi) = -\Pi = s'(\Pi)$, where Π is the set of fundamental roots. Then both s and s' are in the coset of the long word. Hence, their product centralizes $\bar{T}$, as required. We will verify the supposition for s. The same argument will work for s'.

The fundamental roots are precisely those roots affording $\hat{T}$-weight 2 on $L(G)$. The corresponding weight space for t has dimension one more than the rank. This is because the negative of the highest root also affords the same weight and this is the only other root affording the same weight.

Now s inverts t. So if α is a fundamental root, then α^s is either the negative of a fundamental root or the root of highest height.

If Q^s is the opposite unipotent group, then s must send each fundamental root to the negative of a fundamental root. In this case s sends every $\bar{T}$-weight on $L(G)$ to its negative. The result follows.

Now suppose that s does not send all fundamental roots to negatives of fundamental roots. Then s sends some fundamental root to the root of greatest height, while the other fundamental roots are sent to negatives of fundamental roots. The image of the fundamental roots must be a base for the root system. Using the extended Dynkin diagram, we consider the possibilities and easily see that $\langle Q, Q^s\rangle$ is the derived group of a maximal parabolic subgroup, P_0, of type D_5, E_6, according to $G = E_6, E_7$.

Note that $X = \langle u, u^s\rangle < \langle Q, Q^s\rangle$. The preimage of $\langle Q, Q^s\rangle$ in $\hat{G}$ splits over the center, so we may regard this group as a subgroup of $\hat{G}$. Replacing V by its dual in the E_6 case, if necessary, we may assume that $\langle Q, Q^s\rangle$ has a fixed point on V. This is impossible for the E_7 case, since we have already observed that for this case $\hat{X} \cong SL_2(K)$, the involution inducing -1 on V, whereas t fixes a point of V. In the E_6 case let $B = \langle u, t^2\rangle$, a Borel subgroup of X. Considering B as a subgroup of $\hat{X}$

and comparing with the composition factors of $\hat{X}$ we see that the only fixed point is the fixed space of $\hat{X}$. Maximality of P_0 forces $\hat{X} < P_0$. However, the P_0 composition factors on V have dimensions 1, 10, and 16, which are not compatible with the action of $\hat{X}$. This completes the proof of the lemma.

Write $L(\overline{X}) = \langle e, h, f \rangle$, where f is in the Lie algebra of the other $\overline{T}$-invariant unipotent subgroup of $\overline{X}$ and $h = [ef]$.

Let $V = V_0 > V_1 > V_2 > \cdots > V_k > 0$ be an $X\langle t \rangle$-composition series for $L(G)$. We may arrange things so that some V_j is a cyclic module generated by e_δ, a root element corresponding to the root δ of highest height.

Note that since X has a unique irreducible module of a given dimension and since these irreducibles extend to irreducibles of $X\langle t \rangle$, Clifford's theorem implies that X is irreducible on each of the successive quotients in the filtration.

Choose i such that $f \in V_i - V_{i+1}$.

LEMMA 3.4. $V_i = \langle e, h, f \rangle \oplus V_{i+1}$.

Proof. Let bars denote images in V_i/V_{i+1}. First suppose $[u, f] \in V_{i+1}$. Then $\bar{f}$ is a maximal vector of V_i/V_{i+1} and has $\overline{T}$-weight -2. On the other hand, this quotient is an irreducible module for X and so $\bar{f}$ has high weight c with $0 \leq c \leq p - 1$. Comparing the actions of $t^2 \in X$, it follows that $c \equiv -2 \bmod (p - 1)$ and so $c = p - 3$ and $\dim(V_i/V_{i+1}) = p - 2$.

