The parametrization of interior algebras

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Contents

For indecomposable representations of a finite group G in characteristic p, the theory of vertices and sources introduced by J.A. Green in 1959 [G1] is a fundamental tool in modular representation theory. The vertex and source of an indecomposable module are (up to conjugation) two invariants of the module and only finitely many modules (up to isomorphism) have the same invariants. Thus a natural question is the existence of a third invariant which would distinguish further the modules and lead to a bijective parametrization of indecomposable modules using three invariants. When the base field of characteristic p is algebraically closed, the answer lies in the concept of multiplicity module introduced by Puig [P3], although the result is not explicitly stated in Puig's work. It turns out that this third invariant is an indecomposable projective module over a twisted group algebra of the group N/P, where P is a vertex and N is the inertial subgroup of a source. This invariant has been used for the solution of some problems concerning almost split squences of group representations [P5], [T2]. More generally the same question arises for interior G-algebras but further complications appear (essentially because of the existence of outer automorphisms). The purpose of this paper is to give a complete description of the parametrization of primitive interior G-algebras with three invariants, including a description of the special case of indecomposable modules.

Let $\mathcal{O}$ be a complete local commutative ring with residue field k of non-zero characteristic p (allowing the possibility $\mathcal{O} = k$). We assume that k is algebraically closed. By an $\mathcal{O}$-algebra (or simply an algebra), we always mean an $\mathcal{O}$-algebra which is finitely generated as an $\mathcal{O}$-module. Let G be a finite group. Recall that an algebra A is called a G-algebra if it is endowed with an action of G by algebra automorphisms,

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and it is called an interior G-algebra if there is a group homomorphism $G \to A^*$. Any interior G-algebra is a G-algebra, using the conjugation action of G on A. The main two examples of interior G-algebras are on the one hand the algebra $\mathrm{End}_{\mathcal{O}}(M)$ of $\mathcal{O}$-endomorphisms of an $\mathcal{O}G$-module M and on the other hand the group algebra $\mathcal{O}G$, as well as the block algebras $e\mathcal{O}G$.

A G-algebra A is called *primitive* if 1_A is a primitive idempotent of the algebra A^G of G-fixed elements. It is equivalent to require that A^G is a local ring with residue field k. This is the case for instance when $A = \mathrm{End}_{\mathcal{O}}(M)$ where M is an indecomposable $\mathcal{O}G$-module, or when $A = e\mathcal{O}G$ is a block algebra. From the work of Puig [P1], [P3] (extending earlier work of Green [G2]), we know that with any primitive G-algebra A are associated several fundamental objects: a *defect group* P, which is a p-subgroup of G, unique up to G-conjugation; a *source algebra* B, which is a primitive P-algebra, unique up to conjugation (and isomorphism); finally a *multiplicity module* V, which is an indecomposable projective module over a twisted group algebra of the group $\overline{N}_G(P_\gamma) = N_G(P_\gamma)/P$, where $N_G(P_\gamma)$ is the stabilizer of a defect pointed group P_γ of A.

In the case of *interior* algebras, a result of Puig [P3, 9.9] asserts in essence that for any given triple (P, B, V) as above (with B interior), there exists a primitive interior G-algebra A with defect group P, source algebra B and multiplicity module V. For this result one has to consider the right twisted group algebra in order to give the correct meaning to the word "multiplicity module". Now it turns out that two non-isomorphic multiplicity modules may give rise to isomorphic primitive algebras A. Therefore for a parametrization of primitive interior G-algebras in terms of three invariants, the third invariant has to be analyzed in more detail. Our purpose is to provide complete answers to these questions. This will correct the too optimistic statement made in the Appendix of [T1], where the three invariants above were presented as giving a parametrization of primitive interior algebras.

We are forced to attach several multiplicity modules to any given primitive interior G-algebra A. Firstly for a given defect pointed group of A, we prove that one has to consider an orbit of multiplicity modules under some natural action of the group $\mathrm{Out}(B)$ of outer automorphisms of B. Secondly two distinct defect pointed groups may be conjugate under an element of $N_G(P)$ which stabilizes the isomorphism class of B. In other words the inertial subgroup $N_G(P, B)$ of the P-algebra B may be larger than the group $N_G(P_\gamma)$ (but this complication does not occur for $\mathcal{O}G$-lattices). We prove that $N_G(P_\gamma)$ is always a normal subgroup of $N_G(P, B)$, as a by-product of a detailed analysis of the group $\mathrm{Out}(\mathrm{Ind}_P^G(B))$. If we now take into account the conjugation action of $N_G(P, B)$ on $N_G(P_\gamma)$, we obtain that the multiplicity modules have not only to be defined up to the action of $\mathrm{Out}(B)$, but also (in a rough sense) up to the conjugation action of $N_G(P, B)/N_G(P_\gamma)$. In fact we have to consider the action of a single group, namely $\mathrm{Out}(\mathrm{Ind}_P^G(B))$, which has a normal subgroup isomorphic to $\mathrm{Out}(B)$ with quotient $N_G(P, B)/N_G(P_\gamma)$. This group does not depend on G, but only on $N_G(P, B)$.

Thus for any primitive interior G-algebra, we have three invariants (P, B, V) defined up to conjugation by G and also up to conjugation by $\mathrm{Out}(\mathrm{Ind}_P^G(B))$ for the third one. In this way we obtain the parametrization we are looking for, namely a bijection between the set of isomorphism classes of primitive interior G-algebras and the set of equivalence classes of triples as above.

As a by-product of this bijection, we make explicit that the Green correspondence holds for primitive interior algebras with a given defect group P. The proof consists in the mere observation that if $H \geq N_G(P)$, then any triple of invariants (P, B, V) as above also corresponds to a uniquely determined primitive interior H-algebra.

We note that all the complications disappear when $A = \mathrm{End}_{\mathcal{O}}(M)$ is the algebra of endomorphisms of an $\mathcal{O}G$-lattice M, because $\mathrm{Out}(\mathrm{Ind}_P^G(B)) = 1$ by the Skolem-Noether theorem. Thus the multiplicity module of an indecomposable $\mathcal{O}G$-lattice M can be uniquely defined (but we emphasize that one needs to define carefully the multiplicity module structure). In particular we recover the well-known parametrization of trivial source modules.

When B is the source algebra of a block algebra, simple examples show that a non-trivial action of $\mathrm{Out}(B)$ does occur and also that $N_G(P,B)$ can be larger than $N_G(P_\gamma)$. Thus the group $N_G(P,B)$ seems to be a new invariant associated with a block.

In Section 1, we gather all the necessary background of the theory of G-algebras and pointed groups, and in Section 2, we recall the construction of multiplicity modules and twisted group algebras. Then in Section 3, we start with the proof of the main result, which proceeds in several steps, scattered over all sections of the paper. We fix a defect group P and a source algebra B, and we consider the set $\mathcal{A}(G,P,B)$ of isomorphism classes of primitive interior G-algebras with defect group P and source algebra B. We first note that any $A \in \mathcal{A}(G,P,B)$ can be embedded in $\mathrm{Ind}_P^G(B)$ so that the whole proof takes place within this fixed interior G-algebra. Each step consists in showing that $\mathcal{A}(G,P,B)$ is in bijection with some set of orbits of points, or some set of orbits of multiplicity modules, the final description being expressed in terms of B alone, independently of G. In order to facilitate the understanding of the main ideas, a detailed description of the successive steps is given at the end of Section 3.

Acknowledgment. This work would not have existed without Lluis Puig's influence. Not only did he mention to me a few years ago the possibility of using the multiplicity module as a third invariant for the parametrization, but he also contributed directly to this work by making decisive comments on a preliminary version of the paper. I am very grateful to him and I would like in particular to thank him for mentioning to me a significant improvement of the main result. After this paper was written, Lluis Puig communicated to me another approach of the parametrization [P6], which sheds some new light on the question, and which corresponds better to his own view of the subject.

1. Preliminaries

In this section, we fix our notation and review some basic facts in the theory of G-algebras, see [P1], [P3]. If A is a G-algebra, the (left) action of $g \in G$ on $a \in A$ is written ${}^g a$. If A is an interior G-algebra, the image of $g \in G$ in A is denoted $g{\cdot}1_A$, and more generally we write

$$g{\cdot}a = (g{\cdot}1_A)a \qquad \text{and} \qquad a{\cdot}g = a(g{\cdot}1_A)\,.$$

Thus the conjugation action of g becomes ${}^g a = g{\cdot}a{\cdot}g^{-1}$. Note that we cannot identify G with its image in A and this explains why we systematically use a dot for the left and right action of $g \in G$ on A. We write $\mathrm{Inn}(b)$ for the inner automorphism of A defined by $b \in A^*$, that is, $\mathrm{Inn}(b)(a) = bab^{-1}$. This is an automorphism of $\mathcal{O}$-algebras. If $\mathrm{Inn}(b)$ stabilizes a subalgebra X (or a subgroup X of A^*), then we write $\mathrm{Conj}(b) : X \to X$ for the restriciton of $\mathrm{Inn}(b)$, which may not be any longer an inner automorphism. If H is a subgroup of G, then A^H denotes the algebra of H-fixed elements in A. For $K \leq H$, $r_K^H : A^H \to A^K$ is the inclusion map and $t_K^H : A^K \to A^H$ is the relative trace map (i.e. $t_K^H(a) = \sum_{h\in[H/K]} {}^h a$). The notation $[H/K]$ always refers to an arbitrary system of representatives of cosets.

A homomorphism of G-algebras is a homomorphism of $\mathcal{O}$-algebras $f : A \to B$ which is not necessarily unitary and such that $f({}^g a) = {}^g(f(a))$ for every $a \in A$ and $g \in G$. A homomorphism of interior G-algebras is a non necessarily unitary homomorphism of $\mathcal{O}$-algebras $f : A \to B$ such that $f(g{\cdot}a) = g{\cdot}f(a)$ and

$f(a{\cdot}g) = f(a){\cdot}g$ for every $a \in A$ and $g \in G$. Thus in both cases $f(1_A)$ is an idempotent in A, necessarily fixed under G, and the image of f is contained in the subalgebra $f(1_A)Bf(1_A)$ (with identity element $f(1_A)$). Note that f induces for each subgroup H of G a homomorphism of algebras $f^H : A^H \to B^H$.

We can compose a homomorphism of G-algebras $f : A \to B$ with all possible inner automorphisms $\mathrm{Inn}(a)$ of A and $\mathrm{Inn}(b)$ of B, where $a \in (A^G)^*$ and $b \in (B^G)^*$. This yields an equivalence class of homomorphisms, called an *exomorphism* of G-algebras. In fact it suffices to compose with inner automorphisms of B, and as a result, the composition of exomorphisms is well-defined. We use the script letter $\mathcal{F}$ for the exomorphism containing a homomorphism f, and we use the notation $\mathcal{F} : A \to B$. We also write $\mathcal{F}(1_A) = \{ f(1_A) \mid f \in \mathcal{F} \}$; this is always a $(B^G)^*$-conjugacy class of idempotents of B^G. An exomorphism $\mathcal{F} : A \to B$ is called an *embedding* if for some $f \in \mathcal{F}$ (or for every $f \in \mathcal{F}$), f is injective and the image of f is the whole of $f(1_A)Bf(1_A)$. An exomorphism $\mathcal{F} : A \to B$ is called an *exo-isomorphism* if some $f \in \mathcal{F}$ (or every $f \in \mathcal{F}$) is an isomorphism. In case $A = B$, we use the more common term "outer automorphism" (instead of "exo-automorphism") and we write $\mathrm{Out}(A)$ for the group of outer automorphisms of A. We shall also use the notation $\mathrm{Out}(X)$ for the group of outer automorphisms of a group X. Moreover by an *outer action* of a group H on a group X, we mean a group homomorphism $H \to \mathrm{Out}(X)$.

For an interior G-algebra B, an inner automorphism $\mathrm{Inn}(b)$ is a homomorphism of interior G-algebras if and only if $b \in B^G$. Thus an exomorphism is obtained by composing with all inner automorphisms which are homomorphisms of interior G-algebras. However for an arbitrary G-algebra B, an inner automorphism $\mathrm{Inn}(b)$ is an automorphism of G-algebras if and only if the image of b in $B^*/Z(B)^*$ is fixed under G (where $Z(B)$ denotes the centre of B). But b itself need not be fixed under G. Thus an exomorphism is obtained by composing with fewer inner automorphisms than those which are homomorphisms of G-algebras. It turns out that this is the relevant definition of exomorphism for arbitrary G-algebras. The reason is that we do not want to allow an inner automorphism to move the points of B^G.

If H is a subgroup of G, a G-algebra A can be viewed as an H-algebra by restriction and is written $\mathrm{Res}^G_H(A)$. For an exomorphism $\mathcal{F} : A \to B$, its restriction to H is an exomorphism of H-algebras $\mathrm{Res}^G_H(\mathcal{F}) : \mathrm{Res}^G_H(A) \to \mathrm{Res}^G_H(B)$. As it is obtained by composition with all inner automorphisms $\mathrm{Inn}(b)$ with $b \in (B^H)^*$, the set $\mathrm{Res}^G_H(\mathcal{F})$ is in general larger than $\mathcal{F}$. But for interior algebras, one of the key results asserts that the restriction of an exomorphism determines uniquely this exomorphism.

Proposition 1.1 (Puig [P1, 3.7], [P3, 2.12.2]). *Let $\mathcal{F} : A \to B$ and $\mathcal{F}' : A \to B$ be two exomorphisms of interior G algebras. If $\mathrm{Res}^G_H(\mathcal{F}) = \mathrm{Res}^G_H(\mathcal{F}')$ for some subgroup H of G, then $\mathcal{F} = \mathcal{F}'$.*

Recall that a *point* of an $\mathcal{O}$-algebra A is an A^*-conjugacy class of primitive idempotents of A. The set $\mathcal{P}(A)$ of points of A is in bijection with the set of maximal ideals of A (see the discussion of pointed groups below). An important feature of exomorphisms is the following cancellation result. The proof of the second statement uses Proposition 1.1 above to reduce to the case of the trivial group (i.e. $\mathcal{O}$-algebras).

Proposition 1.2. *Let $\mathcal{G} : A \to B$ be an embedding of G-algebras.*

(a) (Puig [P3, 2.3.3]) *Let $\mathcal{F} : C \to A$ and $\mathcal{F}' : C \to A$ be two exomorphisms of G-algebras such that $\mathcal{G}\mathcal{F} = \mathcal{G}\mathcal{F}'$. Then $\mathcal{F} = \mathcal{F}'$.*

(b) (Puig [P1, 2.7], [P3, 2.3.4]) *Assume that A and B are interior G-algebras and that $\mathcal{G}$ is an embedding of interior G-algebras. Assume also that A and B have the same number of points. Let $\mathcal{F} : B \to C$ and $\mathcal{F}' : B \to C$ be two exomorphisms of interior G-algebras such that $\mathcal{F}\mathcal{G} = \mathcal{F}'\mathcal{G}$. Then $\mathcal{F} = \mathcal{F}'$.*

Another feature of interior algebras is *induction*. If H is a subgroup of G and B is an interior H-algebra, the induced algebra $\mathrm{Ind}_H^G(B)$ is the $\mathcal{O}$-module $\mathcal{O}G\otimes_{\mathcal{O}H}B\otimes_{\mathcal{O}H}\mathcal{O}G$, endowed with the $\mathcal{O}$-bilinear extension of the product

$$(x\otimes b\otimes y)(x'\otimes b'\otimes y')=\begin{cases} x\otimes b{\cdot}yx'{\cdot}b'\otimes y' & \text{if } yx'\in H,\\ 0 & \text{otherwise.}\end{cases}$$

Then $\mathrm{Ind}_H^G(B)$ is an $\mathcal{O}$-algebra isomorphic to a matrix algebra $M_{|G:H|}(B)$, with identity element $\sum_{x\in[G/H]} x\otimes 1_B\otimes x^{-1}$. An interior G-algebra structure on $\mathrm{Ind}_H^G(B)$ is defined by mapping $g\in G$ to $\sum_{x\in[G/H]} gx\otimes 1_B\otimes x^{-1}$. Then the left multiplication by $g{\cdot}1$ coincides with the left action of g coming from the natural $\mathcal{O}G$-module structure on $\mathrm{Ind}_H^G(B)$, and similarly on the right. There is a canonical homomorphism $d_H^G : B\to \mathrm{Res}_H^G\,\mathrm{Ind}_H^G(B)$ defined by $d_H^G(b)=1\otimes b\otimes 1$. The exomorphism $\mathcal{D}_H^G : B\to \mathrm{Res}_H^G\,\mathrm{Ind}_H^G(B)$ containing d_H^G is an embedding, called the *canonical embedding* of B into the induced algebra.

A *pointed group* on a G-algebra A is a pair (H,α), always written H_α, where H is a subgroup of G and α is a *point* of A^H (i.e. α is an $(A^H)^*$-conjugacy class of primitive idempotents of A^H). The set of all points of A^H is denoted $\mathcal{P}(A^H)$. It is in bijection with the set $\mathrm{Max}(A^H)$ of all maximal ideals of A^H, thanks to the theorem on lifting idempotents (which holds since $\mathcal{O}$ is complete by assumption). Given a point α, the corresponding maximal ideal $\mathfrak{m}_\alpha$ is characterized by the property $\alpha\not\subseteq\mathfrak{m}_\alpha$. We write $S(\alpha)=A^H/\mathfrak{m}_\alpha$ for the corresponding simple k-algebra and $\pi_\alpha : A^H\to S(\alpha)$ for the canonical surjection. Thus $\pi_\alpha(\alpha)\neq\{0\}$ but $\pi_\beta(\alpha)=\{0\}$ for every $\beta\in\mathcal{P}(A^H)$ with $\beta\neq\alpha$. The simple algebra $S(\alpha)$ is called the *multiplicity algebra* of α. Since k is algebraically closed, $S(\alpha)\cong \mathrm{End}_k(V(\alpha))$ for some finite-dimensional k-vector space $V(\alpha)$. The dimension of $V(\alpha)$ is the multiplicity of α, that is, the number of occurences of primitive idempotents of α in a decomposition of 1_A as an orthogonal sum of primitive idempotents of A^H. The G-algebra A is called *primitive* if 1_A is a primitive idempotent of A^G, so that $\alpha=\{1_A\}$ is the unique point of A^G (with multiplicity one) and $A^G/J(A^G)\cong k$, where $J(A^G)$ is the Jacobson radical of A^G. (This notion of primitivity has nothing to do with the ring-theoretic notion of primitive ring.)

The group G acts by conjugation on the set of pointed groups on A. The stabilizer of H_α is written $N_G(H_\alpha)$. It is a subgroup of the normalizer $N_G(H)$ and it contains H. We write $\overline{N}_G(H_\alpha)=N_G(H_\alpha)/H$. If A is an interior G-algebra, then $N_G(H_\alpha)$ contains $HC_G(H)$, because the centralizer $C_G(H)$ maps into $(A^H)^*$.

There is an essentially unique embedding associated with a pointed group H_α on a G-algebra A. It is defined to be an embedding $\mathcal{F}: B\to \mathrm{Res}_H^G(A)$, where B is an H-algebra, such that $\mathcal{F}(1_B)=\alpha$, that is, $f(1_B)\in\alpha$ for some (or for every) $f\in\mathcal{F}$. Such an embedding associated with H_α will always be written $\mathcal{F}_\alpha : A_\alpha\to \mathrm{Res}_H^G(A)$. As a typical choice, one can take $A_\alpha=iAi$ where $i\in\alpha$ and consider the embedding $\mathcal{F}_\alpha$ containing the inclusion $iAi\to A$. The H-algebra A_α is necessarily primitive and we call it the *localization* of A with respect to α (but we emphasize that this has nothing to do with the notion of local point to be defined below). In fact if A is commutative, A_α^H is the localization of A^H with respect to the maximal ideal $\mathfrak{m}_\alpha$. The essential uniqueness of an embedding associated with H_α is provided by the following result (which is only stated by Puig for interior G-algebras, but the proof carries over without change to arbitrary G-algebras).

Proposition 1.3 (Puig [P3, 2.13.1]). *Let H_α be a pointed group on a G-algebra A and let $\mathcal{F}_\alpha : A_\alpha \to \mathrm{Res}^G_H(A)$ and $\mathcal{F}'_\alpha : A'_\alpha \to \mathrm{Res}^G_H(A)$ be two embeddings associated with H_α. Then there exists a unique exo-isomorphism $\mathcal{H} : A_\alpha \to A'_\alpha$ such that $\mathcal{F}'_\alpha \mathcal{H} = \mathcal{F}_\alpha$.*

Notice that the notions of localization A_α and of embedding associated with H_α are distinct. The second one consists of a pair $(A_\alpha, \mathcal{F}_\alpha)$ where A_α is a localization and $\mathcal{F}_\alpha$ is an embedding. In particular one gets a new embedding associated with H_α by composing $\mathcal{F}_\alpha$ with an outer automorphism of A_α.

Two distinct points may have isomorphic localizations (but this does not occur when $A = \mathrm{End}_{\mathcal{O}}(M)$ for an $\mathcal{O}G$-lattice M). We shall say that H_α and $H_{\alpha'}$ are *isomorphic* if the primitive interior H-algebras A_α and $A_{\alpha'}$ are isomorphic. We also say that the points α and α' are isomorphic. In order to have G_α and $G_{\alpha'}$ isomorphic (assuming $H = G$ for simplicity), an easy sufficient condition is the existence of an outer automorphism of interior G-algebras $\mathcal{F} : A \xrightarrow{\sim} A$ such that $\mathcal{F}(\alpha) = \alpha'$. Indeed if $\mathcal{F}_\alpha : A_\alpha \to A$ and $\mathcal{F}_{\alpha'} : A_{\alpha'} \to A$ denote embeddings associated with α and α' respectively, the composite $\mathcal{F}\mathcal{F}_\alpha : A_\alpha \to A$ is an embedding mapping $\{1_{A_\alpha}\}$ to α', hence is an embedding associated with the point α'. By Proposition 1.3, there exists a unique exo-isomorphism $\mathcal{H} : A_\alpha \to A_{\alpha'}$ such that $\mathcal{F}\mathcal{F}_\alpha = \mathcal{F}_{\alpha'}\mathcal{H}$. In particular G_α and $G_{\alpha'}$ are isomorphic.

We write $\geq$ (and $\leq$) for the order relation between pointed groups. By definition $H_\alpha \geq K_\beta$ if and only if $H \geq K$ and for some $i \in \alpha$, there exists $j \in \beta$ appearing in a primitive decomposition of the idempotent $r^H_K(i)$ (and then this holds for every $i \in \alpha$). The latter condition can be written $\pi_\beta\, r^H_K(i) \neq 0$ for some (or for every) $i \in \alpha$, and this in turn is equivalent to the condition $(r^H_K)^{-1}(\mathfrak{m}_\beta) \subseteq \mathfrak{m}_\alpha$. An important characterization of the order relation uses associated embeddings. Again Puig only proves the result for interior G-algebras, but a careful analysis of the proof shows that it holds for arbitrary G-algebras.

Proposition 1.4 (Puig [P1, p.277], [P3, 2.13.2]). *Let H_α and K_β be two pointed groups on a G-algebra A. Let $\mathcal{F}_\alpha : A_\alpha \to \mathrm{Res}^G_H(A)$ and $\mathcal{F}_\beta : A_\beta \to \mathrm{Res}^G_K(A)$ be associated embeddings. Assume that $H \geq K$. Then $H_\alpha \geq K_\beta$ if and only if there exists an exomorphism $\mathcal{F}^\alpha_\beta : A_\beta \to \mathrm{Res}^H_K(A_\alpha)$ such that $\mathrm{Res}^H_K(\mathcal{F}_\alpha)\,\mathcal{F}^\alpha_\beta = \mathcal{F}_\beta$. If this condition is satisfied, the exomorphism $\mathcal{F}^\alpha_\beta$ is an embedding and is unique.*

A pointed group P_γ on a G-algebra A is called *local* if for every proper subgroup $Q < P$, we have $\gamma \not\subseteq t^P_Q(A^Q)$ (where t^P_Q denotes the relative trace map). This forces P to be a p-subgroup of G. Given a pointed group H_α, a *defect pointed group* of H_α, or simply a *defect* of H_α, is a minimal pointed group P_γ such that $H \geq P$ and $\alpha \subseteq t^H_P(A^P\gamma A^P)$. Then P_γ is necessarily local, and in fact it can also be characterized as a maximal pointed group such that P_γ is local and $H_\alpha \geq P_\gamma$. The subgroup P is called a *defect group* of H_α and it is a p-subgroup of G. All defect pointed groups of H_α are conjugate under the action of H (with stabilizer $N_H(P_\gamma)$). In particular all defect groups of H_α are conjugate under H. When the context is clear we sometimes say that P_γ is a defect of α rather than a defect of H_α. For interior algebras, there is also the following characterization of defect pointed groups in terms of associated embeddings and induced algebras.

