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A_1-Type Overgroups of Elements of Order p in Semisimple Algebraic Groups and the Associated Finite Groups

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0. INTRODUCTION AND NOTATION

This work is a contribution to the study of the overgroups of unipotent elements in finite and algebraic groups of Lie type and to the more general study of the subgroup structure of these groups. We consider the following natural question: Given G a finite or algebraic simple group of Lie type defined over F_l a finite or, respectively, algebraically closed field of nonzero characteristic p, and a unipotent element $u \in G$, when does there exist a subgroup $A \leq G$ with A isomorphic to $SL_2(F)$ or $PSL_2(F)$ and with $u \in A$? An answer to the analogous question for *nilpotent* elements in simple Lie algebras over algebraically closed fields has been available for many years now in the Jacobson–Morozov theorem. It was shown in [8] that in simple Lie algebras defined over algebraically closed fields of characteristic 0 or sufficiently large characteristic p, every nilpotent element can be embedded in an sl_2 subalgebra. In [12], this result was shown to be valid for all algebraically closed fields of "good" characteristic. (See the statement of Theorem 0.1 below for a definition of "good".) We establish here the best possible group theoretic version of the Jacobson–Morozov theorem under the condition that p is a good prime for G; namely, we find that a unipotent element in G lies in a subgroup isomorphic to $(P)SL_2(F)$ precisely when its order allows it to. That is,

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every element of order p is contained in a subgroup isomorphic to (P)$\mathrm{SL}_2(F)$. The algebraic group result is precisely stated as follows:

THEOREM 0.1. *Let G be a semisimple algebraic group defined over an algebraically closed field k of nonzero characteristic p. Assume p is a good prime for G (i.e., $p > 2$ if G has a factor of type B_l, C_l, D_l, $p > 3$ if G has a factor of type G_2, F_4, E_6, E_7; and $p > 5$ if G has a factor of type E_8). Let $u \in G$ such that $u^p = 1$. Then there exists a closed connected subgroup X of G, X isomorphic to $\mathrm{SL}_2(k)$ or $\mathrm{PSL}_2(k)$, with $u \in X$.*

We obtain the finite group result by viewing the finite groups as fixed point subgroups of certain endomorphisms of the algebraic groups and then applying the following strong version of Theorem 0.1.

THEOREM 0.2. *Let G be a semisimple algebraic group defined over an algebraically closed field k of nonzero characteristic p. Assume p is a good prime for G. Let σ be a surjective endomorphism of G with finite fixed-point subgroup G_σ. Let $u \in G_\sigma$ with $u^p = 1$. Then there exists a closed connected subgroup X of G with $\sigma(X) \subseteq X$, X isomorphic to $\mathrm{SL}_2(k)$ or $\mathrm{PSL}_2(k)$ and with $u \in X$.*

In the study of simple algebraic groups, the (P)SL_2 subgroups of these groups have been used to establish many results in the theory. For example, (P)SL_2's arise in the classification of the conjugacy classes of unipotent elements, a fundamental result of [1, 2, 11, and 12]. More recently, Suprunenko has used the existence of (P)SL_2's overlying unipotent elements in determining the minimal polynomials of unipotent elements on certain representation spaces. (See [20] and [21].) With our Theorem 0.1, one will be able to remove certain prime restrictions in that result. The proof of Theorem 0.1 provides as well a construction of the maximal A_1 subgroups in the exceptional simple algebraic groups over algebraically closed fields of nonzero characteristic, with certain mild restrictions on the characteristic. (See [14] and [24].) Thus, we have reason to expect that these results will be quite useful in the study of the subgroup structure and representation theory of simple groups of Lie type. Additionally, one of our preliminary results provides a formula for the order of a unipotent element in a simple algebraic group and can be used to determine the exponent of the Sylow p-subgroup of the finite fixed-point subgroups of G. (See 0.4 and 0.5.) We should mention here that the assumption that p is a good prime is not known to be necessary. However, our proof depends upon the Bala–Carter–Pommerening classification of the conjugacy classes of unipotent elements (see 0.3 below) which requires that p is a good prime.

We make a few comments about the organization of the paper here. In Section 1 we establish the tools which will be used for constructing the A_1 subgroups overlying certain elements of order p in the algebraic groups. We present these results separately because the exponentiating process which we use is not restricted to the construction of subgroups of type A_1.

In Section 2, we establish the order formula for exceptional algebraic groups and construct the A_1 subgroups overlying certain unipotent elements in the exceptional algebraic groups. We do the same thing for the classical groups in Section 3. In fact, Theorem 0.1 for the classical algebraic groups can be deduced from work in the literature. (See the opening remarks of Section 3 for a more complete discussion of this.) We include a proof here both because it is elementary and because the work is necessary for establishing the order formula. The main content of Section 4 is the proofs of Theorems 0.1 and 0.2.

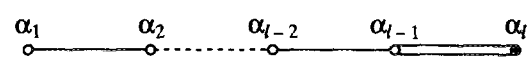
The appendix consists of explicit (matrix) descriptions of the images of certain root groups of E_7 and E_8, including those associated with the roots in a base, under the smallest dimensional nontrivial restricted irreducible rational representations of these groups. For the purposes of this paper, the given matrices were used to obtain the orders of certain distinguished unipotent elements in the exceptional algebraic groups. For further details, see Section 2. We have included this information in detail with the hopes that duplication of elementary but tedious computations can be avoided.

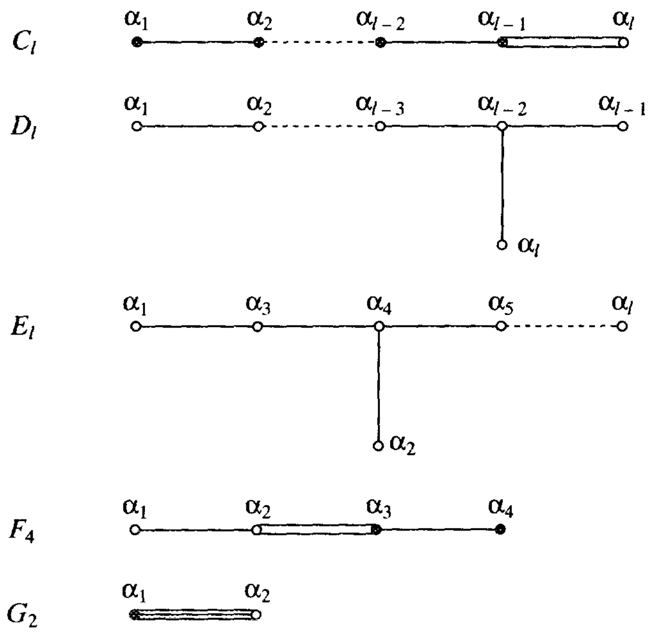
We introduce here the terminology and notation to be used throughout the paper and recall certain basic notions about the conjugacy classes of unipotent elements in simple algebraic groups.

Let G be a semisimple algebraic group defined over an algebraically closed field k of characteristic $p > 0$. We will assume throughout that p is a good prime for G, except where specifically noted. That is, $p > 2$ if G has a factor of type B_l, C_l, D_l; $p > 3$ if G has a factor of type G_2, F_4, E_6, E_7; and $p > 5$ if G has a factor of type E_8. Fix T to be a maximal torus of G and let $\Phi(G)$ denote the root system of G relative to T, and take $\Pi(G) = \{\alpha_1, \alpha_2, \ldots\}$ to be a fundamental system of $\Phi(G)$, with $\Phi^+(G)$ (respectively, $\Phi^-(G)$) the associated set of positive (negative) roots. We fix the following labeling of Dynkin diagrams, where the darkened nodes represent the short roots:

A_l

B_l





Let r_0 be the highest root in $\Phi(G)$ with respect to the partial ordering on $\Phi(G)$ induced by our choice of base $\Pi(G)$. For a subset $J \subseteq \Pi(G)$, let $\Phi(J) = \Phi(G) \cap \sum_{\alpha \in J} \mathbf{Z}\alpha$ and let $\Phi^+(J) = \Phi^+(G) \cap \sum_{\alpha \in J} \mathbf{Z}\alpha$. For $\alpha \in \Phi(G)$, let U_α denote the corresponding T-root subgroup of G and let $\{x_\alpha(t) \mid t \in k\}$ denote the collection of elements in U_α. For $q = p^a$, if σ is the endomorphism of G induced by the map $x_\alpha(t) \to x_\alpha(t^q)$, we will call σ a *p-power Frobenius* endomorphism of G. For an arbitrary field F of characteristic p, let $G(F)$ denote the adjoint Chevalley group of type $\Phi(G)$ defined over F and let T, U_α, $x_\alpha(t)$, $t \in F$, be as in the algebraic group G.

We now recall some notions regarding unipotent conjugacy classes in G. Let P be a parabolic subgroup of G with unipotent radical $R_u(P)$. Then by [13], there exists an open dense subset $\mathscr{O}$ of $R_u(P)$ which is a single P-orbit under the action of P on $R_u(P)$. (The subset $\mathscr{O}$ is called the Richardson dense orbit.) Recall that a unipotent element $u \in G$ is *distinguished* if $C_G(u)^\circ$ contains no nonidentity semisimple element. A parabolic subgroup P of G, with Levi factor L and unipotent radical $R_u(P)$, is *distinguished* if $\dim L = \dim(R_u(P)/R_u(P)')$. The classification of the conjugacy classes of unipotent elements in G can then be stated as follows:

THEOREM 0.3 [1, 2, 11, 12]. *Let G be a simple algebraic group defined over an algebraically closed field k of good characteristic.*

i. *There is a bijective map between conjugacy classes of distinguished unipotent elements of G and conjugacy classes of distinguished parabolic subgroups of G. The unipotent class corresponding to a given parabolic subgroup P contains the dense orbit of P on its unipotent radical.*

ii. *There is a bijective map between conjugacy classes of unipotent elements of G and G-classes of pairs* $(L, P_{L'})$, *where L is a Levi subgroup of G and* $P_{L'}$ *is a distinguished parabolic subgroup of the semisimple part L' of L. The unipotent class corresponding to the pair* $(L, P_{L'})$ *contains the dense orbit of* $P_{L'}$ *on its unipotent radical.*

To work with the above classification one needs to know explicitly which parabolic subgroups are distinguished. As in [5], we will parametrize each parabolic by a labeled Dynkin diagram, where we label with a zero any node corresponding to a simple root in the root system of the semisimple part of the Levi factor and label with a two any simple root for which the corresponding root group lies in the unipotent radical of the parabolic. A description of the distinguished parabolics is given in [5]. When all nodes are labeled with a two, that is, when the distinguished parabolic P is a Borel subgroup, the elements lying in the conjugacy class which contains the dense orbit of P on its radical are called *regular* unipotent elements.

We now introduce a height function on $\Phi(G)$ associated with a subset $J \subseteq \Pi(G)$; this was first defined in [24]:

$$\mathrm{ht}_J\left(\sum_{\gamma \in \Pi(G)} k_\gamma \gamma \right) = \sum_{\gamma \notin J} k_\gamma .$$

Recall, we have the ordinary height function $\mathrm{ht}(\sum k_\gamma \gamma) = \sum k_\gamma$. For a parabolic subgroup P of G corresponding to a subset $J \subseteq \Pi(G)$, we will write $\mathrm{ht}(P) = \mathrm{ht}_J(r_0)$. We can now state the

Order Formula 0.4. Let G be a simple algebraic group defined over an algebraically closed field k of good characteristic $p > 0$. Let P be a distinguished parabolic subgroup of G, and let $u \in R_u(P)$ lie in the dense orbit of P on $R_u(P)$. Then $o(u) = \min\{p^a \mid p^a > \mathrm{ht}(P)\}$.

We note that by applying Theorem 0.3 and Order Formula 0.4, we have a "formula" for the order of every unipotent element in G. We will show as well

COROLLARY 0.5. *Let G be a simple algebraic group defined over an algebraically closed field of arbitrary characteristic* $p > 0$ *and let* σ *be a* *surjective endomorphism of G such that G_σ is finite. Then the exponent of a Sylow p-subgroup of G_σ is* $\min\{p^a \mid p^a > \mathrm{ht}(r_0)\}$.

The order formula and Corollary 0.5 will be established in Sections 2 and 3. All additional notation pertains to the work of a particular section of the paper and will be introduced when necessary.

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1. EXPONENTIATING IN SMALL CHARACTERISTICS

Throughout this section, let G be a simple algebraic group. Let $\mathscr{L}_G(\mathbf{C})$ be a finite-dimensional complex simple Lie algebra of type $\Phi(G)$ with Chevalley basis $\mathscr{B} = \{e_\alpha, f_\alpha, h_\gamma \mid \alpha \in \Phi^+(G),\ \gamma \in \Pi(G)\}$. Let $\mathscr{L}_G(\mathbf{Z})$ be the $\mathbf{Z}$-span of $\mathscr{B}$. Then for any field F, we can form the F-Lie algebra $\mathscr{L}_G(F) = \mathscr{L}_G(\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{F}$. Now suppose F has characteristic $p > 0$. Let $e \in \sum_{\alpha \in \Phi^+(G)} \mathbf{Z}e_\alpha$. Then if $p > 3\,\mathrm{ht}(r_0)$, we can form an automorphism of the Lie algebra $\mathscr{L}_G(F)$ by exponentiating ad te for $t \in F$. (See Chapter 5 of [5].) In this section, we show that the restriction on p need not be this strong. Also, we are able, in certain cases, to describe the one-dimensional subgroup $\{\exp(\mathrm{ad}\, te) \mid t \in F\} \leq \mathrm{Aut}(\mathscr{L}_G(F))$.

We briefly recall some of the ideas discussed in [24]. Fix F as a field of nonzero characteristic p. Let $\mathbf{Z}_{(p)}$ be the localization of $\mathbf{Z}$ at the prime ideal $p\mathbf{Z}$. Let $\mathscr{L}_G(\mathbf{Z}_{(p)})$ be the set of $\mathbf{Z}_{(p)}$ linear combinations of elements in $\mathscr{B}$; so $\mathscr{L}_G(\mathbf{Z}_{(p)}) = \mathscr{L}_G(\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ and $\mathscr{L}_G(F) = \mathscr{L}_G(\mathbf{Z}_{(p)}) \otimes_{\mathbf{Z}_{(p)}} F$. Let $e = \sum c_\alpha e_\alpha$, where α ranges over $\Phi^+(G)$ and $c_\alpha \in \mathbf{Z}$. Suppose $(\mathrm{ad}\, e)^j/j!$ preserves $\mathscr{L}_G(\mathbf{Z}_{(p)})$ for all $j \geq 0$; that is, $[(\mathrm{ad}\, e)^j/j!](\mathscr{L}_G(\mathbf{Z}_{(p)})) \subseteq \mathscr{L}_G(\mathbf{Z}_{(p)})$ for all $j \geq 0$. Then, as described in [24, Lemma 1], we can form $x(t) = \exp(\mathrm{ad}\, te)$, $t \in F$, to obtain automorphisms of $\mathscr{L}_G(F)$. Moreover, under the assumption that p is a good prime for G, $x(t) \in G(F)$, the adjoint Chevalley group of type $\Phi(G)$. Suppose in addition that there exists $f \in \sum \mathbf{Z} f_\alpha$, where α ranges over $\Phi^+(G)$ such that $[(\mathrm{ad}\, f)^j/j!](\mathscr{L}_G(\mathbf{Z}_{(p)})) \subseteq \mathscr{L}_G(\mathbf{Z}_{(p)})$ for all $j \geq 0$ and such that e and f canonically generate an $sl_2(\mathbf{C})$ subalgebra of $\mathscr{L}_G(\mathbf{C})$. Let $y(t)$ be the automorphism of $\mathscr{L}_G(F)$ corresponding to $\exp(\mathrm{ad}\, tf)$ for $t \in F$. Then the subgroup $\langle x(t), y(t) \mid t \in F \rangle$ of $G(F)$ is isomorphic to $\mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$ (Lemma 2 of [24]). We have followed the standard construction of $(\mathrm{P})\mathrm{SL}_2$'s in Chevalley groups as exposed in [6] or [18], with the variation that we require the ad-nilpotent elements to preserve only a $\mathbf{Z}_{(p)}$-form in $\mathscr{L}_G(F)$ rather than a $\mathbf{Z}$-form. This slight weakening of a hypothesis enables us to "exponentiate" nilpotent elements which do not preserve $\mathscr{L}_G(\mathbf{Z})$. For example, in the simple Lie algebra $\mathrm{sl}_3(\mathbf{C})$ with Chevalley basis $\{e_{\alpha_i}, f_{\alpha_i}, e_{\alpha_1+\alpha_2}, f_{\alpha_1+\alpha_2}, h_{\gamma_i} \mid i = 1, 2\}$, if $e = e_{\alpha_1} + e_{\alpha_2}$, then $(\mathrm{ad}\, e)^j/j!$ does not preserve $\mathscr{L}_G(\mathbf{Z})$ for all $j \geq 0$; however, for $p > 2$ it does preserve $\mathscr{L}_G(\mathbf{Z}_{(p)})$ for all $j \geq 0$.

The above remarks lead us to ask the following question: How can one easily check that the condition

$$\left[(\mathrm{ad}\, e)^j/j!\right]\left(\mathscr{L}_G(\mathbf{Z}_{(p)})\right) \subseteq \mathscr{L}_G(\mathbf{Z}_{(p)}) \qquad \text{for all } j \geq 0$$

is satisfied? Before addressing this question, we will describe some characteristics of the (P)SL_2's obtained by exponentiating. In this way, we will restrict the set of nilpotent elements which must be considered in answering this question. We first establish a rather easy result which will be needed for passing to the finite groups.

LEMMA 1.1. *Let $e \in \mathscr{L}_G(\mathbf{C})$, $e \in \sum \mathbf{Z} e_\alpha$, where α ranges over $\Phi^+(G)$, such that*

$$\left[(\mathrm{ad}\, e)^j/j!\right]\left(\mathscr{L}_G(\mathbf{Z}_{(p)})\right) \subseteq \mathscr{L}_G(\mathbf{Z}_{(p)}), \qquad \text{for all } j \geq 0.$$

Assume further that F is an algebraically closed field of characteristic p. Let $x(t)$ be the automorphism of $\mathscr{L}_G(F)$ corresponding to $\exp(\mathrm{ad}\, te)$, *for $t \in F$. Let σ be a p-power Frobenius endomorphism of the algebraic group $G(F)$. Then the one-dimensional subgroup $\{x(t) \mid t \in F\}$ of $G(F)$ is a σ-invariant subgroup and $x(1)$ is fixed by σ.*

Proof. We first recall the construction of $\exp(\mathrm{ad}\, te)$. For $\lambda \in \mathbf{C}$, $\exp(\mathrm{ad}\, \lambda e) \in \mathrm{Aut}(\mathscr{L}_G(\mathbf{C}))$ is represented with respect to the basis $\mathscr{B}$ by a matrix $A(\lambda)$, whose entries lie in $\mathbf{Z}_{(p)}[\lambda]$. Applying the natural homomorphism of matrix rings $\mathbf{Z}_{(p)}[X] \to F[X]$, we obtain a matrix $\overline{A}(X)$ whose entries lie in $F[X]$. Then for $t \in F$, $\exp(\mathrm{ad}\, te)$ is the transformation of $\mathscr{L}_G(F)$ represented by $\overline{A}(t)$ with respect to the basis $\{v \otimes 1 \mid v \in \mathscr{B}\}$. Now view $G(F)$ as a subgroup of the group of matrices $\mathrm{GL}(\mathscr{L}_G(F))$, which itself is a subvariety of $F^{(\dim \mathscr{L}_G(F))^2}$. So for $(a) \in G(F) \leq \mathrm{GL}(\mathscr{L}_G(F))$, $\sigma((a)) = (a_{ij}^q)$ where $q = p^k$ for some $k \in \mathbf{Z}$. Say $\overline{A}(t)$ has (i, j) entry $f_{ij}(t) \in F[t]$. Then, by construction, the coefficients of f_{ij} actually lie in the prime subfield of F. So $\sigma(\overline{A}(t)) = ((f_{ij}(t)^q) = (f_{ij}(t^q))$. So $\sigma(\overline{A}(t)) = \overline{A}(t^q)$ is the matrix corresponding to the transformation $\exp(\mathrm{ad}\, t^q e)$. The result follows.

In the following lemma, we describe to some extent the unipotent elements lying in the one-dimensional group $\{x(t) \mid t \in F\} \leq \mathrm{Aut}(\mathscr{L}_G(F))$.