Now $\bar{f}^s$ is a minimal vector of V_i/V_{i+1} and so $z = [u, \ldots, u, f^s] \notin V_{i+1}$, where there are $p - 3$ commutators. However, Lemma 3.2 implies f^s has $\overline{T}$-weight 2, so that $[u, \ldots, u, f^s]$ involves $\overline{T}$-weight vectors for weights at least $2 + 2(p - 3) = 2p - 4$. A check of weights (use 2.2 and 2.4) shows that the only possibilities are for u regular in E_8, E_7, E_6, F_4, G_2, with $p = 31, 19, 13, 13, 7$, respectively, or u a product of regular elements in the factors of $A_1F_4 < E_7$ and $p = 13$. Moreover, in each case z must be a multiple of e_δ. In particular, z is a weight vector for t.

For the exceptional cases above consider the cyclic module $W = X\langle t \rangle \langle z \rangle$. Since $\langle z \rangle$ is stabilized by the Borel subgroup $B = \langle u, t \rangle$ of $X\langle t \rangle$, and since $|X\langle t \rangle : B| = p + 1$, this module has dimension at most $p + 1$. We have chosen the filtration so that $W = V_j$ for some j. Since $z \in V_i$, we have $j \geq i$. On the other hand, from the above paragraph we have $z \notin V_{i+1}$. It follows that $W = V_i$.

A dimension argument now implies that V_{i+1} has dimension at most 3. Now B is contained in $\overline{X}$ and working in $\overline{X}$ we see that $B\langle f \rangle = \langle e, h, f \rangle$, an indecomposable module for B. Our supposition implies $\langle e, h \rangle \leq V_{i+1}$. It follows that $V_{i+1} = \langle e, h, e^s \rangle$. However, modulo V_{i+1}, z is a multiple of f, as they both afford maximal vectors in V_i/V_{i+1}. Hence, $\langle e_\delta \rangle = \langle z \rangle <$

$\langle e, h, e^s, f\rangle$. But, this subspace is $\bar{T}$-invariant and a consideration of weights gives a contradiction.

Next assume $e \in V_{i+1}$, but $h \notin V_{i+1}$. Here $\bar{h}$ is a maximal vector of V_i/V_{i+1} which is fixed by s. It follows that V_i/V_{i+1} is a trivial module, which is impossible, as a consideration of the action of t^2 shows that $\bar{f} \notin \langle \bar{h}\rangle$. We are left with the case that $\bar{e}$ is a maximal vector of V_i/V_{i+1}. Since $\langle \bar{h}\rangle$ is the unique fixed space of t on V_i/V_{i+1} and $[u, \bar{h}] = \bar{e}$ we conclude that V_i/V_{i+1} is a three dimensional module, which establishes the result.

LEMMA 3.5. *$\langle e, h, f\rangle$ is X-invariant.*

Proof. Lemma 3.3 shows that $V_i = \langle e, h, f\rangle \oplus V_{i+1}$ and the first summand is invariant under $\langle u, t\rangle$, which has p' index in $X\langle t\rangle$. Using (19.5)(ix) of [5] we conclude that there is an $X\langle t\rangle$-invariant complement, say W, to V_{i+1} in V_i.

As p is a good prime, (1.14) of [4] implies $C_{L(G)}(u) = L(C_G(u)) \leq L(Q)$. On the other hand, $C_W(u)$ is a 1-space affording the same weight for t as does e. Except in the case where $u \in A_1F_4 < E_7$, we conclude from Lemma 2.9 and a weight comparison that the corresponding weight space of $L(C_G(u))$ is 1-dimensional and generated by e. (For the weight comparison use the fact that $L(Q')$ is a sum of $\bar{T}$-weight spaces for weights at least 4 and Lemmas 2.2 and 2.4 imply that none of these weights have the same restriction to t as $\bar{T}$-weight 2.)

For the moment exclude the A_1F_4 case. Then $e \in W = X\langle e\rangle$ and by the above paragraph W is the unique irreducible submodule of $L(G)$ of its isomorphism type. Note that W is spanned by e, e^s, and w_0, where the last element is in $C_{L(G)}(t) = C_{L(G)}(\bar{T})$ (this equality was noted in the proof of 3.3). It follows that W is $\bar{T}$-invariant.