Proposition 1.5 (Puig [P1, 3.4], [P3, 2.14.4]). *Let H_α and P_γ be pointed groups on an interior G-algebra A. Assume that $H_\alpha \geq P_\gamma$ and let $\mathcal{F}_\gamma^\alpha : A_\gamma \to \mathrm{Res}_P^H(A_\alpha)$ be a corresponding embedding (Proposition 1.4). Let also $\mathcal{D}_P^H : A_\gamma \to \mathrm{Res}_P^H \mathrm{Ind}_P^H(A_\gamma)$ be the canonical embedding. Then P_γ is a defect of H_α if and only if P_γ is local and there exists an embedding $\mathcal{F} : A_\alpha \to \mathrm{Ind}_P^H(A_\gamma)$ such that $\mathrm{Res}_P^H(\mathcal{F})\,\mathcal{F}_\gamma^\alpha = \mathcal{D}_P^H$. If this condition is satisfied, the embedding $\mathcal{F}$ is unique.*

Proposition 1.5 shows in particular that one can always find A_α embedded into $\mathrm{Ind}_P^H(A_\gamma)$. It turns out that such an embedding is not unique (unless the extra condition of 1.5 is satisfied) and the first purpose of the present paper is in fact to describe all possible embeddings of a given A_α (Section 4). If P_γ is a defect of H_α, the localization A_γ is called a *source algebra* of H_α, or also a source algebra of the primitive algebra A_α. In fact it is often convenient to view the invariants associated with a pointed group H_α as invariants of the primitive algebra A_α (provided these invariants only depend on the isomorphism class of α rather than α itself). In particular if A is a primitive G-algebra, its defect pointed group and its source algebra are by definition those of the pointed group G_α, where $\alpha = \{1_A\}$ is the unique point of A^G. The source algebra A_γ is a primitive P-algebra for a p-group P and the unique point $\{1_{A_\gamma}\}$ of A_γ^P is local. More generally a P-algebra B will be called a *source algebra* if it is primitive and if the unique point $\{1_B\}$ of B^P is local (so that P is a p-group). In fact B is a source algebra of itself.

An important property of embeddings is that they induce maps between pointed groups, preserving the relations between them. We include the full statement, whose proof is straightforward.

Proposition 1.6. *Let $\mathcal{F} : A \to B$ be an embedding of G-algebras.*

(a) For every pointed group H_α on A, the image $\mathrm{Res}_H^G(\mathcal{F})(\alpha)$ is a point of B^H. Thus $\mathcal{F}$ induces an injective map from the set of pointed groups on A to the set of pointed groups on B.

(b) Let H_α, P_γ be two pointed groups on A and let $H_{\alpha'}$, $P_{\gamma'}$ be their images in B under the map defined in (a). Then

  (i) $N_G(P_\gamma) = N_G(P_{\gamma'})$,

  (ii) $H_\alpha \geq P_\gamma$ if and only if $H_{\alpha'} \geq P_{\gamma'}$,

  (iii) P_γ is local if and only if $P_{\gamma'}$ is local,

  (iv) P_γ is a defect of H_α if and only if $P_{\gamma'}$ is a defect of $H_{\alpha'}$.

We shall also need the following property of induction of a source algebra.

Proposition 1.7 (Puig [P1, 3.9], [P3, 2.14.3]). *Let P be a p-subgroup of G, let B be an interior P-algebra which is a source algebra and let γ be the point of $\mathrm{Ind}_P^G(B)^P$ containing $1 \otimes 1_B \otimes 1$. Then P_γ is a local pointed group on $\mathrm{Ind}_P^G(B)$. If moreover P_δ is a local pointed group on $\mathrm{Ind}_P^G(B)$, then P_δ is $N_G(P)$-conjugate to P_γ.*

We consider now a local pointed group P_γ and its multiplicity algebra $S(\gamma) = A^P/\mathfrak{m}_\gamma$. By definition the group $N = N_G(P_\gamma)$ is the stabilizer of $\mathfrak{m}_\gamma$ and therefore N acts by conjugation on $S(\gamma)$. Since P acts trivially on A^P, we obtain a structure of $\overline{N}$-algebra on $S(\gamma)$, where $\overline{N} = N/P$. Thus we can consider pointed groups on $S(\gamma)$, in particular pointed groups $\overline{N}_\delta$ corresponding to the full group $\overline{N}$. In fact we are particularly interested in the pointed groups $\overline{N}_\delta$ which are projective. Here a pointed group is called *projective* if it has the trivial subgroup as a defect group. This means that δ is contained in the image of the trace map from the trivial subgroup $S(\gamma)_1^{\overline{N}} := t_1^{\overline{N}}(S(\gamma))$. Since the canonical surjection $\pi_\gamma : A^P \to S(\gamma)$ is a homomorphism of $\overline{N}$-algebras, the composite $\pi_\gamma\, r_P^G : A^G \to S(\gamma)$ has an image contained in $S(\gamma)^{\overline{N}}$. With this notation, we can now describe the *Puig correspondence* which is an essential tool in the theory.

Proposition 1.8 (Puig correspondence [P1, 1.3], [P3, 2.10.3]). *Let* P_γ *be a local pointed group on a* G*-algebra* A, *let* $S(\gamma)$ *be the multiplicity algebra of* γ, *let* $\pi_\gamma : A^P \to S(\gamma)$ *be the canonical map and let* $\overline{N} = N_G(P_\gamma)/P$. *The map*

$$A^G \overset{\pi_\gamma\, r_P^G}{\longrightarrow} S(\gamma)^{\overline{N}}$$

induces a bijection

$$\{\, \alpha \in \mathcal{P}(A^G) \mid P_\gamma \text{ is a defect of } G_\alpha \,\} \overset{\sim}{\longrightarrow} \{\, \delta \in \mathcal{P}(S(\gamma)^{\overline{N}}) \mid \overline{N}_\delta \text{ is projective } \}\,.$$

If α *corresponds to* δ *under this bijection, the corresponding maximal ideals* $\mathfrak{m}_\alpha$ *and* $\mathfrak{m}_\delta$ *satisfy*

$$\mathfrak{m}_\alpha = (\pi_\gamma\, r_P^G)^{-1}(\mathfrak{m}_\delta)\,.$$

Moreover $\pi_\gamma\, r_P^G$ *induces an isomorphism between the multiplicity algebras*

$$S(\alpha) = A^G/\mathfrak{m}_\alpha \overset{\sim}{\longrightarrow} S(\delta) = S(\gamma)^{\overline{N}}/\mathfrak{m}_\delta\,.$$

If a point $\alpha \in \mathcal{P}(A^G)$ corresponds to a projective point $\delta \in \mathcal{P}(S(\gamma)^{\overline{N}})$ under the Puig correspondence, we shall say that δ is the Puig correspondent of α.

One can say more about the multiplicity algebra $S(\gamma)$ in the special case when A is a primitive G-algebra with defect P_γ.

Proposition 1.9 (Puig [P1, 1.3]). *Let* A *be a primitive* G*-algebra with defect* P_γ, *let* $S(\gamma)$ *be the multiplicity algebra of* γ *and let* $\overline{N} = N_G(P_\gamma)/P$. *Then* $S(\gamma)$ *is a primitive* $\overline{N}$*-algebra and the unique point* $\delta = \{1_{S(\gamma)}\}$ *of* $S(\gamma)^{\overline{N}}$ *is projective, i.e.* $S(\gamma)_1^{\overline{N}} = S(\gamma)^{\overline{N}}$. *Moreover the map* $\pi_\gamma\, r_P^G : A^G \to S(\gamma)^{\overline{N}}$ *is surjective. The Puig correspondence reduces in this case to a bijection between the two singletons* $\{G_\alpha\}$ *and* $\{\overline{N}_\delta\}$ *(where* $\alpha = \{1_A\}$ *denotes the unique point of* A^G *).*

In the situation of Proposition 1.8, it is not difficult to see that the homomorphism $\pi_\gamma\, r_P^G : A^G \to S(\gamma)^{\overline{N}}$ which induces the Puig correspondence behaves well with respect to localization (using Proposition 1.11 below). More precisely if $\delta \in \mathcal{P}(S(\gamma)^{\overline{N}})$ is the Puig correspondent of $\alpha \in \mathcal{P}(A^G)$, then there is an induced map $A_\alpha^G \to S(\gamma)_\delta^{\overline{N}}$ which is exactly the surjective map of Proposition 1.9. In Section 6, we shall come back to this point in an important special case.

We shall also use the following basic property of the relative trace map (which is in fact the cornerstone for the proof of both Propositions 1.8 and 1.9).

Proposition 1.10 (Puig [P1, 1.3]). *Let* A *be a* G*-algebra, let* P_γ *be a local pointed group on* A, *let* $\pi_\gamma : A^P \to S(\gamma)$ *be the canonical map onto the multiplicity algebra of* γ *and let* $\overline{N} = N_G(P_\gamma)/P$. *Then for any* $a \in A^P\gamma A^P$, *we have* $\pi_\gamma\, r_P^G\, t_P^G(a) = t_1^{\overline{N}}\, \pi_\gamma(a)$.

Finally we mention how embeddings induce embeddings between multiplicity algebras. Let $\mathcal{F} : A \to B$ be an embedding. The identification of pointed groups on A with pointed groups on B (Proposition 1.6) does not preserve multiplicities. Let P_γ be a pointed group on A and let $P_{\gamma'}$ be its image. Both multiplicity algebras $S(\gamma)$ and $S(\gamma')$ are $\overline{N}$-algebras where $\overline{N} = N_G(P_\gamma)/P = N_G(P_{\gamma'})/P$. For any $f \in \mathcal{F}$ the composite

$$A^P \overset{f}{\longrightarrow} B^P \overset{\pi_{\gamma'}}{\longrightarrow} S(\gamma')$$

induces an injective homomorphism of $\overline{N}$-algebras $\overline{f} : S(\gamma) \to S(\gamma')$, because by definition of an embedding $\mathrm{Ker}(\pi_\gamma) = f^{-1}(\mathrm{Ker}(\pi_{\gamma'}))$. Thus we have $\pi_{\gamma'}\, f = \overline{f}\, \pi_\gamma$. Clearly if $f, f' \in \mathcal{F}$, then $\overline{f}$ and $\overline{f}'$ belong to the same exomorphism, which we write

$$\overline{\mathcal{F}} : S(\gamma) \to S(\gamma')\,.$$

As $\mathcal{F}$ is an embedding, the image of $\overline{f}$ is the whole of $\overline{f}(1)\, S(\gamma')\, \overline{f}(1)$. Therefore $\overline{\mathcal{F}}$ is an embedding. Summarizing this analysis, we have the following result.

Proposition 1.11. *Let $\mathcal{F} : A \to B$ be an embedding. Then for every pointed group P_γ on A with image $P_{\gamma'}$, there is an induced embedding $\overline{\mathcal{F}} : S(\gamma) \to S(\gamma')$ between the multiplicity algebras of γ and γ'.*

2. Multiplicity modules

In this section we recall how multiplicity modules are attached to multiplicity algebras and how these modules behave under embeddings. We first start with a general setting. Let X be a finite group and let S be an X-algebra over the field k which is simple as a k-algebra. Since k is algebraically closed, $S \cong \mathrm{End}_k(V)$ for some finite dimensional k-vector space V, and therefore $S^* \cong GL(V)$ and $S^*/k^* \cong PGL(V)$. The Skolem-Noether theorem implies that for each $x \in X$, the action of x on S is an inner automorphism $\mathrm{Inn}(s)$ for some $s \in S^*$. Since k^* is the centre of S^*, the element s is only defined up to a scalar and it follows that we obtain a group homomorphism $\rho : X \to S^*/k^*$ mapping x to the class of s. We wish to lift ρ to a group homomorphism $\widehat{\rho} : \widehat{X} \to S^* \cong GL(V)$ (for a suitable group $\widehat{X}$). This will define a representation of the group $\widehat{X}$.

Let $\widehat{X}$ be the central extension of the group X by the central subgroup k^* defined by the following pull-back diagram.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k^* & \xrightarrow{\phi} & \widehat{X} & \xrightarrow{\pi} & X & \longrightarrow & 1 \\ & & \big\downarrow \mathrm{id} & & \big\downarrow \widehat{\rho} & & \big\downarrow \rho & & \\ 1 & \longrightarrow & k^* & \longrightarrow & S^* & \longrightarrow & S^*/k^* & \longrightarrow & 1 \end{array}$$

If $\widehat{X}'$ is another pull-back in the above situation, and if the corresponding maps are $\widehat{\rho}' : \widehat{X}' \to S^*$ and $\pi' : \widehat{X}' \to X$, then there is a unique isomorphism $h : \widehat{X}' \to \widehat{X}$ making the obvious diagrams commute, that is, $\pi h = \pi'$ and $\widehat{\rho} h = \widehat{\rho}'$. Thus the isomorphism type of the pair $(\widehat{X}, \widehat{\rho})$ is unique. But we emphasize that the group $\widehat{X}$ is constructed at the same time as its representation $\widehat{\rho}$. It is convenient in practice to choose $\widehat{X}$ to be the subgroup of $S^* \times X$ consisting of pairs (s, x) such that s and x map to the same element of S^*/k^*.

We define the *twisted group algebra* $k_\sharp \widehat{X}$ to be the quotient of the group algebra $k\widehat{X}$ of the infinite group $\widehat{X}$ by the ideal I generated by the elements $\phi(\lambda) - \lambda{\cdot}1$, where $\lambda \in k^*$. Thus the central subgroup $\phi(k^*) \cong k^*$ is identified with the scalars k^* of the group algebra. Multiplying the generators of I by arbitrary elements $x \in \widehat{X}$, we see that I is the k-linear span of the elements $\phi(\lambda)\, x - \lambda{\cdot}x$, where $\lambda \in k^*$ and $x \in \widehat{X}$. Thus if $\sigma : X \to \widehat{X}$ is a map such that $\sigma\pi = id_X$, the images of the elements $\sigma(x)$, for $x \in X$, form a basis of the algebra $k_\sharp \widehat{X}$ satisfying $\sigma(xy) = \lambda\, \sigma(x)\sigma(y)$ for some $\lambda \in k^*$.

The (infinite) group $\widehat{X}$ has an ordinary representation $\widehat{\rho}$ on the vector space V, but this representation is not arbitrary since it maps the central subgroup k^* to the centre k^* of S^* by the identity map. Taking only into account this special type of representation comes down to the same thing as considering modules over the twisted group algebra $k_\sharp \widehat{X}$, just in the same way as a representation of X over k is the same thing as a kX-module. More precisely $\widehat{\rho}$ extends to an algebra homomorphism $\widehat{\rho} : k\widehat{X} \to S$ and since $\widehat{\rho}(\phi(\lambda)) = \lambda{\cdot}1$, it is clear that the ideal I is in the kernel of $\widehat{\rho}$, so that we obtain an algebra homomorphism $\overline{\rho} : k_\sharp \widehat{X} \to S$. Since $S \cong \mathrm{End}_k(V)$, this provides V with a structure of $k_\sharp \widehat{X}$-module.

We need to modify slightly the previous setting by introducing a normal subgroup Y of X. Suppose that on restriction to Y, the X-algebra structure on S comes from an *interior* Y-algebra structure, so that V is in fact a kY-module. In other words a homomorphism $\widehat{\rho}_Y : Y \to S^*$ is given, which lifts the restriction of ρ to Y. By definition of a pull-back, there is a group homomorphism $Y \to \widehat{X}$ which allows to identify Y with a normal subgroup of $\widehat{X}$. Then the homomorphism $\widehat{\rho} : \widehat{X} \to S^*$ extends the map $\widehat{\rho}_Y$ and this gives a $k_\sharp \widehat{X}$-module structure on V whose restriction to Y is the given kY-module structure.

Now we apply the constructions above to the multiplicity algebra $S(\gamma) \cong \mathrm{End}_k(V(\gamma))$ of a pointed group P_γ on an interior G-algebra A. In order to be able to pass to the quotient by P, we also assume that P is a p-group. Here the group X is $\overline{N} = N/P$ where $N = N_G(P_\gamma)$, and clearly $\overline{N}$ acts on $S(\gamma)$ (even if P is not a p-group). We choose for Y the normal subgroup $\overline{C}_G(P) = PC_G(P)/P$. To prove that we can indeed make this choice, we must show that $S(\gamma)$ has a structure of interior $\overline{C}_G(P)$-algebra. Since $C_G(P){\cdot}1_A \subseteq A^P$, it is clear that there is an interior $C_G(P)$-structure. Now $\overline{C}_G(P) = C_G(P)/Z(P)$ where $Z(P)$ is the centre of P, and $Z(P)$ maps to the centre k^* of $S(\gamma)^*$ because $Z(P) \subseteq P$ acts trivially on $S(\gamma)$. But since $Z(P)$ is a p-group and since there is no non-trivial p-th root of unity in k^*, the image of $Z(P)$ is trivial. Thus we obtain a map $\overline{C}_G(P) \to S(\gamma)^*$ as required. The $\overline{N}$-algebra structure on $S(\gamma) \cong \mathrm{End}_k(V(\gamma))$ is interior on restriction to $\overline{C}_G(P)$. Therefore $V(\gamma)$ is endowed with a structure of module over a twisted group algebra $k_\sharp \widehat{\overline{N}}$, and the restriciton to $\overline{C}_G(P)$ is the canonical $k\overline{C}_G(P)$-module structure on $V(\gamma)$. With this structure, the module $V(\gamma)$ is called the *multiplicity module* of the pointed group P_γ.

We note that $\overline{C}_G(P)$ is not the largest possible normal subgroup of $\overline{N}$ for which $S(\gamma)$ has an interior structure. Indeed let $P{\cdot}1_A$ be the image of P in A^*, let $C_G(P{\cdot}1_A)$ be its centralizer in G (which by definition maps to A^P) and let $C_N(P{\cdot}1_A) = C_G(P{\cdot}1_A) \cap N$. Then $PC_N(P{\cdot}1_A)/P$ is a normal subgroup of $\overline{N}$ for which $S(\gamma)$ has an interior structure. Clearly $C_G(P) \leq C_N(P{\cdot}1_A)$ and if the map $P \to A^*$ is not injective, $C_N(P{\cdot}1_A)$ may be larger than $C_G(P)$. Unfortunately $C_N(P{\cdot}1_A)$ is not invariant under embeddings. In order to avoid such complications and to simplify the approach, we only work with $C_G(P)$. The only cost is that we loose a small portion of canonical interior structure on $S(\gamma)$, coming from the discrepancy between $C_G(P)$ and $C_N(P{\cdot}1_A)$.

We define

$$E_G(P_\gamma) = \overline{N}/\overline{C}_G(P) = N_G(P_\gamma)/PC_G(P)\,. \tag{2.1}$$

Since $\overline{C}_G(P)$ is identified with a normal subgroup of $\widehat{\overline{N}}$, we can also define $\widehat{E}_G(P_\gamma) = \widehat{\overline{N}}/\overline{C}_G(P)$ and consider the central extension

$$1 \longrightarrow k^* \longrightarrow \widehat{E}_G(P_\gamma) \longrightarrow E_G(P_\gamma) \longrightarrow 1\,. \tag{2.2}$$

Then the central extension $\widehat{\overline{N}}$ is in fact obtained from (2.2) by restriction (i.e. pull-back) along the map $\overline{N} \to E_G(P_\gamma)$.

We now recall a result of Puig concerning the behaviour of multiplicity modules with respect to embeddings.

Proposition 2.3 (Puig [P3, 6.18]). *Let $\mathcal{F} : A \to A'$ be an embedding of G-algebras, let P_γ be a pointed group on A, let $P_{\gamma'}$ be its image and let $\overline{N} = \overline{N}_G(P_\gamma) = \overline{N}_G(P_{\gamma'})$. Let $\widehat{\overline{N}}$ and $\widehat{\overline{N}}'$ be the central extensions associated with the $\overline{N}$-algebras $S(\gamma) \cong \mathrm{End}_k(V(\gamma))$ and $S(\gamma') \cong \mathrm{End}_k(V(\gamma'))$ respectively. Then the embedding $\overline{\mathcal{F}}$ induces an isomorphism of central extensions $\overline{\mathcal{F}}^* : \widehat{\overline{N}}' \to \widehat{\overline{N}}$, inducing the identity on both k^* and $\overline{N}$.*

For completeness we recall the definition of the isomorphism $\overline{\mathcal{F}}^*$, which is very natural. Choose $f \in \mathcal{F}$, let $i = f(1_A)$ and let $(s', \overline{x}) \in \widehat{\overline{N}}'$, where $s' \in S(\gamma')^*$ and $\overline{x} \in \overline{N}$. Lifting arbitrarily s' to $a' \in (A')^P$, the element $ia'i$ is the image under the embedding f of a unique element $a \in A^P$. Then $\overline{\mathcal{F}}^*(s', \overline{x}) = (s, \overline{x})$, where $s = \pi_\gamma(a) \in S(\gamma)^*$. Puig proved in [P3, 6.15] that the definition is independent of the choices.

Using the isomorphism $\overline{\mathcal{F}}^* : \widehat{\overline{N}}' \to \widehat{\overline{N}}$ of the proposition, $V(\gamma)$ has a structure of $k_\sharp\widehat{\overline{N}}'$-module (defined by $\widehat{\rho}\,\overline{\mathcal{F}}^*$ where $\widehat{\rho} : \widehat{\overline{N}} \to GL(V(\gamma))$ is the structural map). Endowed with this structure, $V(\gamma)$ is isomorphic to a direct summand of $V(\gamma')$, via the embedding

$$\overline{\mathcal{F}} : S(\gamma) \cong \mathrm{End}_k(V(\gamma)) \longrightarrow S(\gamma') \cong \mathrm{End}_k(V(\gamma'))$$

of Proposition 1.11. Explicitly $V(\gamma)$ is isomorphic to the direct summand $jV(\gamma')$, where $j = \overline{f}(1_{S(\gamma)})$ and $\overline{f} \in \overline{\mathcal{F}}$. Note that the isomorphism $\overline{\mathcal{F}}^* : \widehat{\overline{N}}' \to \widehat{\overline{N}}$ depends on the embedding $\mathcal{F} : A \to A'$ (or more precisely it depends on the embedding $\overline{\mathcal{F}} : S(\gamma) \to S(\gamma')$). Thus a different embedding yields a different isomorphism, hence a different structure of $k_\sharp\widehat{\overline{N}}'$-module on $V(\gamma)$. This remark will be crucial in the sequel.

3. Description of the setting

In this section, we describe the various mathematical objects we are going to work with, we fix our notation, and finally we give an overview of the successive steps of the proof of the main result on the parametrization.

Let A be a primitive interior G-algebra. We know that a defect P_γ of A is unique up to G-conjugation and consequently that a source algebra $B = A_\gamma$ of A is also unique up to G-conjugation. Thus the defect group P and the source algebra B are, up to G-conjugation, two well understood invariants. The first one is a p-group and the second one is an interior P-algebra which is a source algebra (i.e. B is primitive and $P_{\{1_B\}}$ is local). Thus we fix P and B and we want to distinguish up to isomorphism the possible primitive interior G-algebras with defect group P and source algebra B.

To say that B is a source algebra of A means by definition that there exists a point γ of A^P such that $A_\gamma = B$. In that case there is an embedding $\mathcal{F}_\gamma : B \to \mathrm{Res}^G_P(A)$ associated with γ, but $\mathcal{F}_\gamma$ is not unique since it can be composed with any outer automorphism of B. Moreover γ is not uniquely determined by P and B. Indeed let $N_G(P, B)$ be the stabilizer of (P, B) under the action of G, that is, $N_G(P, B)$ is the set of all $x \in N_G(P)$ such that B and ${}^xB$ are isomorphic interior P-algebras. Here ${}^xB$ denotes the conjugate interior P-algebra (defined in the obvious way). Then $N_G(P_\gamma)$ is a subgroup of $N_G(P, B)$ because if $i \in \gamma$ and $g \in N_G(P_\gamma)$, then there exists $a \in (A^P)^*$ such that ${}^gi = aia^{-1}$ and conjugation by a induces an isomorphism of interior P-algebras between $iAi \cong B$ and ${}^giA\,{}^gi \cong {}^gB$. But $N_G(P, B)$ may be larger than $N_G(P_\gamma)$ and it follows that for $g \in N_G(P, B)$ the point ${}^g\gamma$ may be different from γ, whereas the localization $A_{{}^g\gamma}$ is isomorphic to B.

Let A be a primitive interior G-algebra with defect P_γ and source algebra $B = A_\gamma$ and let $\mathcal{F}_\gamma : B \to \mathrm{Res}^G_P(A)$ be an embedding associated with γ. Note that if $\alpha = \{1_A\}$ denotes the unique point of A^G, then

$A_\alpha = A$ and by Proposition 1.4 the relation $G_\alpha \geq P_\gamma$ corresponds to an embedding $\mathcal{F}_\gamma^\alpha : B \to \mathrm{Res}_P^G(A)$, which in this case coincides with $\mathcal{F}_\gamma$. Now by Proposition 1.5, there exists a unique embedding $\mathcal{F} : A \to \mathrm{Ind}_P^G(B)$ such that

$$\mathrm{Res}_P^G(\mathcal{F})\,\mathcal{F}_\gamma^\alpha = \mathcal{D}_P^G \tag{3.1}$$

where $\mathcal{D}_P^G : B \to \mathrm{Res}_P^G \mathrm{Ind}_P^G(B)$ is the canonical embedding. Since A is primitive, $\mathcal{F}$ is an embedding associated with a point of $\mathrm{Ind}_P^G(B)^G$ which we still denote α (namely $\alpha = \mathcal{F}(1_A)$). Thus $A = \mathrm{Ind}_P^G(B)_\alpha$ and $\mathcal{F} = \mathcal{F}_\alpha$.

We use the embedding $\mathcal{F}$ to identify pointed groups on A with pointed groups on $\mathrm{Ind}_P^G(B)$, using Proposition 1.6. If one considers the image of 1_B under both sides of Equation 3.1, we see that the image of the point γ under the embedding $\mathcal{F}$ is the point $\mathcal{D}_P^G(\{1_B\})$ of $\mathrm{Ind}_P^G(B)^P$ which we also denote γ. Thus γ contains $1 \otimes 1_B \otimes 1$ and $\mathcal{D}_P^G$ is an embedding associated with γ. It follows that on the interior G-algebra $\mathrm{Ind}_P^G(B)$, we have two pointed groups G_α and P_γ, with associated embeddings $\mathcal{F}$ and $\mathcal{D}_P^G$ respectively, and the relation $G_\alpha \geq P_\gamma$ corresponds to the embedding $\mathcal{F}_\gamma^\alpha$, thanks to (3.1) and Proposition 1.4. Note that the identification above is compatible with the group $N_G(P_\gamma)$, which is the same when computed with respect to A or to $\mathrm{Ind}_P^G(B)$ by Proposition 1.6.