LEMMA 1.2. *Let $e \in \mathscr{L}_G(\mathbf{Z})$ such that $(\mathrm{ad}\, e)^j/j!$ preserves $\mathscr{L}_G(\mathbf{Z}_{(p)})$ for all $j \geq 0$. Moreover, assume that there exists $J \subseteq \Pi(G)$ with $e = \sum c_\alpha e_\alpha$,* *where the sum ranges over the roots* α *with* $\mathrm{ht}_J(\alpha) = 1$. *Let* F *be an algebraically closed field of characteristic p and let* $t \in F$. *Let* $x(t) \in G(F)$ *be the automorphism of* $\mathscr{L}_G(F)$ *corresponding to* $\exp(\mathrm{ad}\, te)$. *Then there exist an ordering of* $\Phi^+(G)$ *and polynomials* $g_\alpha \in F[\bar{x}]$, *in the indeterminant* $\bar{x}$, *such that with respect to this ordering*

$$x(t) = \prod_{\substack{\alpha \in \Phi^+(G) \\ \mathrm{ht}_J(\alpha) \geq 1}} x_\alpha(g_\alpha(t)) \text{ and } g_\alpha(\bar{x}) = c_\alpha \bar{x}$$

for all α *such that* $\mathrm{ht}_J(\alpha) = 1$.

Proof. For the purposes of this proof, we view $G(F)$ as a subgroup of $\mathrm{Aut}(\mathscr{L}_G(F))$ and make the following simplification of notation. While $\mathscr{L}_G(F)$ has the natural basis $\{v \otimes 1 \mid v \in \mathscr{B}\}$, we will suppress the tensor and use $\mathscr{B}$ itself as a basis for $\mathscr{L}_G(F)$. So $G(F)$ will act on the basis vectors e_α, f_α, h_γ. For $\alpha \in \Phi^+(G)$, let L_α (respectively $L_{-\alpha}$) denote the root-subspace $\langle e_\alpha \rangle$ (respectively, $\langle f_\alpha \rangle$). Let $\mathscr{N}^+$ (respectively, $\mathscr{N}^-$) denote the subspace of $\mathscr{L}_G(F)$ generated by the collection of root-subspaces L_α for $\alpha \in \Phi^+(G)$ (respectively, $\Phi^-(G)$). Recall that we have T a maximal torus of $G(F)$ with corresponding root subgroups U_α, $\alpha \in \Phi(G)$.

Fix $t_0 \in F^*$ and write $u = x(t_0) = b_1 n b_2$ where $b_i \in T\langle U_\alpha \mid \alpha \in \Phi^+(G)\rangle$ and $n \in N_{G(F)}(T)$, the Bruhat expression for u. (See 8.2.3 in [6].)

Claim: $n \in t$. Suppose not. Then if nT represents the nonidentity element w of the Weyl group $W = N_{G(F)}(T)/T$, there exists $\alpha \in \Phi^+(G)$ such that $w(\alpha) \in \Phi^-(G)$. Choose $\alpha_w \in \Phi^+(G)$ of maximal height such that $w(\alpha_w) \in \Phi^-(G)$. We have

$$b_2 e_{\alpha_w} = c e_{\alpha_w} + \sum_{\{\gamma \mid \mathrm{ht}(\gamma) > \mathrm{ht}(\alpha_w)\}} t_\gamma e_\gamma,$$

for some $c \in F^*$ and $t_\gamma \in F$. Then, by our choice of α_w,

$$n b_2 e_{\alpha_w} = d f_{-w(\alpha_w)} + X,$$

for some $d \in F^*$, $X \in \mathscr{N}^+$. Thus,

$$u e_{\alpha_w} = a f_{-w(\alpha_w)} + \left(\sum_{\substack{\gamma \in \Phi^+(G) \\ \gamma \neq -w(\alpha_w)}} a_\gamma f_\gamma \right) + Y,$$

for some $a \in F^*$, $a_\gamma \in F$, $Y \in \mathscr{N}^+$. So $u e_{\alpha_w} \notin \mathscr{N}^+$. However, this contradicts the fact that $[(\mathrm{ad}\, e)^j / j!](e_\delta) \in \mathscr{N}^+$ for all $\delta \in \Phi^+(G)$, for all $j \geq 0$. Thus the claim holds.

By the previous claim, we have u a unipotent element in $T\langle U_\alpha \mid \alpha \in \Phi^+(G)\rangle$, and thus $u \in \langle U_\alpha \mid \alpha \in \Phi^+(G)\rangle$. Now introduce a partial ordering on $\Phi^+(G)$ and hence a partial ordering on $\{U_\alpha \mid \alpha \in \Phi^+(G)\}$, such that $u = u_0 u_1 \cdots u_r$ where $u_i \in \prod\{U_\alpha \mid \alpha \in \Phi^+(G), \mathrm{ht}_J(\alpha) = i\}$.

Claim: $u_0 = 1$. Suppose not. Write $u_0 = \prod_{i=1}^{s} x_{\beta_i}(d_{\beta_i})$, where $\mathrm{ht}_J(\beta_i) = 0$ and $d_{\beta_i} \neq 0$ for all i. Without loss of generality, we may assume that if $i < k$ then $\mathrm{ht}(\beta_i) \leq \mathrm{ht}(\beta_k)$. For $k > 0$, $u_k f_{\beta_1} - f_{\beta_1} \in \langle e_\alpha \mid \alpha \in \Phi^+\rangle$. Also, for $k > 1$, $x_{\beta_k}(d_{\beta_k}) f_{\beta_1} - f_{\beta_1} \in \langle e_\alpha \mid \alpha \in \Phi^+\rangle$ since $\mathrm{ht}(\beta_k - \beta_1) = \mathrm{ht}(\beta_k) - \mathrm{ht}(\beta_1) \geq 0$ and $\beta_k \neq \beta_1$. So $u f_{\beta_1} = f_{\beta_1} + d_{\beta_1}[e_{\beta_1}, f_{\beta_1}] + X$ for some $X \in \mathcal{N}^+$. However, since $e \in \langle e_r \mid \mathrm{ht}_J(r) = 1\rangle$, $[(\mathrm{ad}\, e)^j/j!)](f_{\beta_1}) \in \mathcal{N}^+$ for $j > 0$, so $\exp(\mathrm{ad}\, e) f_{\beta_1} - f_{\beta_1} \in \mathcal{N}^+$. Thus we have arrived at a contradiction and the claim holds.

We have shown that for any choice of $t \in F$, $x(t) \in \langle U_\alpha \mid \alpha \in \Phi^+(G), \mathrm{ht}_J(\alpha) \geq 1\rangle$. Since $\{x(t) \mid t \in F\}$ is a closed one-dimensional unipotent subgroup of $\langle U_\alpha \mid \alpha \in \Phi^+(G), \mathrm{ht}_J(\alpha) \geq 1\rangle$, we have

$$x(t) = \prod_{\substack{\alpha \in \Phi^+(G) \\ \mathrm{ht}_J(\alpha) \geq 1}} x_\alpha(g_\alpha(t))$$

for some polynomials $g_\alpha \in F[\bar{x}]$. Moreover, we may write

$$x(t) = \left(\prod_{\substack{\alpha \in \Phi^+(G) \\ \mathrm{ht}_J(\alpha) = 1}} x_\alpha(g_\alpha(t)) \right) y(t),$$

where $y(t) \in \langle U_\alpha \mid \mathrm{ht}_J(\alpha) > 1\rangle$. Now fix $\alpha \in \Phi^+(G)$ with $\mathrm{ht}_J(\alpha) = 1$ and note that $y(t)(f_\alpha) - f_\alpha \in \mathcal{N}^+$. For $r \in \Phi^+(G)$ with $\mathrm{ht}_J(r) = 1$, $x_r(c) f_\alpha - f_\alpha \in L_{r-\alpha} + \mathcal{N}^+$, if $r \neq \alpha$, and $x_r(c) f_\alpha - f_\alpha = c[e_\alpha, f_\alpha] + X$, for $X \in \mathcal{N}^+$, if $r = \alpha$. Also, for $\beta \in \Phi(G)$ with $\mathrm{ht}_J(\beta) = 0$, and for $r \in \Phi^+(G)$ with $\mathrm{ht}_J(r) = 1$, $x_r(c) L_\beta \subseteq L_\beta + \mathcal{N}^+$. Thus,

$$x(t) f_\alpha - f_\alpha = g_\alpha(t) h_\alpha + X + Y$$
$$\text{where } X \in \mathcal{N}^+ \text{ and } Y \in \langle L_{-\beta} \mid \beta \in \Phi^+(G), \mathrm{ht}_J(\beta) = 0\rangle.$$

By similar reasoning, we have that $\exp(\mathrm{ad}\, te) f_\alpha = f_\alpha + c_\alpha t[e_\alpha, f_\alpha] + X' + Y'$, where $X' \in \mathcal{N}^+$ and $Y' \in \langle L_{-\beta} \mid \beta \in \Phi^+(G), \mathrm{ht}_J(\beta) = 0\rangle$. Thus the lemma follows.

The following result is a group-theoretic version of Proposition 5.8.5 of [5].

LEMMA 1.3. *Let $P \supseteq T\langle U_\alpha \mid \alpha \in \Phi^+(G)\rangle$ be a distinguished parabolic subgroup of G corresponding to a subset $J \subset \Pi(G)$. Let*

$$u = \prod_{\substack{\alpha \in \Phi^+(G) \\ \mathrm{ht}_J(\alpha) \geq 1}} x_\alpha(d_\alpha)$$

lie in the dense orbit of P on $R_u(P)$. Then the $R_u(P)$-orbit containing u consists precisely of elements of the form $\prod x_\alpha(c_\alpha)$ for which $c_\alpha = d_\alpha$ for all α with $\mathrm{ht}_J(\alpha) = 1$. In particular, the element

$$\prod_{\substack{\alpha \in \Phi^+(G) \\ \mathrm{ht}_J(\alpha) = 1}} x_\alpha(d_\alpha)$$

lies in the same $R_u(P)$-orbit as u.

Proof. See (4.5) of [14].

We can now explain how we choose the ad-nilpotent elements we will need to "exponentiate". The classification of unipotent conjugacy classes (0.3) shows that for the proof of Theorem 0.1 it will suffice to construct an A_1 subgroup overlying each distinguished unipotent element of order p in every semisimple algebraic group. Suppose we are given a distinguished unipotent element u lying in the dense orbit of P on its radical $R_u(P)$, for some distinguished parabolic P of G, corresponding to the subset $J \subseteq \Pi(G)$. By (1.3), it will suffice to produce an A_1 overlying a unipotent element $x \in R_u(P)$ where x has coordinates in the root groups U_α, for $\mathrm{ht}_J(\alpha) = 1$, which match the coordinates of u in these root groups. On the other hand, if these coordinates happen to lie in $\mathbf{Z}$, using (1.2) we choose an ad-nilpotent $e \in \mathscr{L}_G(\mathbf{Z})$ such that if $(\mathrm{ad}\, e)^j/j!$ preserves $\mathscr{L}_G(Z_{(p)})$ for all $j \geq 0$, then we can form the exponential of ad e and this exponential will indeed have the same coordinates as u in the root groups U_α with $\mathrm{ht}_J(\alpha) = 1$.

Finally, we address the question of how to determine if $[(\mathrm{ad}\, e)^j/j!](\mathscr{L}_G(\mathbf{Z}_{(p)})) \subseteq \mathscr{L}_G(\mathbf{Z}_{(p)})$ for all $j \geq 0$. This was considered in [24], where we constructed the maximal A_1 subgroups of the exceptional algebraic groups. We include here the crucial result from that paper, along with a sketch of its proof.

LEMMA 1.4. *Let $e \in \sum_{\alpha \in \Phi^+(G) - \Phi(J)} \mathbf{Z}e_\alpha$, for some $J \subseteq \Pi(G)$. Assume*

$$e = e_1 + e_2 \qquad \text{where } e_i \in \sum_{\alpha \in \Phi^+(G) - \Phi(J_i)} \mathbf{Z}e_\alpha$$

for some $J_i \subseteq \Pi(G)$ with $p > \mathrm{ht}_J(r_0)$ and $p > 2\,\mathrm{ht}_{J_i}(r_0)$, for $i = 1, 2$. Then $[(\mathrm{ad}\, e)^j/j!]$ preserves $\mathscr{L}_G(\mathbf{Z}_{(p)})$ for all $j \geq 0$.

Sketch of Proof. Let $n \in \sum_{\alpha \in \Phi^+(G)-\Phi(J)} \mathbf{C}e_\alpha$. We first note that

$$\text{if } j \geq \mathrm{ht}_J(r_0), \text{ then } (\mathrm{ad}\, n)^j(e_\gamma) = 0 \text{ for all } \gamma \in \Phi^+(G) - \Phi(J), \quad (1)$$

and

$$\text{if } j > 2\,\mathrm{ht}_J(r_0) \text{ then } (\mathrm{ad}\, n)^j = 0. \quad (2)$$

Therefore, $(\mathrm{ad}\, n)^{2p} = 0$. Thus, the only possible p-divisible denominators in a term $(\mathrm{ad}\, e)^j/j!$ arise from the term $j = p$ itself. So for e as given, it suffices to show that $(\mathrm{ad}\, e)^p/p!$ preserves $L_G(\mathbf{Z}_{(p)})$.

Let $g(x, y) \in \mathbf{Z}[x, y]$ such that $\sum_{i=1}^{p-1}(1/p)\binom{p}{i}x^i y^{p-i} = (x-y)g(x, y)$. Then in any associative algebra $\mathscr{A}$, we have

$$(a+b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a, b), \quad (3)$$

where for $i = 1, \dots, p-1$, $is_i(a, b)$ is the coefficient of λ^{i-1} in

$$(\mathrm{ad}(\lambda a + b))^{p-1}(a) - pg((\lambda a + b)_L, (\lambda a + b)_R)(a).$$

Here a, $b \in \mathscr{A}$ and, for $A \in \mathscr{A}$, we write A_L, A_R for the left and right multiplications determined by A.

Let $\mathscr{A} = \mathrm{gl}(\mathscr{U})$, where $\mathscr{U}$ is the universal enveloping algebra of $\mathscr{L}_G(\mathbf{C})$. Now take $a = \mathrm{ad}\, e_1$ and $b = \mathrm{ad}\, e_2$ in (3). Then we have

$$(\mathrm{ad}\, e)^p = (\mathrm{ad}\, e_1)^p + (\mathrm{ad}\, e_2)^p + \sum_{i=1}^{p-1} s_i(\mathrm{ad}\, e_1, \mathrm{ad}\, e_2), \quad (4)$$

where $s_i(\mathrm{ad}\, e_1, \mathrm{ad}\, e_2)$ is $(1/i)$ times the coefficient of λ^{i-1} in

$$\begin{aligned}
&(\mathrm{ad}_{\mathscr{A}}(\lambda\,\mathrm{ad}\, e_1 + \mathrm{ad}\, e_2))^{p-1}(\mathrm{ad}\, e_1) \\
&\quad - pg((\lambda\,\mathrm{ad}\, e_1 + \mathrm{ad}\, e_2)_L, (\lambda\,\mathrm{ad}\, e_1 + \mathrm{ad}\, e_2)_R)(\mathrm{ad}\, e_1) \\
&\quad = \mathrm{ad}\big((\mathrm{ad}(\lambda e_1 + e_2))^{p-1}(e_1)\big) \\
&\qquad - pg((\lambda\,\mathrm{ad}\, e_1 + \mathrm{ad}\, e_2)_L, (\lambda\,\mathrm{ad}\, e_1 + \mathrm{ad}\, e_2)_R)(\mathrm{ad}\, e_1).
\end{aligned}$$

But applying (1) to $\lambda e_1 + e_2$ in place of n, and recalling the fact that $p > \mathrm{ht}_J(r_0)$, we have $\mathrm{ad}(\lambda e_1 + e_2)^{p-1}(e_1) = 0$. So for $1 \leq i \leq p-1$,

$$\begin{aligned}
s_i(\mathrm{ad}\, e_1, \mathrm{ad}\, e_2) = -(p/i) &\text{ coefficient of } \lambda^{i-1} \text{ in} \\
&g((\lambda\,\mathrm{ad}\, e_1 + \mathrm{ad}\, e_2)_L, (\lambda\,\mathrm{ad}\, e_1 + \mathrm{ad}\, e_2)_R)(\mathrm{ad}\, e_1).
\end{aligned}$$

So dividing by $p!$ gives something which clearly preserves $\mathscr{L}_G(\mathbf{Z}_{(p)})$. Finally, we note that since $e_i \in \sum_{\alpha \in \Phi^+(G) - \Phi(J)} \mathbf{Z}e_\alpha$ and $p > 2\,\mathrm{ht}_{J_i}(r_0)$, (2) implies that $(\mathrm{ad}\, e_i)^p = 0$, for $i = 1, 2$. So, by (4), $(\mathrm{ad}\, e)^p/p!$ preserves $\mathscr{L}_G(\mathbf{Z}_{(p)})$.

Now as described prior to Lemma 1.4, if we are given a distinguished parabolic subgroup P corresponding to a subset J and u lying in the dense orbit of P on $R_u(P)$, we will choose $e \in \mathscr{L}_G(\mathbf{Z})$ such that $e \in \sum_{\alpha \in \Phi^+(G) - \Phi(J)} \mathbf{Z}e_\alpha$. Then e will satisfy the hypotheses of Lemma 1.4 if

$$p > \mathrm{ht}_J(r_0) \text{ and } p > 2\,\mathrm{ht}_{J_i}(r_0) \text{ for some } J_1, J_2 \subsetneq \Pi(G)$$
$$\text{with } \Pi(G) = J_1 \cup J_2 \text{ and } J_1 \cap J_2 = J. \qquad (\dagger)$$

In fact, the condition (†) does not hold for every $J \subsetneq \Pi(G)$ corresponding to a distinguished parabolic subgroup of G. However, our inductive proof will require the use of Lemma 1.4 for only a few distinguished parabolic subgroups, and in these cases we shall exhibit the appropriate J_1 and J_2.

We point out the following corollary of Lemma 1.4.

PROPOSITION 1.5. *Let $e \in \sum_{\alpha \in \Phi^+(G)} \mathbf{Z}e_\alpha$ and suppose $p > \mathrm{ht}(r_0)$. Then*

$$\left[(\mathrm{ad}\, e)^j/j!\right]\left(\mathscr{L}_G(\mathbf{Z}_{(p)})\right) \subseteq \mathscr{L}_G(\mathbf{Z}_{(p)})$$

for all $j \geq 0$.

Proof. The result is clear if $\Phi(G)$ is a rank 1 root system. Otherwise, this follows from Lemma 1.4, taking J to be the empty set, choosing $J_1 \subsetneq \Pi(G)$, as indicated, and setting $J_2 = \Pi(G) \setminus J_1$.

$G = A_n, B_n, C_n, D_n, \quad J_1 = \{\alpha_1, \alpha_3, \dots\} = \left\{\alpha_{2j-1} \mid 1 \leq j \leq \left[\tfrac{1}{2}(n+1)\right]\right\}.$

$G = G_2, \quad J_1 = \{\alpha_1\}.$

$G = F_4, \quad J_1 = \{\alpha_1, \alpha_2\}.$

$G = E_6, \quad J_1 = \{\alpha_1, \alpha_2, \alpha_3\}.$

$G = E_7, \quad J_1 = \{\alpha_1, \alpha_3, \alpha_4\}.$

$G = E_8, \quad J_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}.$

2. A_1 SUBGROUPS AND UNIPOTENT ELEMENTS IN EXCEPTIONAL ALGEBRAIC GROUPS

In this section we consider distinguished unipotent elements in the exceptional algebraic groups over algebraically closed fields of good characteristic p. We establish (0.4) and the existence of p-power Frobenius invariant A_1 subgroups overlying certain distinguished elements of order p. We first divide the unipotent conjugacy classes into two subsets, each of which will be considered separately.

LEMMA 2.1. *Let G be an exceptional algebraic group defined over an algebraically closed field of good characteristic p. Let u be a distinguished unipotent element in G. Then, either u lies in a maximal rank semisimple subgroup of G or one of the following holds, where $\sim$ denotes G-conjugacy and the given labeled diagram indicates a distinguished parabolic subgroup P for which u lies in the dense orbit of P on $R_u(P)$.*

(i) $G = G_2$, $u \sim x_{\alpha_1}(1)x_{\alpha_2}(1)$; $\begin{array}{cc} 2 & 2 \\ \bullet & \circ \end{array}$.