Let $B = \langle u, t\rangle$, as before. If $f \in W$, then $B\langle f\rangle = \langle e, h, f\rangle \leq W$ and a dimension comparison shows that equality holds, as required.

Suppose $f \notin W$. Here we argue as in the proof of Lemma 3.3. Choose a filtration of V/W (viewed as an $X\langle t\rangle$ module) with $X\langle e_\delta\rangle + W/W$ one of the terms. The uniqueness of W mentioned above and the fact that there is no self extension of W implies that V/W contains no submodule isomorphic to W. The argument of 3.3 together with (4.5) of [1] now show that $X\langle e_\delta\rangle + W/W = X\langle f\rangle + W/W$ and this module is either irreducible of dimension $p-2$ or possibly such an irreducible module extended by a module of dimension at most 3 on which X acts trivially, by the above. In either case a consideration of weights shows that $f \in \langle e_\delta\rangle + W$. But this subspace is $\bar{T}$-invariant, and a weight consideration for $\bar{T}$ gives a contradiction.

Finally we consider the previously excluded case where $\bar{X} < A_1A_1 < A_1F_4$. Let U_1, U_2 be the projection of $\bar{U}$ to the factors, so that $u \in U_1U_2$

and $e \in L(U_1)L(U_2) = \langle e_1 \rangle \oplus \langle e_2 \rangle$. The latter space is contained in a weight space of $\bar{T}$. Using (2.9, ii) we have $C_W(u)$ a 1-space in $\langle e_1 \rangle \oplus \langle e_2 \rangle$. Hence, $C_W(u)$ is $\bar{T}$-invariant. As above, this implies that W is $\bar{T}$-invariant.

Again we consider V/W and use the arguments of Lemma 3.3. If this module does not contain a submodule isomorphic to W, we proceed as above. Otherwise, the preimage of such a space, say D, is the direct sum of two copies of W (use [1] and recall that by Lemma 3.1 we are assuming $p \geq 11$). Lemma 2.9 and arguments as above imply that $C_D(u)$ is a full weight space of $L(C_G(u))$ and so D is not contained in a larger homogeneous space. Hence V/D contains no submodule isomorphic to W. Also, the above argument shows that D is $\bar{T}$-invariant.

If $f \notin D$, then we argue as in the earlier case, working in V/D, the contradiction coming from a comparison of $\bar{T}$-weights. Suppose $f \in D$. Then $B\langle f \rangle = \langle e, h, f \rangle < D$.

Since D is homogeneous we can write $D = A \otimes C$, where A is an irreducible 3-dimensional module for X and C is a 2-dimensional trivial module. Then $X \otimes GL_2(K)$ acts on D. Now t^2 has three distinct eigenvalues on A and $GL_2(K)$ is transitive on the nonidentity elements of any eigenspace. It follows that there is a vector $a \in W$ and an element $g \in GL_2(K)$ such that $ga = f$. Then gW is X-invariant and contains f. As $\langle u \rangle \langle f \rangle = \langle e, h, f \rangle$, we conclude that $gW = \langle e, h, f \rangle$, proving the lemma.

LEMMA 3.6. $X\langle t \rangle \leq \bar{X}$.

Proof. By Lemma 3.5, X stabilizes $\langle e, h, f \rangle = L(\bar{X})$. Part 2 of Lemma 2.3 shows that $C_G(e)^0 \leq Q$. Similarly, $C_G(f)^0 = (C_G(e)^0)^s \leq Q^s$. Thus $C_G(L(\bar{X}))^0 \leq (Q \cap Q^s)$. So $L(C_G(L(X))^0) < (L(Q) \cap (L(Q^s)) = 0$, the last equality holding as the $\bar{T}$-weights of $L(Q)$ are positive and the $\bar{T}$-weights of $L(Q^s)$ are negative. It follows that $C_G(L(\bar{X}))$ is finite. Now $N_G(L(\bar{X}))$ induces a group of Lie algebra automorphisms of $L(\bar{X})$, so that $N_G(L(\bar{X}))^0 = \bar{X}$. Thus, X normalizes $\bar{X}$. Since $u \in \bar{X}$, we have $X = \langle u^X \rangle \leq \bar{X}$. We already have $t \in \bar{X}$, so $X\langle t \rangle \leq \bar{X}$, as required.