By Proposition 1.5, $\mathcal{F}$ is the unique embedding such that (3.1) holds. However $\mathcal{F}$ is not necessarily unique without this condition. If $\mathcal{F}' : A \to \mathrm{Ind}_P^G(B)$ is another embedding, then $\mathcal{F}'$ is an embedding associated with a point α' of $\mathrm{Ind}_P^G(B)^G$, namely $\alpha' = \mathcal{F}'(1_A)$, and thus $A \cong \mathrm{Ind}_P^G(B)_{\alpha'}$. It follows that for the two points α and α' of $\mathrm{Ind}_P^G(B)^G$, the localizations $\mathrm{Ind}_P^G(B)_\alpha$ and $\mathrm{Ind}_P^G(B)_{\alpha'}$ are isomorphic interior G-algebras. In other words G_α and $G_{\alpha'}$ are isomorphic. Therefore we do not only have to find all pointed groups G_α on $\mathrm{Ind}_P^G(B)$ with defect P_γ, but we also have to decide when two such pointed groups are isomorphic.

This analysis shows the importance of the following setting, which will be used as a fixed notation for the largest part of this paper.

Notation 3.2. *Given a finite group G, let P be a p-subgroup of G, let B be an interior P-algebra which is a source algebra, and let $C = \mathrm{Ind}_P^G(B)$.*

Let γ be the point of C^P containing $i = 1 \otimes 1_B \otimes 1$. Let $\mathcal{D}_P^G : B \to \mathrm{Res}_P^G(C)$ be the canonical embedding, containing the map $d_P^G : B \to \mathrm{Res}_P^G(C)$ defined by $d_P^G(b) = 1 \otimes b \otimes 1$.

Let $N = N_G(P_\gamma)$ and $\overline{N} = N/P$. Let $S(\gamma)$ be the multiplicity algebra of γ (which is an $\overline{N}$-algebra) and let $\pi_\gamma : C^P \to S(\gamma)$ be the canonical surjective ring homomorphism. Write $S(\gamma) = \mathrm{End}_k(V(\gamma))$ for some finite dimensional k-vector space $V(\gamma)$. The action of $\overline{N}$ on $S(\gamma)$ defines a central extension $\widehat{\overline{N}}^C$, and $V(\gamma)$ is endowed with a structure of $k_\sharp\widehat{\overline{N}}^C$-module.

Let $\mathcal{A}(G,P,B)$ be the set of isomorphism classes of primitive interior G-algebras with defect group P and source algebra B. Let $\mathcal{P}(C^G)_{P_\gamma}$ be the set of points of C^G with defect P_γ.

The following statement summarizes our analysis.

Proposition 3.3. *With the notation 3.2, write $\sim$ for the isomorphism relation between the points in $\mathcal{P}(C^G)_{P_\gamma}$. Then the set $\mathcal{A}(G,P,B)$ is in bijection with the set of isomorphism classes $\mathcal{P}(C^G)_{P_\gamma}/\sim$.*

We can now give an outline of the different steps leading to the parametrization. In Section 4 we use the (left) action of the group $\mathrm{Out}(C)$ on $\mathcal{P}(C^G)_{P_\gamma}$ and we show that the set of orbits $\mathrm{Out}(C)\backslash\mathcal{P}(C^G)_{P_\gamma}$ coincides with $\mathcal{P}(C^G)_{P_\gamma}/\sim$ (Corollary 4.4).

The Puig correspondence sets up a bijection between $\mathcal{P}(C^G)_{P_\gamma}$ and $\mathcal{P}(S(\gamma)^{\overline{N}})$, because every point of $S(\gamma)^{\overline{N}}$ is projective (Corollary 6.2). Moreover $\mathrm{Out}(C)$ also acts on $\mathcal{P}(S(\gamma)^{\overline{N}})$, via a homomorphism $\mathrm{Out}(C)\to\mathrm{Out}(S(\gamma)^{\overline{N}})$ (Proposition 7.3). We prove that the Puig correspondence commutes with the action of $\mathrm{Out}(C)$, so that $\mathcal{A}(G,P,B)$ is now in bijection with $\mathrm{Out}(C)\backslash\mathcal{P}(S(\gamma)^{\overline{N}})$ (Corollary 8.2).

In this situation, a result of Puig asserts that the $k_\sharp\widehat{\overline{N}}^C$-module $V(\gamma)$ is free of rank one (Proposition 6.1), so that a point of $S(\gamma)^{\overline{N}}$ corresponds to an isomorphism class of indecomposable projective $k_\sharp\widehat{\overline{N}}^C$-modules. Again $\mathrm{Out}(C)$ acts on these isomorphism classes of modules, via a homomorphism $\mathrm{Out}(C)\to\mathrm{Out}_{k^*}(\widehat{\overline{N}}^C)$ (Proposition 7.4), and now $\mathcal{A}(G,P,B)$ is in bijection with the set of orbits (Corollary 8.4).

We need to describe now these orbits directly in terms of B, without reference to G. First $\mathrm{Out}(C)$ is isomorphic to the group $\mathrm{Out}_{\mathrm{skew}}(B)$ of skew outer automorphisms of B (Proposition 5.1) and this only depends on B and $N_G(P,B)$. By a result of Puig (Proposition 9.5), $\widehat{\overline{N}}^C$ is isomorphic to a central extension $(\widehat{\overline{N}}^B)^\circ$ defined in terms of B alone. We prove that this isomorphism commutes with the "outer action" of $\mathrm{Out}_{\mathrm{skew}}(B)$, for some naturally defined group homomorphism $\mathrm{Out}_{\mathrm{skew}}(B)\to(\widehat{\overline{N}}^B)^\circ$ (Proposition 9.7). As a result $\mathcal{A}(G,P,B)$ is in bijection with the set of $\mathrm{Out}_{\mathrm{skew}}(B)$-orbits of indecomposable projective modules over the twisted group algebra $k_\sharp(\widehat{\overline{N}}^B)^\circ$.

Finally the orbit of modules corresponding to some $A\in\mathcal{A}(G,P,B)$ can be described directly from B and its various embeddings into $\mathrm{Res}^G_P(A)$, without going through the process involving C (Proposition 10.1). This defines the third invariant associated with A. The main theorem giving the parametrization follows easily from this (Section 11).

4. Isomorphic points

In this section we show that two points in $\mathcal{P}(C^G)_{P_\gamma}$ are isomorphic if and only if they are in the same orbit of $\mathrm{Out}(C)$.

Recall from Section 1 that if $\mathcal{F}$ is an outer automorphism of C such that $\mathcal{F}(\alpha)=\alpha'$, then α and α' are isomorphic. The fact that the converse holds for points of C^G with defect P_γ is probably one of the most crucial steps in this paper.

Proposition 4.1. *With the notation 3.2, let G_α be a pointed group on C with defect P_γ. A point $\alpha'\in\mathcal{P}(C^G)$ is isomorphic to α if and only if there exists an outer automorphism $\mathcal{F}$ of C such that $\mathcal{F}(\alpha)=\alpha'$. Moreover if this condition is satisfied, then P_γ is also a defect of $G_{\alpha'}$.*

For the proof, we need the following basic construction of automorphisms of C, which works for an arbitrary subgroup P and an arbitrary interior P-algebra B. First recall that for any exomorphism $\mathcal{H} : B \to B'$ of interior P-algebras, there is a unique exomorphism $\mathrm{Ind}_P^G(\mathcal{H}) : \mathrm{Ind}_P^G(B) \to \mathrm{Ind}_P^G(B')$ such that $\mathrm{Res}_P^G \mathrm{Ind}_P^G(\mathcal{H})\, \mathcal{D}_P^G(B) = \mathcal{D}_P^G(B')\, \mathcal{H}$, where $\mathcal{D}_P^G(B) : B \to \mathrm{Res}_P^G \mathrm{Ind}_P^G(B)$ and $\mathcal{D}_P^G(B') : B' \to \mathrm{Res}_P^G \mathrm{Ind}_P^G(B')$ denote the canonical embeddings. For $h \in \mathcal{H}$, the induced homomorphism $\mathrm{Ind}_P^G(h)$ is defined by

$$\mathrm{Ind}_P^G(h)(x \otimes b \otimes y) = x \otimes h(b) \otimes y \quad \text{for all } b \in B \text{ and } x, y \in G \ .$$

Thus in particular for any outer automorphism $\mathcal{H}$ of B, we obtain an outer automorphism $\mathrm{Ind}_P^G(\mathcal{H})$ of C, but there are in general more outer automorphisms of C. Note that if $h = \mathrm{Inn}(b)$ is inner, then $\mathrm{Ind}_P^G(h) = \mathrm{Inn}(c)$ is also inner where $c = \sum_{x \in [G/P]} x \otimes b \otimes x^{-1}$. Thus induction is well defined on exomorphisms.

Extending this construction, we want to define induction of an exo-isomorphism $\mathcal{H}$ between B and some $N_G(P)$-conjugate of B. But as it is more convenient to view $\mathcal{H}$ as an outer automorphism of B which twists the interior P-algebra structure, we make the following definitions. A *skew automorphism* of the interior P-algebra B is an automorphism of $\mathcal{O}$-algebras $h : B \to B$ such that there exists $g \in N_G(P)$ with the following property:

$$h(u{\cdot}1_B) = {}^g u{\cdot}1_B \quad \text{for all } u \in P \ .$$

We shall say that h is a g-skew automorphism of B. It is elementary to translate the condition of the definition into the existence of an (ordinary) isomorphism between B and the conjugate interior P-algebra ${}^{g^{-1}}B$ and therefore g must belong to the group $N_G(P, B)$. In other words $N_G(P, B)$ is the group of all elements $x \in N_G(P)$ such that there exists an x-skew automorphism of B. Note that in the definition of a g-skew automorphism, the element g is only defined up to an element which centralizes the image of P in B^*, that is, up to $C_G(P{\cdot}1_B) \cap N_G(P)$. Thus a g-skew automorphism is also a zg-skew automorphism for every $z \in C_G(P{\cdot}1_B) \cap N_G(P)$. Composing a skew automorphism with all inner automorphisms $\mathrm{Inn}(b)$ where $b \in (B^P)^*$ yields an equivalence class of skew automorphisms called a *skew exo-automorphism* of B. We reserve the name "skew outer automorphism" for another equivalence class of skew automorphisms (see Section 5).

Now we define induction which produces an ordinary outer automorphism of C. Let $\mathcal{H}$ be a skew exo-automorphism of B and let $h \in \mathcal{H}$. Then h is a g-skew automorphism for some $g \in N_G(P, B)$. Consider the automorphism $\widehat{h}$ of C defined by

$$\widehat{h}(x \otimes b \otimes y) = xg^{-1} \otimes h(b) \otimes gy \quad \text{for all } b \in B \text{ and } x, y \in G \ .$$

Using the fact that g normalizes P, it is elementary to check that $\widehat{h}(xu \otimes b \otimes vy) = \widehat{h}(x \otimes ubv \otimes y)$ for $u, v \in P$. Thus $\widehat{h}$ is a well defined map. Moreover $\widehat{h}$ is an automorphism of algebras (with inverse $x \otimes b \otimes y \mapsto xg \otimes h^{-1}(b) \otimes g^{-1}y$). Finally for every $z \in G$ we have $\widehat{h}(z{\cdot}(x \otimes b \otimes y)) = z{\cdot}\widehat{h}(x \otimes b \otimes y)$, and similarly on the right. Thus $\widehat{h}$ is an automorphism of interior G-algebras.

However $\widehat{h}$ is not uniquely defined because it depends on the choice of g. If g is replaced by zg where $z \in C_G(P{\cdot}1_B) \cap N_G(P)$, we obtain an automorphism $\widehat{h}'$ of C defined by $\widehat{h}'(x \otimes b \otimes y) = xg^{-1}z^{-1} \otimes h(b) \otimes zgy$. Then $\widehat{h}'$ is the composition of $\widehat{h}$ with the inner automorphism $\mathrm{Inn}(c)$ where $c = \sum_{x \in [G/P]} xz^{-1} \otimes 1_B \otimes x^{-1}$ (with inverse $c^{-1} = \sum_{x \in [G/P]} x \otimes 1_B \otimes zx^{-1}$). We shall check below that this expression of c is independent of the choice of the coset representatives $[G/P]$ and that $c \in C^G$. Thus $\widehat{h}$ and $\widehat{h}'$ belong to the same exomorphism which we write $\mathrm{Ind}_P^G(\mathcal{H})$. Since we already know that the induction of an inner automorphism is an inner automorphism, it follows that we have associated to the skew exo-automorphism $\mathcal{H}$ of B an outer automorphism $\mathrm{Ind}_P^G(\mathcal{H})$ of C.

We now prove the claim above concerning c. It suffices to prove that $z^{-1}\otimes 1_B\otimes 1$ is fixed under P. Indeed this implies that $c=t_P^G(z^{-1}\otimes 1_B\otimes 1)$ is fixed under G. For all $u\in P$ we have

$$u{\cdot}(z^{-1}\otimes 1_B\otimes 1)=z^{-1}\,{}^z u\otimes 1_B\otimes 1=z^{-1}\otimes {}^z u{\cdot}1_B\otimes 1=z^{-1}\otimes u{\cdot}1_B\otimes 1=(z^{-1}\otimes 1_B\otimes 1){\cdot}u\,,$$

using the fact that z normalizes P and centralizes $P{\cdot}1_B$. This establishes the claim.

Note also that for $z\in C_G(P{\cdot}1_B)\cap N_G(P)$,

$$z^{-1}{\cdot}(1\otimes 1_B\otimes 1){\cdot}z=z^{-1}\otimes 1_B\otimes z=c{\cdot}(1\otimes 1_B\otimes 1){\cdot}c^{-1}\in\gamma\,,$$

so that $z\in N_G(P_\gamma)$. Thus $C_G(P{\cdot}1_B)\cap N_G(P)\le N_G(P_\gamma)=N$ and therefore

$$C_G(P{\cdot}1_B)\cap N_G(P)=C_N(P{\cdot}1_B)\,.$$

An ordinary outer automorphism $\mathcal{H}$ of B has the property $\mathrm{Res}_P^G\,\mathrm{Ind}_P^G(\mathcal{H})\,\mathcal{D}_P^G=\mathcal{D}_P^G\,\mathcal{H}$. Here $\mathcal{D}_P^G: B\to\mathrm{Res}_P^G(C)$ denotes the canonical embedding. Moreover $\mathrm{Ind}_P^G(\mathcal{H})$ is the unique exomorphism with this property. In order to state a similar property in the skew case, we let $\mathcal{C}_g$ denote the conjugation by g viewed as an exomorphism of $\mathrm{Res}_P^G(C)$, that is, $\mathcal{C}_g$ is obtained by composing $\mathrm{Conj}(g{\cdot}1_C)$ with all inner automorphisms $\mathrm{Inn}(c)$ where $c\in C^P$. Note that it suffices to compose on the left with inner automorphisms $\mathrm{Inn}(c)$, because $g{\cdot}c={}^g c{\cdot}g$ and ${}^g c\in C^P$ again since g normalizes P. Clearly $\mathcal{C}_g$ is a g-skew exo-automorphism of $\mathrm{Res}_P^G(C)$.

Lemma 4.2. *With the notation above, let $\mathcal{H}$ be a g-skew exo-automorphism of B for some $g\in N_G(P,B)$. Then*

$$\mathcal{C}_g\,\mathrm{Res}_P^G\,\mathrm{Ind}_P^G(\mathcal{H})\,\mathcal{D}_P^G=\mathcal{D}_P^G\,\mathcal{H}\,,$$

and $\mathrm{Ind}_P^G(\mathcal{H}):C\to C$ is the unique exomorphism with this property.

Proof. Let $h\in\mathcal{H}$ and $\widehat{h}\in\mathrm{Ind}_P^G(\mathcal{H})$ with $\widehat{h}(x\otimes b\otimes y)=xg^{-1}\otimes h(b)\otimes gy$. If $d_P^G\in\mathcal{D}_P^G$ denotes the map $d_P^G(b)=1\otimes b\otimes 1$, then we have

$$\mathrm{Conj}(g{\cdot}1_C)\,\widehat{h}\,d_P^G(b)=g{\cdot}(g^{-1}\otimes h(b)\otimes g){\cdot}g^{-1}=1\otimes h(b)\otimes 1=d_P^G\,h(b)\,,$$

as required. For the proof of uniqueness, we let $\mathcal{F}:C\to C$ be an exomorphism such that $\mathcal{C}_g\,\mathrm{Res}_P^G(\mathcal{F})\,\mathcal{D}_P^G=\mathcal{D}_P^G\,\mathcal{H}$. Using the similar property of $\mathrm{Ind}_P^G(\mathcal{H})$ and cancelling $\mathcal{C}_g$ (by composition with $\mathcal{C}_{g^{-1}}$), we obtain $\mathrm{Res}_P^G(\mathcal{F})\,\mathcal{D}_P^G=\mathrm{Res}_P^G\,\mathrm{Ind}_P^G(\mathcal{H})\,\mathcal{D}_P^G$, which is a relation between ordinary exomorphisms. By Proposition 1.2 we can cancel $\mathcal{D}_P^G$ in this equation because the $\mathcal{O}$-algebras B and C have the same number of points (since C is a matrix algebra over B as an $\mathcal{O}$-algebra). Thus we have $\mathrm{Res}_P^G(\mathcal{F})=\mathrm{Res}_P^G\,\mathrm{Ind}_P^G(\mathcal{H})$ and it follows that $\mathcal{F}=\mathrm{Ind}_P^G(\mathcal{H})$ by Proposition 1.1. □

Remark 4.3. The property in Lemma 4.2 depends on the choice of g. If $\mathcal{H}$ is also a g'-skew exo-automorphism of B, then it satisfies the property obtained from (4.2) by replacing g by g'. But in general $\mathcal{C}_g$ may be different from $\mathcal{C}_{g'}$. Indeed if we write $g'=zg$ with $z\in C_N(P{\cdot}1_B)$, then

$$\mathrm{Conj}(g'{\cdot}1_C)=\mathrm{Conj}(z{\cdot}1_C)\;\mathrm{Conj}(g{\cdot}1_C)\,.$$

But it may happen that $z{\cdot}1_C$ does not belong to C^P, because although z centralizes $P{\cdot}1_B$ it does not necessarily centralize $P{\cdot}1_C$. This problem does not occur if P is mapped injectively in B^* because $C_G(P{\cdot}1_B)=C_G(P{\cdot}1_C)=C_G(P)=C_N(P)$ in that case.

Proof of Proposition 4.1. As observed earlier one implication is easy. So we assume that α and α' are isomorphic and we have to construct an outer automorphism of C mapping α to α'. Since α and α' are isomorphic, we can use the same primitive G-algebra A for the localization with respect to α and α'. Thus we let $\mathcal{F}_\alpha : A \to C$ be an embedding associated with G_α and $\mathcal{F}_{\alpha'} : A \to C$ be an embedding associated with $G_{\alpha'}$. Since B is a primitive P-algebra, the canonical embedding $\mathcal{D}_P^G : B \to \mathrm{Res}_P^G(C)$ is an embedding associated with P_γ. By Proposition 1.4 the relation $G_\alpha \geq P_\gamma$ corresponds to an embedding $\mathcal{E} : B \to \mathrm{Res}_P^G(A)$ such that $\mathrm{Res}_P^G(\mathcal{F}_\alpha)\,\mathcal{E} = \mathcal{D}_P^G$.

Now we prove that the pointed group $G_{\alpha'}$ also has defect P_γ. Let $\delta = \mathcal{E}(\{1_B\})$, a point of A^P. Thus P_δ is a defect pointed group of the primitive G-algebra A and P_δ is mapped to P_γ under the embedding $\mathcal{F}_\alpha$ (but we do not make the identification of δ and γ). The image of P_δ under the other embedding $\mathcal{F}_{\alpha'}$ is a pointed group $P_{\gamma'}$ which is a defect of $G_{\alpha'}$. But by Proposition 1.7, the only local points of C^P are the $N_G(P)$-conjugates of γ. Thus $\gamma' = {}^{g^{-1}}\gamma$ for some $g \in N_G(P)$ (the choice of g^{-1} is more convenient for the sequel). Conjugating by g, we obtain that P_γ is also a defect of $G_{\alpha'}$.

In particular $G_{\alpha'} \geq P_\gamma$ and by Proposition 1.4 again there exists a unique embedding $\mathcal{E}' : B \to \mathrm{Res}_P^G(A)$ such that $\mathrm{Res}_P^G(\mathcal{F}_{\alpha'})\,\mathcal{E}' = \mathcal{D}_P^G$. Thus we have the following commutative diagram of embeddings.

$$\begin{array}{ccccc}
\mathrm{Res}_P^G(C) & & & & \mathrm{Res}_P^G(C) \\
 & \nwarrow^{\mathrm{Res}_P^G(\mathcal{F}_\alpha)} & & {}^{\mathrm{Res}_P^G(\mathcal{F}_{\alpha'})}\nearrow & \\
\mathcal{D}_P^G \uparrow & & \mathrm{Res}_P^G(A) & & \uparrow \mathcal{D}_P^G \\
 & \nearrow_{\mathcal{E}} & & {}_{\mathcal{E}'}\nwarrow & \\
B & & & & B
\end{array}$$

Since $\mathrm{Res}_P^G(\mathcal{F}_{\alpha'})(\delta) = {}^{g^{-1}}\gamma$ and $\mathcal{D}_P^G(\{1_B\}) = \gamma$, we deduce that $\mathcal{E}'(\{1_B\}) = {}^g\delta$ (as both sides map to γ via $\mathcal{F}_{\alpha'}$).

Now we choose $e \in \mathcal{E}$ and we let $i = e(1_B) \in \delta$. Since $\mathcal{E}'(\{1_B\}) = {}^g\delta$ we can choose $e' \in \mathcal{E}'$ such that $i' = e'(1_B) = {}^g i$. Then e, $\mathrm{Conj}(g{\cdot}1_A)$ and e' induce isomorphisms

$$B \xrightarrow{\sim} iAi \xrightarrow{\sim} i'Ai' \xleftarrow{\sim} B\,,$$

and the composite $h : B \to B$ is by construction a g-skew automorphism of B. If $\mathcal{H}$ denotes the skew exo-automorphism containing h, then $\mathcal{E}'\mathcal{H} = \mathcal{C}_g\,\mathcal{E}$, where $\mathcal{C}_g$ is the skew exo-automorphism of $\mathrm{Res}_P^G(A)$ containing the conjugation by $g{\cdot}1_A$. Writing again $\mathcal{C}_g$ for the similar skew exo-automorphism of $\mathrm{Res}_P^G(C)$ and using Lemma 4.2 for the outer automorphism $\mathrm{Ind}_P^G(\mathcal{H})$ of C constructed above, we have

$$\begin{aligned}
\mathcal{C}_g\ \mathrm{Res}_P^G\,\mathrm{Ind}_P^G(\mathcal{H})\ \mathrm{Res}_P^G(\mathcal{F}_\alpha)\,\mathcal{E} &= \mathcal{C}_g\ \mathrm{Res}_P^G\,\mathrm{Ind}_P^G(\mathcal{H})\,\mathcal{D}_P^G = \mathcal{D}_P^G\,\mathcal{H} = \mathrm{Res}_P^G(\mathcal{F}_{\alpha'})\,\mathcal{E}'\,\mathcal{H} \\
&= \mathrm{Res}_P^G(\mathcal{F}_{\alpha'})\,\mathcal{C}_g\,\mathcal{E} = \mathcal{C}_g\ \mathrm{Res}_P^G(\mathcal{F}_{\alpha'})\,\mathcal{E}\,,
\end{aligned}$$

and therefore $\mathrm{Res}_P^G\,\mathrm{Ind}_P^G(\mathcal{H})\ \mathrm{Res}_P^G(\mathcal{F}_\alpha)\,\mathcal{E} = \mathrm{Res}_P^G(\mathcal{F}_{\alpha'})\,\mathcal{E}$. Now by Proposition 1.2 we can cancel $\mathcal{E}$ in this equation, because the $\mathcal{O}$-algebras B and A have the same number of points. Indeed since there are embeddings of B in $\mathrm{Res}_P^G(A)$ and of A in C, we have $|\mathcal{P}(B)| \leq |\mathcal{P}(A)| \leq |\mathcal{P}(C)|$, and since C is a matrix algebra over B (as an $\mathcal{O}$-algebra), we also have $|\mathcal{P}(B)| = |\mathcal{P}(C)|$. Thus we have proved that $\mathrm{Res}_P^G\,\mathrm{Ind}_P^G(\mathcal{H})\ \mathrm{Res}_P^G(\mathcal{F}_\alpha) = \mathrm{Res}_P^G(\mathcal{F}_{\alpha'})$ and it follows that $\mathrm{Ind}_P^G(\mathcal{H})\,\mathcal{F}_\alpha = \mathcal{F}_{\alpha'}$ by Proposition 1.1. Finally

$$\mathrm{Ind}_P^G(\mathcal{H})(\alpha) = \mathrm{Ind}_P^G(\mathcal{H})\,\mathcal{F}_\alpha(\{1_A\}) = \mathcal{F}_{\alpha'}(\{1_A\}) = \alpha'\,.$$

Thus the outer automorphism $\mathrm{Ind}_P^G(\mathcal{H})$ maps α to α', as required. □

Combining Proposition 4.1 with Proposition 3.3, we obtain the following result.

Corollary 4.4. *With the notation 3.2, the set* $\mathcal{A}(G,P,B)$ *is in bijection with the set of orbits* $\mathrm{Out}(C)\backslash\mathcal{P}(C^G)_{P_\gamma}$. $\blacksquare$

5. Automorphisms of source algebras

In the previous section we have seen the importance of the group of outer automorphisms $\mathrm{Out}(C)$. Continuing with our notation 3.2, we now analyze in detail the structure of this group and show that it is an extension, with a normal subgroup $\mathrm{Out}(B)$ and a quotient group $N_G(P,B)/N_G(P_\gamma)$. Thus in particular it only depends on B and $N_G(P,B)$.