(ii) $G = F_4$, $u \sim x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)$; $\begin{array}{cccc} 2 & 2 & 2 & 2 \\ \circ & \circ & \bullet & \bullet \end{array}$.

(iii) $G = E_6$, $u \sim x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1)$; $\begin{array}{ccccc} & & 2 & & \\ 2 & 2 & \circ & 2 & 2 \\ \circ & \circ & \circ & \circ & \circ \\ & & 2 & & \end{array}$.

(iv) $G = E_6$, $u \sim x_{\alpha_2}(1)x_{\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1)x_{\alpha_1+\alpha_3}(1)x_{\alpha_3+\alpha_4+\alpha_5}(1) \sim x_{\alpha_2}(1)x_{\alpha_3+\alpha_4}(-1)x_{\alpha_5}(1)x_{\alpha_6}(1)x_{\alpha_1}(1)x_{\alpha_4+\alpha_5}(-1)$; $\begin{array}{ccccc} & & 0 & & \\ 2 & 2 & \circ & 2 & 2 \\ \circ & \circ & \circ & \circ & \circ \\ & & 2 & & \end{array}$.

(v) $G = E_7$, $u \sim x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1)x_{\alpha_7}(1)$;

$\begin{array}{cccccc} & & 2 & & & \\ 2 & 2 & \circ & 2 & 2 & 2 \\ \circ & \circ & \circ & \circ & \circ & \circ \\ & & 2 & & & \end{array}$.

(vi) $G = E_7$, $u \sim x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_3+\alpha_4}(1)x_{\alpha_2+\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1)x_{\alpha_7}(1)$;

$\begin{array}{cccccc} & & 0 & & & \\ 2 & 2 & \circ & 2 & 2 & 2 \\ \circ & \circ & \circ & \circ & \circ & \circ \\ & & 2 & & & \end{array}$.

(vii) $G = E_7$, $u \sim x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_2+\alpha_4}(1)x_{\alpha_4+\alpha_5}(1)x_{\alpha_5+\alpha_6}(1)x_{\alpha_6+\alpha_7}(1)$;

$\begin{array}{cccccc} & & 0 & & & \\ 2 & 2 & \circ & 2 & 0 & 2 \\ \circ & \circ & \circ & \circ & \circ & \circ \\ & & 2 & & & \end{array}$.

(viii) $G = E_8$, $u \sim x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1)x_{\alpha_7}(1)x_{\alpha_8}(1)$;

$\begin{array}{ccccccc} & & 2 & & & & \\ 2 & 2 & \circ & 2 & 2 & 2 & 2 \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ & & 2 & & & & \end{array}$;

(ix) $G = E_8$, $u \sim x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_2+\alpha_4}(1)x_{\alpha_3+\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1)x_{\alpha_7}(1)x_{\alpha_8}(1)$;

$\begin{array}{ccccccc} & & 0 & & & & \\ 2 & 2 & \circ & 2 & 2 & 2 & 2 \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ & & 2 & & & & \end{array}$.

(x) $G = E_8$,

$$u \sim x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_2}(1)x_{\alpha_2+\alpha_4}(1)x_{\alpha_4+\alpha_5}(1)x_{\alpha_5+\alpha_6}(1)x_{\alpha_6+\alpha_7}(1)x_{\alpha_8}(1);$$

```
         0
2   2    ○   2   0   2   2
○   ○    ○   ○   ○   ○   ○ .
         2
```

Proof. This follows directly from [15] for type F_4, from [19] for type G_2, and from [9] and [10] for type E_n except for (iv). The first representative in (iv) is listed in [9] and the second can be obtained by conjugating by $(n_{\alpha_3})^{-1}$, an element in $N_G(T)$ corresponding to the Weyl group reflection w_{α_3}. (It is necessary to use the structure constants in [7] and refer to [6] for the relevant formulae.)

We now consider the classes (i)–(x) described in the above lemma.

PROPOSITION 2.2. *Let G be a simple algebraic group defined over an algebraically closed field of arbitrary (that is, not necessarily good) positive characteristic p.*

(i) *Let $G = G_2$. Then $o(x_{\alpha_1}(1)x_{\alpha_2}(1)) = 2^3, 3^2, 5^2, p$, for $p = 2, 3, 5$, $p > 5$, respectively.*

(ii) *Let $G = F_4$. Then $o(x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)) = 2^4, 3^3, 5^2, 7^2, 11^2, p$, for $p = 2, 3, 5, 7, 11$, $p > 11$, respectively.*

(iii) *Let $G = E_6$. Then $o(x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1)) = 2^4, 3^3, 5^2, 7^2, 11^2, p$, for $p = 2, 3, 5, 7, 11$, $p > 11$, respectively.*

(iv) *Let $G = E_6$. Then $o(x_{\alpha_2}(1)x_{\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1)x_{\alpha_1+\alpha_3}(1)x_{\alpha_3+\alpha_4+\alpha_5}(1)) = 2^4, 3^2, 5^2, 7^2, p$, for $p = 2, 3, 5, 7$, $p > 7$, respectively.*

(v) *Let $G = E_7$. Then $o(x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1) \times x_{\alpha_7}(1)) = 2^5, 3^3, 5^2, 7^2, 11^2, 13^2, 17^2, p$, for $p = 2, 3, 5, 7, 11, 13, 17$, $p > 17$, respectively.*

(vi) *Let $G = E_7$. Then $o(x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_3+\alpha_4}(1)x_{\alpha_2+\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1) \times x_{\alpha_7}(1)) = 2^4, 3^3, 5^2, 7^2, 11^2, 13^2, p$, for $p = 2, 3, 5, 7, 11, 13$, $p > 13$, respectively.*

(vii) *Let $G = E_7$. Then $o(x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_2+\alpha_4}(1)x_{\alpha_4+\alpha_5}(1)x_{\alpha_5+\alpha_6}(1) \times x_{\alpha_6+\alpha_7}(1)) = 2^4, 3^3, 5^2, 7^2, 11^2, p$, for $p = 2, 3, 5, 7, 11$, $p > 11$, respectively.*

(viii) *Let $G = E_8$. Then $o(x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1)x_{\alpha_7}(1) \times x_{\alpha_8}(1)) = 2^5, 3^4, 5^3, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, p$, for $p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29$, $p > 29$, respectively.*

(ix) *Let $G = E_8$. Then $o(x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_2+\alpha_4}(1)x_{\alpha_3+\alpha_4}(1)x_{\alpha_5}(1)x_{\alpha_6}(1)x_{\alpha_7}(1) \times x_{\alpha_8}(1)) = 2^5, 3^3, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, p$, for $p = 2, 3, 5, 7, 11, 13, 17, 19, 23$, $p > 23$, respectively.*

(x) *Let $G = E_8$. Then $o(x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_2}(1)x_{\alpha_2+\alpha_4}(1)x_{\alpha_4+\alpha_5}(1)x_{\alpha_5+\alpha_6}(1) \times x_{\alpha_6+\alpha_7}(1)x_{\alpha_8}(1)) = 2^5, 3^3, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, p$, for $p = 2, 3, 5, 7, 11, 13, 17, 19$, $p > 19$, respectively.*

Proof. We first note that if we view F_4 as lying naturally in E_6, the two elements described in (ii) and (iii) above are in fact conjugate in E_6. (That is, the regular unipotent elements in F_4 will also be regular in E_6.) Thus, we need not consider F_4 at all. For each of the remaining groups, we choose a low-dimensional nontrivial rational representation $\phi: G \to GL(V)$. In particular, for G of type G_2, E_6, E_7, E_8, V is the Weyl module of high weight λ_1, λ_1, λ_7, λ_8, respectively (where λ_i is the fundamental dominant weight corresponding to α_i). For $\alpha \in \Phi(G)$, such that $x_\alpha(t)$ occurs in the factorization of one of these elements, we compute the action of $\phi(x_\alpha(t))$ on V, and fixing an ordered basis of V, obtain a matrix with entries in $\mathbf{Z}[t]$ corresponding to $\phi(x_\alpha(t))$. For G of type G_2 these 7×7 matrices are given in [23, p. 43]. For G of type E_6, the computation was carried out for the work of [22]. We include the necessary matrices here. For the ordered basis of V, see [22, p. 316]. Here E_{ij} is the 27×27 matrix whose kl entry is $\delta_{ik}\delta_{jl}$ and I denotes the 27×27 identity matrix.

$$\phi(x_{\alpha_1}(t)) = I + t(E_{1,2} + E_{12,15} + E_{13,16} + E_{17,18} + E_{19,20} + E_{21,22});$$

$$\phi(x_{\alpha_2}(t)) = I + t(-E_{4,5} - E_{6,8} - E_{7,9} - E_{19,21} - E_{20,22} - E_{23,24});$$

$$\phi(x_{\alpha_3}(t)) = I + t(E_{2,3} - E_{10,12} - E_{11,13} - E_{14,17} + E_{20,23} + E_{22,24});$$

$$\phi(x_{\alpha_4}(t)) = I + t(E_{3,4} - E_{8,10} - E_{9,11} - E_{17,19} - E_{18,20} + E_{24,25});$$

$$\phi(x_{\alpha_5}(t)) = I + t(E_{4,6} + E_{5,8} - E_{11,14} - E_{13,17} - E_{16,18} + E_{25,26});$$

$$\phi(x_{\alpha_6}(t)) = I + t(E_{6,7} + E_{8,9} + E_{10,11} + E_{12,13} + E_{15,16} + E_{26,27});$$

$$\phi(x_{\alpha_1+\alpha_3}(t)) = I + t(E_{1,3} + E_{10,15} + E_{11,16} + E_{14,18} + E_{19,23} + E_{21,24});$$

$$\phi(x_{\alpha_3+\alpha_4+\alpha_5}(t)) = I + t(E_{2,6} + E_{5,12} + E_{9,17} + E_{11,19} + E_{16,23} + E_{22,26}).$$

For G of type E_7 and E_8, the ordered bases and matrices are given in the appendix of this paper.

We then use the matrices $\phi(x_\alpha(t))$ as input for a computer program written by Voirol. Given an element of the form $u = \Pi_{\alpha \in S} x_\alpha(1)$, for some subset $S \subseteq \Phi(G)$, this program:

(1) forms the appropriate product of matrices to yield a matrix $A(u)$ representing the action of $\phi(u)$ on V, and

(2) computes $\overline{A}(u)^{p^k}$ for all primes p and for all $k \geq 1$ such that $p^k \leq \dim V$, where $\overline{A}(u)$ is the image of $A(u)$ under the natural homomorphism $\mathrm{GL}_{\dim V}(\mathbf{Z}) \to \mathrm{GL}_{\dim V}(\mathbf{F}_p)$, where $\mathbf{F}_p$ is the finite field of order p.

For primes p and integers $l \in \mathbf{Z}$ such that $p^l > \dim V$, we know that $\overline{A}(u)^{p^l} = I$, since the maximal order of a unipotent element in $\mathrm{GL}(V)$ is the order of the regular unipotent element, that is, the minimal p^a such that $p^a \geq \dim V$. Thus, we have (i)–(x).

We include here the highest roots for each of the exceptional-type root systems:

G_2: $3\alpha_1 + 2\alpha_2$
F_4: $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
E_6: $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7: $2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8: $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$

Then an immediate consequence of (2.1) and (2.2) is

PROPOSITION 2.3. *Let G be an exceptional algebraic group defined over an algebraically closed field of good characteristic p. Let u be one of the distinguished unipotent elements listed in* (2.1), *which lies in the dense orbit of a distinguished parabolic subgroup P on its radical $R_u(P)$. Then* $o(u) = \min\{p^a \mid p^a > \mathrm{ht}(P)\}$.

We now turn to the construction of the A_1 subgroups overlying the 10 distinguished elements of (2.1). At this moment it is not clear why we need only construct A_1 subgroups overlying these 10 elements. It will be shown in Section 4 that each of the other distinguished unipotent elements lies in a σ-invariant connected reductive subgroup of G.

PROPOSITION 2.4. *Let G be an exceptional algebraic group of adjoint type defined over an algebraically closed field k of good characteristic p and let σ be a p-power Frobenius endomorphism of G. Let u be one of the* 10 *elements listed in* (2.1) *and assume $o(u) = p$. Then there exists a closed connected subgroup X of G with X of type A_1, $\sigma(X) \subseteq X$, and $u \in X$.*

Proof. We will apply the exponentiating process described in Section 1. For this, we first describe for each u an appropriate $\mathrm{sl}_2(\mathbf{C})$ subalgebra of $\mathscr{L}_G(\mathbf{C})$. Again, we remark that since the regular unipotent element in F_4 is regular in E_6 and we may take $F_4 \subset E_6$ to be σ-invariant, we will not need to consider the regular unipotent element in E_6, that is, the element of Lemma (2.1)(iii). Recall that $\mathscr{B} = \{e_\alpha, f_\alpha, h_\gamma \mid \alpha \in \Phi^+(G), \gamma \in \Pi(G)\}$ is a Chevalley basis of $\mathscr{L}_G(\mathbf{C})$. We also note that we use a set of structure constants for $\mathscr{L}_{E_8}(\mathbf{C})$ constructed by Gilkey and Seitz for [7]. It is available from the authors upon request.

Given u as in Lemma 2.1, let $J \subseteq \Pi(G)$ be the set of roots with a zero labeling in the diagram given for u. We then take $e = \sum c_\alpha e_\alpha$, where the sum ranges over α with $\mathrm{ht}_J(\alpha) = 1$. Moreover, we choose $c_\alpha \in \mathbf{Z}$ to be exactly the coordinate of u in the root group U_α. We then set $f = \sum d_\alpha f_\alpha$, where again the sum ranges over the α with $\mathrm{ht}_J(\alpha) = 1$ and use the identities $[[e, f], e] = 2e$ and $[[e, f], f] = -2f$ to solve for the d_α. One checks that the e, f given in each case do indeed "canonically" generate an $\mathrm{sl}_2(\mathbf{C})$ subalgebra of $\mathscr{L}_G(\mathbf{C})$. For the "nonregular" cases, we give as well subsets J, J_1, $J_2 \subseteq \Pi(G)$, satisfying the condition (†) following Lemma 1.4, which will be needed in what follows.

$G = G_2$ and u is as in Lemma 2.1(i):
$e = e_{\alpha_1} + e_{\alpha_2}$,
$f = 6f_{\alpha_1} + 10f_{\alpha_2}$.

$G = F_4$ and u is as in Lemma 2.1(ii):
$e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4}$,
$f = 22f_{\alpha_1} + 42f_{\alpha_2} + 30f_{\alpha_3} + 16f_{\alpha_4}$.

$G = E_6$ and u is as in Lemma 2.1(iv):
$e = e_{\alpha_1} + e_{\alpha_2} - e_{\alpha_3+\alpha_4} - e_{\alpha_4+\alpha_5} + e_{\alpha_5} + e_{\alpha_6}$,
$f = 12f_{\alpha_1} + 16f_{\alpha_2} - 8f_{\alpha_2+\alpha_4} + 14f_{\alpha_3} - 22f_{\alpha_3+\alpha_4} - 8f_{\alpha_4+\alpha_5} + 14f_{\alpha_5} + 12f_{\alpha_6}$,
$J = \{\alpha_4\}$, $J_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $J_2 = \{\alpha_4, \alpha_5, \alpha_6\}$.

$G = E_7$ and u is as in Lemma 2.1(v):
$e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7}$
$f = 34f_{\alpha_1} + 49f_{\alpha_2} + 66f_{\alpha_3} + 96f_{\alpha_4} + 75f_{\alpha_5} + 52f_{\alpha_6} + 27f_{\alpha_7}$.

$G = E_7$ and u is as in Lemma 2.1(vi):
$e = e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_3+\alpha_4} + e_{\alpha_2+\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7}$,
$f = 26f_{\alpha_1} - 15f_{\alpha_2} + 37f_{\alpha_2+\alpha_4} + 15f_{\alpha_3} + 35f_{\alpha_3+\alpha_4}$
$\quad + 57f_{\alpha_5} + 35f_{\alpha_4+\alpha_5} + 40f_{\alpha_6} + 21f_{\alpha_7}$,
$J = \{\alpha_4\}$, $J_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $J_2 = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7\}$.

$G = E_7$ and u is as in Lemma 2.1(vii):
$e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_2+\alpha_4} + e_{\alpha_4+\alpha_5} + e_{\alpha_5+\alpha_6} + e_{\alpha_6+\alpha_7}$,
$f = 22f_{\alpha_1} + 3f_{\alpha_2} + 42f_{\alpha_3} - 28f_{\alpha_3+\alpha_4} + 28f_{\alpha_2+\alpha_4} + 32f_{\alpha_4+\alpha_5}$
$\quad + 3f\alpha_5 + 15f_{\alpha_5+\alpha_6} + 17f_{\alpha_6+\alpha_7} + 3f_{\alpha_7}$
$J = \{\alpha_4, \alpha_6\}$, $J_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$, $J_2 = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$.

$G = E_8$ and u is as in Lemma 2.1(viii):
$e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7} + e_{\alpha_8}$,
$f = 92f_{\alpha_1} + 136f_{\alpha_2} + 182f_{\alpha_3} + 270f_{\alpha_4} + 220f_{\alpha_5} + 168f_{\alpha_6} + 114f_{\alpha_7} + 58f_{\alpha_8}$.

$G = E_8$ and u is as in Lemma 2.1(ix):
$e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_2+\alpha_4} + e_{\alpha_3+\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7} + e_{\alpha_8}$,
$f = 72f_{\alpha_1} + 38f_{\alpha_2} + 68f_{\alpha_2+\alpha_4} - 38f_{\alpha_3} + 142f_{\alpha_3+\alpha_4}$
$\quad + 172f_{\alpha_5} + 68f_{\alpha_4+\alpha_5} + 132f_{\alpha_6} + 90f_{\alpha_7} + 46f_{\alpha_8}$,
$J = \{\alpha_4\}$, $J_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $J_2 = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$.

$G = E_8$ and u is as in Lemma 2.1(x):
$e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_2+\alpha_4} + e_{\alpha_4+\alpha_5} + e_{\alpha_5+\alpha_6} + e_{\alpha_6+\alpha_7} + e_{\alpha_8}$,
$f = 60f_{\alpha_1} + 22f_{\alpha_2} + 66f_{\alpha_2+\alpha_4} + 118f_{\alpha_3} - 66f_{\alpha_3+\alpha_4} + 22f_{\alpha_5}$
$\quad + 108f_{\alpha_4+\alpha_5} + 34f_{\alpha_5+\alpha_6} + 22f_{\alpha_7} + 74f_{\alpha_6+\alpha_7} + 38f_{\alpha_8}$,
$J = \{\alpha_4, \alpha_6\}$, $J_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6\}$, $J_2 = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$.

Now by (2.3), $o(u) = p$ implies that $p > \mathrm{ht}_J(r_0)$ in each case. So for the cases where u is a regular element in G, that is, cases 2.1(i), (ii), (v), and (viii), Proposition 1.5 shows that $(\mathrm{ad}\ e)^j/j!$ and $(\mathrm{ad}\ f)^j/j!$ preserve $\mathscr{L}_G(\mathbf{Z}_{(p)})$ for all $j \geq 0$. For the remaining cases, where J, J_1, J_2 are given, $p > \mathrm{ht}_J(r_0)$ implies $p > 2\,\mathrm{ht}_{J_i}(r_0)$ for $i = 1, 2$. Therefore, the condition (†) following Lemma 1.4 is satisfied, and again $(\mathrm{ad}\ e)^j/j!$ and $(\mathrm{ad}\ f)^j/j!$ preserve $\mathscr{L}(\mathbf{Z}_{(p)})$ for all $j \geq 0$. So we may apply Lemmas 1 and 2 of [24] to exponentiate ad te and ad tf to form X, a $\mathrm{PSL}_2(k)$ or $\mathrm{SL}_2(k)$ subgroup of the adjoint-type Chevalley group $G(k)$. Moreover, by (1.1), $\sigma(X) \subseteq X$ and by (1.2) and (1.3), there exists $x \in G(k)$ with $u^x \in X$ and $\sigma(u^x) = u^x$. Finally, we note that for each of the classes we are considering, $C_G(u)$ is connected. (See [9, 10, 15, and 19].) So by (E,I,3.4) of [4], G-conjugacy implies G_σ-conjugacy. So, in fact, there exists $y \in G_\sigma$ such that $u^x = u^y$. Then yXy^{-1} is a σ-invariant subgroup of G of type A_1 containing u, as desired.

We now turn our attention to the distinguished unipotent elements which lie in a maximal rank semisimple subgroup of G. We first establish a result which "identifies" the F_4 distinguished classes.