To complete the proof of Theorem 1.1 we must show that $N_G(X) \cong PGL_2(p)$. For this, it will suffice to show $C_G(X) = 1$. As u is semiregular, we have $C_G(X) \leq C_G(u) \leq Q$. So $C_G(X) \leq Q \cap Q^s = J$. Now J is $\bar{T}$-invariant and the arguments of the last paragraph showed that $L(J) = 0$. Hence, J is finite and so $\bar{T}$ acts trivially on J. However, $\bar{T}$ has no fixed points on Q, so $J = 1$ and thus $C_G(X) = 1$.

The following result shows that Theorem 1.1 does not hold for classical groups.

PROPOSITION 3.1. *Let $p > 3$ be prime. There is a subgroup $Y = PGL_2(p)$ contained in $PGL_{p-1}(K)$ which contains a regular unipotent element. The preimage of this group in $GL_{p-1}(K)$ acts reducibly and indecomposably on* *the natural module with composition factors of dimension* $1,\ p-2$. *Moreover,* $Y' = PSL_2(p)$ *is not contained in a connected subgroup of* $PSL_{p-1}(K)$ *of type* A_1.

Proof. By (4.5) of [1] $X = PSL_2(p)$ has an indecomposable extension of the trivial module by the irreducible module of dimension $p-2$. Let V denote such a module and consider $X < SL(V) = SL_{p-1}(K)$.

We first observe that X is contained in a connected subgroup of $SL(V)$ of type A_1. Suppose otherwise. From the results in [1] we see that the only possibility would be if this A_1 acted irreducibly as a tensor product. But all irreducible representations of A_1 are self dual, whereas V is not self dual upon restriction to X. So this is a contradiction.

We next show that X contains a regular element of $SL(V)$. Let F be the fixed space of X on V. Let $B = \langle u\rangle\langle h\rangle$ be a Borel subgroup of X, where u is unipotent and h is semisimple. Then u is represented as a regular element on V/F. So, by way of contradiction, we may assume that under the action of u, $V = W \oplus M$, where u is trivial on W and M is a Jordan block of dimension $p-2$.

Then B leaves invariant the subspace $D = [\langle u\rangle, V]$, a subspace of M having codimension 2 in V. So u acts trivially on V/D, while h acts as a semisimple element. By considering commutators of u it is easy to see that there is a B-invariant 1-space of V/D, whose preimage, say S, is a Jordan block for u. If $F \nleq S$, then $V = F \oplus S$ under the action of B and hence V cannot be indecomposable under the action of X (by (19.5)(ix) of [5]).

Therefore $F \leq S$. Then $F \leq D = [\langle u\rangle, S]$. But V/F is a Jordan block for u, whereas from the decomposition $V = W \oplus M$ we see that this is not the case. So this is a contradiction, showing that u is a regular element.

Finally, we must produce a subgroup of type $PGL_2(p)$. Let j be an automorphism of X with $X\langle j\rangle = PGL_2(p)$. Consider the representation $X \to X \to GL(V)$, where the first map is given by the action of j and the second is the above representation. The composition is another indecomposable representation, which by (4.5) of [1] is equivalent to the first. It follows that $GL(V)$ contains an element r which normalizes X and induces j. From the action of X on V one checks that $C_{GL(V)}(X) = Z(GL(V))$, so that the image of $X\langle g\rangle$ in $PGL_{p-1}(K)$ is isomorphic to $PGL_2(p)$, as required.

The above proposition can be used to establish the following result for orthogonal groups.