Recall from Section 4 that a skew exo-automorphism $\mathcal{H}$ of B is g-skew for some $g \in N_G(P,B)$ and that g is defined up to an element of $C_G(P{\cdot}1_B) \cap N_G(P) = C_N(P{\cdot}1_B)$. Let $\mathrm{Aut}_{\mathrm{skew}}(B)$ be the group of skew automorphisms of B and $\widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B)$ be the group of skew exo-automorphisms of B (we reserve the notation $\mathrm{Out}_{\mathrm{skew}}(B)$ for another version of outer automorphisms to be defined below). By definition of $\widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B)$ and $N_G(P,B)$, there is a surjective group homomorphism

$$\pi : \widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B) \longrightarrow N_G(P,B)/C_N(P{\cdot}1_B)\,,$$

mapping a g-skew exo-automorphism to the class of g . Moreover $\mathrm{Ker}(\pi) = \mathrm{Out}(B)$, the group of ordinary outer automorphisms of B . This is the first normal subgroup of interest in $\widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B)$. The second one is the subgroup $I_{\mathrm{skew}}(B)$ of all skew exo-automorphisms of B consisting of *inner* automorphisms $\mathrm{Inn}(b)$ for some $b \in B^*$. The unit element of $\widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B)$ contains all inner automorphisms $\mathrm{Inn}(b)$ for some $b \in (B^P)^*$, but here in the definition of $I_{\mathrm{skew}}(B)$ we do not require b to P-invariant. In fact if $\mathrm{Inn}(b)$ is a g-skew automorphism, then b satisfies $b{\cdot}u{\cdot}b^{-1} = {}^g u{\cdot}1_B$ for all $u \in P$. Passing now to the quotient by all possible inner automorphisms, we obtain the group $\mathrm{Out}_{\mathrm{skew}}(B) = \widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B)/I_{\mathrm{skew}}(B)$, which we call the group of *skew outer automorphisms* of B . We now show how these groups are related. For simplicity the results are only stated for a source algebra B , but in fact some of the statements below hold for an arbitrary interior P-algebra.

Proposition 5.1. *With the notation 3.2, the following statements hold.*

(a) *The group homomorphism* $\pi : \widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B) \to N_G(P,B)/C_N(P{\cdot}1_B)$ *is surjective with kernel* $\mathrm{Out}(B)$.

(b) *The group homomorphism* $\mathrm{Ind}_P^G : \widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B) \to \mathrm{Out}(C)$ *is surjective with kernel* $I_{\mathrm{skew}}(B)$. *Thus* Ind_P^G *induces an isomorphism* $\mathrm{Out}_{\mathrm{skew}}(B) \cong \mathrm{Out}(C)$.

(c) $I_{\mathrm{skew}}(B) \cap \mathrm{Out}(B) = \{1\}$. *Thus* $I_{\mathrm{skew}}(B) \times \mathrm{Out}(B)$ *is a normal subgroup of* $\widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B)$.

(d) $I_{\mathrm{skew}}(B){\times}\mathrm{Out}(B) = \pi^{-1}\big(N_G(P_\gamma)/C_N(P{\cdot}1_B)\big)$. *In particular* $N_G(P_\gamma)$ *is a normal subgroup of* $N_G(P,B)$. *Moreover* $N_G(P_\gamma)$ *only depends on* P , B *and* $N_G(P,B)$.

(e) $\mathrm{Out}_{\mathrm{skew}}(B)$ *has a normal subgroup isomorphic to* $\mathrm{Out}(B)$, *with quotient isomorphic to* $N_G(P,B)/N_G(P_\gamma)$. *More precisely,* π *induces a surjective group homomorphism* $\overline{\pi} : \mathrm{Out}_{\mathrm{skew}}(B) \to N_G(P,B)/N_G(P_\gamma)$ *with kernel* $\mathrm{Out}(B)$.

(f) *The image of* $\mathrm{Out}(B)$ *under the isomorphism* $\mathrm{Out}_{\mathrm{skew}}(B) \cong \mathrm{Out}(C)$ *is the stabilizer* $\mathrm{Out}(C)_\gamma$ *of the point* γ . *Thus* $\mathrm{Out}(C)_\gamma$ *is a normal subgroup of* $\mathrm{Out}(C)$, *with quotient isomorphic to* $N_G(P,B)/N_G(P_\gamma)$. $\blacksquare$

Proof. Statement (a) holds by definition. Statement (c) is clear since a g-skew inner automorphism $\mathrm{Inn}(b)$ is an ordinary automorphism if and only if $b \in B^P$, which means that $\mathrm{Inn}(b)$ represents the trivial element of $\widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B)$. Part (e) immediately follows from (a) and (d). Part (f) follows from (b) and (e), together with the fact that if $\mathcal{H}$ is a skew outer automorphism and $h \in \mathcal{H}$ is g-skew, then the automorphism $\widehat{h} \in \mathrm{Ind}_P^G(\mathcal{H})$ defined by $\widehat{h}(x \otimes a \otimes y) = xg^{-1} \otimes h(a) \otimes gy$ maps the idempotent $i = 1 \otimes 1_B \otimes 1$ to $g^{-1} \otimes 1_B \otimes g = {}^{g^{-1}}i$ and therefore $\mathrm{Ind}_P^G(\mathcal{H})(\gamma) = {}^{g^{-1}}\gamma$.

We show together the equalities

$$I_{\mathrm{skew}}(B) = \mathrm{Ker}(\mathrm{Ind}_P^G) \quad \text{and} \quad \pi(I_{\mathrm{skew}}(B)) = N_G(P_\gamma)/C_N(P{\cdot}1_B)$$

(the second one is equivalent to (d)), for both proofs use exactly the same arguments. Let $\mathcal{H} \in \widetilde{\mathrm{Aut}}_{\mathrm{skew}}(B)$ and let $h \in \mathcal{H}$ be g-skew. Then $\mathrm{Ind}_P^G(\mathcal{H})$ is represented by the automorphism $\widehat{h}$ defined by $\widehat{h}(x \otimes a \otimes y) = xg^{-1} \otimes h(a) \otimes gy$ for $a \in B$.

Assume first that $\mathcal{H} \in I_{\mathrm{skew}}(B)$, so that $h = \mathrm{Inn}(b)$ for some $b \in B^*$ with $b{\cdot}u{\cdot}b^{-1} = {}^{g}u{\cdot}1_B$ for all $u \in P$. Let

$$c = \sum_{x \in [G/P]} xg^{-1} \otimes b \otimes x^{-1} \in C\,.$$

This expression is independent of the choice of cosets representatives $[G/P]$, for if we replace x by xu_x where $u_x \in P$, then

$$\sum_{x \in [G/P]} xu_xg^{-1} \otimes b \otimes u_x^{-1}x^{-1} = \sum_{x \in [G/P]} xg^{-1} \otimes {}^{g}u_x{\cdot}b{\cdot}u_x^{-1} \otimes x^{-1} = \sum_{x \in [G/P]} xg^{-1} \otimes b \otimes x^{-1} = c\,.$$

It follows that c is fixed under G because if $z \in G$, then

$$z{\cdot}c{\cdot}z^{-1} = \sum_{x \in [G/P]} zxg^{-1} \otimes b \otimes (zx)^{-1} = c\,.$$

Moreover c is invertible with inverse $c^{-1} = \sum_{x \in [G/P]} x \otimes b^{-1} \otimes gx^{-1}$. Now we have

$$\mathrm{Inn}(c)(x \otimes a \otimes y) = xg^{-1} \otimes bab^{-1} \otimes gy = \widehat{h}(x \otimes a \otimes y)\,,$$

proving that $\widehat{h}$ is inner, that is, $\mathrm{Ind}_P^G(\mathcal{H}) = 1$. Thus $I_{\mathrm{skew}}(B) \subseteq \mathrm{Ker}(\mathrm{Ind}_P^G)$.

On the other hand $c(1 \otimes 1_B \otimes 1)c^{-1} = g^{-1} \otimes 1_B \otimes g = {}^{g^{-1}}(1 \otimes 1_B \otimes 1)$ and since $c \in C^P$, this proves that ${}^{g^{-1}}(1 \otimes 1_B \otimes 1)$ belongs to the point γ. Thus $g \in N_G(P_\gamma)$ and so $\pi(I_{\mathrm{skew}}(B)) \subseteq N_G(P_\gamma)/C_N(P{\cdot}1_B)$.

Assume now that $\mathcal{H} \in \mathrm{Ker}(\mathrm{Ind}_P^G)$ so that $\widehat{h} = \mathrm{Inn}(c^{-1})$ for some $c \in C^G$. We choose c^{-1} instead of c for notational convenience because this allows to consider g rather than g^{-1} in the proof below. We have in particular $g^{-1} \otimes h(a) \otimes g = c^{-1}(1 \otimes a \otimes 1)c$ for all $a \in B$. Writing $c = \sum_{x,y \in [G/P]} x \otimes c_{x,y} \otimes y$ and choosing a system of representatives $[G/P]$ containing 1 and g, we have

$$\sum_{x \in [G/P]} x \otimes c_{x,g}h(a) \otimes g = c\,(g^{-1} \otimes h(a) \otimes g) = (1 \otimes a \otimes 1)\,c = \sum_{y \in [G/P]} 1 \otimes ac_{1,y} \otimes y\,.$$

It follows that $c_{x,g} = 0$ for $x \neq 1$, $c_{1,y} = 0$ for $y \neq g$ and $c_{1,g}h(a) = ac_{1,g}$. Viewing C as a matrix algebra over B, we see that $c_{1,g}$ is the only non-zero entry in the first row and in the g-th column. As c is invertible, it follows that $c_{1,g} \in B^*$. Then $h(a) = c_{1,g}^{-1}ac_{1,g} = \mathrm{Inn}(c_{1,g}^{-1})(a)$, proving that $\mathcal{H} \in I_{\mathrm{skew}}(B)$. Thus $\mathrm{Ker}(\mathrm{Ind}_P^G) \subseteq I_{\mathrm{skew}}(B)$. Together with the other inclusion, this proves half of the statement in (b).

Assume now that $\pi(\mathcal{H}) \in N_G(P_\gamma)/C_N(P{\cdot}1_B)$, which simply means that the class of g belongs to $N_G(P_\gamma)/C_N(P{\cdot}1_B)$, that is, $g \in N_G(P_\gamma)$. Therefore there exists $c \in (C^P)^*$ such that ${}^{g^{-1}}(1 \otimes 1_B \otimes 1) = c^{-1}(1 \otimes 1_B \otimes 1)c$. The same argument as above (with $a = 1$) shows that $c_{1,g}$ is the only non-zero entry in the first row and in the g-th column of c , so that $c_{1,g}$ is invertible. Since c is fixed under P , we have $c{\cdot}u = u{\cdot}c$ for all $u \in P$. The $(1,g)$-entry of $u{\cdot}c$ is $1 \otimes u{\cdot}c_{1,g} \otimes g$ and that of $c{\cdot}u$ is $1 \otimes c_{1,g}{\cdot}{}^g u \otimes g$. Therefore $u{\cdot}c_{1,g} = c_{1,g}{\cdot}{}^g u$ and $\mathrm{Inn}(c_{1,g}^{-1})$ is a g-skew inner automorphism of B . As h is also g-skew, $\mathrm{Inn}(c_{1,g})h$ is an ordinary automorphism of B . This proves that $\mathcal{H} \in I_{\mathrm{skew}}(B) \times \mathrm{Out}(B)$, as required. Thus (d) is established since we already proved the other inclusion.

We are left with the proof of the other statement in (b), namely the surjectivity of Ind_P^G . Let $\mathcal{F} \in \mathrm{Out}(C)$. As the local points of C^P are just the $N_G(P)$-conjugates of γ by Proposition 1.7 and since $\mathcal{F}$ necessarily permutes the local points of C^P , we must have $\mathrm{Res}_P^G(\mathcal{F})(\gamma) = {}^{g^{-1}}\gamma$ for some $g \in N_G(P)$. Recall that the canonical embedding $\mathcal{D}_P^G : B \to \mathrm{Res}_P^G(C)$ is associated with the point γ . Since $\mathrm{Res}_P^G(\mathcal{F})\,\mathcal{D}_P^G$ is an embedding which maps $\{1_B\}$ to ${}^{g^{-1}}\gamma$, it is an embedding associated with ${}^{g^{-1}}\gamma$ and so B is isomorphic to the conjugate algebra ${}^{g^{-1}}B$, proving that $g \in N_G(P,B)$. If $\mathcal{C}_g$ denotes the skew exo-automorphism of $\mathrm{Res}_P^G(C)$ containing the conjugation by $g{\cdot}1_C$, the composition $\mathcal{C}_g\ \mathrm{Res}_P^G(\mathcal{F})$ fixes γ and therefore induces by "restriction" a skew exo-automorphism $\mathcal{H}$ of the localization B , that is, $\mathcal{C}_g\ \mathrm{Res}_P^G(\mathcal{F})\,\mathcal{D}_P^G = \mathcal{D}_P^G\,\mathcal{H}$. But since $\mathrm{Ind}_P^G(\mathcal{H})$ is the unique exomorphism satisfying (4.2), we must have $\mathcal{F} = \mathrm{Ind}_P^G(\mathcal{H})$. This completes the proof of (b). □

Remarks 5.2. (a) Among the numerous reasons for considering exomorphisms rather than homomorphisms, we note that the surjectivity of induction proved in the proposition does not hold in general for automorphisms, since an inner automorphism may not be in the image of induction.

(b) Part (d) of Proposition 5.1 is not new. It says that an element $g \in N_G(P)$ belongs to $N_G(P_\gamma)$ if and only if there exists $b \in B^*$ such that $b{\cdot}u{\cdot}b^{-1} = {}^g u{\cdot}1_B$ for all $u \in P$ (that is, $\mathrm{Inn}(b)$ is g-skew). This is a special case of a result of Puig which relates the group $E_G(P_\gamma) = N_G(P_\gamma)/PC_G(P)$ with the group of fusions $F_B(P_\gamma)$ (see [P2] for details).

(c) Given $\mathcal{F} \in \mathrm{Out}(C)$, we have $\mathcal{F}(\gamma) = {}^g\gamma$ for some $g \in N_G(P,B)$. It is easy to check directly that the map $\mathrm{Out}(C) \to N_G(P,B)/N_G(P_\gamma)$ sending $\mathcal{F}$ to the class of g^{-1} is a group homomorphism with kernel $\mathrm{Out}(C)_\gamma$. For another way of seeing this, notice that if $\mathcal{H} \in \mathrm{Out}_{\mathrm{skew}}(B)$ is g-skew, then $\mathcal{F} = \mathrm{Ind}_P^G(\mathcal{H})$ maps γ to ${}^{g^{-1}}\gamma$ by construction.

Example 5.3. This example shows that $N_G(P,B)$ can be larger than $N_G(P_\gamma)$, or equivalently that $\mathrm{Out}_{\mathrm{skew}}(B)$ can be larger than $\mathrm{Out}(B)$. Take $B = \mathcal{O}P$, the group algebra. As P is a p-group, B is primitive. Then $\mathrm{Out}(B) = 1$ because any automorphism which is the identity on P is the identity on B . Moreover $N_G(P,B) = N_G(P)$ because for $g \in N_G(P)$, the map $u \mapsto {}^g u$ (for $u \in P$) extends by linearity to a g-skew automorphism of B . Finally $N_G(P_\gamma) = PC_G(P)$. This is easy to see when P is abelian because B cannot have any non-trivial inner automorphism so that $I_{\mathrm{skew}}(B) = 1$; this implies that $N_G(P_\gamma) = C_G(P)$ by part (d) of Proposition 5.1. The equality $N_G(P_\gamma) = PC_G(P)$ still holds if P is not necessarily abelian by a result of Coleman [C]. It follows that $N_G(P,B)/N_G(P_\gamma) = N_G(P)/PC_G(P)$ which is often non-trivial. In particular $\mathrm{Out}_{\mathrm{skew}}(\mathcal{O}P) = N_G(P)/PC_G(P)$.

This example appears in some cases of nilpotent blocks. Therefore when B is the source algebra of a block algebra, the group $N_G(P,B)$ can be larger than the inertial group $N_G(P_\gamma)$, and so could be an invariant of interest in block theory. An explicit example occurs when G is the dihedral group of order $6p$ where $p \geq 5$. Then $P = C_p$ is cyclic of order p and $C_G(P) = C_p \times C_3$. The two non-trivial characters of C_3 are permuted by C_2 , thus correspond to a block of G (the non-principal block), which is in fact nilpotent [P4]. Its source algebra is $B = \mathcal{O}P$ and so we are in the situation described above.

6. Defect multiplicity modules

In this section we recall that any $A \in \mathcal{A}(G,P,B)$ has a defect multiplicity module W_γ, which is an indecomposable projective module over a twisted group algebra $k_\sharp \widehat{\overline{N}}^A$. We also recall a result of Puig which says that V_γ is free of rank one over $k_\sharp \widehat{\overline{N}}^C$, and we discuss the relationship between those modules.

Let A be a primitive interior G-algebra with defect group P and source algebra B. Then there is a local point γ of A^P such that P_γ is a defect of A and $B \cong A_\gamma$, but we know that γ is not unique, for any $N_G(P,B)$-conjugate of γ has the same property.

By Proposition 1.5 there is an embedding $\mathcal{F}: A \to C$, where $C = \mathrm{Ind}_P^G(B)$. Since A is primitive, $\mathcal{F}$ is an embedding associated with a point α of C^G, namely $\alpha = \mathcal{F}(1_A)$, and thus $A \cong C_\alpha$ and $\mathcal{F} = \mathcal{F}_\alpha$. As in Section 3, we use the embedding $\mathcal{F}_\alpha$ to identify pointed groups on A with pointed groups on C. Thus γ also denotes the point of C^P which contains $1 \otimes 1_B \otimes 1$ and so the notation is coherent with that of (3.2).

Viewed as a point on C^P, the point γ has a multiplicity algebra $S(\gamma)$ with canonical map $\pi_\gamma : C^P \to S(\gamma)$, and viewed as a point on A^P, it has a multiplicity algebra $T(\gamma)$ with canonical map $\rho_\gamma : A^P \to T(\gamma)$. By Proposition 1.11, the embedding $\mathcal{F}_\alpha$ induces an embedding $\overline{\mathcal{F}}_\alpha : T(\gamma) \to S(\gamma)$. Since A is a primitive G-algebra, $T(\gamma)$ is a primitive $\overline{N}$-algebra (Proposition 1.9) and therefore $\overline{\mathcal{F}}_\alpha$ is an embedding associated with a point $\overline{\alpha} \in \mathcal{P}(S(\gamma)^{\overline{N}})$. Thus $T(\gamma) = S(\gamma)_{\overline{\alpha}}$ and $\overline{\mathcal{F}}_\alpha = \mathcal{F}_{\overline{\alpha}}$. Finally $\overline{\alpha}$ is the Puig correspondent of α. Indeed the Puig correspondent of α is $\pi_\gamma\, r_P^G(\alpha) = \pi_\gamma(\alpha)$ (as r_P^G is just the inclusion map) and we have

$$\pi_\gamma(\alpha) = \pi_\gamma\, \mathcal{F}_\alpha(\{1_A\}) = \overline{\mathcal{F}}_\alpha\, \rho_\gamma(\{1_A\}) = \overline{\mathcal{F}}_\alpha(\{1_{T(\gamma)}\}) = \overline{\alpha}\,.$$

Therefore to the embedding of (interior) G-algebras $\mathcal{F}_\alpha : A \to C$ associated with a pointed group G_α with defect P_γ corresponds an embedding of $\overline{N}$-algebras $\mathcal{F}_{\overline{\alpha}} : T(\gamma) \to S(\gamma)$ associated with the Puig correspondent $\overline{\alpha}$ of α.

We let $T(\gamma) \cong \mathrm{End}_k(W(\gamma))$ and $S(\gamma) \cong \mathrm{End}_k(V(\gamma))$ and we use Proposition 2.3 to relate the multiplicity module structures of $W(\gamma)$ and $V(\gamma)$. Let us write $\widehat{\overline{N}}^A$ for the central extension of $\overline{N}$ associated with $T(\gamma)$ and $\widehat{\overline{N}}^C$ for the central extension of $\overline{N}$ associated with $S(\gamma)$, so that $W(\gamma)$ is a $k_\sharp \widehat{\overline{N}}^A$-module and $V(\gamma)$ is a $k_\sharp \widehat{\overline{N}}^C$-module. By Proposition 2.3 both central extensions are isomorphic via an isomorphism $(\mathcal{F}_{\overline{\alpha}})^*$ induced by $\mathcal{F}_{\overline{\alpha}}$, and this induces a structure of $k_\sharp \widehat{\overline{N}}^C$-module on $W(\gamma)$. With respect to this new structure, $W(\gamma)$ is isomorphic to a direct summand of $V(\gamma)$. We now analyze both modules in more detail.

By Proposition 1.9, the $\overline{N}$-algebra $T(\gamma)$ is primitive and projective (that is, $T(\gamma)_1^{\overline{N}} = T(\gamma)^{\overline{N}}$). Thus by Higman's criterion the $k_\sharp \widehat{\overline{N}}^A$-module $W(\gamma)$ is indecomposable projective. It is called a *defect multiplicity module* of A. (This notion depends on the choice of defect P_γ, and therefore is uniquely defined up to conjugation). We note immediately that this canonical structure of module on $W(\gamma)$ is not the one which can be used for the parametrization of G-algebras: we shall first need the other central extension $\widehat{\overline{N}}^C$, and later in Section 10 we shall view $W(\gamma)$ as a module over a central extension $(\widehat{\overline{N}}^B)^\circ$ defined in terms of B alone.

By a result of Puig, the $k_\sharp \widehat{\overline{N}}^C$-module structure on $V(\gamma)$ turns out to be extremely easy to describe.

Proposition 6.1 (Puig [P3, 9.12]). *With the notation 3.2, consider* $V(\gamma)$ *with its structure of* $k_\sharp\widehat{\overline{N}}^{C}$*-module. Then* $V(\gamma)$ *is free of rank one.*

Proof. For the convenience of the reader, we sketch a proof of the proposition. Consider the canonical map $\pi_\gamma : C^P \to S(\gamma)$ and let $i = 1 \otimes 1_B \otimes 1 \in \gamma$, so that $\pi_\gamma(i)$ is a primitive idempotent of $S(\gamma)$. By construction of induced algebras, we have $1_C = t_P^G(i)$ and the decomposition $1_C = \sum_{g\in[G/P]} {}^g i$ is orthogonal. Since $\{1_B\}$ is local by assumption, γ is also local and therefore Proposition 1.10 applies. We obtain

$$1_{S(\gamma)} = \pi_\gamma\, r_P^G(1_C) = \pi_\gamma\, r_P^G(t_P^G(i)) = t_1^{\overline{N}}(\pi_\gamma(i))\,.$$

The decomposition $1_{S(\gamma)} = \sum_{x\in\overline{N}} {}^x(\pi_\gamma(i))$ is primitive and orthogonal. Therefore the multiplicity module $V(\gamma)$ decomposes as k-vector space as a direct sum of one-dimensional subspaces

$$V(\gamma) = \bigoplus_{x\in\overline{N}} {}^x(\pi_\gamma(i))V(\gamma)\,.$$

Let us write the one-dimensional space $\pi_\gamma(i)V(\gamma) = kw$ for some $w \in \pi_\gamma(i)V(\gamma)$. Since, by definition, the action of $\widehat{\overline{N}}^{C}$ on $V(\gamma)$ lifts the conjugation action of $\overline{N}$ on $S(\gamma)$, it follows that

$$V(\gamma) = \bigoplus_{x\in\overline{N}} \widehat{x}\cdot kw\,,$$

where $\widehat{x} \in \widehat{\overline{N}}^{C}$ lifts $x \in \overline{N}$. This proves that $V(\gamma)$ is generated as a module over the twisted group algebra $k_\sharp\widehat{\overline{N}}^{C}$ by the single element w. Moreover the surjective homomorphism of $k_\sharp\widehat{\overline{N}}^{C}$-modules

$$k_\sharp\widehat{\overline{N}}^{C} \to V(\gamma)\,, \qquad \widehat{x} \mapsto \widehat{x}\cdot w$$

is an isomorphism because both modules are k-vector spaces of the same dimension, namely $|\overline{N}|$. □

Note that Proposition 6.1 provides another proof of the fact that any defect multiplicity module $W(\gamma)$ is projective, because by the analysis above, we know that $W(\gamma)$ is isomorphic to a direct summand of $V(\gamma)$ as a $k_\sharp\widehat{\overline{N}}^{C}$-module, hence is projective as a module over either $k_\sharp\widehat{\overline{N}}^{C}$ or $k_\sharp\widehat{\overline{N}}^{A}$. Moreover Proposition 6.1 also implies that every direct summand of $V(\gamma)$ is projective, that is, every point of $S(\gamma)^{\overline{N}}$ is projective. Let us write $\mathcal{P}(S(\gamma)^{\overline{N}})_{\rm proj} = \{\, \delta \in \mathcal{P}(S(\gamma)^{\overline{N}}) \mid \delta \text{ is projective } \}$.

Corollary 6.2. $\mathcal{P}(S(\gamma)^{\overline{N}})_{\rm proj} = \mathcal{P}(S(\gamma)^{\overline{N}})$.