LEMMA 2.5. *Let G be a simple algebraic group of type F_4, defined over an algebraically closed field k of characteristic $p > 3$. Then there are four classes of distinguished unipotent elements with representatives and labelled diagrams as follows*:

$$u_1 = x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)_{\alpha_4}(1),\ \overset{2}{\circ}\ \overset{2}{\circ}\ \overset{2}{\bullet}\ \overset{2}{\bullet};$$

$$u_2 = x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_2+2\alpha_3+2\alpha_4}(1),\ \overset{2}{\circ}\ \overset{2}{\circ}\ \overset{0}{\bullet}\ \overset{2}{\bullet};$$

$$u_3 = x_{\alpha_1+\alpha_2}(1)x_{\alpha_2+\alpha_3}(1)x_{\alpha_1+\alpha_2+2\alpha_3}(1)x_{\alpha_3+\alpha_4}(1),\ \overset{0}{\circ}\ \overset{2}{\circ}\ \overset{0}{\bullet}\ \overset{2}{\bullet};$$

$$u_4 = x_{\alpha_1+\alpha_2}(1)x_{\alpha_2+2\alpha_3}(1)x_{\alpha_2+2\alpha_3+2\alpha_4}(-1)$$
$$\times\ x_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4}(-1),\ \overset{0}{\circ}\ \overset{2}{\circ}\ \overset{0}{\bullet}\ \overset{0}{\bullet}.$$

Proof. That the distinguished classes in G have representatives u_i for $i = 1, 2, 3, 4$ follows directly from [15]. As well, Shoji describes the last three elements as regular unipotent elements in maximal rank subgroups of types B_4, $C_3 \times A_1$, and $A_2 \times \tilde{A}_2$, respectively.

We will identify the labelled diagram corresponding to the last three elements. (It is clear that u_1 is a regular unipotent element in G and so has labelled diagram $\overset{2}{\circ}\ \overset{2}{\circ}\ \overset{2}{\bullet}\ \overset{2}{\bullet}$.)

Consider first the labelled diagram $\overset{2}{\circ}\ \overset{2}{\circ}\ \overset{0}{\bullet}\ \overset{2}{\bullet}$ and let $P_1 \supseteq T\langle U_\alpha \mid \alpha$

$\in \Phi^+(G)\rangle$ be the corresponding distinguished parabolic subgroup. Set $J = \{\alpha_3\}$. Note that both u_3 and u_4 lie in $R_u(P_1)$. According to (1.3), if u_3 lies in the dense orbit of P_1 on $R_u(P_1)$, then u_3 is $R_u(P_1)$-conjugate to $x_{\alpha_2+\alpha_3}(1)x_{\alpha_3+\alpha_4}(1)$. But this last element is clearly not in the dense orbit of P_1 on its radical as its P_1-orbit lies in the closed set $\{\Pi_{\mathrm{ht}_J(\alpha)\geq 1}\, x_\alpha(c_\alpha) \mid c_{\alpha_1} = 0\}$. So u_3 does not have labeled diagram $\overset{2}{\circ}\ \overset{2}{\circ}\ \overset{0}{\bullet}\ \overset{2}{\bullet}$. Similarly, we see that u_4 does not lie in the dense orbit of P_1 on its radical. Thus this leaves u_2 as the distinguished element with labeled diagram $\overset{2}{\circ}\ \overset{2}{\circ}\ \overset{0}{\bullet}\ \overset{2}{\bullet}$.

Now consider the labeled diagram $\overset{0}{\circ}\ \overset{2}{\circ}\ \overset{0}{\bullet}\ \overset{2}{\bullet}$, and let $P_2 \supseteq T\langle U_\alpha \mid \alpha \in \Phi^+(G)\rangle$ be the distinguished parabolic corresponding to this diagram. Then $u_4 \in R_u(P_2)$. Again, if u_4 lies in the dense orbit of P_2 on its radical, then u_4 is $R_u(P_2)$-conjugate to $x_{\alpha_1+\alpha_2}(1)x_{\alpha_2+2\alpha_3}(1)$, an element which is clearly not in the dense orbit of P_2 on its radical. Thus u_3 must have labelled diagram $\overset{0}{\circ}\ \overset{2}{\circ}\ \overset{0}{\bullet}\ \overset{2}{\bullet}$, which completes the proof of the lemma.

We can now establish the order formula for all distinguished unipotent elements in exceptional groups.

PROPOSITION 2.6. *Let G be an exceptional algebraic group defined over an algebraically closed field of good characteristic p. Let u be a distinguished unipotent element of G lying in the dense orbit of the distinguished parabolic P on its radical. Then* $o(u) = \min\{p^a \mid p^a > \mathrm{ht}(P)\}$.

Proof. By (2.3), we may assume u is not conjugate to any of the 10 elements listed in (2.1). Then, we refer to [9, 10, 15, 19] and the above lemma for the information contained in the following table. Namely, in the first column we give the type of G. In column 2 we give the labeled diagram corresponding to the distinguished parabolic P. Column 3 gives the type of a maximal rank semisimple subgroup $D \leq G$ containing u. Moreover, in each case, since u is a distinguished unipotent element in D, there is a distinguished parabolic P_D of D for which u lies in the dense orbit of P_D on its radical $R_u(P_D)$. In column 4 we give the labeled diagram for the parabolic P_D.

G	Diagram of P	D	Diagram of P_D
G_2	0 2	A_2	2 2
F_4	2 2 0 2	B_4	2 2 2 2
F_4	0 2 0 2	$A_1 \times C_3$	2 × 2 2 2
F_4	0 2 0 0	$A_2 \times \tilde{A}_2$	2 2 × 2 2
E_6	2 0 $\substack{2\\0}$ 0 2	$A_5 \times A_1$	2 2 2 2 2 × 2
E_7	2 0 $\substack{2\\0}$ 0 2 2	$D_6 \times A_1$	2 2 2 2 $\substack{2\\2}$ × 2

G	Diagram of P	D	Diagram of P_D
E_7	$2\ 0\ \genfrac{}{}{0pt}{}{2}{0}\ 0\ 0\ 2$	$D_6 \times A_1$	$2\ 2\ 2\ 0\ \genfrac{}{}{0pt}{}{2}{2}\ \times\ 2$
E_7	$0\ 0\ \genfrac{}{}{0pt}{}{2}{0}\ 0\ 0\ 2$	$D_6 \times A_1$	$2\ 0\ 2\ 0\ \genfrac{}{}{0pt}{}{2}{2}\ \times\ 2$
E_8	$2\ 0\ \genfrac{}{}{0pt}{}{2}{0}\ 0\ 2\ 2\ 2$	$E_7 \times A_1$	$2\ 2\ \genfrac{}{}{0pt}{}{2}{2}\ 2\ 2\ 2\ \times\ 2$
E_8	$2\ 0\ \genfrac{}{}{0pt}{}{2}{0}\ 0\ 2\ 0\ 2$	D_8	$2\ 2\ 2\ 2\ 2\ 2\ \genfrac{}{}{0pt}{}{2}{2}$
E_8	$2\ 0\ \genfrac{}{}{0pt}{}{2}{0}\ 0\ 0\ 2\ 0$	D_8	$2\ 2\ 2\ 2\ 2\ 0\ \genfrac{}{}{0pt}{}{2}{2}$
E_8	$0\ 0\ \genfrac{}{}{0pt}{}{2}{0}\ 0\ 0\ 2\ 0$	A_8	$2\ 2\ 2\ 2\ 2\ 2\ 2\ 2$
E_8	$0\ 0\ \genfrac{}{}{0pt}{}{0}{0}\ 2\ 0\ 0\ 0$	$A_4 \times A_4$	$2\ 2\ 2\ 2\ \times\ 2\ 2\ 2\ 2$
E_8	$2\ 0\ \genfrac{}{}{0pt}{}{2}{0}\ 0\ 0\ 2\ 2$	$E_7 \times A_1$	$2\ 2\ \genfrac{}{}{0pt}{}{0}{2}\ 2\ 2\ 2\ \times\ 2$
E_8	$0\ 0\ \genfrac{}{}{0pt}{}{2}{0}\ 0\ 0\ 2\ 2$	$E_7 \times A_1$	$2\ 2\ \genfrac{}{}{0pt}{}{0}{2}\ 2\ 0\ 2\ \times\ 2$
E_8	$0\ 0\ \genfrac{}{}{0pt}{}{2}{0}\ 0\ 0\ 0\ 2$	D_8	$2\ 0\ 2\ 0\ 2\ 0\ \genfrac{}{}{0pt}{}{2}{2}$

Given the information in the above table, together with (3.4) (the order formula for the classical groups), one checks that in each case $o(u) = \min\{p^a \mid p^a > \mathrm{ht}(P)\}$.

We close this section with a lemma which identifies a representative of one of the distinguished classes in E_6 (listed in (2.1)) as lying in a proper connected reductive subgroup. In Section 4, we will use this result to establish the σ-invariance of one of the algebraic A_1 subgroups constructed in Proposition 2.4.

LEMMA 2.7. *Let G have type E_6 and assume $p > 3$. Let τ be the graph automorphism of G, let $x = h_{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6}(-1)$, and let i_x be the inner automorphism corresponding to x. Then $G_{\tau i_x}$ is a closed connected subgroup of G of type C_4 and if u is a regular unipotent element in $G_{\tau i_x}$ then u lies in the E_6 conjugacy class having the labeled diagram*

$$\begin{matrix} & & 0 & & \\ 2 & 2 & \circ & 2 & 2 \\ \circ & \circ & \circ & \circ & \circ \\ & & 2 & & \end{matrix}.$$

Proof. We first refer to (5.0) of [23] to see that u is conjugate to

$$x_{\alpha_1}(1)x_{\alpha_6}(1)x_{\alpha_3}(1)x_{\alpha_5}(1)x_{\alpha_4}(1)x_{\alpha_2+\alpha_3+\alpha_4}(1)x_{\alpha_2+\alpha_4+\alpha_5}(-1),$$

and so u is not a regular unipotent element in G. We have that $G_{\tau i_x}$ acts irreducibly on the restricted 27-dimensional modules for E_6; moreover, the high weight of these modules as C_4-modules is μ_2 where μ_2 is the fundamental dominant weight corresponding to the simple root α_2. This

C_4-module can be realized as a direct summand of $W \wedge W$ where W is the natural module for C_4 and the complement is a one-dimensional submodule of $W \wedge W$. Thus, we will compare the action of the unipotent elements in E_6 on the 27-dimensional modules and the action of u on $W \wedge W$. One checks that on $W \wedge W$ u has a four-dimensional fixed point space so on the 27-dimensional E_6 irreducible module u has a fixed point space of dimension 3. But one can check by using [9] that each of the nonregular unipotent elements in E_6 which do not lie in the E_6 class with given labeled diagram have fixed point space on the 27-dimensional module of dimension strictly greater than 3. In most cases, the element is regular in a Levi factor of G of type A_n or D_n for some n and the 27-dimensional module decomposes as a direct sum of natural modules or spin modules or wedge products for the Levi factor. For the class having labeled diagram $2\ 0\ \overset{2}{\underset{0}{}}\ 0\ 2$, the element is a regular unipotent element in a maximal rank subgroup of type $A_5 \times A_1$ which acts on the 27-dimensional E_6 module as the wedge product of the natural module for A_5 plus the natural module for $A_5 \times A_1$. One can then compute the fixed point spaces of the regular unipotent elements on these modules and see that in all cases the fixed point space on the 27-dimensional module is strictly greater than 3. Thus, we are left with the indicated class.

3. A_1 SUBGROUPS AND UNIPOTENT ELEMENTS IN CLASSICAL ALGEBRAIC GROUPS

In this section we consider distinguished unipotent elements in the classical algebraic groups over algebraically closed fields of good characteristic p. We establish the order formula and the existence of p-power Frobenius invariant A_1 subgroups overlying certain distinguished unipotent elements of order p.

As we mentioned in the Introduction, Theorem 0.1 can in fact be deduced from the literature. For this, it is necessary to combine information from [4, 11, 12, 16, and 25] concerning the relation between the Jordan canonical forms of unipotent elements and the labeled diagrams. We have included a proof here because it is elementary, it lends completeness to the paper, and it is necessary for Theorem 0.2.

PROPOSITION 3.1. *Let G be an adjoint-type algebraic group of type C_l defined over an algebraically closed field k of characteristic $p \neq 2$. There is a bijection between the distinguished unipotent elements of G and the partitions of l which have distinct parts. Namely, let $n_1 > n_2 > \cdots > n_r > 0$ be integers with $\sum_{i=1}^{r} n_i = l$. Let $H \leq G$ be a maximal rank subgroup of type $C_{n_1} \times \cdots \times C_{n_r}$. Let u be a regular unipotent element in H. Then u is a* *distinguished unipotent element in G and all distinguished unipotent elements in G can be obtained in this manner.*

Proof. By (5.1.1) of [5], it will suffice to work inside the group $\mathrm{Sp}(2l, k)$. Let $\{n_1, n_2, \dots, n_r\}$ be a partition of l with distinct parts. Say $n_1 > n_2 > \cdots > n_r > 0$. For $j = 1, \dots, r$, let V_j be a $2n_j$-dimensional vector space over k. Fix $1 \le j \le r$. We will define a bilinear form on V_j and certain elements of $\mathrm{SL}(V_j)$ which preserve this form. Fix a basis $\{v_{j1}, \dots, v_{j,2n_j}\}$ of V_j. Define a bilinear form $f_j(\ ,\ )$ on V_j by

$$f_j(v_{jk}, v_{ji}) = \delta_{i,2n_j-k+1}(-1)^{k+1}, \qquad \text{for } 1 \le k, i \le 2n_j.$$

Now for $1 \le i \le n_j$, define $g_{ij} \in \mathrm{SL}(V_j)$ as follows:

$$\begin{aligned} g_{ij}(v_{j,i+1}) &= v_{j,i} + v_{j,i+1}; \\ g_{ij}(v_{j,2n_j-i+1}) &= v_{j,2n_j-i+1} + v_{j,2n_j-i}; \\ g_{ij}(v_{j,k}) &= v_{j,k}, \text{for } k \notin \{i+1, 2n_j - i + 1\}. \end{aligned} \tag{1}$$

Then let

$$u_j = g_{1j} \cdots g_{n_j j}. \tag{2}$$

Then one easily checks that the following holds.

Claim. f_j is a nondegenerate symplectic form on V_j and, for $1 \le i \le n_j$, g_{ij} preserves f_j. In particular, u_j preserves f_j. Moreover, u_j is a regular unipotent element in $\mathrm{Sp}(V_j)$.

Now, let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$, a $2l$-dimensional vector space over k. Define a nondegenerate symplectic form $f(\ ,\ )$ on V as the orthogonal sum of the forms f_j. So

$$f(v_{ij}, v_{km}) = \delta_{ik}\delta_{m,2n_i-j+1}(-1)^{j+1}. \tag{3}$$

Extend the action of u_j to all of V by letting u_j act as the identity on V_i for $i \ne j$. Then set

$$u = u_1 \cdots u_r. \tag{4}$$

By the above claim, $u \in \mathrm{Sp}(V)$.

We now define a flag of totally isotropic subspaces of V. For $k = 1, \dots, n_1$, let

$$W_k = \langle v_{st} \mid n_s - t = n_1 - k \rangle.$$

Then

$$\mathcal{F}: W_1 < W_1 + W_2 < \cdots < W_1 + \cdots + W_{n_1} \tag{5}$$

is a flag of totally isotropic subspaces in V. Let

$$P = \mathrm{stab}_{\mathrm{Sp}(V)}(\mathcal{F}), \tag{6}$$

a parabolic subgroup of $\mathrm{Sp}(V)$.

Claim. Let u and P be as defined above in (4) and (6). Then $C_{\mathrm{Sp}(V)}(u)^\circ$ lies in the unipotent radical of P. In particular, u is a distinguished unipotent element of $\mathrm{Sp}(V)$. Moreover, P is a distinguished parabolic subgroup of $\mathrm{Sp}(V)$.

Proof of Claim. We will first examine $C_{\mathrm{SL}(V)}(u)$. Let $x = u - 1$ in $\mathrm{gl}(V)$. Then $C_{\mathrm{SL}(V)}(u)$ must leave invariant $\ker(x^i) = \langle v_{j1}, \ldots, v_{ji} \mid 1 \le j \le r\rangle$ and $\mathrm{im}(x^i) = \langle v_{j1}, \ldots, v_{j,2n_j-i} \mid 1 \le j \le r\rangle$, for all i. Now let $g \in C_{\mathrm{SL}(V)}(u)$ and for a fixed $1 \le p \le r$ and $1 \le q \le 2n_p$, set

$$gv_{pq} = \sum_{\substack{1 \le s \le r \\ 1 \le t \le 2n_s}} b_{st} v_{st}.$$

Since $v_{pq} \in \ker(x^q)$, gv_{pq} also lies in $\ker(x^q)$ and so $b_{st} = 0$ for $t > q$. Similarly, since $v_{pq} \in \mathrm{im}(x^{2n_p - q})$, $b_{st} = 0$ for $t > 2n_s - (2n_p - q)$.

Now suppose $b_{st} \neq 0$; so $t \le q$ and $q - t \ge 2n_p - 2n_s$. If $n_p - n_s = 0$, then $p = s$ and $2n_s - 2t + 1 = 2n_p - 2t + 1 \ge 2n_p - 2q + 1$, with equality only if $t = q$. If $n_p - n_s < 0$ then since $q - t \ge 0$, $q - t > n_p - n_s$. So $2n_s - 2t + 1 > 2n_p - 2q + 1$. If $n_p - n_s > 0$, then $q - t \ge 2n_p - 2n_s$ implies $q - t > n_p - n_s$ and again $2n_s - 2t + 1 > 2n_p - 2q + 1$. So

$$gv_{pq} = c_{pq}v_{pq} + \sum \{a_{st}v_{st} \mid 2n_s - 2t + 1 > 2n_p - 2q + 1\}, \quad \text{for some } c_{pq}. \tag{$*$}$$

That is, $g(W_k) \subseteq \sum_{i \le k} W_i$.

Now fix $1 \le i \le r$ and again let $g \in C_{\mathrm{SL}(V)}(u)$. Comparing the coefficients of $v_{i,j-1}$ in $gu(v_{ij})$ and $ug(v_{ij})$, we see that $c_{ij} = c_{i,j-1}$ for all $j > 1$. Thus,

$$c_{ik} = c_{ij}, \qquad \text{for } 1 \le j, k \le 2n_i. \tag{$**$}$$

Finally, we restrict our attention to the centralizer of u in $\mathrm{Sp}(V)$. Let $g \in C_{\mathrm{Sp}(V)}(u)$ and let c_{pq} be as in ($*$) above. Suppose $f(v_{ij}, v_{km}) \neq 0$, so $i = k$ and $m = 2n_i - j + 1$. Then one checks that $f(gv_{ij}, gv_{km}) = c_{ij}c_{i,2n_i-j+1}(-1)^{j+1}$. But this gives $(-1)^{j+1} = c_{ij}c_{i,2n_i-j+1}(-1)^{j+1}$;

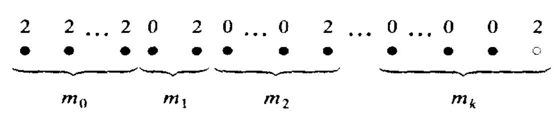
so $c_{ij}c_{i,2n_i-j+1} = 1$. Together with $(**)$, we then have $c_{ij}^2 = 1$, for all i, j. So for $g \in C_{\mathrm{Sp}(V)}(u)$ and for $1 \le p \le r$ and $1 \le q \le 2n_p$,

$$gv_{pq} = c_{pq}v_{pq} + \sum \{a_{st}v_{st} \mid 2n_s - 2t + 1 > 2n_p - 2q + 1\}$$

$$\text{and} \qquad c_{pq}^2 = 1.$$

Thus, $C_{\mathrm{Sp}(V)}(u)^\circ$ lies in the unipotent radical of P. Thus, u is a distinguished unipotent element of $\mathrm{Sp}(V)$. Moreover, P is a distinguished parabolic subgroup of $\mathrm{Sp}(V)$. For note that the Levi factor of the parabolic P has structure $(\mathrm{GL}(W_1) \times \cdots \times \mathrm{GL}(W_{n_1}) \times T_0)^\circ$, for some central torus T_0, and that $\dim W_j = 1$ for $j = 1, \ldots, n_1 - n_2$, $\dim W_{n_1 - n_2 + 1} = 2$ and $\dim W_{k+1} = \dim W_k$ or $\dim W_k + 1$ for $k \ge n_1 - n_2 + 1$. This is precisely the condition which ensures that P is a distinguished parabolic subgroup of $\mathrm{Sp}(V)$. See [1, Chap. 5]. We note here that we have made no claims that u lies in the dense orbit of P on its radical. We will discuss this further in the proof of (3.3).