PROPOSITION 3.2. *Let* $p > 3$ *be a prime. There is a subgroup* $Y = PGL_2(p)$ *contained in* $SO_p(K)$ *which contains a regular unipotent element. This group acts reducibly and indecomposably on the natural orthogonal module with composition factors of dimension* $1,\ p-2, 1$. *Moreover,* $Y' = PSL_2(p)$ *is not contained in a connected subgroup of* $SO_p(K)$ *of type* A_1.

Proof. Consider the group $SO_p(K)$ and let P be a parabolic subgroup stabilizing a singular one space of the usual orthogonal module, V. Then $P = QL$, where $L' \cong SO_{p-2}$. Moreover, Q has the structure of a KL'-module which can be identified with the usual orthogonal module for L'. Now L' contains an irreducible connected subgroup of type A_1 which in turn contains a group $E \cong PGL_2(p)$. By the above proposition, the action of E on Q can be extended by the trivial module. Consequently, QE contains a complement, say Y, to Q which is not Q-conjugate to E. It follows that the action of Y on the natural orthogonal module for $SO_p(K)$ is uniserial with composition factors of dimension $1, p-2, 1$.

From the last result we see that unipotent elements of Y have just one Jordan block when restricted to the invariant hyperplane, say H. We claim that unipotent elements of Y must have a single Jordan block on the usual orthogonal module for $SO_p(K)$. For otherwise, $V = H' \oplus E$, where H' is a hyperplane, affording a single Jordan block for u, and E is a u-invariant 1-space. Let F be the fixed space of u within H'. Then F is the unique fixed space in $[u, V]$. But H has such a fixed space, so $F \leq H$ and is the 1-space invariant under P. Then V/F is the dual of H, so also affords a single Jordan block for u. However, a consideration of the decomposition $V = H' \oplus E$ shows that V/F has two Jordan blocks. This contradiction completes the proof.

4. SUBGROUPS OF TYPE $PSL_2(q)$, $q > p$

In this section we consider subgroups X of G of type $PSL_2(q)$, where $q > p$. Assume u is a semiregular unipotent element of G which is contained in X. Our goal is to prove Theorem 1.2. Recall that Lemma 2.1 allows us to take p a good prime. The first lemma settles the result for classical groups.

LEMMA 4.1. *Assume G is of classical type. Then X is contained in a connected group of type A_1 and $N_G(X) \cong PGL_2(q)$.*

Proof. Replace G with the appropriate classical group and let V be the corresponding classical module. In some cases, this replacement may require us to consider subgroups $X = SL_2(q)$. We first establish the existence of a connected group of type A_1 containing X.

First assume that u has a single Jordan block on V. This is the case if G is of type A_s, B_s, or C_s. Then $\dim(V) \leq p$. Using (4.5) of [1] we see that for $q > p$, $SL_2(q)$ has no indecomposable representations of dimension at most p, except for the irreducible ones. Hence, X acts completely reducibly on V. Since u has a single Jordan block on V, this action must be irreducible. Further, if the representation is tensor decomposable, then an easy check shows that unipotent elements have fixed space of dimension greater than 1. Hence, the action is a twist of a restricted representation. The results of [11] or [13] show that X is contained in a connected subgroup of G of type A_1.

Now assume that u has more than one Jordan block. Here $G = D_s$ for $s \geq 4$ and $u \in B_j B_k$, where $j > k$ and $j + k = s - 1$. In particular, $\dim(V) \leq p + (p - 2) = 2p - 2$. Since $\dim V \geq 8$, we have $p \geq 5$.

We claim that X acts completely reducibly on V. Otherwise, there is a 2-step indecomposable section S/L of V and we choose such a section with L of minimal dimension. Then (4.5) of [1] and the dimension restriction show that the composition factors of S/L have high weights which are a common field twist of a and $b + p$, where $a + b = p - 2$. Applying a field automorphism, we may ignore the twist. If there are two composition factors of V having high weight a, then $\dim(V) \geq 2(a + 1) + 2(b + 1) = 2p$, a contradiction. So there is just one such factor.