The distinction between $k_\sharp\widehat{\overline{N}}^{C}$ and $k_\sharp\widehat{\overline{N}}^{A}$ is crucial in many respects. Firstly the isomorphism $(\mathcal{F}_{\overline{\alpha}})^* : k_\sharp\widehat{\overline{N}}^{C} \xrightarrow{\sim} k_\sharp\widehat{\overline{N}}^{A}$ depends on the embedding $\mathcal{F}_\alpha : A \to C$ and therefore a different embedding may yield a different structure of $k_\sharp\widehat{\overline{N}}^{C}$-module on $W(\gamma)$. But there is another phenomenon which appears even if the embedding $\mathcal{F}_\alpha$ is unique (and this is the case for instance for $\mathcal{O}G$-lattices). Two non-isomorphic primitive interior G-algebras A and A' may have the same defect group P, the same source algebra B and isomorphic defect multiplicity algebras (even with a canonical isomorphism). Since the defect multiplicity module is uniquely constructed from the defect multiplicity algebra, one may be tempted to conclude that the defect multiplicity modules $W(\gamma)$ of A, and $W'(\gamma)$ of A', are isomorphic, and this would make impossible the parametrization we are looking for. Explicitly the isomorphism of defect multiplicity algebras induces an isomorphism of central extensions $\widehat{\overline{N}}^A \cong \widehat{\overline{N}}^{A'}$, which could be used to identify the corresponding multiplicity modules $W(\gamma)$ and $W'(\gamma)$. But we do not do so: we use rather the two isomorphisms $\widehat{\overline{N}}^C \cong \widehat{\overline{N}}^A$ and $\widehat{\overline{N}}^C \cong \widehat{\overline{N}}^{A'}$ to view $W(\gamma)$ and $W'(\gamma)$ as $k_\sharp\widehat{\overline{N}}^C$-modules, in which case they are no longer isomorphic. Thus it is absolutely essential to view all defect multiplicity modules as modules over a single uniquely defined twisted group algebra, namely $k_\sharp\widehat{\overline{N}}^C$. We shall see in Section 9 an even better way of doing this by defining yet another central extension, directly in terms of the source algebra B.

We wish to illustrate these important points with two easy examples.

Example 6.3. Let $G = S_3$ be the symmetric group on 3 letters, generated by an element u of order 3 and an element x of order 2. We take a field k of characteristic 3. There are two indecomposable kG-modules L and L' of dimension 2. The top composition factor of L is the trivial representation and its socle is the sign representation, while the opposite holds for L'. Both L and L' restrict to the same 2-dimensional module M for $P = < u >$, which is a source of both modules. In matrix terms, we have

$$u \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad , \qquad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for L, and the same for L' with a change of sign for the image of x. Let $A = \mathrm{End}_k(L)$ and $A' = \mathrm{End}_k(L')$. In both cases the unique point $\gamma = \{1\}$ of $A^P = A'^P$ (corresponding to the source module M) has a multiplicity algebra $\mathrm{End}_k(W(\gamma))$ isomorphic to k, and we have $N_G(P_\gamma) = G$ and $\overline{N}_G(P_\gamma) = C_2$, the cyclic group of order 2. Thus in both cases we have $GL(W(\gamma)) = k^*$ and $PGL(W(\gamma)) = 1$, so that both central extensions are determined by the pull-back diagram

$$\begin{array}{ccc} \widehat{C}_2 = k^* \times C_2 & \longrightarrow & C_2 \\ {\scriptstyle \widehat{\rho}}\big\downarrow & & \big\downarrow \\ k^* & \longrightarrow & 1\,. \end{array}$$

Despite the fact that our two one-dimensional k-algebras are canonically isomorphic, we do not identify the corresponding central extensions, but we use two different isomorphisms with the central extension $\widehat{C}_2^C$ determined by $C = \mathrm{Ind}_P^G(\mathrm{End}_k(M)) = \mathrm{End}_k(\mathrm{Ind}_P^G(M))$ (corresponding to the two embeddings $L \to \mathrm{Ind}_P^G(M)$ and $L' \to \mathrm{Ind}_P^G(M)$). With their structure of module over $k_\sharp\widehat{C}_2^C$, the two multiplicity modules are now distinguished by a sign: since the central extension $\widehat{C}_2^C$ splits, the twisted group algebra $k_\sharp\widehat{C}_2^C$ is isomorphic to the ordinary group algebra kC_2 and the two possible multiplicity modules are the trivial and the sign representations of C_2 (which are indeed projective modules since the characteristic is 3). One of these corresponds to L and the other one to L'.

There is another subtle point which we want to emphasize. The two kG-modules L and L' now correspond respectively to each of the two distinct one-dimensional representations of $k_\sharp\widehat{C}_2^C$, in a uniquely determined fashion. However one cannot say which is the trivial and which is the sign representation, because this depends on the isomorphism $k_\sharp\widehat{C}_2^C \cong kC_2$. Indeed the twisted group algebra $k_\sharp\widehat{C}_2^C$ has no canonical basis and is isomorphic to the ordinary group algebra kC_2 in two different ways, which swap the role of the trivial and the sign representations. This phenomenon is in fact not surprising in view of the complete symmetry between L and L'.

Example 6.4. Consider again $G = S_3$, generated by u and x, and k a field of characteristic 3. Take $A = kG$, the group algebra, which is a primitive G-algebra. Then A has defect group $P =< u >$, source algebra $B = \mathrm{Res}^G_P(A)$ and multiplicity algebra of dimension 1. As in the previous example we have a twisted group algebra of the group C_2 and there are two possible multiplicity modules. But this time A is (up to isomorphism) the unique primitive G-algebra with these invariants. Our aim is to explain this by showing that the two multiplicity modules are in the same orbit under some natural action of $\mathrm{Out}(B)$. This corresponds to the fact that there are two distinct embeddings of A into C, which determine two isomorphic points of C^G, hence in the same orbit under the action of $\mathrm{Out}(C) \cong \mathrm{Out}(B)$.

7. Automorphisms of central extensions

In this section, we describe two group homomorphisms $\sigma : \mathrm{Out}(C) \to \mathrm{Out}(S(\gamma)^{\overline{N}})$ and $\tau : \mathrm{Out}(C) \to \mathrm{Out}(\widehat{\overline{N}}^C)$, and we prove some properties of τ.

Given a central extension $1 \to k^* \to \widehat{X} \xrightarrow{\pi} X \to 1$, we let $\mathrm{Aut}_{k^*}(\widehat{X})$ be the group of all automorphisms of $\widehat{X}$ inducing the identity on k^*, which we call k^**-automorphisms* of $\widehat{X}$. Any k^*-automorphism of $\widehat{X}$ induces an automorphism of the quotient X and we let $\mathrm{Aut}^0_{k^*}(\widehat{X})$ be the normal subgroup consisting of all automorphisms which induce the identity on X. Thus we have an exact sequence

$$1 \longrightarrow \mathrm{Aut}^0_{k^*}(\widehat{X}) \longrightarrow \mathrm{Aut}_{k^*}(\widehat{X}) \longrightarrow \mathrm{Aut}(X) .$$

Any $h \in \mathrm{Aut}^0_{k^*}(\widehat{X})$ necessarily maps $x \in \widehat{X}$ to λx where $\lambda \in k^*$ only depends on the image $\pi(x)$ of x in X. Therefore $h(x) = \chi(\pi(x))\, x$ for a uniquely determined group homomorphism $\chi : X \to k^*$ and we obtain an isomorphism

$$\mathrm{Aut}^0_{k^*}(\widehat{X}) \xrightarrow{\sim} \mathrm{Hom}(X, k^*) , \qquad h \mapsto \chi ,$$

where $\mathrm{Hom}(X, k^*)$ is the group of k^*-valued characters of X.

Continuing with the notation 3.2, we specialize to the case $X = \overline{N}$ and we consider the central extension $\widehat{\overline{N}}^C$ associated with the multiplicity algebra $S(\gamma)$ of the point γ of C^P. Recall that $\overline{C}_G(P) = PC_G(P)/P$ is identified with a normal subgroup of $\widehat{\overline{N}}^C$ mapping by the identity to the corresponding normal subgroup of $\overline{N}$ and that we have defined $E_G(P_\gamma) = \overline{N}/\overline{C}_G(P)$. Let $\mathrm{Aut}_{k^*}(\widehat{\overline{N}}^C)_{\overline{C}_G(P)}$ be the stabilizer of $\overline{C}_G(P)$. Its intersection with $\mathrm{Aut}^0_{k^*}(\widehat{\overline{N}}^C)$ induces the identity on $\overline{N}$, hence fixes $\overline{C}_G(P)$ pointwise, and we write $\mathrm{Aut}^0_{k^*\overline{C}_G(P)}(\widehat{\overline{N}}^C)$ for this intersection. In other words $\mathrm{Aut}^0_{k^*\overline{C}_G(P)}(\widehat{\overline{N}}^C)$ is the subgroup of $\mathrm{Aut}^0_{k^*}(\widehat{\overline{N}}^C)$ consisting of automorphisms inducing the identity on $\overline{C}_G(P)$. These automorphisms correspond to characters of $\overline{N}$ vanishing on $\overline{C}_G(P)$, under the isomorphism $\mathrm{Aut}^0_{k^*}(\widehat{\overline{N}}^C) \cong \mathrm{Hom}(\overline{N}, k^*)$, so that we obtain by restriction an isomorphism $\mathrm{Aut}^0_{k^*\overline{C}_G(P)}(\widehat{\overline{N}}^C) \cong \mathrm{Hom}(E_G(P_\gamma), k^*)$. Thus there is an exact sequence

$$1 \longrightarrow \mathrm{Hom}(E_G(P_\gamma), k^*) \longrightarrow \mathrm{Aut}_{k^*}(\widehat{\overline{N}}^C)_{\overline{C}_G(P)} \longrightarrow \mathrm{Aut}(\overline{N})_{\overline{C}_G(P)} .$$

By Proposition 5.1, N is a normal subgroup of $N_G(P, B)$ and we consider the conjugation action of $N_G(P, B)$ on $\overline{N}$ (which stabilizes $\overline{C}_G(P)$). The group $N_G(P, B)$ does not act directly on the central extension $\widehat{\overline{N}}^C$. Thus we let $\mathrm{Aut}^{N(P,B)}(\overline{N})$ be the image of $N_G(P, B)$ in $\mathrm{Aut}(\overline{N})_{\overline{C}_G(P)}$ and we let $\mathrm{Aut}^{N(P,B)}_{k^*}(\widehat{\overline{N}}^C)$ be the inverse image of $\mathrm{Aut}^{N(P,B)}(\overline{N})$ in $\mathrm{Aut}_{k^*}(\widehat{\overline{N}}^C)_{\overline{C}_G(P)}$. In other words $\mathrm{Aut}^{N(P,B)}_{k^*}(\widehat{\overline{N}}^C)$

is the group of all k^*-automorphisms of $\widehat{\overline{N}}^{C}$ stabilizing $\overline{C}_G(P)$ and inducing on $\overline{N}$ an automorphism which is the conjugation by some element of $N_G(P,B)$. Thus there is an exact sequence

$$(7.1) \qquad 1 \longrightarrow \mathrm{Hom}(E_G(P_\gamma),k^*) \longrightarrow \mathrm{Aut}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^{C}) \longrightarrow \mathrm{Aut}^{N(P,B)}(\overline{N})\,.$$

We shall see below that this is a short exact sequence, that is, the last map is surjective.

We introduce the analogous sequence for outer automorphisms. Let I be the normal subgroup of $\mathrm{Aut}(\widehat{\overline{N}}^{C})$ consisting of inner automorphisms. Then $I \subseteq \mathrm{Aut}_{k^*}(\widehat{\overline{N}}^{C})$ and we define $\mathrm{Out}_{k^*}(\widehat{\overline{N}}^{C}) = \mathrm{Aut}_{k^*}(\widehat{\overline{N}}^{C})/I$. We also consider the images in $\mathrm{Out}_{k^*}(\widehat{\overline{N}}^{C})$ of various subgroups of $\mathrm{Aut}_{k^*}(\widehat{\overline{N}}^{C})$, in particular $\mathrm{Out}^0_{k^*}(\widehat{\overline{N}}^{C})$, $\mathrm{Out}^0_{k^*\overline{C}_G(P)}(\widehat{\overline{N}}^{C})$, $\mathrm{Out}_{k^*}(\widehat{\overline{N}}^{C})_{\overline{C}_G(P)}$ and $\mathrm{Out}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^{C})$. In other words $\mathrm{Out}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^{C})$ is the subgroup of $\mathrm{Out}_{k^*}(\widehat{\overline{N}}^{C})_{\overline{C}_G(P)}$ whose image in $\mathrm{Out}(\overline{N})$ is contained in the image of $N_G(P,B)$ (written $\mathrm{Out}^{N(P,B)}(\overline{N})$). Since the image of N in $\mathrm{Out}(\overline{N})$ consists of inner automorphisms, the group $\mathrm{Out}^{N(P,B)}(\overline{N})$ is in fact the image of $N_G(P,B)/N$ in $\mathrm{Out}(\overline{N})$. We also write $\widetilde{\mathrm{Hom}}(E_G(P_\gamma),k^*)$ for the quotient of $\mathrm{Hom}(E_G(P_\gamma),k^*)$, that is, $\widetilde{\mathrm{Hom}}(E_G(P_\gamma),k^*) \cong \mathrm{Out}^0_{k^*\overline{C}_G(P)}(\widehat{\overline{N}}^{C})$. The group $\mathrm{Out}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^{C})$ is described by the following exact sequence (in which again the last map is surjective, as we shall see below)

$$(7.2) \qquad 1 \longrightarrow \widetilde{\mathrm{Hom}}(E_G(P_\gamma),k^*) \longrightarrow \mathrm{Out}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^{C}) \longrightarrow \mathrm{Out}^{N(P,B)}(\overline{N})\,.$$

Now we want to show that $\mathrm{Out}(C)$ has an outer action on $S(\gamma)^{\overline{N}}$, as well as on the central extension $\widehat{\overline{N}}^{C}$. Let $\mathcal{F} \in \mathrm{Out}(C)$ and choose $f \in \mathcal{F}$. Then f acts on C^P, but it does not necessarily fix γ. In fact by Proposition 5.1 (f) (see also Remark 5.2 (c)), we have $f(\gamma) = {}^{g^{-1}}\gamma$ for some $g \in N_G(P,B)$ and g is well-defined up to an element of $N = N_G(P_\gamma)$. The automorphism $f_g = \mathrm{Conj}(g{\cdot}1_C)\, f$ leaves invariant C^P and γ, hence induces an automorphism of $S(\gamma)$, still written f_g.

Proposition 7.3. *Let $\mathcal{F} \in \mathrm{Out}(C)$, let $f \in \mathcal{F}$, and let $g \in N_G(P,B)$ be such that $f(\gamma) = {}^{g^{-1}}\gamma$.*
(a) The automorphism $f_g = \mathrm{Conj}(g{\cdot}1_C)\, f$ of $S(\gamma)$ induces by restriction an automorphism $\overline{f} : S(\gamma)^{\overline{N}} \to S(\gamma)^{\overline{N}}$ which only depends on f (not on g).
(b) There is a group homomorphism $\sigma : \mathrm{Out}(C) \to \mathrm{Out}(S(\gamma)^{\overline{N}})$ mapping $\mathcal{F}$ to the outer automorphism of $S(\gamma)^{\overline{N}}$ containing $\overline{f}$.

Proof. (a) If $x \in N$ and $c \in C^P$, we have

$$f_g(x{\cdot}c{\cdot}x^{-1}) = gx{\cdot}f(c){\cdot}x^{-1}g^{-1} = (gxg^{-1}){\cdot}f_g(c){\cdot}(gxg^{-1})^{-1}\,.$$

Applying $\pi_\gamma : C^P \to S(\gamma)$ and setting $s = \pi_\gamma(c)$, we obtain $f_g({}^x s) = {}^{gxg^{-1}} f_g(s)$, for all $s \in S(\gamma)$. If now s is fixed under $\overline{N}$, then $f_g(s)$ is also fixed under $\overline{N}$ (because $N \triangleleft N_G(P,B)$ by Proposition 5.1), and therefore f_g restricts to an automorphism $\overline{f}$ of $S(\gamma)^{\overline{N}}$. If g is replaced by gx where $x \in N$, then $\overline{f}$ has to be composed with the action of x, which by definition is the identity on $S(\gamma)^{\overline{N}}$. Therefore $\overline{f}$ only depends on f, not on the choice of g.

(b) If we change the choice of $f \in \mathcal{F}$, we have to compose f with an inner automorphism $\mathrm{Inn}(c)$ where $c \in C^G$. Since c commutes with g, we have to compose f_g with $\mathrm{Inn}(c)$, but since $c \in C^N$, $\pi_\gamma(c) \in S(\gamma)^{\overline{N}}$ and so $\overline{f}$ has to be composed with $\mathrm{Inn}(\pi_\gamma(c))$, which is an inner automorphism of $S(\gamma)^{\overline{N}}$. It follows that $\mathcal{F}$ induces a well-defined outer automorphism of the k-algebra $S(\gamma)^{\overline{N}}$. Thus the map $\sigma : \mathrm{Out}(C) \to \mathrm{Out}(S(\gamma)^{\overline{N}})$ is well-defined.

If $\mathcal{F}' \in \mathrm{Out}(C)$ and if $f' \in \mathcal{F}'$ with $f'(\gamma) = {}^{g'^{-1}}\gamma$, then since g' commutes with f we have

$$f_g\, f'_{g'} = \mathrm{Conj}(g{\cdot}1_C)\, f\; \mathrm{Conj}(g'{\cdot}1_C)\, f' = \mathrm{Conj}(gg'{\cdot}1_C)\, ff' = (ff')_{gg'}\,,$$

and the class of gg' is indeed the element of $N_G(P,B)/N$ associated with ff' (see Remark 5.2 (c)). It follows that σ is a group homomorphism. □

Now we show that $\mathrm{Out}(C)$ acts as a group of outer automorphisms of the central extension $\widehat{\overline{N}}^C$.

Proposition 7.4. *Let $\mathcal{F} \in \mathrm{Out}(C)$, let $f \in \mathcal{F}$, let $g \in N_G(P,B)$ be such that $f(\gamma) = {}^{g^{-1}}\gamma$, and let $f_g = \mathrm{Conj}(g{\cdot}1_C)\, f$, viewed as an automorphism of $S(\gamma)$.*

(a) There is a k^*-automorphism $\tau(f_g) : \widehat{\overline{N}}^C \to \widehat{\overline{N}}^C$, inducing $\mathrm{Conj}(g)$ on $\overline{N}$ and satisfying $\widehat{\rho}\,\tau(f_g) = f_g\,\widehat{\rho}$, where $\widehat{\rho} : \widehat{\overline{N}}^C \to S(\gamma)^*$ is the structural map. Moreover the outer automorphism of $\widehat{\overline{N}}^C$ containing $\tau(f_g)$ only depends on $\mathcal{F}$.

(b) There is a group homomorphism $\tau : \mathrm{Out}(C) \to \mathrm{Out}_{k^*}(\widehat{\overline{N}}^C)$ mapping $\mathcal{F}$ to the outer automorphism of $\widehat{\overline{N}}^C$ containing $\tau(f_g)$.

(c) The image of τ is contained in $\mathrm{Out}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^C)$, and the image of the normal subgroup $\mathrm{Out}(C)_\gamma$ is contained in $\widetilde{\mathrm{Hom}}(E_G(P_\gamma), k^*)$.

Proof. The automorphism f_g of C acts on $N{\cdot}1_C$ by conjugation by g because f is the identity on $G{\cdot}1_C$ (since it is an automorphism of interior G-algebras). It follows that the conjugation action of g on $\overline{N}$ satisfies $\rho\,\mathrm{Conj}(g) = \overline{f}_g\,\rho$ where $\overline{f}_g$ is the automorphism of $S(\gamma)^*/k^*$ induced by f_g, and where $\rho : \overline{N} \to S(\gamma)^*/k^*$ is the structural map (obtained from the action of $\overline{N}$ on $S(\gamma)$). It follows that the automorphism $\mathrm{Conj}(g)$ of $\overline{N}$ together with the automorphism f_g of $S(\gamma)^*$ induce by pull-back an automorphism $\tau(f_g)$ of $\widehat{\overline{N}}^C$ with the required properties.

This construction depends on the choice of $g \in N_G(P,B)$. If g is replaced by xg with $x \in N$, then $\tau(f_{xg})$ is the composite of $\tau(f_g)$ with the automorphism of $\widehat{\overline{N}}^C$ defined by the conjugation action of x on $\overline{N}$ and the action of $\mathrm{Conj}(x{\cdot}1_C)$ on $S(\gamma)^*$. Since $x \in N$, the conjugation action of x on $\overline{N}$ is the inner automorphism $\mathrm{Inn}(\overline{x})$. On the other hand $\mathrm{Conj}(x{\cdot}1_C) = \mathrm{Inn}(a)$ for some $a \in S(\gamma)^*$ and by definition of the map $\rho : \overline{N} \to S(\gamma)^*/k^*$, the image of $\overline{x}$ is equal to the class of a. Therefore the pair $(a, \overline{x})$ defines an element $\widehat{\overline{x}} \in \widehat{\overline{N}}^C$ and it is clear that $\mathrm{Inn}(\widehat{\overline{x}})$ is the automorphism induced by $\mathrm{Inn}(\overline{x})$ and $\mathrm{Conj}(x{\cdot}1_C)$. Therefore $\tau(f_{xg})$ differs from $\tau(f_g)$ by an inner automorphism and we have a well-defined outer automorphism of $\widehat{\overline{N}}^C$, which we write $\tau(f)$.

If we modify the choice of f in $\mathcal{F}$, this does not change $\tau(f)$. Indeed we now show that an inner automorphism $\mathrm{Inn}(c)$ with $c \in C^G$ acts trivially on $\widehat{\overline{N}}^C$. The element $\overline{c} = \pi_\gamma r_P^G(c)$ belongs to $S(\gamma)$ and is fixed under $\overline{N}$. Therefore if $(a, \overline{x}) \in \widehat{\overline{N}}^C$, where $\overline{x} \in \overline{N}$ and $a \in S(\gamma)^*$ lifts the action of $\overline{x}$, then $\overline{c}$ commutes with a and c fixes $x{\cdot}1_C$. Thus the automorphism of $\widehat{\overline{N}}^C$ induced by $\mathrm{Inn}(c)$ fixes $(a, \overline{x})$, that is, it is the identity.

(b) We have already noticed in the proof of the previous proposition that $f_g f'_{g'} = (ff')_{gg'}$. Together with the equation $\mathrm{Conj}(gg') = \mathrm{Conj}(g)\,\mathrm{Conj}(g')$, this implies that τ is a group homomorphism.

(c) The action of $\tau(f)$ on $\widehat{\overline{N}}^C$ stabilizes the normal subgroup $\overline{C}_G(P) = PC_G(P)/P$. Clearly $\mathrm{Conj}(g)$ stabilizes $\overline{C}_G(P)$. The structure of interior $\overline{C}_G(P)$-algebra on $S(\gamma)$ is given by the composite of the map $C_G(P) \to C^P$, $x \mapsto x{\cdot}1_C$, and the surjection $\pi_\gamma : C^P \to S(\gamma)$. It suffices to show that f_g, viewed as an automorphsim of C^P, stabilizes $C_G(P){\cdot}1_C$. But this is clear since, for $x \in C_G(P)$, we have

$$f_g(x{\cdot}1_C) = \mathrm{Conj}(g) f(x{\cdot}1_C) = \mathrm{Conj}(g)(x{\cdot}1_C) = {}^g x{\cdot}1_C\,.$$

In fact this proves that $\tau(f)$ acts on $\overline{C}_G(P)$ by conjugation by g. Therefore the image of τ is contained in the group $\mathrm{Out}_{k^*}(\widehat{\overline{N}}^C)_{\overline{C}_G(P)}$ and its image in $\mathrm{Out}(\overline{N})$ consists of conjugations by elements of $N_G(P,B)/N$. Thus we have $\tau(\mathrm{Out}(C)) \subseteq \mathrm{Out}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^C)$ by definition (see 7.2).

If $\mathcal{F}$ belongs to the normal subgroup $\mathrm{Out}(C)_\gamma = \mathrm{Ker}\big(\mathrm{Out}(C) \to N_G(P,B)/N\big)$, then we can choose $g = 1$, and by construction $\tau(\mathcal{F})$ induces the identity on the quotient $\overline{N}$. Thus $\tau(\mathcal{F}) \in \mathrm{Out}_{k^*}^0(\widehat{\overline{N}}^C)$, and since $\tau(\mathcal{F})$ fixes $\overline{C}_G(P)$ pointwise, $\tau(\mathcal{F})$ belongs to $\mathrm{Out}_{k^*\overline{C}_G(P)}^0(\widehat{\overline{N}}^C) \cong \widetilde{\mathrm{Hom}}(E_G(P_\gamma), k^*)$. □

Note that the argument given in the proof of part (a) shows that it is only the consideration of $g \in N_G(P,B)/N$ which forces to introduce outer automorphisms of $\widehat{\overline{N}}^C$. Thus the normal subgroup $\mathrm{Out}(C)_\gamma$ acts in fact on $\widehat{\overline{N}}^C$ as a group of genuine automorphisms. Therefore the restriction $\tau^0 : \mathrm{Out}(C)_\gamma \to \widetilde{\mathrm{Hom}}(E_G(P_\gamma), k^*)$ factorizes through a map $\tau' : \mathrm{Out}(C)_\gamma \to \mathrm{Hom}(E_G(P_\gamma), k^*)$.

The results of this section are summarized in the following result.

Proposition 7.5. *There is a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Out}(C)_\gamma & \longrightarrow & \mathrm{Out}(C) & \xrightarrow{\pi} & N_G(P,B)/N & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \tau^0} & & \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \text{conjugation}} & & \\
1 & \longrightarrow & \widetilde{\mathrm{Hom}}(E_G(P_\gamma), k^*) & \longrightarrow & \mathrm{Out}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^C) & \longrightarrow & \mathrm{Out}^{N(P,B)}(\overline{N}) & \longrightarrow & 1\,.
\end{array}$$

In particular both sequences 7.1 and 7.2 are short exact.