We have described a method of starting with a partition of l (having distinct parts) and a corresponding element u and obtaining a distinguished parabolic P having $u \in R_u(P)$. We next describe how to choose a partition corresponding to a given distinguished diagram. Let



be the given labeled diagram. So by [1], $m_0 > 0$, $m_1 = 2$, and $m_{i+1} = m_i$ or $m_i + 1$ for $i > 1$. Now set

$$r = m_k;$$
$$n_1 = m_0 + k;$$
$$n_2 = k;$$
$$n_t - n_{t+1} = \#\{j \mid m_j = t\} \qquad \text{for } 2 \le t \le r - 1.$$

Then it is clear that $n_1 > n_2 > \cdots > n_r > 0$. Note that $\#\{j \ge 1 \mid m_j = r\} = k - \#\{j \ge 1 \mid m_j > r\} = k - (n_{r-1} - n_r + n_{r-2} - n_{r-1} + \cdots + n_2 - n_3) = n_r$. Also,

$$\begin{aligned} l &= m_0 + m_1 + \cdots + m_k \\ &= m_0 + 2(\#\{j \mid m_j = 2\}) + \cdots + r(\#\{j \mid m_j = r\}) \\ &= m_0 + 2(n_2 - n_3) + \cdots + (r-1)(n_{r-1} - n_r) + m_r \\ &= n_1 + n_2 + \cdots + n_r. \end{aligned}$$

So $\{n_1, n_2, \ldots, n_r\}$ is a partition of l with distinct parts. Moreover, it is straightforward to check that the above described processes of starting with a partition and obtaining a labeled diagram and the reverse are inverses. Finally, we point out that given two distinct partitions of l, having distinct parts, the corresponding distinguished unipotent elements are not conjugate, as they have different Jordan block sizes. The proposition follows.

We now sketch the analogous result for G of type B_l and D_l.

PROPOSITION 3.2. *Let G be an adjoint-type algebraic group of type B_l or D_l over an algebraically closed field k of characteristic $p \neq 2$. Set $N = 2l + 1$ or $2l$, according to whether G has type B_l or D_l. Then there is a bijection between the conjugacy classes of distinguished unipotent elements of G and the partitions of N having distinct odd parts. Namely, let $n_1 > n_2 > \cdots > n_r \geq 0$ be integers with $\sum_{i=1}^{r}(2n_i + 1) = N$. Let $H \leq G$ be a subgroup of type $B_{n_1} \times \cdots \times B_{n_r}$. Let u be a regular unipotent element in H. Then u is a distinguished unipotent element in G and all distinguished unipotent elements in G can be obtained in this manner.*

Proof. By (5.1.1) of [5], it will suffice to work in $SO(N, k)$. Let $n_1 > n_2 > \cdots > n_r \geq 0$ be integers with $\sum_{i=1}^{r}(2n_i + 1) = N$. For $j = 1, \ldots, r$, let V_j be a $(2n_j + 1)$-dimensional vector space over k. Fix $1 \leq j \leq r$. Fix a basis $\{v_{j1}, \ldots, v_{j,2n_j+1}\}$ of V_j. We define a bilinear form $f_j(\ ,\ )$ on v_j as follows:

$$f_j(v_{ji}, v_{jk}) = \delta_{k,2n_j+2-i}(-1)^i, \qquad \text{for } 1 \leq i, k \leq 2n_j + 1.$$

Then one checks that f_j is a nondegenerate, symmetric bilinear form on V_j. Now for each $1 \leq i \leq n_j$, we define $g_{ij} \in \mathrm{SL}(V_j)$:

$$g_{ij}(v_{j,i+1}) = v_{j,i+1} + v_{ji};$$
$$g_{ij}(v_{j,2n_j+2-i}) = v_{j,2n_j+2-i} + v_{j,2n_j+1-i};$$
$$g_{ij}(v_{jk}) = v_{jk}, \qquad \text{if } k \notin \{i+1, 2n_j+2-i\}$$

and

$$g_{n_jj}(v_{jk}) = v_{jk}, \qquad \text{for } k \leq n_j \text{ and for } k \geq n_j + 3;$$
$$g_{n_jj}(v_{j,n+1}) = v_{j,n+1} + v_{jn};$$
$$g_{n_jj}(v_{j,n+2}) = v_{j,n+2} + v_{j,n+1} + \tfrac{1}{2}v_{jn}.$$

Now one checks that g_{ij} preserves the form f_j for all $1 \le i \le n_j$. Now set $u_j = g_{1j} \cdot g_{2j} \cdots g_{n_j j}$. Then u_j preserves the form f_j. Moreover, u_j is a regular unipotent element in $\mathrm{SO}(V_j)$.

Let $V = V_1 \oplus \cdots \oplus V_r$, an N-dimensional vector space over k. Define a nondegenerate symmetric bilinear form f on V as the orthogonal sum of the forms f_j. So

$$f(v_{ij}, v_{st}) = \delta_{is}\delta_{t,2n_i+2-j}(-1)^j.$$

Extend the action of u_j to all of V by letting u_j act as the identity on V_i for $i \ne j$. Now set $u = u_1 \cdots u_r$, so $u \in \mathrm{SO}(V)$.

We now define a flag of totally isotropic subspaces in V. For $1 \le k \le n_1$ set $W_k = \langle v_{st} \mid 2n_s + 2 - 2t = 2n_1 + 2 - 2k \rangle$. Then

$$\mathscr{F}: W_1 \le W_1 + W_2 \le \cdots \le W_1 + W_2 + \cdots + W_{n_1}$$

is a flag of totally isotropic subspaces. Let $P = \mathrm{stab}_{\mathrm{SO}(V)}(\mathscr{F})$, a parabolic subgroup of $\mathrm{SO}(V)$. One checks (basically as in the $\mathrm{Sp}(V)$ case) that $C_{\mathrm{SO}(V)}(u)^\circ$ lies in the unipotent radical of P, which implies that u is a distinguished unipotent element of $\mathrm{SO}(V)$.

We will now show that P is in fact a distinguished parabolic subgroup of $\mathrm{SO}(V)$ (by describing its labeled diagram) and show that all distinguished diagrams are obtained in this way. We first note that $(W_1 + \cdots + W_{n_1})^\perp = \langle v_{st} \mid 2n_s + 2 - 2t \ge 0 \rangle$. Let $W_{n_1+1} = \langle v_{st} \mid 2n_s + 2 - 2t = 0 \rangle$. Then the Levi factor of P has structure $(\mathrm{GL}(W_1) \times \cdots \times \mathrm{GL}(W_{n_1}) \times SO(W_{n_1+1}) \times T_0)^\circ$, for some central torus T_0. We note that

(1) $\dim W_j = 1$ for $j = 1, \ldots, n_1 - n_2$,

(2) $\dim W_{n_1 - n_2 + 1} = 2$, and

(3) $\dim W_{k+1} = \dim W_k$ or $\dim W_k + 1$ for $k \ge n_1 - n_2 + 1$.

As in type C_l, these are precisely the conditions which ensure that P is a distinguished parabolic subgroup of G. (See [1, Chap. 5].) Thus we have described how to start with a partition of N having distinct odd parts and obtain a distinguished diagram. It is described in [1] how to start with a distinguished diagram of G and obtain a partition of the appropriate type. Moreover, one can check that these two processes are inverses. As in the previous proposition, the unipotent elements thus obtained represent distinct conjugacy classes. The proposition follows.

We are now ready to establish the order formula for the classical groups. This will follow from the previous two propositions and the following lemma which was pointed out to me by Ross Lawther.

LEMMA 3.3. *Let G be a simple algebraic group and let $P_1, \ldots, P_k$ be a complete list of nonconjugate distinguished parabolic subgroups of G. Let*

$u_i \in R_u(P_i)$ be such that $u_1, \ldots, u_k$ are nonconjugate distinguished unipotent elements of G. Then u_i lies in the dense orbit of P_i on $R_u(P_i)$.

Proof. Let $\mathscr{C}_i, \ldots, \mathscr{C}_k$ be the unipotent conjugacy classes of G with $u_i \in \mathscr{C}_i$ for all i. Let $\mathrm{Rich}(P_i)$ denote the unipotent class containing the dense orbit of P_i on $R_u(P_i)$. Then since the P_i form a complete set of distinguished parabolics and the u_i a complete set of representatives of the distinguished unipotent conjugacy classes, either $\mathscr{C}_i = \mathrm{Rich}(P_i)$ for all i as desired, or after renumbering we may assume that there exists k such that $\mathscr{C}_1 = \mathrm{Rich}(P_2), \ldots, \mathscr{C}_{k-1} = \mathrm{Rich}(P_k)$, $\mathscr{C}_k = \mathrm{Rich}(P_1)$. Now for $x \in \mathrm{Rich}(P_j)$, $\dim C_G(x) = \dim C_{P_j}(x) = \dim P_j - \dim R_u(P_j)$, while for $x' \in R_u(P_j)$ with $x' \notin \mathrm{Rich}(P_j)$, $\dim C_{P_j}(x') > \dim P_j - \dim R_u(P_j)$. So $\dim C_G(x') > \dim C_G(x)$. Applying this to our situation, when $\mathscr{C}_i \neq \mathrm{Rich}(P_i)$ for some i, we get $\dim C_G(u_1) < \dim C_G(u_2) < \cdots < \dim C_G(u_k) < \dim C_G(u_1)$, a contradiction.

Thus the bijection we had in Propositions 3.1 and 3.2 between distinguished unipotent classes and classes of distinguished parabolic subgroups actually gives us a representative in the dense orbit of the parabolic on its radical. We now establish the order formula for the classical groups.

PROPOSITION 3.4. *Let G be a simple algebraic group defined over an algebraically closed field k of characteristic p. Assume that G is of type A_l, B_l, C_l, or D_l, and assume $p > 2$ when G has type B_l, C_l, or D_l. Let P be a distinguished parabolic subgroup of G and let u lie in the dense orbit of P on $R_u(P)$. Then $o(u) = \min\{p^a \mid p^a > \mathrm{ht}(P)\}$.*

Proof. In type A_l, the only distinguished parabolic (up to conjugacy) is B, a Borel subgroup, and the height of B is $\mathrm{ht}(r_0) = l$. Moreover, the elements lying in the dense orbit of B on its radical are regular unipotent elements, which are easily seen to have order $\min\{p^a \mid p^a > l\}$.

Consider the group $\mathrm{Sp}(V) = \mathrm{Sp}(2l, k)$. Recall from (3.1) that we have a distinguished parabolic P as the stabilizer of a flag of isotropic subspaces $W_1 < W_1 + W_2 < \cdots < W_1 + \cdots + W_{n_1}$, where $W_1 + \cdots + W_{n_1}$ is a maximal isotropic subspace of V. Also, we have $u \in R_u(P)$ where u is a regular unipotent element in $\mathrm{Sp}(V_1) \times \cdots \times \mathrm{Sp}(V_r)$ and $\dim V_i = 2n_i$ and $n_1 > n_2 > \cdots > n_r > 0$ is a partition of l. Moreover, by Lemma 3.3, u lies in the dense orbit of P on $R_u(P)$. Since u has a maximal Jordan block size of $2n_1$, $o(u) = \min\{p^a \mid p^a \geq 2n_1\} = \min\{p^a \mid p^a > 2n_1 - 1\}$. But we have that the Levi factor of P has structure $(\mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \times \cdots \times \mathrm{GL}(W_{n_1}) \times T_0)^\circ$, so the number of zeroes in the labeled diagram for P is $\dim W_1 - 1 + \cdots + \dim W_{n_1} - 1 = \dim W_1 + \cdots + \dim W_{n_1} - n_1 = l - n_1$ and the zero labelings occur on short root nodes of the Dynkin diagram. So $\mathrm{ht}(P) = 2n_1 - 1$. So the order of the unipotent elements lying in the dense orbit is $\min\{p^a \mid p^a > \mathrm{ht}(P)\}$ as claimed.

Now consider the group $SO(V)$, where V is a $(2l+1)$- or $2l$-dimensional vector space over k. We recall from (3.2) that P is the stabilizer of a flag of isotropic subspaces $W_1 < W_1 + W_2 < \cdots < W_1 + \cdots + W_{n_1}$, where again $W_1 + \cdots + W_{n_1}$ is a maximal isotropic subspace of V. On the other hand, we have as above x in the dense orbit of P on $R_u(P)$ with x conjugate to a regular unipotent element in $SO(V_1) \times \cdots \times SO(V_r)$ where $\dim(V_i) = 2n_i + 1$ and $n_1 > n_2 > \cdots > n_r \geq 0$ and $\Sigma 2n_i + 1 = 2l + 1$ (respectively, $2l$). So x has a maximal Jordan block size of $2n_1 + 1$. So the order of the unipotent elements lying in the dense orbit of P on $R_u(P)$ is $\min\{p^a \mid p^a \geq 2n_1 + 1\} = \min\{p^a \mid p^a > 2n_1\}$. But we have that the Levi factor of P has structure $(GL(W_1) \times GL(W_2) \times \cdots \times GL(W_{n_1}) \times SO(W_{n_1+1}) \times T_0)^{\circ}$, where $W_{n_1+1} = \langle v_{st} \mid 2n_s + 2 - 2t = 0 \rangle$. (Note that $(W_1 + \cdots + W_{n_1})^{\perp} = W_1 + \cdots + W_{n_1} + W_{n_1+1}$. Also note that $\dim W_{n_1+1} = r$ so is odd if $\dim V$ is odd and even otherwise.)

Case I. $\dim(W_{n_1+1}) \geq 3$. Then, as in C_l, we count the number of zeroes in the labeled diagram for P and obtain

$$\dim(W_1) - 1 + \cdots + \dim(W_{n_1}) - 1 + \begin{cases} \frac{1}{2}(\dim W_{n_1+1} - 1), & \text{if } \dim V \text{ is odd;} \\ \frac{1}{2}(\dim W_{n_1+1}), & \text{if } \dim V \text{ is even.} \end{cases}$$

Now using the fact that

$$\dim V = 2(\dim W_1 + \cdots + \dim W_{n_1}) + \dim W_{n_1+1},$$

we get that the number of zeroes is $l - n_1$. So there are n_1 nodes labeled with a 2, namely the nodes corresponding to the roots $\alpha_1, \ldots, \alpha_{l-1}$ if $\dim V$ is odd and those corresponding to the roots $\alpha_1, \ldots, \alpha_{l-2}$ if $\dim V$ is even. So $\mathrm{ht}(P) = 1 + 2(n_1 - 1) = 2n_1 - 1$. Since $p \neq 2$, $\min\{p^a \mid p^a > 2n_1 - 1\} = \min\{p^a \mid p^a > 2n_1\}$. So the order of the unipotent elements lying in the dense orbit is $\min\{p^a \mid p^a > \mathrm{ht}(P)\}$ as claimed.

Case II. $\dim(W_{n_1+1}) < 3$. In this case, if $\dim V$ is odd then $\dim(W_{n_1+1}) = 1$, and if $\dim V$ is even then $\dim(W_{n_1+1}) = 2$. In particular, if $\dim V$ is odd, then $r = 1$ and $n_1 = l$ and the corresponding unipotent element is the regular unipotent element and the order formula clearly holds. So consider the case where $\dim V$ is even and $r = 2$. So $SO(W_{n_1+1}) = \pm 1$ and the Levi factor of the parabolic is $GL(W_1) \times GL(W_2) \times \cdots \times GL(W_{n_1}) \times T_0$, and by counting the zeroes we obtain

$$\dim W_1 - 1 + \cdots + \dim W_{n_1} - 1.$$

But now, $2l = \dim V = 2(\dim W_1 + \cdots + \dim W_{n_1}) + 2$. So the number of zeroes is $l - n_1 - 1$. So there are $n_1 + 1$ nodes labeled with a 2. Moreover, the nodes α_{l-1}, α_l, and α_1 are labelled with a 2. So $\mathrm{ht}(P) = 2n_1 - 1$ as in case I. This completes the proof of the proposition.

We close the section with the following:

PROPOSITION 3.5. *Let G be an adjoint simple algebraic group of type A_l, B_l, or C_l, $l > 1$, defined over an algebraically closed field k of characteristic $p > 0$, where $p > 2$ if G has type B_l or C_l. Let σ be a fixed p-power Frobenius endomorphism of G and let u be a regular unipotent element of G. If $o(u) = p$ there exists a closed connected subgroup $X \leq G$, X of type A_1, with $u \in X$. Moreover, if $\sigma(u) = u$, X can be chosen so that $\sigma(X) \subseteq X$.*

Proof. The proof is exactly the same as the Proof of Proposition 2.4 for the regular unipotent elements. Thus, we need only exhibit the appropriate sl_2 subalgebras of the complex Lie algebras $\mathscr{L}_G(\mathbf{C})$ of types A_l, B_l, and C_l.

In $\mathscr{L}_{A_l}(\mathbf{C})$, let $e = \sum_{i=1}^{l} e_{\alpha_i}$ and let $f = \sum_{j=1}^{l} j(l-j+1)f_{\alpha_j}$.

In $\mathscr{L}_{B_l}(\mathbf{C})$, let $e = \sum_{i=1}^{l} e_{\alpha_i}$ and let $f = \sum_{j=1}^{l-1} j(2l+1-j)f_{\alpha_j} + \frac{1}{2}l(l+1)f_{\alpha_l}$.

In $\mathscr{L}_{C_l}(\mathbf{C})$, let $e = \sum_{i=1}^{l} e_{\alpha_i}$ and let $f = \sum_{j=1}^{l} j(2l-j)f_{\alpha_j}$.

Then one checks that in each case the arguments of (2.4) go through.

We point out the following alternative proof for the case G of type A_l. Since $o(u) = p$, $p > l$. Let $\mathrm{SL}_2(k)$ act on the homogeneous polynomials of degree l in two variables, a k-vector space of dimension $l+1$. Let $\phi : \mathrm{SL}_2(k) \to \mathrm{SL}_{l+1}(k)$ be the corresponding representation. Then, $\phi\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right)_{ij} = a_{ij}t^{k_{ij}}$ for some a_{ij} in the prime subfield of k and for some integer k_{ij}. Then as $p > l$, ϕ is the unique restricted irreducible rational representation of A_1 of dimension $l+1$. Moreover, $\phi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a regular unipotent element in A_l. As $C_G(u)$ is connected, G-conjugacy implies G_σ-conjugacy, and therefore we may assume that $u = \phi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Now let σ be the Frobenius endomorphism of A_l induced by the map $(\sigma(M))_{ij} = (m_{ij})^q$, for any $M \in \mathrm{GL}(l+1, k)$. Then $\sigma\left(\phi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \phi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\sigma\left(\phi\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \phi\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ and $\sigma\left(\phi\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \phi\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ so $\phi(\mathrm{SL}_2(k))$ is the desired closed connected σ-invariant subgroup of type A_1.

4. PROOFS OF THEOREMS 0.1 AND 0.2

This section contains the proofs of Theorems 0.1 and 0.2 and Corollary 0.5. Theorems 0.1 and 0.2 will be established together in the following.

THEOREM 4.1. *Let G be a semisimple algebraic group defined over an algebraically closed field k of good characteristic $p > 0$. Let σ be a surjective endomorphism of G such that either $\sigma = 1$ or G_σ is finite. Let $u \in G_\sigma$ such that $o(u) = p$. Then there exists a closed connected subgroup $X \leq G$, X of type A_1 with $u \in X$. Moreover, if $\sigma \neq 1$ then X can be chosen so that $\sigma(X) \subseteq X$.*

Proof. We will use induction on $\dim G$, the case where $\dim G = 3$ being clear. Thus, whenever we have u lying in a proper closed connected reductive σ-invariant subgroup, the induction hypothesis ensures the existence of the σ-invariant A_1.

Claim 1. We may assume G is simple.