Let $R = \mathrm{rad}(L)$. Minimality of L implies that L is completely reducible under the action of X so we can write $L = R \perp N$, where N is nondegenerate and X-invariant. Suppose $N \neq 0$. Then $V = N \perp N^{\perp}$, $S = N \perp (S \cap N^{\perp})$, and $(S \cap N^{\perp})/R \cong S/L$, contradicting minimality of L. Hence, $N = 0$ and L is singular.

Let C/L be the socle of S/L. As S/L is nonsplit, we see that C is singular, so by the next to the last paragraph, C/L has high weight $b + p$. So both C and $V/C^{\perp}$ have composition factors of high weight $b + p$, while $C^{\perp}/C$ has a unique composition factor of high weight a.

The minimality of $\dim L$ implies that C must contain a submodule, say Z of high weight $b + p$. Now u has two Jordan blocks on this module, so $Z = \mathrm{soc}(C)$. Indeed, $Z = \mathrm{soc}(V)$ and so Z is singular. Let I/Z be a simple submodule of $Z^{\perp}/Z$. If the high weight of this composition factor is a, then by uniqueness, I/Z is nondegenerate in $Z^{\perp}/Z$. For this to be an orthogonal space, a must be even. Since V has even dimension, there must be another irreducible submodule in $Z^{\perp}/Z$. So we may assume that I/Z does not have high weight a.

Another application of (4.5) of [1] and the dimension restriction shows that I is completely reducible, whereas $Z = \mathrm{soc}(V)$. This contradiction establishes the claim.

If the representation is reducible, then there exist just two summands and u induces a Jordan block on each of sizes $2j + 1$, $2k + 1$. It follows that both summands are nondegenerate and X is contained in a conjugate of $B_j B_k$. We now embed each projection of X in a corresponding group of type A_1, using the first paragraph. Taking a suitable diagonal subgroup, possibly involving field twists, we obtain a connected group of type A_1 containing X.

The final case is when X acts irreducibly on V. The representation is given by a tensor product of twists of restricted representations [13]. Now u induces a single Jordan block on each restricted representation of X. An easy computation shows that u can have the correct action on V only if there are two tensor factors, one of which has dimension 2. That is, V is a twist of $L(c) \otimes L(1)^{p^e}$, for $0 \leq c \leq p-1$. In order to get an orthogonal representation, c must be odd. In particular, $c < p - 1$. Restricting this representation to $PSL_2(p)$ we find that V is the direct sum of irreducibles of dimensions c and $c + 2$. Moreover, unipotent elements act as a Jordan block on each summand, so this provides an example. The representation does extend to an orthogonal representation of a connected group of type A_1.

To complete the proof we must establish the assertion about normalizers. The containment $N_G(X) \geq PGL_2(q)$ follows from the embedding of X in a connected group of type A_1. The equality follows from representation theory, since a nontrivial field automorphism cannot preserve the above representation.

From now on we take G to be an exceptional group. Let B be a Borel subgroup of X with Cartan subgroup $\langle t \rangle$, where t is an element of order $\frac{1}{2}(q-1)$. Fix a nonidentity unipotent element $u \in B$.

By hypothesis u is semiregular and as in Section 2 we let P be the corresponding parabolic subgroup and $Q = R_u(P)$.

The result of Testerman [14] shows that u is contained in a connected group $D = A_1$. Let U denote the unipotent subgroup of D containing u and T a 1-dimensional torus of D normalizing U.

Recall the integer δ given in Lemma 2.7. If $\delta > 1$, recall also the 1-dimensional tori T_1, T_2 discussed after Lemma 2.7.