Proof. By construction of τ and τ^0, all diagrams are commutative. The top sequence is exact by Proposition 5.1. Thus we are left with the proof that the bottom sequence is right exact. This an immediate consequence of the surjectivity of both the maps $N_G(P,B)/N \to \mathrm{Out}^{N(P,B)}(\overline{N})$ and $\pi : \mathrm{Out}(C) \to N_G(P,B)/N$ (which was proved by inducing to C a g-skew automorphism of B). This proves that the sequence 7.2 is right exact.

In order to show that the sequence 7.1 is right exact, we note that an inner automorphism $\mathrm{Inn}(x)$ of $\overline{N}$ lifts to an inner automorphism $\mathrm{Inn}(\widehat{x})$ of $\widehat{\overline{N}}^C$ by surjectivity of $\widehat{\overline{N}}^C \to \overline{N}$, and $\mathrm{Inn}(\widehat{x})$ belongs necessarily to $\mathrm{Aut}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^C)$. Therfore the surjectivity of $\mathrm{Aut}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^C) \to \mathrm{Aut}^{N(P,B)}(\overline{N})$ is equivalent to the surjectivity of the map $\mathrm{Out}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^C) \to \mathrm{Out}^{N(P,B)}(\overline{N})$. □

Example 7.6. The examples given at the end of Section 6 do not illustrate the phenomenon analyzed here, because we have $N_G(P,B) = N$ in these examples, so that $\mathrm{Aut}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^C) = \mathrm{Hom}(E_G(P_\gamma), k^*)$. On the other hand we have $N < N_G(P;B)$ in Example 5.3, but $\mathrm{Hom}(E_G(P_\gamma), k^*) = 1$ because $E_G(P_\gamma) = 1$. For an example where the short exact sequence 7.1 is not trivial, take $G = S_4$, the symmetric group, P the normal four-subgroup, and $B = \mathcal{O}A_4$, the group algebra of the alternating group. Then $N = A_4$, $N_G(P,B) = G$ and $E = A_4/P \cong C_3$. Then in the exact sequence 7.1, the middle group is isomorphic to S_3, with kernel C_3 and quotient C_2.

8. Reduction to the defect multiplicity algebra

In this section we prove that the Puig correspondence commutes with the action of $\mathrm{Out}(C)$ and we deduce that $\mathcal{A}(G,P,B)$ is in bijection with the set of $\mathrm{Out}(C)$-orbits of indecomposable projective $k_\sharp\widehat{\overline{N}}^{C}$-modules.

Continuing with the notation 3.2, we know from Corollary 4.4 that the set $\mathcal{A}(G,P,B)$ which we are interested in is in bijection with $\mathrm{Out}(C)\backslash\mathcal{P}(C^G)_{P_\gamma}$. The Puig correspondence is a bijection between the set $\mathcal{P}(C^G)_{P_\gamma}$ and the set $\mathcal{P}(S(\gamma)^{\overline{N}})_{\mathrm{proj}}$ of projective points of $S(\gamma)^{\overline{N}}$, and by Corollary 6.2 $\mathcal{P}(S(\gamma)^{\overline{N}})_{\mathrm{proj}} = \mathcal{P}(S(\gamma)^{\overline{N}})$. We have seen in Proposition 7.3 that $\mathrm{Out}(C)$ has an outer action on $S(\gamma)^{\overline{N}}$. It follows that $\mathrm{Out}(C)$ acts on the set of points $\mathcal{P}(S(\gamma)^{\overline{N}})$, because inner automorphisms act trivially on points.

Lemma 8.1. *The map $\pi_\gamma\, r_P^G : C^G \to S(\gamma)^{\overline{N}}$ commutes with the outer action of $\mathrm{Out}(C)$. In particular the Puig correspondence $\mathcal{P}(C^G)_{P_\gamma} \xrightarrow{\sim} \mathcal{P}(S(\gamma)^{\overline{N}})_{\mathrm{proj}} = \mathcal{P}(S(\gamma)^{\overline{N}})$ commutes with the action of $\mathrm{Out}(C)$.*

Proof. Let $\mathcal{F} \in \mathrm{Out}(C)$, choose $f \in \mathcal{F}$, let $g \in N_G(P,B)$ be such that $f(\gamma) = {}^{g^{-1}}\gamma$, let $f_g = \mathrm{Conj}(g{\cdot}1_C)\, f$, and let $\overline{f}$ be the restriction of f_g to $S(\gamma)^{\overline{N}}$. Then by Proposition 7.3, $\overline{f}$ represents the outer action of $\mathcal{F}$ on $S(\gamma)^{\overline{N}}$. If $c \in C^G$, then $f(c) \in C^G$ commutes with g and therefore

$$\pi_\gamma\, r_P^G(f(c)) = \pi_\gamma\, r_P^G(\mathrm{Conj}(g{\cdot}1_C) f(c)) = \pi_\gamma\, r_P^G(f_g(c)) = \overline{f}(\pi_\gamma\, r_P^G(c)) .$$

The second statement follows immediately since the Puig correspondence is induced by the map $\pi_\gamma\, r_P^G$ and since $\mathcal{P}(S(\gamma)^{\overline{N}})_{\mathrm{proj}} = \mathcal{P}(S(\gamma)^{\overline{N}})$ by Corollary 6.2. □

Combining this lemma with Corollary 4.4, we obtain the following form of the parametrization.

Corollary 8.2. *With the notation 3.2, the set $\mathcal{A}(G,P,B)$ is in bijection with the set of orbits $\mathrm{Out}(C)\backslash\mathcal{P}(S(\gamma)^{\overline{N}})$.*

Our aim is now to view the points of $S(\gamma)^{\overline{N}}$ as indecomposable direct summands of the multiplicity module $V(\gamma)$ (where $S(\gamma) = \mathrm{End}_k(V(\gamma))$), and then to use the action of $\mathrm{Out}(C)$ on these modules. We have constructed in Proposition 7.4 an outer action $\tau : \mathrm{Out}(C) \to \mathrm{Out}_{k^*}(\widehat{\overline{N}}^{C})$. Since any k^*-automorphism is by definition the identity on k^*, the map τ extends to a homomorphism $\tau : \mathrm{Out}(C) \to \mathrm{Out}(k_\sharp\widehat{\overline{N}}^{C})$ and we deduce an action of $\mathrm{Out}(C)$ on isomorphism classes of $k_\sharp\widehat{\overline{N}}^{C}$-modules in the following way. For every $k_\sharp\widehat{\overline{N}}^{C}$-module W and $\mathcal{F} \in \mathrm{Out}(C)$ we write ${}^{\mathcal{F}}W$ for the isomorphism class of modules obtained by first applying the outer automorphism $\tau(\mathcal{F})^{-1}$ and then the structural map $k_\sharp\widehat{\overline{N}}^{C} \to \mathrm{End}_k(W)$. Note that this is well defined since for any inner automorphism $\mathrm{Inn}(r)$ of a ring R and for any R-module W given by a map $\rho : R \to \mathrm{End}(W)$, the two module structures on W given by the structural maps ρ and $\rho\,\mathrm{Inn}(r)$ are isomorphic (via the isomorphism $\rho(r)$). Note also that one needs the inverse $\tau(\mathcal{F})^{-1}$ for a left action.

Since $S(\gamma) = \mathrm{End}_k(V(\gamma))$, we can identify a point δ in $\mathcal{P}(S(\gamma)^{\overline{N}})$ with an isomorphism class of indecomposable direct summands W_δ of the $k_\sharp\widehat{\overline{N}}^{C}$-module $V(\gamma)$. For simplicity we do not distinguish between a direct summand W_δ and its isomorphism class. By Proposition 6.1, $V(\gamma)$ is free of rank one over $k_\sharp\widehat{\overline{N}}^{C}$, so that W_δ runs over the set of isomorphism classes of indecomposable projective $k_\sharp\widehat{\overline{N}}^{C}$-modules.

We have to prove that the action of $\mathrm{Out}(C)$ on $\mathcal{P}(S(\gamma)^{\overline{N}})$ and on the corresponding $k_\sharp\widehat{\overline{N}}^{C}$-modules are related in the expected fashion.

Lemma 8.3. *Let $\delta, \delta' \in \mathcal{P}(S(\gamma)^{\overline{N}})$ and let respectively W_δ and $W_{\delta'}$ be the corresponding isomorphism classes of direct summands of $V(\gamma)$. Suppose that $\delta' = \mathcal{F}(\delta)$ for some $\mathcal{F} \in \mathrm{Out}(C)$. Then $W_{\delta'} = {}^{\mathcal{F}}(W_\delta)$.*

Proof. Choose $f \in \mathcal{F}$. We have $f(\gamma) = {}^{g^{-1}}\gamma$ for some $g \in N_G(P,B)$ and for the description of the action of $\mathcal{F}$ on $\widehat{\overline{N}}^C$, we have to consider the action of $f_g = \mathrm{Conj}(g{\cdot}1_C)\, f$ on $S(\gamma)$. The restriction of f_g to $S(\gamma)^{\overline{N}}$ is the automorphism $\overline{f}$ which defines the action of $\mathcal{F}$ on $S(\gamma)^{\overline{N}}$ and therefore $f_g(\delta) = \delta'$ by assumption.

The $k_\sharp\widehat{\overline{N}}^C$-module structure on $V(\gamma)$ is given by a homomorphism $\widehat{\rho} : \widehat{\overline{N}}^C \to S(\gamma)^*$ lifting the map $\overline{N} \to S(\gamma)^*/k^*$. Now the outer action of $\mathcal{F}$ on $\widehat{\overline{N}}^C$ is represented by the automorphism $\tau(f_g)$ of $\widehat{\overline{N}}^C$ induced by f_g and therefore $\widehat{\rho}\,\tau(f_g) = f_g\,\widehat{\rho}$. Moreover the $k_\sharp\widehat{\overline{N}}^C$-module structure on ${}^{\mathcal{F}}(V(\gamma))$ corresponds to the homomorphism $\widehat{\rho}' : \widehat{\overline{N}}^C \to S(\gamma)^*$, where $\widehat{\rho}' = \widehat{\rho}\,\tau(f_g)^{-1} = f_g^{-1}\,\widehat{\rho}$. By the Skolem-Noether theorem, the automorphism f_g of $S(\gamma) = \mathrm{End}_k(V(\gamma))$ is equal to $\mathrm{Inn}(s)$ for some k-linear automorphism s of $V(\gamma)$. Moreover s is an isomorphism between ${}^{\mathcal{F}}(V(\gamma))$ and $V(\gamma)$ because for each $x \in \widehat{\overline{N}}^C$ we have $\widehat{\rho}'(x) = f_g^{-1}\,\widehat{\rho}(x) = s^{-1}\,\widehat{\rho}(x)\, s$, hence $s\,\widehat{\rho}'(x) = \widehat{\rho}(x)\, s$.

For the direct summand W_δ of $V(\gamma)$, we can choose $jV(\gamma)$ where $j \in \delta$. Since $f_g(\delta) = \delta'$, we let $j' = f_g(j) \in \delta'$ and choose $W_{\delta'} = j'V(\gamma)$. Then $sjs^{-1} = j'$ and therefore s maps the summand $jV(\gamma)$ to the summand $j'V(\gamma)$. It follows that the direct summand ${}^{\mathcal{F}}(jV(\gamma))$ of ${}^{\mathcal{F}}(V(\gamma))$ is mapped by the $k_\sharp\widehat{\overline{N}}^C$-linear isomorphism s to the summand $j'V(\gamma)$ of $V(\gamma)$, as was to be shown. □

Combining this lemma with Corollary 8.2, we obtain the first version of the parametrization we are looking for.

Corollary 8.4. *With the notation 3.2, the set $\mathcal{A}(G,P,B)$ is in bijection with the set of orbits of isomorphism classes of indecomposable projective $k_\sharp\widehat{\overline{N}}^C$-modules under the action of $\mathrm{Out}(C)$.*

9. The central extension associated with the source algebra

In this section we describe a central extension $\widehat{\overline{N}}^B$, defined in terms of B alone, and an outer action of $\mathrm{Out}_{\mathrm{skew}}(B)$ on it.

Corollary 8.4 shows how primitive interior G-algebras can be parametrized using $\mathrm{Out}(C)$-orbits of $k_\sharp\widehat{\overline{N}}^C$-modules. We need to show that this action can be described directly from B and $N_G(P,B)$, because our final goal is to parametrize primitive interior algebras with three invariants which are independent of G. We can obviously replace $\mathrm{Out}(C)$ by its isomorphic group $\mathrm{Out}_{\mathrm{skew}}(B)$, and consequently the normal subgroup $\mathrm{Out}(C)_\gamma$ by $\mathrm{Out}(B)$ (cf. Proposition 5.1). Also by Proposition 5.1, we know that the group $N = N_G(P_\gamma)$ only depends on P, B and $N_G(P,B)$. But we still need to have on the one hand a definition of a central extension of $\overline{N}$ in terms of B and on the other hand a direct description of the action of $\mathrm{Out}_{\mathrm{skew}}(B)$ on it. The first problem has been solved by Puig and we use his result to handle the second.

In [P3, Section 6] Puig describes a central extension $\widehat{\overline{N}}^B$ entirely in terms of B and shows that the opposite extension $(\widehat{\overline{N}}^B)^\circ$ is isomorphic to the central extension $\widehat{\overline{N}}^C$. Puig's result holds for an arbitrary pointed p-group on an interior G-algebra and for later use we state the result in this generality.

Let D be an interior G-algebra and let P_γ be a pointed p-group on D, with an associated embedding $\mathcal{F} : D_\gamma \to \mathrm{Res}^G_P(D)$, and write $B = D_\gamma$. From the multiplicity algebra $S(\gamma)$ is constructed a central extension

$$(9.1) \qquad 1 \longrightarrow k^* \longrightarrow \widehat{\overline{N}}^D \longrightarrow \overline{N} \longrightarrow 1\,,$$

where as usual $N = N_G(P_\gamma)$ and $\overline{N} = N/P$.

We now describe the other central extension of $\overline{N}$. Let $N_B(P)$ be the set of all $b \in B^*$ which normalize $P{\cdot}1_B$. Then $(B^P)^*$ is the normal subgroup of $N_B(P)$ which centralizes $P{\cdot}1_B$ and so $N_B(P)/(B^P)^*$ is a group of automorphisms of $P{\cdot}1_B$ (hence it is finite). Since B is primitive, $B^P/J(B^P) \cong k$ where $J(B^P)$ denotes the Jacobson radical of B^P. Therefore $(B^P)^*/(1+J(B^P)) \cong k^*$ and it follows that there is a central extension

$$1 \longrightarrow k^* \longrightarrow N_B(P)/(1+J(B^P)) \longrightarrow N_B(P)/(B^P)^* \longrightarrow 1\,.$$

The normal p-subgroup $P{\cdot}1_B$ of $N_B(P)$ intersects k^* trivially because there are no non-trivial p-th roots of unity in a field of characteristic p. Thus we obtain a central extension

$$(9.2) \qquad 1 \longrightarrow k^* \longrightarrow N_B(P)/P{\cdot}(1+J(B^P)) \longrightarrow N_B(P)/P{\cdot}(B^P)^* \longrightarrow 1\,,$$

and $N_B(P)/P{\cdot}(B^P)^*$ is now a group of outer automorphisms of $P{\cdot}1_B$.

Recall that $E_G(P_\gamma) = N_G(P_\gamma)/PC_G(P) = \overline{N}/\overline{C}_G(P)$ is a group of outer automorphisms of P. There is a natural map

$$q : E_G(P_\gamma) \longrightarrow N_B(P)/P{\cdot}(B^P)^*\,, \qquad \sigma \mapsto \overline{\sigma}$$

described as follows. If an outer automorphism σ of P belongs to $E_G(P_\gamma)$, then it turns out that σ necessarily induces an outer automorphism $\overline{\sigma}$ of $P{\cdot}1_B$ (in other words the kernel of $P \to P{\cdot}1_B$ is necessarily invariant under σ). Moreover it also turns out that $\overline{\sigma}$ necessarily belongs to the subgroup $N_B(P)/P{\cdot}(B^P)^*$ (provided we identify it with a subgroup of $\mathrm{Out}(P{\cdot}1_B)$). Then by definition $\overline{\sigma}$ is the image of σ. To make clear why these properties hold, note that the embedding $\mathcal{F} : B \to \mathrm{Res}^G_P(D)$ induces an isomorphism $f : B \cong iDi$ where $i = f(1_B)$ and $f \in \mathcal{F}$. If σ is represented by $x \in N = N_G(P_\gamma)$, then ${}^x i \in \gamma$, so that there exists $a \in (D^P)^*$ such that ${}^{ax} i = i$. Then it is easy to check that the element $i{\cdot}ax{\cdot}i \in iDi$ corresponds under the isomorphism f^{-1} to an element of B which belongs to $N_B(P)$ and whose image in $N_B(P)/P{\cdot}(B^P)^*$ is the outer automorphism $\overline{\sigma}$.

The restriction (i.e. the pull-back) of the central extension 9.2 along the group homomorphism $q : E_G(P_\gamma) \to N_B(P)/P{\cdot}(B^P)^*$ defines a central extension

$$(9.3) \qquad 1 \longrightarrow k^* \longrightarrow \widehat{E}^B_G(P_\gamma) \longrightarrow E_G(P_\gamma) \longrightarrow 1\,,$$

and by restriction along the projection $\overline{N} \to E_G(P_\gamma)$, this defines in turn a central extension

$$(9.4) \qquad 1 \longrightarrow k^* \longrightarrow \widehat{\overline{N}}^B \longrightarrow \overline{N} \longrightarrow 1\,.$$

Recall that the opposite of a central extension $1 \to k^* \to \widehat{X} \to X \to 1$ is the extension $1 \to k^* \stackrel{-1}{\to} \widehat{X} \to X \to 1$, with the same map $\widehat{X} \to X$, but where the embedding of the central subgroup k^* is the composite of the automorphism $\lambda \mapsto \lambda^{-1}$ with the given embedding. We write $(\widehat{X})^\circ$ for this opposite extension. Thus a homomorphism of central extensions $\phi : \widehat{X}' \to (\widehat{X})^\circ$ is a group homomorphism $\phi : \widehat{X}' \to \widehat{X}$ which induces on the central subgroup k^* the map $\lambda \mapsto \lambda^{-1}$.

Puig proved that the extensions 9.1 and 9.4 are related by a homomorphism of central extensions $\psi_{\mathcal{F}} : \widehat{\overline{N}}^D \to (\widehat{\overline{N}}^B)^\circ$ induced by the embedding $\mathcal{F} : B \to \mathrm{Res}^G_P(D)$. It is obtained by pull-back from a map

$$\psi'_{\mathcal{F}} : \widehat{\overline{N}}^D \longrightarrow \big(N_B(P)/P{\cdot}(1 + J(B^P))\big)^\circ .$$

In fact we describe a homomorphism

$$\psi''_{\mathcal{F}} : \widehat{N}^D \longrightarrow \big(N_B(P)/(1 + J(B^P))\big)^\circ$$

where $\widehat{N}^D$ is the extension obtained by restriction from $\widehat{\overline{N}}^D$ along the map $N \to \overline{N}$, so that P is identified with a normal subgroup of $\widehat{N}^D$ and $\widehat{N}^D/P = \widehat{\overline{N}}^D$. Choosing $f \in \mathcal{F}$ (where $\mathcal{F} : B \to \mathrm{Res}^G_P(D)$ is the given embedding), there is an isomorphism of interior P-algebras $f : B \cong iDi$ where $i = f(1_B)$. An element of $\widehat{N}^D$ is a pair $(\overline{a}, x) \in S(\gamma)^* \times N$ where x and $\overline{a}$ have the same action on $S(\gamma)$ (i.e. map to the same element of $S(\gamma)^*/k^*$). Let $a \in (D^P)^*$ be any element such that $\pi_\gamma(a) = \overline{a}$, where $\pi_\gamma : D^P \to S(\gamma)$ is the canonical map (which is also surjective on invertible elements). Then $ia^{-1}{\cdot}x{\cdot}i \in iDi$ corresponds under the isomorphism f^{-1} to an element $b \in B$ which in fact belongs to $N_B(P)$. Then by definition $\psi''_{\mathcal{F}}(\overline{a}, x) = \overline{b}$, where $\overline{b}$ denotes the class of b in $N_B(P)/(1 + J(B^P))$.

Puig proved that the map $\psi''_{\mathcal{F}}$ is well-defined, independent of the choice of $f \in \mathcal{F}$, and is a group homomorphism (see Proposition 6.10 of [P3]). Moreover $\psi''_{\mathcal{F}}$ maps P to $P{\cdot}1_B$ by the obvious map and so induces the required map

$$\psi'_{\mathcal{F}} : \widehat{\overline{N}}^D \longrightarrow \big(N_B(P)/P{\cdot}(1 + J(B^P))\big)^\circ .$$

We need the opposite central extension because $\psi'_{\mathcal{F}}$ maps a scalar $\lambda \in k^*$ to its inverse $\lambda^{-1} \in N_B(P)/P{\cdot}(1+J(B^P))$. Passing to the quotient by k^*, we obtain the natural map $q : \overline{N} \to N_B(P)/P{\cdot}(B^P)^*$ described above. Therefore $\psi'_{\mathcal{F}}$ induces $\psi_{\mathcal{F}} : \widehat{\overline{N}}^D \to (\widehat{\overline{N}}^B)^\circ$, which is necessarily an isomorphism since it induces the identity on $\overline{N}$.

Both central extensions are obtained by restriction from central extensions of $E_G(P_\gamma)$. It is an easy exercise to check that the isomorphism $\psi_{\mathcal{F}}$ induces also an isomorphism between the opposite of the central extension 9.3 and the extension 2.2.

We now summarize this discussion and state Puig's result.

Proposition 9.5 Puig [P3, 6.10-6.12]**).** *The map $\psi_{\mathcal{F}} : \widehat{\overline{N}}^D \to (\widehat{\overline{N}}^B)^\circ$ described above is a k^*-isomorphism of central extensions inducing the identity on $\overline{N}$.*

We now return to outer automorphisms and to our usual setting described in the notation 3.2. The canonical embedding $\mathcal{D}_P^G : B \to \mathrm{Res}_P^G(C)$ induces an isomorphism $\psi_{\mathcal{D}_P^G} : \widehat{\overline{N}}^C \to (\widehat{\overline{N}}^B)^\circ$ and we want to prove that it commutes with some natural outer action of the group $\mathrm{Out}_{\mathrm{skew}}(B)$. We know that $\mathrm{Out}(C)$ has an outer action on $\widehat{\overline{N}}^C$ and we first replace $\mathrm{Out}(C)$ by its isomorphic group $\mathrm{Out}_{\mathrm{skew}}(B)$. The isomorphism is given by induction (Proposition 5.1) and we define

$$\theta : \mathrm{Out}_{\mathrm{skew}}(B) \longrightarrow \mathrm{Out}_{k^*}(\widehat{\overline{N}}^C)$$

to be the composite of $\mathrm{Ind}_P^G : \mathrm{Out}_{\mathrm{skew}}(B) \xrightarrow{\sim} \mathrm{Out}(C)$ and $\tau : \mathrm{Out}(C) \to \mathrm{Out}_{k^*}(\widehat{\overline{N}}^C)$, where τ is the map defined in Proposition 7.4. We write $\theta^0 : \mathrm{Out}(B) \to \mathrm{Out}_{k^*}^0(\widehat{\overline{N}}^C)$ for the restriction of θ. In fact we know from Proposition 7.4 that the image of θ is contained in $\mathrm{Out}_{k^*}^{N(P,B)}(\widehat{\overline{N}}^C)$ and that the image of θ^0 is contained in $\widetilde{\mathrm{Hom}}(E_G(P_\gamma), k^*)$. We shall come back to the map θ^0 at the end of this section.

The group $\mathrm{Out}_{\mathrm{skew}}(B)$ also has a natural outer action on the extensions 9.2 and 9.4.

Proposition 9.6. *Let $\mathcal{H} \in \mathrm{Out}_{\mathrm{skew}}(B)$ and let $g \in N_G(P,B)$ be a representative of the image of $\mathcal{H}$ in $N_G(P,B)/N$.*

(a) The outer action of $\mathcal{H}$ on B induces an outer action of $\mathcal{H}$ on the central extension 9.2.

(b) The outer action of $\mathcal{H}$ on the central extension 9.2 together with the conjugation action of g on $\overline{N}$ induce an outer action of $\mathcal{H}$ on $\widehat{\overline{N}}^B$. This procedure defines a group homomorphism $\eta : \mathrm{Out}_{\mathrm{skew}}(B) \to \mathrm{Out}(\widehat{\overline{N}}^B)$.

Proof. Let $h \in \mathcal{H}$ be g-skew.

(a) Since h normalizes $P{\cdot}1_B$ (because $h(u{\cdot}1_B) = {}^g u{\cdot}1_B$ by definition), h leaves invariant $N_B(P)$, $(B^P)^*$ and $1+J(B^P)$, and it is the identity on k^*. Hence h induces an action on the central extension 9.2.

We now modify the choice of $h \in \mathcal{H}$ and $g \in N_G(P,B)$. It suffices to consider a modification by an arbitrary skew inner automorphism of B because by part (d) of Proposition 5.1, any change of the choice of g can be realized by a skew inner automorphism. Thus we have to replace h by $\mathrm{Inn}(b)\,h$ where $\mathrm{Inn}(b)$ is a g'-skew inner automorphism of B for some g' (and $g' \in N$ by Proposition 5.1). Then clearly the action of $\mathrm{Inn}(b)$ on the central extension 9.2 is by the inner automorphism $\mathrm{Inn}(\bar{b})$ of $N_B(P)/P{\cdot}(1+J(B^P))$, where $\bar{b}$ denotes the class of b (since $b \in N_B(P)$ by definition). It follows that we have defined a natural outer action of $\mathrm{Out}_{\mathrm{skew}}(B)$ on the central extension 9.2.