This is clear if $\sigma = 1$. Suppose $\sigma \neq 1$ and suppose we have established the result for simple groups. Now suppose $G = G_1 \circ G_2 \circ \cdots \circ G_k$, for simple algebraic groups G_i. By induction, we may assume $\langle \sigma \rangle$ has just one orbit on the $\{G_i\}$. Reorder this so that $G_i = \sigma^{i-1}(G_1)$. Then G_1 is σ^k-invariant and $u = u_1\sigma(u_1)\cdots \sigma^{k-1}(u_1)$ for some $u_1 \in G_1$ with $o(u_1) = p$; moreover, $u = \sigma(u)$ implies that $\sigma^k(u_1) = u_1$. By induction, there exists a σ^k-invariant subgroup $X_1 < G_1$ of type A_1 containing u_1. Then u lies in a diagonally embedded σ-invariant A_1 subgroup of $X_1\sigma(X_1)\cdots \sigma^{k-1}(X_1)$.

Claim 2. We may assume G has adjoint type.

We suppose we have established the result for simple groups of adjoint type. Let G be an arbitrary simple algebraic group and let $u \in G_\sigma$ with $o(u) = p$. Now $\sigma(Z(G)) \subseteq Z(G)$, so σ induces a surjective endomorphism $\bar{\sigma}$ of the adjoint group $G/Z(G)$. If $\sigma = 1$ then $\bar{\sigma} = 1$, and if G_σ is finite then $(G/Z(G))_\sigma$ is finite. So by our assumption there exists a $\bar{\sigma}$-invariant closed connected $\bar{X} < G/Z(G)$ with $\bar{X}$ of type A_1 and $uZ(G) \in \bar{X}$. The lift of $\bar{X}$ to G gives the desired subgroup X.

Claim 3. We may assume u is a distinguished unipotent element of G.

Suppose u is not distinguished. Then by (7.3.3) of [17], $C_G(u)^\circ$ contains a semisimple element, and so contains a torus. Then by Lang's theorem, there exists a σ-invariant maximal torus T_0 of $C_G(u)^\circ$. So $C_G(T_0)$ is a σ-invariant reductive group containing u, properly contained in G, and so by induction we are done.

Claim 4. Let $\sigma = i_x\tau q$ where i_x is conjugation by x, τ is a graph automorphism of G, and q is a p-power Frobenius endomorphism of G. Then we may assume that $i_x = 1$.

Suppose that the result is true for all endomorphisms of the form $\sigma' = \tau q'$. Then via a direct application of Lang's theorem we establish the result for the endomorphism $i_x \sigma'$.

We now will reduce to the case where $C_G(u)$ is connected, thereby avoiding difficulties with the splitting of classes in G_σ. This will involve a series of claims.

Claim 5. Let $Q \leq G$ be a connected unipotent group and let $a \in N_G(Q)$ with $o(a) = r$ for some prime $r \neq p$. Let $b \in \langle a \rangle Q$ such that $o(b) = o(a)$. Then there exists $x \in Q$ with $\langle b \rangle = \langle a \rangle^x$.

Let $Q^{(k)} = [Q^{(k-1)}, Q^{(k-1)}]$ and $Q^{(0)} = Q$ and suppose $Q^{(N)} = 1$ but $Q^{(N-1)} \neq 1$. We will use induction on N.

Suppose $N = 1$ so Q is abelian. Say $b = a^k q_0$ for some $q_0 \in Q$. Now by (9.3) of [3], the product map induces an isomorphism of varieties:

$$Q \simeq [a^k, Q] \times C_Q(a^k).$$

So

$$b = a^k q_0 = a^k a^{-k} q_1^{-1} a^k q_1 q_2 = q_1^{-1} a^k q_1 q_2,$$

where $q_1 \in Q$ and $q_2 \in C_Q(a^k)$. Now, $o(q_1^{-1} a^k q_1 q_2) = o(a^k) o(q_2)$ implies that $q_2 = 1$. So $b = (a^k)^{q_1}$ as desired.

Now suppose $N > 1$. By the abelian case, there exists $x \in Q$ with $\langle bQ' \rangle = \langle aQ' \rangle^{xQ'}$. So $b = (a^k)^x q_0$ for some $q_0 \in Q'$. Now Q' is a connected unipotent group and $a \in N_G(Q')$ and $b \in (a^k)^x Q'$ so, by induction, there exists $y \in Q'$ with $\langle b \rangle = \langle a^x \rangle^y$, which is the desired result.

Claim 6. Let Q, $D \leq G$, with Q a normal connected unipotent subgroup of D with D/Q an elementary abelian r-group for some prime $r \neq p$. Then there exists $A \leq D$ with $A \cap Q = 1$ and $D = AQ$. Moreover, if $D = BQ$ for some $B \leq D$ then $B = A^{q_0}$ for some $q_0 \in Q$.

Suppose $|D/Q| = r^t$. We will use induction on t, the case $t = 1$ following from the previous claim.

Let $a \in D$ with $o(a) = r$. Then $\langle a \rangle Q$ is a normal subgroup of D. So for $d \in D$, $\langle a \rangle^d Q = \langle a \rangle Q$. So by the previous claim $\langle a \rangle^d = \langle a \rangle^{q_0}$ for some $q_0 \in Q$. So $dq_0^{-1} \in N_D(\langle a \rangle)$, and hence $D = N_D(\langle a \rangle)Q$. If $N_D(\langle a \rangle) \cap Q = 1$, then we have the desired complement. Otherwise, we have $N_D(\langle a \rangle)/\langle a \rangle$ with normal p-subgroup $(N_D(\langle a \rangle) \cap Q)\langle a \rangle/\langle a \rangle$ and the quotient is an elementary abelian group of order r^{t-1}. So, by induction, there exists $B \leq N_D(\langle a \rangle)$ with $\langle a \rangle \leq B$ and $B \cap Q = 1$ and $N_D(\langle a \rangle) = B(N_D(\langle a \rangle) \cap Q)\langle a \rangle$. But $D = N_D(\langle a \rangle)Q$ so $D = BQ$ as desired.

Now we must show that all such complements are Q-conjugate. If $t = 1$ this follows from the previous claim. So suppose $t > 1$ and $D = AQ = BQ$

with $|A| = r^t = |B|$. Let $a \in A$. Then there exists $b \in B$ with $a \in \langle b \rangle Q$. Then, by the first claim, there exists $q_0 \in Q$ with $\langle a \rangle = \langle b \rangle^{q_0}$. As above, we work inside $N_D(\langle a \rangle)$. We have in $N_D(\langle a \rangle)/\langle a \rangle$ the complements $A/\langle a \rangle$ and $B^{q_0}/\langle a \rangle$ to the p-group $(Q \cap N_D(\langle a \rangle))\langle a \rangle/\langle a \rangle$. So, by induction, there exists $q_1 \in Q$ with $(A/\langle a \rangle)^{q_1 \langle a \rangle} = B^{q_0}/\langle a \rangle$, which gives the desired conjugacy statement.

Now we are ready to show

Claim 7. We may assume $C_G(u)$ is connected.

Recall that we have $C_G(u)^\circ$, a unipotent group, by Claim 3. Suppose $C_G(u)$ is not connected. Then $C_G(u)/C_G(u)^\circ$ is either an elementary abelian r-group for some prime $r \neq p$ or Sym_3, Sym_4, or Sym_5. (See [9, 10, 15, and 19].)

If $C_G(u)/C_G(u)^\circ$ is an elementary abelian r-group for some prime $r \neq p$, we apply the above claims and Lang's theorem to get A, a σ-invariant complement to $C_G(u)^\circ$ in $C_G(u)$. Then $u \in C_G(A)^\circ$, a proper connected reductive maximal rank σ-invariant subgroup of G. So in this case we are done by induction.

Now suppose $C_G(u)/C_G(u)^\circ \cong \mathrm{Sym}_3$, which occurs only if G is an exceptional group in which case we are assuming $p > 3$. Then the above claims imply that for all $x, y \in C_G(u)$ with $o(x) = 3 = o(y)$, there exists $q_0 \in C_G(u)^\circ$ with $\langle y \rangle = \langle x \rangle^{q_0}$. So by Lang's theorem there exists $z \in C_G(u)$ such that $o(z) = 3$ and $\sigma(\langle z \rangle) \subseteq \langle z \rangle$. Then $u \in C_G(\langle z \rangle)^\circ$. Again, we are done by induction.

Now suppose $C_G(u)/C_G(u)^\circ \cong \mathrm{Sym}_4$, which occurs only if G has type F_4. By [15], there exists $S \leq C_G(u)$ with $C_G(u) = SC_G(u)^\circ$ and $S \cong \mathrm{Sym}_4$. Let $A \leq S$ be the alternating subgroup. Then $AC_G(u)^\circ$ is σ-invariant. Also, taking $V \leq A$, $o(V) = 4$, we have $VC_G(u)^\circ$ a σ-invariant group. Now we are in the elementary abelian case as above.

This leaves us with the case $C_G(u)/C_G(u)^\circ \cong \mathrm{Sym}_5$, which occurs for exactly one class in the group E_8. Now E_8 has no nontrivial graph automorphism, so by Claim 4, $\sigma = q$, a p-power Frobenius endomorphism. By Lemma 70 in [10], there exists $v \in G$, v conjugate to u such that σ acts trivially on $C_G(v)/C_G(v)^\circ$ and $C_G(v) = AC_G(v)^\circ$, where $\sigma|_A = \mathrm{id}$.

Now for $g \in G$ such that $g^{-1}vg \in G_\sigma$, we claim that we have an isomorphism (of groups)

$$C_{G_\sigma}(g^{-1}vg)/C_{G_\sigma}(g^{-1}vg) \cap C_G(g^{-1}vg)^\circ$$
$$\cong C_{C_G(v)/C_G(v)^\circ}\big(g\sigma(g)^{-1}C_G(v)^\circ\big).$$

Now recall that for $g^{-1}vg$, $v \in G_\sigma$, then $g\sigma(g)^{-1}$ centralizes v. Define

$$\theta : C_{G_\sigma}(g^{-1}vg) \to C_{C_G(v)/C_G(v)^\circ}\big(g\sigma(g)^{-1}C_G(v)^\circ\big)$$

via $\theta(z) = gzg^{-1}C_G(v)^\circ$. Now note that for $z \in C_{G_\sigma}(g^{-1}vg)$,

$$\begin{aligned} gzg^{-1}g\sigma(g)^{-1}gz^{-1}g^{-1} &= gz\sigma(g)^{-1}gz^{-1}g^{-1} \\ &= g\sigma(g)^{-1}\sigma(g)\sigma(z)\sigma(g)^{-1}gz^{-1}g^{-1} \\ &= g\sigma(g)^{-1}\sigma(gzg^{-1})gz^{-1}g^{-1}. \end{aligned}$$

So modulo $C_G(v)^\circ$, and recalling that σ acts trivially on $C_G(v)/C_G(v)^\circ$, we have

$$\begin{aligned} &= g\sigma(g)^{-1}C_G(v)^\circ\sigma(gzg^{-1})C_G(v)^\circ gz^{-1}g^{-1}C_G(v)^\circ \\ &= g\sigma(g)^{-1}C_G(v)^\circ gzg^{-1}C_G(v)^\circ gz^{-1}g^{-1}C_G(v)^\circ = g\sigma(g)^{-1}C_G(v)^\circ \end{aligned}$$

as desired. Thus θ is a well-defined map into $C_{C_G(v)/C_G(v)^\circ}(g\sigma(g)^{-1}C_G(v)^\circ)$.

Note that θ is clearly a homomorphism and that $\ker(\theta) = C_{G_\sigma}(g^{-1}vg) \cap C_G(g^{-1}vg)^\circ$. Also θ is onto. For if $x \in C_G(v)$ such that $x^{-1}g\sigma(g)^{-1}x \in g\sigma(g)^{-1}C_G(v)^\circ$, then by (2.7) of [4, E.I], there exists $y \in g^{-1}xgC_G(g^{-1}vg)^\circ$ with $\sigma(y) = y$. Then $\theta(y) = gyg^{-1}C_G(v)^\circ$. But $y = g^{-1}xga$ for some $a \in C_G(g^{-1}vg)^\circ$. So $gyg^{-1} = xgag^{-1} \in xC_G(g^{-1}vg)^\circ$. So $\theta(y) = xC_G(v)^\circ$ and we have that θ is also onto.

Now we use this isomorphism to show that there exists a semisimple element in $C_{G_\sigma}(g^{-1}vg)$. Let $s \in A\setminus\{1\}$, so $(o(s), p) = 1$. Let $z \in C_{G_\sigma}(g^{-1}vg)$ such that $\theta(z) = sC_G(v)^\circ$. Note that $z \notin C_G(v)^\circ$, so z is not unipotent. Write $z = tn$, the Jordan decomposition of z with $t \neq 1$ semisimple and n unipotent. Then $z^{p^k} = t^{p^k}$ for some integer k large enough. So $t^{p^k} \neq 1$ is a semisimple element in $C_{G_\sigma}(g^{-1}vg)$ as desired. Now we could choose g such that $u = g^{-1}vg$ and we would have a semisimple element in $C_{G_\sigma}(u)$ and we are again done by induction. This completes the proof of the claim.

As a result of the previous claim, if $u \in G_\sigma$ has G-class $\mathscr{C}$, then $G_\sigma \cap \mathscr{C}$ is the G_σ-class of u. In particular, we are left with the following G-classes of elements (refer to Prop 3.2, [4, 9, 10, 11, 12, 15, and 19]).

G_2: regular.

F_4: regular.

E_6: regular, and the distinguished class with diagram.

$$2\ 2\ \overset{0}{\underset{2}{}}\ 2\ 2.$$

E_7: regular, and the distinguished classes with diagrams

$$2\ 2\ \overset{0}{\underset{2}{}}\ 2\ 2\ 2$$

and

$$2\ 2\ \overset{0}{\underset{2}{}}\ 2\ 0\ 2.$$

E_8: regular, and the distinguished classes with diagrams

$$2\ 2\ \overset{0}{\underset{2}{}}\ 2\ 2\ 2\ 2$$

and

$$2\ 2\ \overset{0}{\underset{2}{}}\ 2\ 0\ 2\ 2.$$

A_l, B_l, C_l: regular.

D_l: distinguished classes corresponding to the partitions of $2l$ consisting of two distinct odd parts.

Claim 8. Theorem 4.1 holds for G of type D_l, $l \geq 4$.

Consider first the case where $\sigma = q\tau$, q a p-power Frobenius endomorphism and τ a (possibly trivial) graph automorphism of G such that $\tau^2 = 1$. Fix a partition $\{2m + 1, 2n + 1\}$, $m > n \geq 0$ of $2l$. Then the unipotent class corresponding to this partition has representative a regular unipotent element in a closed connected reductive subgroup of type $B_n \times B_m$. (See the proof of Proposition 3.2.) We will construct such a subgroup of G.

Set $\beta = \alpha_n + \alpha_{n+1} + \cdots + \alpha_{l-2}$, and then set

$$H_1 = \langle U_{\pm\alpha_1}, U_{\pm\alpha_2}, \ldots, U_{\pm\alpha_{n-1}}, x_{\beta+\alpha_{l-1}}(t)x_{\beta+\alpha_l}(t), x_{-\beta-\alpha_{l-1}}(t)x_{-\beta-\alpha_l}(t) \mid t \in k\rangle,$$

and

$$H_2 = \langle U_{\pm\alpha_{n+1}}, U_{\pm\alpha_{n+2}}, \ldots, U_{\pm\alpha_{l-2}}, x_{\alpha_{l-1}}(t)x_{\alpha_l}(t), x_{-\alpha_{l-1}}(t)x_{-\alpha_l}(t) \mid t \in k\rangle.$$

Now it is easy to see that $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \beta + \alpha_{l-1}, \beta + \alpha_l\}$ (respectively, $\{\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{l-2}, \alpha_{l-1}, \alpha_l\}$) is the base of a D_{n+1}- (respectively, D_{m+1}-) type root subsystem of $\Phi(G)$. Thus, taking fixed points of the involuntary graph automorphisms of the corresponding closed connected subgroups, we obtain the subgroups H_1 and H_2, which therefore have types B_n and B_m respectively.

We now show that $[H_1, H_2] = 1$. For this, we must show that, for all t, $u \in k$,

(i) $[x_{\beta+\alpha_{l-1}}(t)x_{\beta+\alpha_l}(t), x_{\alpha_{l-1}}(u)x_{\alpha_l}(u)] = 1$,

(ii) $[x_{\beta+\alpha_{l-1}}(t)x_{\beta+\alpha_l}(t), x_{-\alpha_{l-1}}(u)x_{-\alpha_l}(u)] = 1$,

(iii) $[x_{-\beta-\alpha_{l-1}}(t)x_{-\beta-\alpha_l}(t), x_{\alpha_{l-1}}(u)x_{\alpha_l}(u)] = 1$,
(iv) $[x_{-\beta-\alpha_{l-1}}(t)x_{-\beta-\alpha_l}(t), x_{-\alpha_{l-1}}(u)x_{-\alpha_l}(u)] = 1$.

For $\alpha, \delta \in \Phi(G)$ such that $\alpha + \delta \in \Phi(G)$, we write $[e_\alpha, e_\delta] = N_{\alpha,\delta}e_{\alpha+\delta}$. Then, using the relations in 4.1.2 of [6], it is straightforward to show that (i)–(iv) are satisfied if and only if the following hold:

(I) $N_{\beta+\alpha_{l-1},\alpha_l} + N_{\beta+\alpha_l,\alpha_{l-1}} = 0$,
(II) $N_{\beta,\alpha_l} + N_{\beta,\alpha_{l-1}} = 0$.

Using the algorithm given in [6] for choosing a set of structure constants, we see that we can arbitrarily choose the values for the structure constants $N_{\beta+\alpha_{l-1},\alpha_l}$, N_{β,α_l}, and $N_{\beta,\alpha_{l-1}}$. Moreover, again using 4.1.2 of [6], we have that

$$N_{\alpha_{l-1},\beta+\alpha_l}N_{\alpha_l,\beta+\alpha_{l-1}} = N_{\alpha_l,\beta}N_{\alpha_{l-1},\beta}.$$

So we choose $N_{\alpha_l,\beta} = 1$, $N_{\alpha_{l-1},\beta} = -1$, $N_{\alpha_l,\beta+\alpha_{l-1}} = 1$, and get $N_{\alpha_{l-1},\beta+\alpha_l} = -1$, which ensures that (I) and (II) are satisfied.

We next show that H_i is invariant under σ for $i = 1, 2$. This is clear if $\tau = 1$. Suppose $\tau \neq 1$; so, $\sigma(x_{\pm\alpha_i}(t)) = x_{\pm\alpha_i}(t^q)$, for $1 \leq i \leq l-2$, for all $t \in k$, and $\sigma(x_{\alpha_{l-1}}(t)) = x_{\alpha_l}(t^q)$, $\sigma(x_{-\alpha_{l-1}}(t)) = x_{-\alpha_l}(t^q)$, $\sigma(x_{\alpha_l}(t)) = x_{\alpha_{l-1}}(t^q)$, and $\sigma(x_{-\alpha_l}(t)) = x_{\alpha_{l-1}}(t^q)$, for all $t \in k$. Then it is clear that $\sigma(H_2) \subseteq H_2$. Also,

$$\sigma\left(x_{\beta+\alpha_{l-1}}(t)\right) = \sigma\left(\left[x_\beta(t), x_{\alpha_{l-1}}(1)\right]\right) = \left[x_\beta(t^q), x_{\alpha_l}(1)\right] = x_{\beta+\alpha_l}(-t^q),$$

$$\begin{aligned}\sigma\left(x_{\beta+\alpha_l}(t)\right) &= \sigma\left(\left[x_\beta(t), x_{\alpha_l}(-1)\right]\right)\\ &= \left[x_\beta(t^q), x_{\alpha_{l-1}}(-1)\right] = x_{\beta+\alpha_{l-1}}(-t^q),\end{aligned}$$

$$\begin{aligned}\sigma\left(x_{-\beta-\alpha_{l-1}}(t)\right) &= \sigma\left(\left[x_{-\beta}(t), x_{-\alpha_{l-1}}(-1)\right]\right)\\ &= \left[x_{-\beta}(t^q), x_{-\alpha_l}(-1)\right] = x_{-\beta-\alpha_l}(-t^q),\end{aligned}$$

and

$$\begin{aligned}\sigma\left(x_{-\beta-\alpha_l}(t)\right) &= \sigma\left(\left[x_{-\beta}(t), x_{-\alpha_l}(1)\right]\right)\\ &= \left[x_{-\beta}(t^q), x_{-\alpha_{l-1}}(1)\right] = x_{-\beta-\alpha_{l-1}}(-t^q).\end{aligned}$$

So $\sigma(H_1) \subseteq H_1$ as well.