LEMMA 4.2. *$t \in P$. Moreover*

(i) *If $\delta = 1$, then t is Q-conjugate to an element of T.*

(ii) *If $\delta = 2$, then t is Q-conjugate to an element of T_1T_2.*

Proof. Let $v = u^t$. Then $v \in O_p(B) \leq C_G(u) \leq Q$. The third paragraph of the proof of 2.8 shows that $v \notin Q'$. If $\delta = 1$, then 2.7 implies that T acts transitively on the nonidentity elements of $C_G(u)Q'/Q'$, so there is an element $h \in T$, such that $vQ' = u^hQ'$. Suppose $\delta = 2$. Here we claim that the same holds with $h \in T_1T_2$. For this, note that T_1T_2 is transitive on $U_1U_2 - (U_1 \cup U_2)$. So we have the claim, using 2.7, unless $vQ' \in U_iQ'$ for $i = 1$ or 2. But in this case U_iQ' contains a conjugate of u and so (4.5) of [10] implies that U_i contains a conjugate of u. This is impossible as these elements centralize one of the simple factors of the group A_1F_4.

By 2.8, $h \in P$ and by the above paragraph $u^t \in u^hQ'$. So (4.5) of [10] shows that $u^t = u^{hq_0}$ for some $q_0 \in Q$. Then $tq_0^{-1}h^{-1} \in C_G(u) \leq Q$. In particular, $t \in P$, establishing the first assertion. To complete the proof we now apply Claim 5 of [14] which shows that t is Q-conjugate to h.

In view of the above lemma, we will from now on assume $t \in T$ or T_1T_2, according to whether $\delta = 1$ or 2. The conjugation required to achieve this may send u outside $\bar{X}$. However, the conjugation is accomplished by an element of Q, so the coset uQ' is unchanged.

It is shown in [3] that Q/Q' has the structure of a vector space over K. In this action T induces scalars (corresponding to weight 2), so that a T-invariant subgroup is actually a subspace. Consequently, UQ'/Q' (respectively U_1U_2Q'/Q') is a subspace containing uQ'.

LEMMA 4.3. *Let t be the image of $h(\beta)$ under the map $SL_2(q) \to X$ and let γ be a root of level* 1 (*with respect to a maximal torus of P containing T or T_1T_2*) *for which uQ' projects nontrivially to $U_\beta Q'/Q'$. Then the eigenvalue of t on $U_\gamma Q'/Q'$ has the form β^{2q_0}, where q_0 is a power of p strictly less than q.*

Proof. Let $V = Q/Q'$ and for γ a root, set $V_\gamma = U_\gamma Q'/Q'$. Let $\gamma_1, \ldots, \gamma_k$ be the roots of level 1 for which the projection of uQ is nontrivial and write $uQ' = v_1 + \cdots + v_k$, where $v_i \in V_{\beta_i}$. Let $f(x)$ be the minimal polynomial of β^2 over the prime field $\mathbf{F}_p$. View V as an $\mathbf{F}_p$-space. Then working in B we see that $f(t)(uQ') = 0$. Hence, $f(t)v_i = 0$ for each i. Now t induces a scalar on V_{β_i} and so it follows that this scalar is a root of $f(x)$. These roots are the Galois conjugates of β^2, so the result follows.

For $\gamma \in K$, let $T(\gamma)$ denote the image of $h(\gamma)$ in $T < D = PSL_2(K)$. If $T < T_1T_2$, let $T_i(\gamma)$, for $i = 1, 2$, denote the projection of $T(\gamma)$ to T_i.

LEMMA 4.4. *Conjugating by a field automorphism of X, if necessary, we may assume that $t = T(\beta)$ or $T_1(\beta)T_2(\beta^{q_0})$, where q_0 is a nontrivial power of p and $q_0 < q$.*

Proof. We have $t \in T \leq T_G$ or $t \in T_1T_2 \leq T_G$. In the former case, T induces scalars on Q/Q' corresponding to weight 2, so the previous lemma implies that $t = T(\beta^{q_0})$ for some power q_0 of p. Conjugating by a Frobenius automorphism of X we may assume $q_0 = 1$.