(b) We want to show that the action of h on $N_B(P)/P{\cdot}(1+J(B^P))$ is compatible with the action of g by conjugation on $\overline{N}$, in the sense that they induce the same action on $N_B(P)/P{\cdot}(B^P)^*$. Let σ_g be the automorphism of $P{\cdot}1_B$ defined by $\sigma_g(u{\cdot}1_B) = {}^g u{\cdot}1_B$ (that is, σ_g is the restriction of h to $P{\cdot}1_B$) and let $\overline{\sigma}_g$ be the class of σ_g in the group of outer automorphisms of $P{\cdot}1_B$. Then the action of h on $N_B(P)/P{\cdot}(B^P)^*$ is by conjugation by $\overline{\sigma}_g$. We check that at the level of $N_B(P)/(B^P)^*$ (which is a group of automorphisms of $P{\cdot}1_B$). If $b \in N_B(P)$, then $b{\cdot}u{\cdot}b^{-1} = \phi(u{\cdot}1_B)$ for every $u \in P$, where $\phi \in \mathrm{Aut}(P{\cdot}1_B)$, and therefore $h(b{\cdot}u{\cdot}b^{-1}) = h(b){\cdot}{}^g u{\cdot}h(b)^{-1} = {}^g(\phi(u{\cdot}1_B))$. This means that

$$h(b){\cdot}u{\cdot}h(b)^{-1} = {}^g(\phi({}^{g^{-1}}u{\cdot}1_B)) = \sigma_g\,\phi\,\sigma_g^{-1}(u{\cdot}1_B)$$

as claimed. Now the natural map $\overline{N} \to N_B(P)/P{\cdot}(B^P)^*$ commutes with the action of g (that is, conjugation by g on $\overline{N}$ and conjugation by $\overline{\sigma}_g$ on $N_B(P)/P{\cdot}(B^P)^*$). It follows from all this that the action of h on the central extension 9.2 together with the conjugation action of g on $\overline{N}$ induce an action on the central extension $\widehat{\overline{N}}^B$.

Again if we modify h by $\mathrm{Inn}(b)$ where $\mathrm{Inn}(b)$ is a g'-skew inner automorphism of B, then $g' \in N$ and we let $\overline{g}'$ be the image of g' in $\overline{N}$. Let also $\overline{b}$ be the image of b in $N_B(P)/P{\cdot}(1+J(B^P))$. Then the pair $(\overline{b}, \overline{g}')$ defines an element $z \in \widehat{\overline{N}}^B$ and we have modified the action of h on $\widehat{\overline{N}}^B$ by the inner automorphism $\mathrm{Inn}(z)$. $\square$

We have now defined an outer action of $\mathcal{H} \in \mathrm{Out}_{\mathrm{skew}}(B)$ on the central extension $\widehat{\overline{N}}^B$, such that the induced outer action on $\overline{N}$ is by conjugation by g if $\mathcal{H}$ is g-skew. In particular if $\mathcal{H}$ is an ordinary automorphism (so that we can choose $g = 1$), then $\mathcal{H}$ induces the identity on $\overline{N}$.

We now show that the isomorphism $\psi_{\mathcal{D}_P^G}$ commutes with the action of $\mathrm{Out}_{\mathrm{skew}}(B)$. Note that the central extensions $\widehat{\overline{N}}^B$ and $(\widehat{\overline{N}}^B)^\circ$ have the same automorphisms.

Proposition 9.7. *For every skew outer automorphism $\mathcal{H}$ of B, the isomorphism of central extensions $\psi_{\mathcal{D}_P^G} : \widehat{\overline{N}}^C \xrightarrow{\sim} (\widehat{\overline{N}}^B)^\circ$ commutes with the outer action of $\mathcal{H}$, that is, the exomorphisms $\psi_{\mathcal{D}_P^G}\,\theta(\mathcal{H})$ and $\eta(\mathcal{H})\,\psi_{\mathcal{D}_P^G}$ are equal.*

Proof. Since $\psi_{\mathcal{D}_P^G}$ is constructed from $\psi'_{\mathcal{D}_P^G} : \widehat{\overline{N}}^C \to \big(N_B(P)/P{\cdot}(1+J(B^P))\big)^\circ$ and since the outer action of $\mathrm{Out}_{\mathrm{skew}}(B)$ is also constructed from $N_B(P)/P{\cdot}(1+J(B^P))$, it suffices to show that $\psi'_{\mathcal{D}_P^G}$ commutes with the action of $\mathcal{H} \in \mathrm{Out}_{\mathrm{skew}}(B)$. Let $h \in \mathcal{H}$ be g-skew, where $g \in N_G(P,B)$. Let $(\overline{a}, \overline{x}) \in \widehat{\overline{N}}^C$ where $\overline{x} \in \overline{N}$ and $\overline{a} \in S(\gamma)^*$ (with $\mathrm{Inn}(\overline{a})$ equal to the action of $\overline{x}$ on $S(\gamma)$). Let $a \in (C^P)^*$ lifting $\overline{a}$ (i.e. $\overline{a} = \pi_\gamma(a)$) and let $x \in N$ lifting $\overline{x}$. Write $i = 1 \otimes 1_B \otimes 1$. Then $ia^{-1}{\cdot}x{\cdot}i = 1 \otimes b \otimes 1$ where $b \in N_B(P)$ and $\psi'_{\mathcal{D}_P^G}(\overline{a}, \overline{x}) = \overline{b}$ by definition. Therefore

$$h\,\psi'_{\mathcal{D}_P^G}(\overline{a}, \overline{x}) = h(\overline{b}) = \overline{h(b)}\,.$$

For the action of h on $\widehat{\overline{N}}^C$, recall that $\mathrm{Ind}_P^G(\mathcal{H}) \in \mathrm{Out}(C)$ is represented by the automorphism $\widehat{h}$ defined by $\widehat{h}(x \otimes b \otimes y) = xg^{-1} \otimes h(b) \otimes gy$. Moreover the action on $\widehat{\overline{N}}^C$ is obtained from the action of $\widehat{h}_g = \mathrm{Conj}(g{\cdot}1_C)\,\widehat{h}$ on $S(\gamma)^*$ and the conjugation action of g on $\overline{N}$. Therefore the action of h on $(\overline{a}, \overline{x})$ is the element $(\widehat{h}_g(\overline{a}), \overline{gxg^{-1}})$ and its image under $\psi_{\mathcal{D}_P^G}$ is determined by the following element of $iCi = 1 \otimes B \otimes 1$:

$$\begin{aligned} i\widehat{h}_g(a)^{-1}{\cdot}gxg^{-1}{\cdot}i &= i\widehat{h}_g(a^{-1})\widehat{h}_g(x{\cdot}1_C)i = \widehat{h}_g(ia^{-1}{\cdot}x{\cdot}i) = \widehat{h}_g(1 \otimes b \otimes 1) \\ &= g(g^{-1} \otimes h(b) \otimes g)g^{-1} = 1 \otimes h(b) \otimes 1\,, \end{aligned}$$

using $\widehat{h}_g(i) = i$ and $\widehat{h}_g(x{\cdot}1_C) = gxg^{-1}{\cdot}1_C$ (because $\widehat{h}(x{\cdot}1_C) = x{\cdot}1_C$ since it is an automorphism of interior G-algebras). Therefore $\psi'_{\mathcal{D}_P^G}\,h(\overline{a}, \overline{x}) = \overline{h(b)}$, proving the result. $\square$

We end this section with a few more facts about the map $\theta^0 : \mathrm{Out}(B) \to \mathrm{Out}^0_{k^*}(\widehat{\overline{N}}^C)$ defined earlier in this section, as well as the map $\eta^0 : \mathrm{Out}(B) \to \mathrm{Out}^0_{k^*}(\widehat{\overline{N}}^B)$ defined similarly from η. We have already noticed that $\mathrm{Out}(B)$ induces in fact genuine automorphisms of $\widehat{\overline{N}}^C$ and from the argument of Section 2, we know that $\mathrm{Aut}^0_{k^*}(\widehat{\overline{N}}^C)$ is isomorphic to the group of k^*-valued characters $\mathrm{Hom}(\overline{N}, k^*)$. Moreover by

Proposition 7.5 θ^0 factorizes through a homomorphism $\theta' : \mathrm{Out}(B) \to \mathrm{Hom}(E_G(P_\gamma), k^*)$ and similarly η^0 factorizes through a homomorphism $\eta' : \mathrm{Out}(B) \to \mathrm{Hom}(E_G(P_\gamma), k^*)$. We give here a direct description of these maps and we prove that they are "opposite". The description of η' is due to Puig [P3, 6.9 and 14.8].

Since the action of $\mathrm{Out}(B)$ on $\widehat{\overline{N}}^B$ is obtained by restriction from the action on the central extension 9.2, we consider instead the group $\mathrm{Hom}(N_B(P)/P{\cdot}(B^P)^*, k^*)$. In fact the natural map $q : E_G(P_\gamma) \to N_B(P)/P{\cdot}(B^P)^*$ induces a group homomorphism

$$q^* : \mathrm{Hom}(N_B(P)/P{\cdot}(B^P)^*, k^*) \longrightarrow \mathrm{Hom}(E_G(P_\gamma), k^*)$$

and we describe a map $\eta_B : \mathrm{Out}(B) \to \mathrm{Hom}(N_B(P)/P{\cdot}(B^P)^*, k^*)$ such that η' is the composite

$$\mathrm{Out}(B) \xrightarrow{\eta_B} \mathrm{Hom}(N_B(P)/P{\cdot}(B^P)^*, k^*) \xrightarrow{q^*} \mathrm{Hom}(E_G(P_\gamma), k^*)\,.$$

Puig's description of η_B is the following. Let $\mathcal{H} \in \mathrm{Out}(B)$ and choose $h \in \mathcal{H}$. Since the action of $\mathcal{H}$ on the extension 9.2 induces the identity on the quotient $N_B(P)/P{\cdot}(B^P)^*$, we have $h(\overline{b})\overline{b}^{-1} \in k^*$ for every $\overline{b} \in N_B(P)/P{\cdot}(1 + J(B^P))$ and this defines a scalar $\eta_B(\mathcal{H})(\overline{b}) \in k^*$. In other words η_B is characterized by the property

$$h(b) \equiv \eta_B(\mathcal{H})(\overline{b}){\cdot}b \mod \big(P{\cdot}(1 + J(B^P))\big) \quad \text{for every } b \in N_B(P) \text{ and } h \in \mathcal{H}\,.$$

We turn now to the description of $\theta' : \mathrm{Out}(B) \to \mathrm{Hom}(E_G(P_\gamma), k^*)$. We use the action of $\mathrm{Out}(B)$ on $\widehat{\overline{N}}^C$, via the isomorphism $\mathrm{Out}(B) \cong \mathrm{Out}(C)_\gamma$ given by induction and the action of $\mathrm{Out}(C)_\gamma$, which we now recall. If $\mathcal{H} \in \mathrm{Out}(B)$ and $h \in \mathcal{H}$, then $\mathrm{Ind}_P^G(h)$ fixes γ and hence induces an action on $S(\gamma)$. Moreover since $\mathrm{Ind}_P^G(h)$ is an automorphism of interior G-algebras, the induced automorphism of $S(\gamma)$ is an automorphism of $N_G(P_\gamma)$-algebras and also an automorphism of interior $PC_G(P)$-algebras. Since $S(\gamma)$ is simple, this automorphism is of the form $\mathrm{Inn}(s)$ and we have $\overline{s} \in (S(\gamma)^*/k^*)^{N_G(P_\gamma)}$ and $s \in (S(\gamma)^*)^{PC_G(P)}$. This defines an element $\overline{s} \in \big((S(\gamma)^*)^{PC_G(P)}/k^*\big)^{E_G(P_\gamma)}$. Taking fixed points under $E_G(P_\gamma)$ in the short exact sequence

$$1 \longrightarrow k^* \longrightarrow (S(\gamma)^*)^{PC_G(P)} \longrightarrow (S(\gamma)^*)^{PC_G(P)}/k^* \longrightarrow 1\,,$$

one obtains a connecting homomorphism in group cohomology

$$(S(\gamma)^*)^{N_G(P_\gamma)} \longrightarrow \big((S(\gamma)^*)^{PC_G(P)}/k^*\big)^{E_G(P_\gamma)} \longrightarrow H^1(E_G(P_\gamma), k^*) = \mathrm{Hom}(E_G(P_\gamma), k^*)\,.$$

We have just seen that the automorphism $\mathrm{Ind}_P^G(h)$ defines an element of the middle group and it is easy to check that its image in $\mathrm{Hom}(E_G(P_\gamma), k^*)$ is independent of the choice of $h \in \mathcal{H}$. This defines a map $\theta'' : \mathrm{Out}(B) \to \mathrm{Hom}(E_G(P_\gamma), k^*)$ which turns out to be the "opposite" of the map θ', that is, $\theta''(\mathcal{H}) = \theta'(\mathcal{H})^{-1}$ for all $\mathcal{H} \in \mathrm{Out}(B)$. The proof of this fact is left to the reader.

The isomorphism $\psi_{\mathcal{D}_P^G} : \widehat{\overline{N}}^C \cong (\widehat{\overline{N}}^B)^\circ$ of Proposition 9.7 commutes with the action of $\mathrm{Out}_{\mathrm{skew}}(B)$, hence in particular with the action of $\mathrm{Out}(B)$, which in both cases maps to $\mathrm{Hom}(E_G(P_\gamma), k^*) \subseteq \mathrm{Hom}(\overline{N}, k^*)$. $\blacksquare$ Now $\psi_{\mathcal{D}_P^G}$ induces an isomorphism ψ_* from $\mathrm{Hom}(\overline{N}, k^*) \cong \mathrm{Aut}^0_{k^*}(\widehat{\overline{N}}^C)$ to $\mathrm{Hom}(\overline{N}, k^*) \cong \mathrm{Aut}^0_{k^*}(\widehat{\overline{N}}^B)$. Since $\psi_{\mathcal{D}_P^G}$ maps a scalar $\lambda \in k^*$ to its inverse, it is easy to see that ψ_* consists simply in taking inverses. But as $\psi_{\mathcal{D}_P^G}$ commutes with the action of $\mathcal{H} \in \mathrm{Out}(B)$, clearly ψ_* maps $\theta'(\mathcal{H})$ to $\eta'(\mathcal{H})$, from which it follows that

$$\eta'(\mathcal{H}) = \theta'(\mathcal{H})^{-1} = \theta''(\mathcal{H})\,.$$

Therefore we obtain the following result.

Proposition 9.8. *The maps η' and θ'' described above are equal.*

Note that we have only worked with $C = \mathrm{Ind}_P^G(B)$, but the proposition holds for an arbitrary interior G-algebra C, a local pointed group P_γ on C with associated embedding $B \to \mathrm{Res}_P^G(C)$.

10. Description of the third invariant

We finally come to the exact definition of the orbit of multiplicity modules needed for the parametrization.

Let A be a primitive interior G-algebra with defect group P and source algebra B. We have associated with A its various embeddings in $C = \mathrm{Ind}_P^G(B)$ and proved that the corresponding points of C^G form a single orbit under $\mathrm{Out}(C)$ (Corollary 4.4). Then we have used the Puig correspondence to obtain an $\mathrm{Out}(C)$-orbit of points of $S(\gamma)^{\overline{N}}$ (Corollary 8.2) and we have interpreted this as an $\mathrm{Out}(C)$-orbit of indecomposable projective $k_\sharp\widehat{\overline{N}}^C$-modules (Corollary 8.4). We replace $\mathrm{Out}(C)$ by its isomorphic group $\mathrm{Out}_{\mathrm{skew}}(B)$ (the isomorphism being given by induction). By the previous section the isomorphism $\psi_{\mathcal{D}_P^G} : \widehat{\overline{N}}^C \xrightarrow{\sim} (\widehat{\overline{N}}^B)^\circ$ commutes with the outer action of $\mathrm{Out}_{\mathrm{skew}}(B)$, and therefore we obtain an $\mathrm{Out}_{\mathrm{skew}}(B)$-orbit of indecomposable projective $k_\sharp(\widehat{\overline{N}}^B)^\circ$-modules, which we temporarily call the $\mathrm{Out}_{\mathrm{skew}}(B)$-orbit of multiplicity modules of A *determined by* C. This is a precise definition of the third invariant we are looking for, but we now want to describe this invariant directly from A, without going through this process involving C.

To say that A has defect group P and source algebra B means that there is an embedding $\mathcal{G} : B \to \mathrm{Res}_P^G(A)$ such that $\mathcal{G}(1_B) = \gamma$ is a source point of A. But $\mathcal{G}$ is not unique for two reasons. First one can compose $\mathcal{G}$ with an outer automorphism of B. Second the point γ is not uniquely determined by B since it can be replaced by any isomorphic point ${}^g\gamma$ where $g \in N_G(P, B)$.

For a given choice of $\mathcal{G}$, we have a central extension $\widehat{\overline{N}}^A$ associated with the multiplicity algebra $T(\gamma) \cong \mathrm{End}_k(W(\gamma))$ of γ, and the $k_\sharp\widehat{\overline{N}}^A$-module $W(\gamma)$ is a defect multiplicity module of A, hence indecomposable projective. Here $N = N_G(P_\gamma)$ as usual. (Note that the notation $\widehat{\overline{N}}^A$ does not say that the central extension in fact depends on γ, which is not uniquely determined by A, but this will not create any problem for the following arguments). By Proposition 9.5, there exists an isomorphism $\psi_{\mathcal{G}} : \widehat{\overline{N}}^A \xrightarrow{\sim} (\widehat{\overline{N}}^B)^\circ$ induced by the embedding $\mathcal{G}$, and this allows to view $W(\gamma)$ as a module over $k_\sharp(\widehat{\overline{N}}^B)^\circ$. We write $W_{\mathcal{G}}$ for this indecomposable projective $k_\sharp(\widehat{\overline{N}}^B)^\circ$-module, and we call it again a defect multiplicity module of A (although the canonical structure of module of $W(\gamma)$ is the $k_\sharp\widehat{\overline{N}}^A$-module structure). We have already noticed at the end of Section 6 that it is crucial to distinguish between $\widehat{\overline{N}}^A$ and $\widehat{\overline{N}}^C$. The same remarks hold here for $\widehat{\overline{N}}^A$ and $(\widehat{\overline{N}}^B)^\circ$, since we simply use the uniquely defined isomorphism $\psi_{\mathcal{D}_P^G} : \widehat{\overline{N}}^C \xrightarrow{\sim} (\widehat{\overline{N}}^B)^\circ$ to pass from $\widehat{\overline{N}}^C$ and $(\widehat{\overline{N}}^B)^\circ$. In particular the whole discussion in Example 6.3 applies with $(\widehat{\overline{N}}^B)^\circ$ instead of $\widehat{\overline{N}}^C$.

When $\mathcal{G}$ varies in the set $\mathcal{E}(B, \mathrm{Res}_P^G(A))$ of all embeddings of B into $\mathrm{Res}_P^G(A)$, the module $W_{\mathcal{G}}$ varies in the set of isomorphism classes of indecomposable projective $k_\sharp(\widehat{\overline{N}}^B)^\circ$-modules. We call the set

$$\{\, W_{\mathcal{G}} \mid \mathcal{G} \in \mathcal{E}(B, \mathrm{Res}_P^G(A)) \,\}$$

the *set of defect multiplicity modules of* A. As usual we do not distinguish between a module and its isomorphism class. Note that this set depends on the pair (P, B), which is unique up to G-conjugation.

Proposition 10.1. *Let A be a primitive interior G-algebra with defect group P and source algebra B. Then the set of defect multiplicity modules of A is equal to the $\mathrm{Out}_{\mathrm{skew}}(B)$-orbit of multiplicity modules of A determined by C.*

The proof is an immediate consequence of the following two results of Puig. The first result allows to identify the $k_\sharp(\widehat{\overline{N}}^B)^\circ$-module structures, and the second one implies that both sets of multiplicity modules coincide. We let $\mathcal{E}(A,C)$ be the set of all embeddings of A into C.

Lemma 10.2. *Let $C = \mathrm{Ind}_P^G(B)$ and let $\mathcal{D}_P^G : B \to \mathrm{Res}_P^G(C)$ be the canonical embedding. Let $\mathcal{G} \in \mathcal{E}(B, \mathrm{Res}_P^G(A))$ and $\mathcal{F} \in \mathcal{E}(A,C)$ be such that $\mathrm{Res}_P^G(\mathcal{F})\,\mathcal{G} = \mathcal{D}_P^G$. Let $\psi_{\mathcal{G}} : \widehat{\overline{N}}^A \xrightarrow{\sim} (\widehat{\overline{N}}^B)^\circ$ be the isomorphism induced by $\mathcal{G}$, let $\psi_{\mathcal{D}_P^G} : \widehat{\overline{N}}^C \xrightarrow{\sim} (\widehat{\overline{N}}^B)^\circ$ be the isomorphism induced by $\mathcal{D}_P^G$, and let $\overline{\mathcal{F}}^* : \widehat{\overline{N}}^C \xrightarrow{\sim} \widehat{\overline{N}}^A$ be the isomorphism induced by $\mathcal{F}$ (see Proposition 2.3). Then $\psi_{\mathcal{G}}\,\overline{\mathcal{F}}^* = \psi_{\mathcal{D}_P^G}$.*

Proof. If $\gamma = \mathcal{G}(1_B)$, then $\mathcal{G}$ is an embedding associated with the pointed group P_γ, and $\mathcal{D}_P^G$ is an embedding associated with the image of P_γ under $\mathcal{F}$. Proposition 6.21 in [P3] says that the isomorphisms ψ behave well with respect to embeddings. Applying this to the case of the embedding $\mathcal{F}$, one gets the statement of the lemma. □

Lemma 10.3. *Let $C = \mathrm{Ind}_P^G(B)$ and let $\mathcal{D}_P^G : B \to \mathrm{Res}_P^G(C)$ be the canonical embedding. The set $\mathcal{E}(B, \mathrm{Res}_P^G(A))$ is in bijection with the set $\mathcal{E}(A,C)$, as follows: $\mathcal{G} \in \mathcal{E}(B, \mathrm{Res}_P^G(A))$ corresponds to $\mathcal{F} \in \mathcal{E}(A,C)$ if and only if $\mathrm{Res}_P^G(\mathcal{F})\,\mathcal{G} = \mathcal{D}_P^G$.*

Proof. Given $\mathcal{G} : B \to \mathrm{Res}_P^G(A)$, the existence of a unique $\mathcal{F} : A \to C$ such that $\mathrm{Res}_P^G(\mathcal{F})\,\mathcal{G} = \mathcal{D}_P^G$ follows from Proposition 1.5. Given $\mathcal{F} : A \to C$, we choose a defect P_δ of A and also write P_δ for its image under $\mathcal{F}$. Then we have $G_\alpha \geq P_\delta$ where $\alpha = \mathcal{F}(1_A)$. By Proposition 1.6 P_δ is local and so by Proposition 1.7 P_δ is conjugate to P_γ, where γ denotes the point of C^P containing $1 \otimes 1_B \otimes 1$. Therefore we also have $G_\alpha \geq P_\gamma$. Since $\mathcal{F}$ is an embedding associated with the pointed group G_α and $\mathcal{D}_P^G$ is an embedding associated with the pointed group P_γ, the existence of a unique $\mathcal{G} : B \to \mathrm{Res}_P^G(A)$ such that $\mathrm{Res}_P^G(\mathcal{F})\,\mathcal{G} = \mathcal{D}_P^G$ follows from Proposition 1.4. □

Remark 10.4. We emphasize that each module in the set of defect multiplicity modules of A has a structure obtained by using a specific isomorphism $\psi_{\mathcal{G}}$ between two central extensions. This isomorphism is not unique since it can be composed with an arbitrary k^*-automorphism of $\widehat{\overline{N}}^B$ inducing the identity on $\overline{N}$, that is, an element of the group $\mathrm{Aut}^0_{k^*}(\widehat{\overline{N}}^B) \cong \mathrm{Hom}(\overline{N}, k^*)$. The use of another isomorphism would produce another set of modules (and hence another parametrization). However the isomorphism $\psi_{\mathcal{G}}$ constructed by Puig is very “natural”, so that the procedure which defines the set of defect multiplicity modules is the most “natural” one. It seems to be an interesting open problem to find some “natural” extra properties of $\psi_{\mathcal{G}}$ which would make it unique.

11. The parametrization of interior algebras and modules

We have now paved the way for the main result giving the parametrization of primitive interior G-algebras, for a fixed finite group G. Let $\mathcal{A}(G)$ be the set of isomorphism classes of primitive interior G-algebras. Let $\Omega(G)$ be the set of triples $(P,B,\mathcal{W})$ where P is a p-subgroup of G, B is an interior P-algebra which is a source algebra, and $\mathcal{W}$ is an $\mathrm{Out}_{\mathrm{skew}}(B)$-orbit of isomorphism classes of projective indecomposable $k_\sharp(\widehat{\overline{N}}{}^{B}_{G}(P_\gamma))^\circ$-modules. By Proposition 5.1, the group $N_G(P_\gamma)$ only depends on P, B and $N_G(P,B)$. The group G acts by conjugation on $\Omega(G)$ and we are interested in the set of orbits $\Omega(G)/G$. The group $N_G(P,B)$ is the stabilizer of a triple $(P,B,\mathcal{W})$.