Finally, we show that there exists a regular unipotent element of $H_1 \times H_2$ which is fixed by σ. Fix $t \in k$ such that if $\tau = 1$ then $t = 1$ and if $\tau \neq 1$ then $t^{q-1} = -1$. Then set

$$u = x_{\alpha_1}(1)\cdots x_{\alpha_{n-1}}(1)x_{\beta+\alpha_{l-1}}(t)x_{\beta+\alpha_l}(t)x_{\alpha_{n+1}}(1)\cdots x_{\alpha_{l-2}}(1)x_{\alpha_{l-1}}(1)x_{\alpha_l}(1).$$

Clearly u is a regular unipotent element in $H_1 \times H_2$ and one checks that $\sigma(u) = u$. We have shown that there exists a representative of the class corresponding to the partition $\{2m+1, 2n+1\}$ which lies in G_σ, and moreover that this representative lies in a closed connected reductive σ-invariant subgroup of G. Thus we are done by induction.

Finally, consider the case where G has type D_4 and $\sigma = q\tau$ for τ a nontrivial graph automorphism of G with $\tau^3 = 1$. For the regular class, we take the representative $u = x_{\alpha_2}(1)x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)$. Then $\sigma(u) = u$ and $\tau(u) = u$. So $u \in G_\tau$, which is a proper closed connected reductive σ-invariant subgroup of G.

It remains to consider the unique nonregular semiregular class of unipotent elements in G of type D_4. The assumption that the elements in this class have order p implies that $p > 3$. (See Propositions 3.2 and 3.3.) Let $\phi : A_2 \to D_4$ be the (irreducible) adjoint representation of A_2 and set $Y = \phi(A_2)$. We note that the regular unipotent elements in Y have Jordan blocks of sizes 5 and 3 on the natural module for D_4 and hence are representatives of the nonregular semiregular class in D_4. As well, we note that $\sigma(Y)$ acts irreducibly on the natural module for G. But there exists a unique conjugacy class of irreducible A_2-type subgroups of D_4, so there exists $g \in G$ such that $\sigma(Y) = g^{-1}Yg$. So, by Lang's theorem, there exists $x \in D_4$ with $\sigma(x^{-1}Yx) \subseteq x^{-1}Yx$. Again by Lang's theorem, there exists $u \in x^{-1}Yx$ with $\sigma(u) = u$ and u a regular unipotent element in $x^{-1}Yx$. Thus u is a nonregular semiregular unipotent element in G, is fixed by σ, and lies in a closed connected reductive σ-invariant subgroup of G. Thus we are once again done by induction. This completes the proof of Claim 8.

Claim 9. We may assume $\sigma = q$, a p-power Frobenius endomorphism.

By Claim 8, we need only consider the groups E_6 and A_l as these are the only remaining groups having a nontrivial graph automorphism. Suppose $\sigma = \tau q$, where τ is a nontrivial graph automorphism and q is as above. If u is a regular unipotent element in A_l where l is odd, then we may assume

$$u = x_{\alpha_1}(1)x_{\alpha_l}(1)x_{\alpha_2}(1)x_{\alpha_{l-1}}(1)\cdots x_{\alpha_{(l-1)/2}}(1)x_{\alpha_{(l+3)/2}}(1)x_{\alpha_{(l+1)/2}}(1).$$

Then $u \in G_\tau \cap G_q$; in particular, $u \in G_\tau$, a σ-invariant closed connected reductive subgroup of G. So we are done by induction. If u is regular unipotent in A_l where l is even, then $o(u) = p$ implies $p > 2$ and we may assume

$$u = x_{\alpha_1}(1)x_{\alpha_l}(1)x_{\alpha_2}(1)x_{\alpha_{l-1}}(1)\cdots x_{\alpha_{l/2}}(1)x_{\alpha_{(l+2)/2}}(1)x_{\alpha_{l/2}+\alpha_{(l+2)/2}}(-\tfrac{1}{2}).$$

Again one may check that $u \in G_\tau \cap G_q$ and argue as above. Suppose G has type E_6 and u is a regular unipotent element. Then we may assume

$$u = x_{\alpha_1}(1)x_{\alpha_6}(1)x_{\alpha_3}(1)x_{\alpha_5}(1)x_{\alpha_2}(1)x_{\alpha_4}(1)$$

and argue as in the A_l case.

Finally, consider the case where G has type E_6 and u is a distinguished unipotent element in the E_6 class with diagram $2\ 2\ \overset{0}{2}\ 2\ 2$. Then by (2.7), $u \sim v$, where v is a regular unipotent element in $G_{\tau i_x}$ for some $x \in G$. Moreover, from the expression for v given in (2.7), we see that $q(v) = v$. So if $\sigma' = q\tau i_x$ then $\sigma'(v) = v$ and v lies in the σ'-invariant closed connected subgroup $G_{\tau i_x}$ and, by induction, there exists a σ' invariant subgroup $X \leq G$ with $v \in X$ and X of type A_1. But if $\sigma(a)a^{-1} = \sigma(x)$ for $a \in G$ (given by Lang's theorem), then since $\sigma'(v) = v$, $\sigma(a^{-1}va) = a^{-1}va$ and $a^{-1}va \in a^{-1}Xa$, a subgroup of type A_1 and $\sigma(a^{-1}Xa) \subseteq a^{-1}Xa$. But since $C_G(u)$ is connected, $a^{-1}va$ is G_σ-conjugate to u and we have the desired σ-invariant A_1 subgroup overlying u.

We may now complete the Proof of Theorem 4.1. By the above series of claims, it remains to establish the existence of A_1 subgroups in each of the adjoint simple groups, excepting type D_l, overlying the elements listed prior to Claim 8, and show that the A_1 subgroups are invariant under p-power Frobenius endomorphisms. This was done in (2.4) and (3.4).

We close this section with the

Proof of Corollary 0.5. Let G be a simple algebraic group defined over an algebraically closed field of arbitrary positive characteristic p. Let $x \in G$ be an arbitrary unipotent element. We first note that $o(x) \leq o(u)$ where u is a regular unipotent element of G. For otherwise, we fix B, a Borel subgroup of G with $u \in R_u(B)$, and $g \in G$ such that $g^{-1}xg \in R_u(B)$. Then the B-orbit of u, being dense in $R_u(B)$, must intersect the nonempty open set $\{v \in R_u(B) \mid v^{o(u)} \neq 1\}$. But clearly the elements in the B-orbit of u all have order $o(u)$.

We now recall that G_σ nontrivially intersects the class of regular unipotent elements in G. Thus to establish Corollary 0.5 it remains to show that the regular unipotent elements in G have order $\min\{p^a \mid p^a > \mathrm{ht}(r_0)\}$. This was established in Proposition 2.2 for G of exceptional type and in Proposition 3.3 for G of type A_l. For G of type C_l, the regular unipotent element is a regular unipotent in A_{2l-1} (using the natural representation of C_l) and so has order $\min\{p^a \mid p^a > 2l - 1\}$ as desired. For G of type B_l, the corollary follows from Proposition 3.3 if $p > 2$. If $p = 2$, there exist bijective morphisms $\theta_1 : G \to H$ and $\theta : H \to G$, where H is a simple algebraic group of type C_l, and u_G, u_H are regular unipotent elements in G, H, respectively, such that $\theta_1(u_G) = u_H$ and $\theta_2(u_H) = u_G$. (See Theorem 28 of [18].) Thus $o(u_G) = o(u_H) = \min\{p^a \mid p^a > 2l - 1\}$ as desired. Finally, for G of type D_l, there exists a regular unipotent element which lies in the fixed point subgroup of the involuntary graph automorphism, a simple algebraic group of type B_{l-1}. Moreover, this element is a regular unipotent element in B_{l-1}. So, by the above, it has order $\min\{p^a \mid p^a > 2(l - 1) - 1\}$ as desired.

APPENDIX

In this section we describe certain matrix representations of the Chevalley groups associated with the finite-dimensional Lie algebras of types E_7 and E_8, over an arbitrary field $\mathscr{K}$. (We denote these groups $E_7(\mathscr{K})$ and $E_8(\mathscr{K})$, respectively.) We have included more than enough information to check the statements about orders of unipotent elements in (2.2). Indeed, we have given a complete description of the images of the fundamental root groups under the representation, with the hopes that this computation itself may be of use in the field. For the purposes of this computation, we used a set of structure constants for the complex Lie algebra of type E_8 constructed by Gilkey and Seitz for their work [7]. It is available from them upon request.

Let $\mathscr{L}(E_7)$ be a finite-dimensional complex simple Lie algebra of type E_7. Let $\{e_\alpha, f_\alpha, h_\gamma \mid \alpha \in \Phi^+, \gamma \in \Pi\}$ be a Chevalley basis for $\mathscr{L}(E_7)$. Let $\Pi = \{\gamma_i \mid 1 \leq i \leq 7\}$ with the ordering chosen to be consistent with the labeling of Dynkin diagrams given in the introduction. For $\beta \in \Phi^+$, if $\beta = \sum_{i=1}^{7} a_i\gamma_i$ we will write $f_{a_1a_2\cdots a_7}$ for f_β. Let W be the irreducible $\mathscr{L}(E_7)$ module with high weight λ_7. Choose $0 \neq w^+ \in W$ such that $e_\alpha w^+ = 0$ for all $\alpha \in \Phi^+$. Fix the following (Kostant) basis of W:

$w_1 = w^+$
$w_2 = f_{0000001}w^+$
$w_3 = f_{0000011}w^+$
$w_4 = f_{0000111}w^+$
$w_5 = f_{0001111}w^+$
$w_6 = f_{0101111}w^+$
$w_7 = f_{0011111}w^+$
$w_8 = f_{1011111}w^+$
$w_9 = f_{0111111}w^+$
$w_{10} = f_{1111111}w^+$
$w_{11} = f_{0112111}w^+$
$w_{12} = f_{1112111}w^+$
$w_{13} = f_{0112211}w^+$
$w_{14} = f_{1112211}w^+$
$w_{15} = f_{0112221}w^+$
$w_{16} = f_{1112221}w^+$
$w_{17} = f_{1122111}w^+$
$w_{18} = f_{1122211}w^+$
$w_{19} = f_{1122221}w^+$
$w_{20} = f_{0000001}f_{0112221}w^+$
$w_{21} = f_{0000001}f_{1112221}w^+$
$w_{22} = f_{0000001}f_{1122221}w^+$
$w_{23} = f_{1123211}w^+$
$w_{24} = f_{1223211}w^+$
$w_{25} = f_{1123221}w^+$
$w_{26} = f_{1223221}w^+$
$w_{27} = f_{0000001}f_{1123221}w^+$
$w_{28} = f_{0000001}f_{1223221}w^+$
$w_{29} = f_{1123321}w^+$
$w_{30} = f_{1223321}w^+$
$w_{31} = f_{0000001}f_{1123321}w^+$
$w_{32} = f_{0000001}f_{1223321}w^+$
$w_{33} = f_{0000011}f_{1123321}w^+$
$w_{34} = f_{0000011}f_{1223321}w^+$
$w_{35} = f_{1224321}w^+$
$w_{36} = f_{1234321}w^+$
$w_{37} = f_{2234321}w^+$
$w_{38} = f_{0000001}f_{1224321}w^+$
$w_{39} = f_{0000011}f_{1224321}w^+$
$w_{40} = f_{0000111}f_{1224321}w^+$
$w_{41} = f_{0000001}f_{1234321}w^+$
$w_{42} = f_{0000001}f_{2234321}w^+$
$w_{43} = f_{0000011}f_{1234321}w^+$
$w_{44} = f_{0000011}f_{2234321}w^+$
$w_{45} = f_{0000111}f_{1234321}w^+$
$w_{46} = f_{0000111}f_{2234321}w^+$
$w_{47} = f_{0001111}f_{1234321}w^+$
$w_{48} = f_{0001111}f_{2234321}w^+$
$w_{49} = f_{0101111}f_{1234321}w^+$
$w_{50} = f_{0101111}f_{2234321}w^+$
$w_{51} = f_{0011111}f_{2234321}w^+$
$w_{52} = f_{0111111}f_{2234321}w^+$
$w_{53} = f_{0112111}f_{2234321}w^+$
$w_{54} = f_{0112211}f_{2234321}w^+$
$w_{55} = f_{0112221}f_{2234321}w^+$
$w_{56} = f_{0000001}f_{0112221}f_{2234321}w^+$

Let $\mathscr{B} = \{w_i \mid 1 \leq i \leq 56\}$ (an ordered basis) and set $M = \sum \mathbf{Z}w_i$. It is well known that M is invariant under $\{(e_\alpha^n)/n!, (f_\alpha^n)/n! \mid \alpha \in \Phi^+, n \in \mathbf{Z}^+\}$

and that e_α^n and f_α^n act as zero on W for sufficiently large values of n. Set $W(\mathscr{K}) = M \otimes_{\mathbf{Z}} \mathscr{K}$. Then, for $t \in \mathscr{K}$, we have an action of $\exp(te_\alpha) = 1 + \sum_{n=1}^{\infty}(te_\alpha)^n/n!$ and $\exp(tf_\alpha)$ on $W(\mathscr{K})$. We may then define a faithful representation $\phi : E_7(\mathscr{K}) \to \mathrm{SL}(W(\mathscr{K}))$ on certain elements of $E_7(\mathscr{K})$ by $\phi(x_\beta(t)) = \exp(te_\beta)$ for $\beta \in \Phi^+$ and $t \in \mathscr{K}$. We identify $\mathrm{SL}(W(\mathscr{K}))$ with $\mathrm{SL}_{56}(\mathscr{K})$ via the ordered basis $\mathscr{B} = \{w_i \otimes 1 \mid 1 \le i \le 56\}$. A description of ϕ is given below, with $E_{i,j}$ denoting the 56×56 matrix whose (k, l) entry is $\delta_{ik}\delta_{jl}$ and I denoting the 56×56 identity matrix.

$\phi(x_{\alpha_1}(t)) = I + t(-E_{7,8} - E_{9,10} - E_{11,12} - E_{13,14} - E_{15,16} - E_{20,21} - E_{36,37} - E_{41,42} - E_{43,44} - E_{45,46} - E_{47,48} - E_{49,50})$.

$\phi(x_{\alpha_2}(t)) = I + t(-E_{5,6} - E_{7,9} - E_{8,10} - E_{23,24} - E_{25,26} - E_{27,28} - E_{29,30} - E_{31,32} - E_{33,34} - E_{47,49} - E_{48,50} - E_{51,52})$.

$\phi(x_{\alpha_3}(t)) = I + t(-E_{5,7} - E_{6,9} - E_{12,17} - E_{14,18} - E_{16,19} - E_{21,22} - E_{35,36} - E_{38,41} - E_{39,43} - E_{40,45} - E_{48,51} - E_{50,52})$.

$\phi(x_{\alpha_4}(t)) = I + t(-E_{4,5} - E_{9,11} - E_{10,12} - E_{18,23} - E_{19,25} - E_{22,27} - E_{30,35} - E_{32,38} - E_{34,39} - E_{45,47} - E_{46,48} - E_{52,53})$.

$\phi(x_{\alpha_5}(t)) = I + t(-E_{3,4} - E_{11,13} - E_{12,14} - E_{17,18} - E_{25,29} - E_{26,30} - E_{27,31} - E_{28,32} - E_{39,40} - E_{43,45} - E_{44,46} - E_{53,54})$.

$\phi(x_{\alpha_6}(t)) = I + t(-E_{2,3} - E_{13,15} - E_{14,16} - E_{18,19} - E_{23,25} - E_{24,26} - E_{31,33} - E_{32,34} - E_{38,39} - E_{41,43} - E_{42,44} - E_{54,55})$.

$\phi(x_{\alpha_7}(t)) = I + t(E_{1,2} + E_{15,20} + E_{16,21} + E_{19,22} + E_{25,27} + E_{26,28} + E_{29,31} + E_{30,32} + E_{35,38} + E_{36,41} + E_{37,42} + E_{55,56})$.

$\phi(x_{\alpha_3+\alpha_4}(t)) = I + t(-E_{4,7} + E_{6,11} - E_{10,17} + E_{14,23} + E_{16,25} + E_{21,27} - E_{30,36} - E_{32,41} - E_{34,43} + E_{40,47} - E_{46,51} + E_{50,53})$.

$\phi(x_{\alpha_2+\alpha_4}(t)) = I + t(-E_{4,6} + E_{7,11} + E_{8,12} - E_{18,24} - E_{19,26} - E_{22,28} + E_{29,35} + E_{31,38} + E_{33,39} - E_{45,49} - E_{46,50} + E_{51,53})$.

$\phi(x_{\alpha_4+\alpha_5}(t)) = I + t(-E_{3,5} + E_{9,13} + E_{10,14} - E_{17,23} + E_{19,29} + E_{22,31} - E_{26,35} - E_{28,38} + E_{34,40} - E_{43,47} - E_{44,48} + E_{52,54})$.

$\phi(x_{\alpha_5+\alpha_6}(t)) = I + t(-E_{2,4} + E_{11,15} + E_{12,16} + E_{17,19} - E_{23,29} - E_{24,30} + E_{27,33} + E_{28,34} - E_{38,40} - E_{41,45} - E_{42,46} + E_{53,55})$.

$\phi(x_{\alpha_6+\alpha_7}(t)) = I + t(E_{1,3} - E_{13,20} - E_{14,21} - E_{18,22} - E_{23,27} - E_{24,28} + E_{29,33} + E_{30,34} + E_{35,39} + E_{36,43} + E_{37,44} - E_{54,56})$.

This completes our description of the representation of $E_7(\mathscr{K})$. We will now describe a representation of $E_8(\mathscr{K})$.