Now assume that $t \notin T$. Write $u = u_1 + u_2$ with $u_i \in U_i$. Since T_2 centralizes u_1 it centralizes each root group U_δ such that u_1Q' involves nonidentity elements of $U_\delta Q'$. If u_2Q' also involved a nonidentity element of $U_\delta Q'$, then T_1 would also centralize U_δ. But then $T < T_1T_2$ would centralize U_δ, whereas we know that T induces nonidentity scalars on Q/Q'. It follows that u_1Q' and u_2Q' involve different root groups for roots of level 1. The former are centralized by T_2 and the latter by T_1.

Since $T(\gamma)$ induces γ^2 on $U_\delta Q'$ for each root δ of level 1, T_1 (respectively T_2) induces scalars on those root groups, $U_\delta Q'$, involved in the expression of u_1Q' (respectively u_2Q'). The previous lemma now implies

$t = T_1(\beta^{q_1})T_2(\beta^{q_2})$, where q_1, q_2 are distinct powers of p. Conjugating by an appropriate field automorphism of X we obtain the result.

LEMMA 4.5. *X is contained in a connected group of type A_1.*

Proof. By Lemma 4.2 we have $t \in T$ or in T_1T_2, according to whether $\delta = 1$ or 2. The latter case occurs only for $D < F_4A_1 = D_1D_2$, with $G = E_7$. Here we use Lemma 4.4 to find a 1-dimensional torus $\hat{T}$ containing t (set $\hat{T} = \{T_1(\beta)T_2(\beta^{q_0})\colon \beta \in K\}$). Now a comparison of the eigenvalues of t with the weights of T or $\hat{T}$ shows that any t-invariant subspace of $L(G)$ is also T-invariant (respectively, $\hat{T}$-invariant).

Let $J = \langle T^X \rangle$ (respectively $\langle \hat{T}^X \rangle$), a closed connected subgroup of $\langle X, T \rangle$ (respectively $\langle X, \hat{T} \rangle$). Then J leaves invariant all X-invariant subgroups of $L(G)$ and $X = \langle t^X \rangle < J$.

Note that t is trivial on the Lie algebra of a maximal torus of G. On the other hand, the fixed space of t on an irreducible module for X has dimension 1 or 3, the latter only in the case of the Steinberg module of dimension q. Consequently, X cannot be irreducible on $L(G)$, although the restrictions on p imply that $L(G)$ affords an irreducible module for G. So $J < G$. If $R_u(J) \neq 1$, then we can embed J in a parabolic subgroup of G. In view of our information on p, the argument of Theorem 9 of [8] shows that X is contained in a Levi factor of this parabolic. But then X is centralized by the central torus of the Levi factor, whereas u is centralized only by unipotent elements. We conclude that J is reductive. Similarly J is semisimple and Theorem 2 of [8] implies that it must be the commuting product of several groups of type A_1. Working in J it is now an easy matter to show that X extends to a connected group of type A_1.

LEMMA 4.6. *$N_G(X) \cong PGL_2(q)$ and this group is contained in a connected group of type A_1.*

Proof. For classical groups this is established in Lemma 4.1. Suppose G is an exceptional group. We have seen that X is contained in a connected group, say E, of type A_1. As u centralizes no nonidentity semisimple element, E is necessarily of type $PGL_2(K)$ and it follows that $N_G(X)$ contains $PGL_2(q)$. The argument at the end of the proof of Theorem 1.1 shows that $C_G(X) = 1$.

If the assertion of the lemma is false then $N_G(X)$ must contain an element g of prime order r inducing a field automorphism of X. We can choose this element to normalize B and centralize u. It follows that $g \in Q$ and so $|g| = r = p$. But $t \in N_G(Q)$, so $[g, t]$ is unipotent. However, this implies that g centralizes $B/O_p(B)$, which is not the case. This completes the proof of the lemma.

Theorem 1.2 now follows from Lemma 4.7.

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