We define a map

$$\Delta : \mathcal{A}(G) \longrightarrow \Omega(G)/G$$

by sending a primitive interior G-algebra A to the G-orbit of the triple $(P,B,\mathcal{W})$, where P is a defect group of A, B is a source algebra of A, and $\mathcal{W}$ is the set of defect multiplicity modules of A (as defined in Section 10). In the other direction we define a map

$$\Gamma : \Omega(G)/G \longrightarrow \mathcal{A}(G)$$

by sending the G-orbit of a triple $(P,B,\mathcal{W})$ to the primitive interior G-algebra C_α, where $C = \mathrm{Ind}_P^G(B)$ and $\alpha \in \mathcal{P}(C^G)$ is defined as follows. Let γ be the point of C^P containing $1 \otimes 1_B \otimes 1$, let $S(\gamma) = \mathrm{End}_k(V(\gamma))$ be its multiplicity algebra, let $N = N_G(P_\gamma)$, let $\mathcal{D}_P^G : B \to \mathrm{Res}_P^G(C)$ be the canonical embedding, and let $\psi_{\mathcal{D}_P^G} : \widehat{\overline{N}}{}^{C} \xrightarrow{\sim} (\widehat{\overline{N}}{}^{B})^\circ$ be the isomorphism of central extensions induced by $\mathcal{D}_P^G$. Now choose a module W in the set $\mathcal{W}$, view it as a $\widehat{\overline{N}}{}^{C}$-module by means of the isomorphism $\psi_{\mathcal{D}_P^G}$. This defines an indecomposable direct summand of the multiplicity module $V(\gamma)$ (because $V(\gamma)$ is free of rank one by Proposition 6.1), hence a point $\overline{\alpha}$ of $S(\gamma)^{\overline{N}}$. Then the Puig correspondent of the point $\overline{\alpha}$ is the point α of C^G. Since we started with an $\mathrm{Out}_{\mathrm{skew}}(B)$-orbit of multiplicity modules and since we have proved that each step commutes with the action of $\mathrm{Out}_{\mathrm{skew}}(B) \cong \mathrm{Out}(C)$, only the $\mathrm{Out}(C)$-orbit of such points α is well-defined, but all the localizations C_α are isomorphic. On the other hand if we choose the triple ${}^g(P,B,\mathcal{W})$ instead of $(P,B,\mathcal{W})$, for some $g \in G$, then the whole situation has to be conjugated by g; but since $\mathrm{Ind}_{{}^gP}^G({}^gB)$ can clearly be identified with $\mathrm{Ind}_P^G(B)$, we end up with the g-conjugate of the point α, which is equal to α since C is an interior G-algebra. Thus the map Γ is well-defined.

We now apply the analysis of the previous sections to each subset of the disjoint union

$$\mathcal{A}(G) = \bigcup_{\substack{(P,B) \\ \text{up to } G-\text{conjugacy}}} \mathcal{A}(G,P,B)\,.$$

Using in particular Corollary 8.4 and Proposition 10.1, we obtain our main result.

Theorem 11.1. *The maps* Δ *and* Γ *are inverse bijections.*

We end this section with a discussion of the parametrization of indecomposable $\mathcal{O}G$-lattices. Recall that an $\mathcal{O}G$-lattice M is a finitely generated $\mathcal{O}G$-module which is free as an $\mathcal{O}$-module. The algebra $A = \mathrm{End}_{\mathcal{O}}(M)$ is an interior G-algebra which is $\mathcal{O}$-*simple*, that is, A is isomorphic to a matrix algebra over $\mathcal{O}$. Conversely any interior G-algebra which is $\mathcal{O}$-simple has the form $A = \mathrm{End}_{\mathcal{O}}(M)$ for some $\mathcal{O}G$-lattice M. Moreover M and M' are isomorphic $\mathcal{O}G$-lattices if and only if A and $A' = \mathrm{End}_{\mathcal{O}}(M')$ are isomorphic interior G-algebras. Since $A^G = \mathrm{End}_{\mathcal{O}G}(M)$, the lattice M is indecomposable if and only if A is primitive, and we assume this in the sequel. The defect group of A is known as a *vertex* of M. If P_γ is a defect of A and if $i \in \gamma$, then the source algebra $B = iAi$ of A is isomorphic to $\mathrm{End}_{\mathcal{O}}(iM)$ and the indecomposable $\mathcal{O}P$-module $Q = iM$ is known as a *source* of M. Thus in this case the first two invariants are just a vertex P and a source Q of M. Given $B = \mathrm{End}_{\mathcal{O}}(Q)$, the induced algebra $C = \mathrm{Ind}_P^G(B)$ is by construction isomorphic to $\mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_P^G(Q))$, where $\mathrm{Ind}_P^G(Q)$ denotes the induced module. Thus we do not quit the category of modules.

By the Skolem-Noether theorem, every automorphism of the $\mathcal{O}$-simple algebra A is an inner automorphism $\mathrm{Inn}(a)$, and $\mathrm{Inn}(a)$ is an automorphism of G-algebras if and only if $a \in A^G$. Thus every automorphism of the G-algebra A is inner and $\mathrm{Out}(A) = 1$. This also holds for B and C, and it follows that all $\mathrm{Out}(C)$-orbits (or equivalently all $\mathrm{Out}_{\mathrm{skew}}(B)$-orbits) are trivial. Therefore Proposition 4.1 asserts that two distinct points of C^G with defect P_γ can never be isomorphic. However this can be proved in the following much more elementary way.

Lemma 11.2. *Let* D *be an interior* G*-algebra which is* $\mathcal{O}$*-simple. If two points* α *and* α' *of* D^G *are isomorphic, then* $\alpha = \alpha'$.

Proof. By assumption $D = \mathrm{End}_{\mathcal{O}}(X)$ for some $\mathcal{O}G$-lattice X. The points α and α' correspond to isomorphism classes of indecomposable $\mathcal{O}G$-direct summands Y and Y' of X. We can choose $Y = iX$ where $i \in \alpha$ and $Y' = i'X$ where $i' \in \alpha'$. Since α and α' are isomorphic, the localizations $D_\alpha \cong iDi \cong \mathrm{End}_{\mathcal{O}}(iX)$ and $D_{\alpha'} \cong i'Di' \cong \mathrm{End}_{\mathcal{O}}(i'X)$ are isomorphic, and therefore the $\mathcal{O}G$-modules iX and $i'X$ are isomorphic. By the Krull-Schmidt theorem, the complementary modules $(1-i)X$ and $(1-i')X$ are also isomorphic. Thus there is an automorphism $h \in \mathrm{End}_{\mathcal{O}G}(X) = D^G$ such that $h(iX) = i'X$ and $h((1-i)X) = (1-i')X$. Then the idempotent hih^{-1} has the same image and the same kernel as i'. Therefore $hih^{-1} = i'$ and this proves that $\alpha = \alpha'$. □

The fact that $\mathrm{Out}(C) = 1$ also implies that its quotient group $N_G(P,B)/N_G(P_\gamma)$ is trivial. Thus in the case of lattices we have $N_G(P,B) = N_G(P_\gamma)$. This group is known as the *inertial subgroup* of the source Q and we write it $N_G(P,Q)$. Since $N_G(P,B) = N_G(P_\gamma)$ and since $\mathrm{Out}(B) = 1$, there is a unique embedding $B \to A$, and therefore the set of defect multiplicity modules of M (that is, of A) is a singleton. We write $\widehat{\overline{N}}{}_G^Q(P,Q)$ instead of $\widehat{\overline{N}}{}_G^B(P,Q)$.

We can now describe the parametrization of $\mathcal{O}G$-lattices by restricting the parametrization of interior algebras to $\mathcal{O}$-simple algebras. Since the restrictions of the maps Δ and Γ can be defined directly without considering $\mathrm{Out}_{\mathrm{skew}}(B)$-orbits, the parametrization of $\mathcal{O}G$-lattices does not depend on the results of this paper, but only on the Puig correspondence and Puig's Proposition 6.1. Let $\mathcal{M}(G)$ be the set of isomorphism classes of indecomposable $\mathcal{O}G$-lattices. Let $\Pi(G)$ be the set of triples (P,Q,W) where P is a p-subgroup of G, Q is an indecomposable $\mathcal{O}P$-module which is its own source (up to isomorphism), and W is an indecomposable projective $k_\sharp(\widehat{\overline{N}}^Q_G(P,Q))^\circ$-module (up to isomorphism). The group G acts by conjugation on $\Pi(G)$ and the group $N_G(P,Q)$ is the stabilizer of a triple (P,Q,W).

If an $\mathcal{O}G$-lattice $M\in\mathcal{M}(G)$ is mapped to the triple (P,Q,W) (up to conjugation), where P is a vertex of M, Q is a source of M, and W is a defect multiplicity module of M, this defines a map

$$\Delta_{\mathrm{mod}}:\mathcal{M}(G)\longrightarrow \Pi(G)/G\,,\qquad M\mapsto (P,Q,W)\,,$$

which is the restriction of the map Δ defined for arbitrary primitive interior G-algebras (up to suitable identifications of $\mathcal{O}G$-lattices with their corresponding interior algebras). Thus we have the following result.

Corollary 11.3. *The map $\Delta_{\mathrm{mod}}:\mathcal{M}(G)\to\Pi(G)/G$ is a bijection.*

Example 11.4. In the special case of trivial source $\mathcal{O}G$-lattices, we have $Q=\mathcal{O}$, the trivial module. Then $N_G(P,Q)=N_G(P)$ and the twisted group algebra $k_\sharp\widehat{\overline{N}}^Q_G(P)$ turns out to be canonically isomorphic to the ordinary group algebra $k\overline{N}_G(P)$. Indeed since the source algebra is the trivial algebra $\mathcal{O}$, the central extension 9.2 is the trivial sequence $1\to k^*\to k^*\to 1\to 1$, and so the restriction of this sequence along the map $\overline{N}_G(P)\to 1$ splits canonically. Therefore trivial source lattices are parametrized by pairs (P,W) up to conjugation, where P is a p-subgroup of G and W is an indecomposable projective $k\overline{N}_G(P)$-module (up to isomorphism). Thus we recover the well-known parametrization of trivial source modules. Since a trivial source $\mathcal{O}G$-lattice M has the same two invariants as the trivial source kG-module $k\otimes_{\mathcal{O}}M$, one deduces also the well-known fact that trivial source kG-modules lift to $\mathcal{O}$.

12. The Green correspondence

An important consequence of the Puig correspondence is another bijection between pointed groups, which is a first form of the *Green correspondence*. We provide a proof because the result is not explicitly stated in Puig's work. Then we shall discuss the Green correspondence for primitive interior algebras.

Proposition 12.1 (Green correspondence for pointed groups)**.** *Let P_γ be a local pointed group on a G-algebra A and let H be a subgroup of G containing $N_G(P_\gamma)$. There is a bijection*

$$\{\,\alpha\in\mathcal{P}(A^G)\mid P_\gamma \text{ is a defect of } G_\alpha\,\}\ \stackrel{\sim}{\longrightarrow}\ \{\,\beta\in\mathcal{P}(A^H)\mid P_\gamma \text{ is a defect of } H_\beta\,\}\,,$$

which is characterized by the following property: if α corresponds to β under this bijection, then β is the unique point of A^H such that $G_\alpha\ge H_\beta\ge P_\gamma$.

Proof. Let $S(\gamma)$ be the multiplicity algebra of γ. Since $H\ge N_G(P_\gamma)$ by assumption, we have $N_H(P_\gamma)=N_G(P_\gamma)$ and we set $\overline{N}=\overline{N}_H(P_\gamma)=\overline{N}_G(P_\gamma)$. Instead of working with points, it is here more convenient to work with the corresponding maximal ideals. Recall that a point α of A^G corresponds to a maximal ideal $\mathfrak{m}_\alpha\in\mathrm{Max}(A^G)$. Consider the following sets:

$$\begin{aligned}X&=\{\,\mathfrak{m}_\alpha\in\mathrm{Max}(A^G)\mid P_\gamma \text{ is a defect of } G_\alpha\,\}\,,\\ Y&=\{\,\mathfrak{m}_\beta\in\mathrm{Max}(A^H)\mid P_\gamma \text{ is a defect of } H_\beta\,\}\,,\\ Z&=\{\,\mathfrak{m}_\delta\in\mathrm{Max}(S(\gamma)^{\overline{N}})\mid \overline{N}_\delta \text{ is projective }\,\}\,.\end{aligned}$$

By the Puig correspondence (Proposition 1.8), X is in bijection with Z via $(\pi_\gamma\, r_P^G)^{-1}$ and similarly Y is in bijection with Z via $(\pi_\gamma\, r_P^H)^{-1}$. Thus it is clear that X is in bijection with Y via $(r_H^G)^{-1}$. If $\mathfrak{m}_\alpha \in X$ corresponds to $\mathfrak{m}_\beta \in Y$, we have $(r_H^G)^{-1}(\mathfrak{m}_\beta) = \mathfrak{m}_\alpha$ and in particular $G_\alpha \geq H_\beta$.

Let $\beta' \in \mathcal{P}(A^H)$ such that $G_\alpha \geq H_{\beta'}$ and $\beta' \neq \beta$. Since we have the two relations $G_\alpha \geq H_\beta$ and $G_\alpha \geq H_{\beta'}$, then for $i \in \alpha$ there is an orthogonal decomposition $r_H^G(i) = j + j' + e$ where $j \in \beta$, $j' \in \beta'$ and e is some idempotent in A^H. By the construction of the bijection, we have $\pi_\gamma\, r_P^G(\alpha) = \delta = \pi_\gamma\, r_P^H(\beta)$, where δ is the Puig correspondent of both α and β, and therefore $\pi_\gamma\, r_P^G(i)$ and $\pi_\gamma\, r_P^H(j)$ are primitive idempotents. Now the orthogonal decomposition

$$\pi_\gamma\, r_P^G(i) = \pi_\gamma\, r_P^H(j) + \pi_\gamma\, r_P^H(j') + \pi_\gamma\, r_P^H(e)$$

forces to have $\pi_\gamma\, r_P^H(j') = 0 = \pi_\gamma\, r_P^H(e)$ and the first of these equalities means that $H_{\beta'} \not\geq P_\gamma$. This proves that β is the unique point of A^H such that $G_\alpha \geq H_\beta \geq P_\gamma$. □

Remark 12.2. If α corresponds to β under the Green correspondence, then the following properties also hold:
(a) $\mathfrak{m}_\alpha = (r_H^G)^{-1}(\mathfrak{m}_\beta) = A^G \cap \mathfrak{m}_\beta$.
(b) r_H^G induces an isomorphism $S(\alpha) = A^G/\mathfrak{m}_\alpha \xrightarrow{\sim} S(\beta) = A^H/\mathfrak{m}_\beta$.
(c) $\alpha \subseteq t_H^G(A^H \beta A^H)$.
Only the proof of the third statement requires some work. In the case of interior algebras (and under the assumption $G_\alpha \geq H_\beta$), (c) is equivalent to the assertion that the localization A_α can be embedded in $\mathrm{Ind}_H^G(A_\beta)$, in such a way that the pointed group H_β on A_α is identified with the pointed group $H_{\beta'}$ on $\mathrm{Ind}_H^G(A_\beta)$, where β' corresponds canonically to β (i.e. contains $1 \otimes \beta \otimes 1$). The proof of this equivalence follows the same line as that of Proposition 1.5.

The Green correspondence for primitive interior algebras is essentially a consequence of the parametrization. Let P be a p-subgroup of G, let B be an interior P-algebra which is a source algebra and let H be a subgroup of G containing $N_G(P,B)$. As before we set $N = N_G(P_\gamma)$, where γ is the point of $\mathrm{Ind}_P^G(B)^P$ containing $1 \otimes 1_B \otimes 1$, and we know by Proposition 5.1 that in fact N only depends on P and B. Consequently if γ' is the point of $\mathrm{Ind}_P^H(B)^P$ containing $1 \otimes 1_B \otimes 1$, then we also have $N_H(P_{\gamma'}) = N$. Since $N_H(P,B) = N_G(P,B)$ by the choice of H, the group $\mathrm{Out}_{\mathrm{skew}}(B)$ is the same when computed with respect to G or to H. It follows that both sets $\mathcal{A}(G,P,B)$ and $\mathcal{A}(H,P,B)$ are parametrized by the set of $\mathrm{Out}_{\mathrm{skew}}(B)$-orbits of isomorphism classes of indecomposable projective $k_\sharp(\widehat{\overline{N}}^B)^\circ$-modules. Therefore by composing the parametrization for G with the inverse of the parametrization for H, we obtain a bijection between $\mathcal{A}(G,P,B)$ and $\mathcal{A}(H,P,B)$.

Proposition 12.3 (Green correspondence for interior algebras). *Let P be a p-subgroup of G, let B be an interior P-algebra which is a source algebra, and let H be a subgroup of G containing $N_G(P,B)$. Then there is a bijection between $\mathcal{A}(G,P,B)$ and $\mathcal{A}(H,P,B)$ mapping a primitive interior G-algebra $A \in \mathcal{A}(G,P,B)$ to the unique primitive interior H-algebra $A' \in \mathcal{A}(H,P,B)$ having the same set of defect multiplicity modules as A.*

We can also fix the defect group P and allow the source B to vary. Let $\mathcal{A}(G,P)$ be the set of isomorphism classes of primitive interior G-algebras with defect group P. For each source B, the subgroup $N_G(P,B)$ is contained in $N_G(P)$ and is equal to it in some cases (Examples 5.3 and 11.4). Thus the disjoint union of the Green correspondences for each B yields an overall correspondence, provided the subgroup H contains $N_G(P)$.

Corollary 12.4. *Let P be a p-subgroup of G and let $H \geq N_G(P)$. Then there is a source-preserving bijection between $\mathcal{A}(G,P)$ and $\mathcal{A}(H,P)$.*

We now briefly indicate how the Green correspondence can be described directly and how it is related with the Green correspondence for pointed groups. Clearly $\mathrm{Ind}_P^H(B)$ can be identified with a subalgebra of $\mathrm{Ind}_P^G(B)$. In other words there is a canonical embedding

$$\mathcal{D}_H^G : \mathrm{Ind}_P^H(B) \longrightarrow \mathrm{Res}_H^G \mathrm{Ind}_P^G(B)$$

containing the inclusion d_H^G. By the embedding $\mathcal{D}_H^G$, the pointed group $P_{\gamma'}$ is identified with P_γ, because it is clear that $\mathrm{Res}_P^H(\mathcal{D}_H^G)\mathcal{D}_P^H = \mathcal{D}_P^G$ where $\mathcal{D}_P^H : B \to \mathrm{Res}_P^H \mathrm{Ind}_P^H(B)$ and $\mathcal{D}_P^G : B \to \mathrm{Res}_P^G \mathrm{Ind}_P^G(B)$ are the canonical embeddings. In general an embedding does not preserve multiplicities, but induces an embedding between corresponding multiplicity algebras (Proposition 1.11). But here we have the following result, which shows that we can identify the multiplicity algebras $S(\gamma)$ and $S(\gamma')$, as well as the multiplicity modules $V(\gamma)$ and $V(\gamma')$.

Lemma 12.5. *With the notation above, the embedding $\overline{\mathcal{D}_H^G} : S(\gamma') \to S(\gamma)$ induced by $\mathcal{D}_H^G$ is an exo-isomorphism.*

Proof. It suffices to note that, by Proposition 6.1, $S(\gamma')$ and $S(\gamma)$ have the same dimension (namely $|\overline{N}|$) since both are the algebra of k-endomorphism of a free module of rank one over a suitable twisted group algebra $k_\sharp \widehat{\overline{N}}$. □

By Corollary 4.4, $\mathcal{A}(G,P,B)$ is in bijection with $\mathrm{Out}_{\mathrm{skew}}(B)\backslash\mathcal{P}(\mathrm{Ind}_P^G(B)^G)_{P_\gamma}$, and similarly $\mathcal{A}(H,P,B)$ is in bijection with $\mathrm{Out}_{\mathrm{skew}}(B)\backslash\mathcal{P}(\mathrm{Ind}_P^H(B)^H)_{P_{\gamma'}}$. By Proposition 5.1 (b), the outer action of $\mathrm{Out}_{\mathrm{skew}}(B)$ is obtained by inducing up to G (respectively up to H) the skew outer automorphisms of B.

Lemma 12.6. *The embedding $\mathcal{D}_H^G : \mathrm{Ind}_P^H(B) \to \mathrm{Res}_H^G \mathrm{Ind}_P^G(B)$ commutes with the outer action of $\mathrm{Out}_{\mathrm{skew}}(B)$. Moreover $\mathcal{D}_H^G\big(\mathcal{P}(\mathrm{Ind}_P^H(B)^H)_{P_{\gamma'}}\big) = \mathcal{P}(\mathrm{Ind}_P^G(B)^H)_{P_\gamma}$ and consequently $\mathcal{D}_H^G\big(\mathrm{Out}_{\mathrm{skew}}(B)\backslash\mathcal{P}(\mathrm{Ind}_P^H(B)^H)_{P_{\gamma'}}\big) = \mathrm{Out}_{\mathrm{skew}}(B)\backslash\mathcal{P}(\mathrm{Ind}_P^G(B)^H)_{P_\gamma}$.*

Proof. Let $\mathcal{H} \in \mathrm{Out}_{\mathrm{skew}}(B)$ and let $h \in \mathcal{H}$ be g-skew, for some $g \in N_G(P,B)$. Then $\mathrm{Ind}_P^G(\mathcal{H})$ is represented by $\widehat{h}$, where $\widehat{h}(x \otimes b \otimes y) = xg^{-1} \otimes h(b) \otimes gy$ for $x,y \in G$ and $b \in B$. The same formula describes $\mathrm{Ind}_P^H(\mathcal{H})$ for $x,y \in H$ (note that $g \in H$ by assumption on H). Thus it is clear that the inclusion $d_H^G : \mathrm{Ind}_P^H(B) \to \mathrm{Res}_H^G \mathrm{Ind}_P^G(B)$ commutes with the action of $\widehat{h}$.

For the second assertion, we note that the set $\mathcal{P}(\mathrm{Ind}_P^H(B)^H)_{P_{\gamma'}}$ is in bijection by the Puig correspondence with the set $\mathcal{P}(S(\gamma')^{\overline{N}})$ (which consists of projective points by Corollary 6.2). Similarly $\mathcal{P}(\mathrm{Ind}_P^G(B)^H)_{P_\gamma}$ is in bijection with $\mathcal{P}(S(\gamma)^{\overline{N}})$. By Lemma 12.5 the inclusion d_H^G induces an isomorphism $\overline{d_H^G} : S(\gamma') \to S(\gamma)$. Thus we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ind}_P^H(B)^H & \xrightarrow{r_P^H} & \mathrm{Ind}_P^H(B)^P & \xrightarrow{\pi_{\gamma'}} & S(\gamma') \\
\big\downarrow{\scriptstyle d_H^G} & & \big\downarrow{\scriptstyle d_H^G} & & \big\downarrow{\scriptstyle \overline{d_H^G}} \\
\mathrm{Ind}_P^G(B)^H & \xrightarrow{r_P^H} & \mathrm{Ind}_P^G(B)^P & \xrightarrow{\pi_\gamma} & S(\gamma)
\end{array}$$

and since the Puig correspondence is induced by $\pi_{\gamma'}\, r_P^H$ (respectively $\pi_\gamma\, r_P^H$), it follows that $d_H^G\big(\mathcal{P}(\mathrm{Ind}_P^H(B)^H)_{P_{\gamma'}}\big) = \mathcal{P}(\mathrm{Ind}_P^G(B)^H)_{P_\gamma}$. □

It follows that we can identify $\mathcal{P}(\mathrm{Ind}_P^H(B)^H)_{P_{\gamma'}}$ and $\mathcal{P}(\mathrm{Ind}_P^G(B)^H)_{P_\gamma}$, as well as the $\mathrm{Out}_{\mathrm{skew}}(B)$-orbits on these sets. Now the Green correspondence for pointed groups (Proposition 12.1) is a bijection between $\mathcal{P}(\mathrm{Ind}_P^G(B)^G)_{P_\gamma}$ and $\mathcal{P}(\mathrm{Ind}_P^G(B)^H)_{P_\gamma}$, and since it is induced by the restriction map r_H^G, the bijection commutes with the action of $\mathrm{Out}_{\mathrm{skew}}(B)$. Therefore the Green correpondence for pointed groups induces a bijection

$$\mathrm{Out}_{\mathrm{skew}}(B)\backslash\mathcal{P}(\mathrm{Ind}_P^G(B)^G)_{P_\gamma} \xrightarrow{\sim} \mathrm{Out}_{\mathrm{skew}}(B)\backslash\mathcal{P}(\mathrm{Ind}_P^G(B)^H)_{P_\gamma},$$

which is essentially the Green correspondence for primitive interior algebras in view of the canonical bijections of Corollary 4.4

$$\begin{aligned} &\mathcal{A}(G,P,B) \cong \mathrm{Out}_{\mathrm{skew}}(B)\backslash\mathcal{P}(\mathrm{Ind}_P^G(B)^G)_{P_\gamma} \\ \text{and}\quad &\mathcal{A}(H,P,B) \cong \mathrm{Out}_{\mathrm{skew}}(B)\backslash\mathcal{P}(\mathrm{Ind}_P^H(B)^H)_{P_{\gamma'}} \cong \mathrm{Out}_{\mathrm{skew}}(B)\backslash\mathcal{P}(\mathrm{Ind}_P^G(B)^H)_{P_\gamma}. \end{aligned}$$

Remarks 12.7. (a) Let $A \in \mathcal{A}(G,P,B)$ and let $D \in \mathcal{A}(H,P,B)$ be its Green correspondent. By the third property of the Green correspondence mentioned in Remark 12.2, we also have an embedding of A into $\mathrm{Ind}_H^G(D)$ (which "commutes" with the embedding of the source algebra B).

(b) The characterization of the Green correspondence given in Proposition 12.1 includes the Burry-Carlson-Puig theorem. We state it here for interior algebras. Let A be a primitive interior G-algebra and let $N_G(P,B) \le H \le G$ for some p-subgroup P and source algebra B. Suppose that a primitive interior H-algebra D embeds into $\mathrm{Res}_H^G(A)$ and that D has defect group P and source algebra B. Then A has defect group P and source algebra B, and D is the Green correspondent of A. One can easily deduce a version of this result for a subgroup $H \ge N_G(P)$ without any mention to B, as in Corollary 12.4 above.

(c) We warn the reader that the Green correspondent of a block algebra $\mathcal{O}Gb$ is not the block algebra of the Brauer correspondent of b. It is a primitive interior algebra which has no reason to be a block algebra.

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