Let $\mathscr{L}(E_8)$ be a finite-dimensional complex simple Lie algebra of type E_8. Let $\{e_\alpha, f_\alpha, h_\gamma \mid \alpha \in \Phi^+, \gamma \in \Pi\}$ be a Chevalley basis for $\mathscr{L}(E_8)$. Let $\Pi = \{\gamma_i \mid 1 \le i \le 8\}$ with the ordering chosen to be consistent with the labeling of Dynkin diagrams given in the introduction. For $\beta \in \Phi^+$, with $\beta = \sum_{i=1}^{8} a_i\gamma_i$, we will write $(a_1a_2 \cdots a_8)$ for β. Fix the following ordering of Φ^+:

$\beta_{1} = (23465432)$	$\beta_{31} = (12232221)$	$\beta_{61} = (01122210)$	$\beta_{91} = (11111000)$
$\beta_{2} = (23465431)$	$\beta_{32} = (12243210)$	$\beta_{62} = (11222100)$	$\beta_{92} = (00001111)$
$\beta_{3} = (23465421)$	$\beta_{33} = (12232211)$	$\beta_{63} = (11122110)$	$\beta_{93} = (00011110)$
$\beta_{4} = (23465321)$	$\beta_{34} = (11233211)$	$\beta_{64} = (11221110)$	$\beta_{94} = (00111100)$
$\beta_{5} = (23464321)$	$\beta_{35} = (11232221)$	$\beta_{65} = (01121111)$	$\beta_{95} = (01011100)$
$\beta_{6} = (23454321)$	$\beta_{36} = (12233210)$	$\beta_{66} = (11122100)$	$\beta_{96} = (01111000)$
$\beta_{7} = (22454321)$	$\beta_{37} = (12232111)$	$\beta_{67} = (01122110)$	$\beta_{97} = (10111000)$
$\beta_{8} = (23354321)$	$\beta_{38} = (11232211)$	$\beta_{68} = (11221100)$	$\beta_{98} = (11110000)$
$\beta_{9} = (22354321)$	$\beta_{39} = (11222221)$	$\beta_{69} = (11121110)$	$\beta_{99} = (00000111)$
$\beta_{10} = (13354321)$	$\beta_{40} = (11233210)$	$\beta_{70} = (11111111)$	$\beta_{100} = (00001110)$
$\beta_{11} = (22344321)$	$\beta_{41} = (12232210)$	$\beta_{71} = (01122100)$	$\beta_{101} = (00011100)$
$\beta_{12} = (12354321)$	$\beta_{42} = (11232111)$	$\beta_{72} = (11221000)$	$\beta_{102} = (00111000)$
$\beta_{13} = (12344321)$	$\beta_{43} = (11222211)$	$\beta_{73} = (01121110)$	$\beta_{103} = (01011000)$
$\beta_{14} = (22343321)$	$\beta_{44} = (11122221)$	$\beta_{74} = (11121100)$	$\beta_{104} = (01110000)$
$\beta_{15} = (12244321)$	$\beta_{45} = (12232110)$	$\beta_{75} = (10111111)$	$\beta_{105} = (10110000)$
$\beta_{16} = (12343321)$	$\beta_{46} = (11232210)$	$\beta_{76} = (01111111)$	$\beta_{106} = (00000011)$
$\beta_{17} = (22343221)$	$\beta_{47} = (11122211)$	$\beta_{77} = (11111110)$	$\beta_{107} = (00000110)$
$\beta_{18} = (12243321)$	$\beta_{48} = (11222111)$	$\beta_{78} = (01121100)$	$\beta_{108} = (00001100)$
$\beta_{19} = (22343211)$	$\beta_{49} = (01122221)$	$\beta_{79} = (11121000)$	$\beta_{109} = (00011000)$
$\beta_{20} = (12343221)$	$\beta_{50} = (12232100)$	$\beta_{80} = (00111111)$	$\beta_{110} = (00110000)$
$\beta_{21} = (12233321)$	$\beta_{51} = (11232110)$	$\beta_{81} = (01011111)$	$\beta_{111} = (01010000)$
$\beta_{22} = (12243221)$	$\beta_{52} = (11222210)$	$\beta_{82} = (01111110)$	$\beta_{112} = (10100000)$
$\beta_{23} = (12343211)$	$\beta_{53} = (11122111)$	$\beta_{83} = (10111110)$	$\beta_{113} = (00000001)$
$\beta_{24} = (22343210)$	$\beta_{54} = (11221111)$	$\beta_{84} = (11111100)$	$\beta_{114} = (00000010)$
$\beta_{25} = (11233321)$	$\beta_{55} = (11222110)$	$\beta_{85} = (01121000)$	$\beta_{115} = (00000100)$
$\beta_{26} = (12243211)$	$\beta_{56} = (11122210)$	$\beta_{86} = (00011111)$	$\beta_{116} = (00001000)$
$\beta_{27} = (12233221)$	$\beta_{57} = (11232100)$	$\beta_{87} = (01011110)$	$\beta_{117} = (00010000)$
$\beta_{28} = (12343210)$	$\beta_{58} = (01122211)$	$\beta_{88} = (00111110)$	$\beta_{118} = (00100000)$
$\beta_{29} = (12233211)$	$\beta_{59} = (11121111)$	$\beta_{89} = (01111100)$	$\beta_{119} = (01000000)$
$\beta_{30} = (11233221)$	$\beta_{60} = (01122111)$	$\beta_{90} = (10111100)$	$\beta_{120} = (10000000)$

We now fix an ordered basis of $\mathscr{L}(E_8)$ as follows: For $1 \le i \le 120$, set $v_i = e_{\beta_i}$; for $129 \le i \le 248$, set $v_i = f_{\beta_{249-i}}$; for $121 \le i \le 128$, set $v_i = h_{\gamma_{(i-120)}}$. Let $\mathscr{B} = \{v_i \mid 1 \le i \le 248\}$ and set $M = \sum \mathbf{Z} v_i$. As above, we obtain a matrix representation $\psi : E_8(\mathscr{K}) \to \mathrm{SL}(\mathscr{V}(\mathscr{K}))$, where $V(\mathscr{K}) = \mathscr{M} \otimes_{\mathbf{Z}} \mathscr{K}$. We identify $\mathrm{SL}(V(\mathscr{K}))$ with $\mathrm{SL}_{248}(\mathscr{K})$ via the ordered basis $\overline{\mathscr{B}} = \{v_i \otimes 1 \mid 1 \le i \le 248\}$. The description of the images under ψ of certain root group elements $x_\beta(t)$, $\beta \in \Phi^+$, $t \in \mathscr{K}$, is given below, where $E_{i,j}$ now denotes the 248×248 matrix whose (k, l) entry is $\delta_{ik}\delta_{jl}$ and I denotes the 248×248 identity matrix.

$$\psi(x_{\alpha_1}(t)) = I + t(E_{8,10} + E_{9,12} + E_{11,13} + E_{14,16} + E_{17,20} + E_{19,23} + E_{24,28} + E_{44,49} + E_{47,58} + E_{53,60} + E_{56,61} + E_{59,65} + E_{63,67} + E_{66,71} + E_{69,73} + E_{70,76} + E_{74,78} + E_{75,80} + E_{77,82} + E_{79,85} + E_{83,88} + E_{84,89} + E_{90,94} + E_{91,96} + E_{97,102} + E_{98,104} + E_{105,110} + E_{112,118} - 2E_{120,121} + E_{120,123} + E_{121,129} - E_{131,137} - E_{139,144} - E_{145,151} - E_{147,152} - E_{153,158} - E_{155,159} - E_{160,165} - E_{161,166} - E_{164,170} - E_{167,172} - E_{169,174} - E_{171,175} - E_{173,179} - E_{176,180} - E_{178,183} - E_{182,186} - E_{184,190} - E_{188,193} - E_{189,196} - E_{191,202} - E_{200,205} - E_{221,225} - E_{226,230} - E_{229,232} - E_{233,235} - E_{236,238} - E_{237,240} - E_{239,241}) - t^2 E_{120,129}.$$

$$\psi(x_{\alpha_2}(t)) = I + t(E_{6,7} + E_{8,9} + E_{10,12} + E_{21,25} + E_{27,30} + E_{29,34} + E_{31,35} + E_{33,38} + E_{36,40} + E_{37,42} + E_{41,46} + E_{45,51} + E_{50,57} + E_{70,75} + E_{76,80} + E_{77,83} + E_{81,86} + E_{82,88} + E_{84,90} + E_{87,93} + E_{89,94} + E_{91,97} + E_{95,101} + E_{96,102} + E_{98,105} + E_{103,109} + E_{104,110} + E_{111,117} - 2E_{119,122} + E_{119,124} + E_{122,130} - E_{132,138} - E_{139,145} - E_{140,146} - E_{144,151} - E_{147,153} - E_{148,154} - E_{152,158} - E_{155,160} - E_{156,162} - E_{159,165} - E_{161,167} - E_{163,168} - E_{166,172} - E_{169,173} - E_{174,179} - E_{192,199} - E_{198,204} - E_{203,208} - E_{207,212} - E_{209,213} - E_{211,216} - E_{214,218} - E_{215,220} - E_{219,222} - E_{224,228} - E_{237,239} - E_{240,241} - E_{242,243}) - t^2 E_{119,130}.$$

$$\psi(x_{\alpha_2+\alpha_4}(t)) = I + t(-E_{5,7} + E_{8,11} + E_{10,13} - E_{18,25} - E_{22,30} - E_{26,34} - E_{32,40} + E_{31,39} + E_{33,43} + E_{37,48} + E_{41,52} + E_{45,55} + E_{50,62} - E_{59,75} - E_{65,80} - E_{69,83} - E_{73,88} - E_{74,90} - E_{78,94} - E_{79,97} + E_{81,92} - E_{85,102} + E_{87,100} + E_{95,108} - E_{98,112} + E_{103,116} - E_{104,118} - E_{111,122} + E_{111,123} - E_{111,124} + E_{111,125} - E_{117,130} + E_{119,132} + E_{122,138} + E_{124,138} + E_{131,145} - E_{133,146} + E_{137,151} - E_{141,154} - E_{149,162} - E_{157,168} + E_{147,164} + E_{152,170} + E_{155,171} + E_{159,175} + E_{161,176} + E_{166,180} + E_{169,184} + E_{174,190} - E_{187,199} - E_{194,204} - E_{197,208} - E_{201,212} - E_{206,216} + E_{209,217} - E_{210,218} + E_{215,223} + E_{219,227} + E_{224,231} - E_{236,239} - E_{238,241} + E_{242,244}) - t^2 E_{111,138}.$$

$$\psi(x_{\alpha_3}(t)) = I + t(E_{6,8} + E_{7,9} + E_{13,15} + E_{16,18} + E_{20,22} + E_{23,26} + E_{28,32} + E_{39,44} + E_{43,47} + E_{48,53} + E_{52,56} + E_{54,59} + E_{55,63} + E_{62,66} + E_{64,69} + E_{68,74} + E_{72,79} + E_{76,81} + E_{80,86} + E_{82,87} + E_{88,93} + E_{89,95} + E_{94,101} + E_{96,103} + E_{102,109} + E_{104,111} + E_{110,117} - E_{112,120} + E_{118,121} - 2E_{118,123} + E_{118,124} + E_{123,131} + E_{129,137} - E_{132,139} - E_{138,145} - E_{140,147} - E_{146,153} - E_{148,155} - E_{154,160} - E_{156,161} - E_{162,167} - E_{163,169} - E_{168,173} - E_{170,177} - E_{175,181} - E_{180,185} - E_{183,187} - E_{186,194} - E_{190,195} - E_{193,197} - E_{196,201} - E_{202,206} - E_{205,210} - E_{217,221} - E_{223,226} - E_{227,229} - E_{231,233} - E_{234,236} - E_{240,242} - E_{241,243}) - t^2 E_{118,131}.$$

$$\psi(x_{\alpha_4}(t)) = I + t(E_{5,6} + E_{9,11} + E_{12,13} + E_{18,21} + E_{22,27} + E_{26,29} + E_{32,36} + E_{35,39} + E_{38,43} + E_{42,48} + E_{46,52} + E_{51,55} + E_{57,62} + E_{59,70} + E_{65,76} + E_{69,77} + E_{73,82} + E_{74,84} + E_{78,89} + E_{79,91} + E_{86,92} + E_{85,96} + E_{93,100} + E_{101,108} - E_{105,112} + E_{109,116} - E_{110,118} - E_{111,119} + E_{117,122} + E_{117,123} - 2E_{117,124} + E_{117,125} + E_{124,132} + E_{130,138} + E_{131,139} - E_{133,140} + E_{137,144} - E_{141,148} - E_{149,156} - E_{157,163} - E_{153,164} - E_{158,170} - E_{160,171} - E_{165,175} - E_{167,176} - E_{172,180} - E_{173,184} - E_{179,190} - E_{187,192} - E_{194,198} - E_{197,203} - E_{201,207} - E_{206,211} - E_{210,214} - E_{213,217} - E_{220,223} - E_{222,227} - E_{228,231} - E_{236,237} - E_{238,240} - E_{243,244}) - t^2 E_{117,132}.$$

$$\psi(x_{\alpha_3+\alpha_4}(t)) = I + t(-E_{5,8} + E_{7,11} - E_{12,15} + E_{16,21} + E_{20,27} + E_{23,29} + E_{28,36} - E_{35,44} - E_{38,47} - E_{42,53} - E_{46,56} - E_{51,63} + E_{54,70} - E_{57,66} + E_{64,77} - E_{65,81} + E_{68,84} + E_{72,91} - E_{73,87} - E_{78,95} + E_{80,92} - E_{85,103} + E_{88,100} + E_{94,108} + E_{102,116} - E_{104,119} - E_{105,120} + E_{110,122} - E_{110,123} - E_{110,124} + E_{110,121} + E_{110,125} - E_{117,131} + E_{118,132} + E_{123,139} + E_{124,139} + E_{129,144} + E_{130,145} - E_{133,147} - E_{141,155} + E_{146,164} - E_{149,161} + E_{154,171} - E_{157,169} - E_{158,177} + E_{162,176} - E_{165,181} + E_{168,184} - E_{172,185} - E_{179,195} + E_{183,192} + E_{186,198} + E_{193,203} + E_{196,207} + E_{202,211} + E_{205,214} - E_{213,221} - E_{220,226} - E_{222,229} - E_{228,233} + E_{234,237} - E_{238,242} + E_{241,244}) - t^2 E_{110,139}.$$

$$\psi(x_{\alpha_5}(t)) = I + t(E_{4,5} + E_{11,14} + E_{13,16} + E_{15,18} + E_{27,31} + E_{29,33} + E_{30,35} + E_{34,38} + E_{36,41} + E_{40,46} + E_{48,54} + E_{53,59} + E_{55,64} + E_{60,65} + E_{62,68} + E_{63,69} + E_{67,73} + E_{66,74} + E_{71,78} - E_{91,98} + E_{92,99} - E_{96,104} - E_{97,105} + E_{100,107} - E_{102,110} - E_{103,111} + E_{108,115} - E_{109,117} + E_{116,124} - 2E_{116,125} + E_{116,126} + E_{125,133} + E_{132,140} - E_{134,141} + E_{138,146} + E_{139,147} - E_{142,149} + E_{144,152} + E_{145,153} - E_{150,157} + E_{151,158} - E_{171,178} - E_{176,182} - E_{175,183} - E_{180,186} - E_{181,187} - E_{184,189} - E_{185,194} - E_{190,196} - E_{195,201} - E_{203,209} - E_{208,213} - E_{211,215} - E_{214,219} - E_{216,220} - E_{218,222} - E_{231,234} - E_{233,236} - E_{235,238} - E_{244,245}) - t^2 E_{116,133}.$$

$$\psi(x_{\alpha_4+\alpha_5}(t)) = I+t(-E_{4,6}+E_{9,14}+E_{12,16}-E_{15,21}+E_{22,31}+E_{26,33}-E_{30,39}+E_{32,41}-E_{34,43}-E_{40,52}+E_{42,54}+E_{51,64}-E_{53,70}+E_{57,68}-E_{60,76}-E_{63,77}-E_{66,84}-E_{67,82}-E_{71,89}-E_{79,98}-E_{85,104}+E_{86,99}+E_{93,107}-E_{97,112}+E_{101,115}-E_{102,118}-E_{103,119}-E_{109,124}+E_{109,122}+E_{109,123}-E_{109,125}+E_{109,126}-E_{116,132}+E_{117,133}+E_{124,140}+E_{125,140}+E_{130,146}+E_{131,147}-E_{134,148}+E_{137,152}-E_{142,156}+E_{145,164}-E_{150,163}+E_{151,170}+E_{160,178}+E_{165,183}+E_{167,182}+E_{172,186}+E_{173,189}+E_{179,196}-E_{181,192}-E_{185,198}-E_{195,207}+E_{197,209}+E_{206,215}-E_{208,217}+E_{210,219}-E_{216,223}-E_{218,227}+E_{228,234}-E_{233,237}-E_{235,240}+E_{243,245})-t^2E_{109,140}.$$

$$\psi(x_{\alpha_6}(t)) = I+t(E_{3,4}+E_{14,17}+E_{16,20}+E_{18,22}+E_{21,27}+E_{25,30}+E_{33,37}+E_{38,42}+E_{41,45}+E_{43,48}+E_{46,51}+E_{47,53}+E_{52,55}+E_{58,60}+E_{56,63}+E_{61,67}-E_{68,72}-E_{74,79}-E_{78,85}-E_{84,91}-E_{89,96}-E_{90,97}-E_{94,102}-E_{95,103}+E_{99,106}-E_{101,109}+E_{107,114}-E_{108,116}+E_{115,125}-2E_{115,126}+E_{115,127}+E_{126,134}+E_{133,141}-E_{135,142}+E_{140,148}-E_{143,150}+E_{146,154}+E_{147,155}+E_{152,159}+E_{153,160}+E_{158,165}+E_{164,171}+E_{170,175}+E_{177,181}-E_{182,188}-E_{189,191}-E_{186,193}-E_{194,197}-E_{196,202}-E_{198,203}-E_{201,206}-E_{204,208}-E_{207,211}-E_{212,216}-E_{219,224}-E_{222,228}-E_{227,231}-E_{229,233}-E_{232,235}-E_{245,246})-t^2E_{115,134}.$$

$$\psi(x_{\alpha_5+\alpha_6}(t)) = I+t(-E_{3,5}+E_{11,17}+E_{13,20}+E_{15,22}-E_{21,31}-E_{25,35}+E_{29,37}+E_{34,42}+E_{36,45}+E_{40,51}-E_{43,54}-E_{47,59}-E_{52,64}-E_{56,69}-E_{58,65}-E_{61,73}-E_{62,72}-E_{66,79}-E_{71,85}-E_{84,98}-E_{89,104}-E_{90,105}+E_{92,106}-E_{94,110}-E_{95,111}+E_{100,114}-E_{101,117}+E_{108,124}-E_{108,125}-E_{108,126}+E_{108,127}-E_{115,133}+E_{116,134}+E_{125,141}+E_{126,141}+E_{132,148}-E_{135,149}+E_{138,154}+E_{139,155}-E_{143,157}+E_{144,159}+E_{145,160}+E_{151,165}+E_{164,178}+E_{170,183}+E_{176,188}+E_{177,187}+E_{180,193}+E_{184,191}+E_{185,197}+E_{190,202}+E_{195,206}-E_{198,209}-E_{204,213}-E_{207,215}-E_{212,220}+E_{214,224}+E_{218,228}-E_{227,234}-E_{229,236}-E_{232,238}+E_{244,246})-t^2E_{108,141}.$$

$$\psi(x_{\alpha_7}(t)) = I+t(E_{2,3}+E_{17,19}+E_{20,23}+E_{22,26}+E_{27,29}+E_{31,33}+E_{30,34}+E_{35,38}+E_{39,43}+E_{44,47}-E_{45,50}-E_{51,57}+E_{49,58}-E_{55,62}-E_{63,66}-E_{64,68}-E_{67,71}-E_{69,74}-E_{73,78}-E_{77,84}-E_{82,89}-E_{83,90}-E_{88,94}-E_{87,95}-E_{93,101}-E_{100,108}+E_{106,113}-E_{107,115}+E_{114,126}-2E_{114,127}+E_{114,128}+E_{127,135}+E_{134,142}-E_{136,143}+E_{141,149}+E_{148,156}+E_{155,161}+E_{154,162}+E_{159,166}+E_{160,167}+E_{165,172}+E_{171,176}+E_{175,180}+E_{178,182}+E_{181,185}+E_{183,186}+E_{187,194}+E_{192,198}-E_{191,200}+E_{199,204}-E_{202,205}-E_{206,210}-E_{211,214}-E_{216,218}-E_{215,219}-E_{220,222}-E_{223.227}-E_{226,229}-E_{230,232}-E_{246,247})-t^2E_{114,135}.$$

$$\psi(x_{\alpha_6+\alpha_7}(t)) = I+t(-E_{2,4}+E_{14,19}+E_{16,23}+E_{18,26}+E_{21,29}+E_{25,34}-E_{31,37}-E_{35,42}-E_{39,48}-E_{41,50}-E_{44,53}-E_{46,57}-E_{49,60}-E_{52,62}-E_{56,66}-E_{61,71}-E_{64,72}-E_{69,79}-E_{73,85}-E_{77,91}-E_{82,96}-E_{83,97}-E_{87,103}-E_{88,102}-E_{93,109}+E_{99,113}-E_{100,116}-E_{107,126}-E_{107,127}+E_{107,125}+E_{107,128}-E_{114,134}+E_{115,135}+E_{126,142}+E_{127,142}+E_{133,149}-E_{136,150}+E_{140,156}+E_{146,162}+E_{147,161}+E_{152,166}+E_{153,167}+E_{158,172}+E_{164,176}+E_{170,180}+E_{177,185}+E_{178,188}+E_{183,193}+E_{187,197}+E_{189,200}+E_{192,203}+E_{196,205}+E_{199,208}+E_{201,210}+E_{207,214}+E_{212,218}-E_{215,224}-E_{220,228}-E_{223,231}-E_{226,233}-E_{230,235}+E_{245,247})-t^2E_{107,142}.$$

$$\psi(x_{\alpha_8}(t)) = I+t(E_{1,2}-E_{19,24}-E_{23,28}-E_{26,32}-E_{29,36}-E_{34,40}-E_{33,41}-E_{37,45}-E_{38,46}-E_{42,51}-E_{43,52}-E_{48,55}-E_{47,56}-E_{58,61}-E_{53,63}-E_{54,64}-E_{60,67}-E_{59,69}-E_{65,73}-E_{70,77}-E_{76,82}-E_{75,83}-E_{81,87}-E_{80,88}-E_{86,93}-E_{92,100}-E_{99,107}-E_{106,114}+E_{113,127}-2E_{113,128}+E_{128,136}+E_{135,143}+E_{142,150}+E_{149,157}+E_{156,163}+E_{162,168}+E_{161,169}+E_{167,173}+E_{166,174}+E_{172,179}+E_{176,184}+E_{182,189}+E_{180,190}+E_{188,191}+E_{185,195}+E_{186,196}+E_{194,201}+E_{193,202}+E_{197,206}+E_{198,207}+E_{203,211}+E_{204,212}+E_{209,215}+E_{208,216}+E_{213,220}+E_{217,223}+E_{221,226}+E_{225,230}-E_{247,248})-t^2E_{113,136}.$$

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