Most finite groups are p-nilpotent

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The statement of the title is a recent result of H.W. Henn and S. Priddy [HP]. The proof breaks up into two quite distinct parts and the purpose of the present note is to provide an elementary proof of one of these parts. Explicitly Henn and Priddy show the following two statements:

(a) For most finite p-groups P, the series of subgroups $\Omega_k(P)$ is central and $\mathrm{Out}(P)$ is a p-group (see the example below for the definition of $\Omega_k(P)$).

(b) Any finite group having a Sylow p-subgroup P satisfying the properties of part (a) is p-nilpotent (that is, P has a normal complement).

The meaning of the word "most" is the following. Fix a Frattini length $n \geq 2$. Then among all p-groups P with Frattini length n, the proportion of those having the required properties tends to 1 when the minimal number of generators of P tends to infinity. See [HP] for more details.

The proof of part (b) given by Henn and Priddy uses the double Burnside ring and its connection with classifying spaces, and so is essentially of a topological nature (although it can be made purely algebraic, as H.W. Henn pointed out to us). We present here a direct proof based on the notion of control of fusion and the classical theorem of Frobenius giving a p-local criterion for p-nilpotence. In fact we do not use the specific series of subgroups $\Omega_k(P)$ considered by Henn and Priddy, but we prove a slightly more general result by taking an arbitrary series of strongly characteristic subgroups.

We say that a subgroup Q of a finite p-group P is *strongly characteristic* in P if every subgroup of P isomorphic to Q is equal to Q. This implies that Q is a characteristic subgroup, hence a normal subgroup.

EXAMPLE. For every positive integer k, the subgroup $\Omega_k(P)$ generated by all elements of P of order dividing p^k is strongly characteristic. Indeed if $R \leq P$ is isomorphic to $\Omega_k(P)$, then R is generated by elements of order dividing p^k, which all lie in $\Omega_k(P)$, so that $R \leq \Omega_k(P)$, hence $R = \Omega_k(P)$.

A series of subgroups $1 = P_0 < P_1 < \ldots < P_n = P$ will be called *strongly characteristic* if each P_k is strongly characteristic in P. It is called *central* if $[P, P_k] \leq P_{k-1}$ for all $k \geq 1$.

THEOREM. *Let G be a finite group with a Sylow p-subgroup P having a strongly characteristic central series. Then $N_G(P)$ controls p-fusion in G.*

Recall that a subgroup H of G is said to *control p-fusion* in G if H contains a Sylow p-subgroup and whenever $Q, gQg^{-1} \leq H$ for some p-subgroup Q and some $g \in G$, then $g = hc$ where $h \in H$ and $c \in C_G(Q)$.

From now on P always denotes a Sylow p-subgroup of a finite group G. Also we write ${}^gQ = gQg^{-1}$. We shall need Alperin's fusion theorem [Go, Section 7.2] which describes the fusion of p-subgroups of P from p-local information (that is, information about normalizers of non-trivial p-subgroups). We need the following form of the result.

ALPERIN'S FUSION THEOREM. *If* $Q, {}^gQ \leq P$ *for some* $g \in G$, *then it is possible to write* $g = g_1 g_2 \ldots g_n$ *with* $g_i \in N_G(Q_i)$, *where each* $Q_i \leq P$ *is a tame intersection* $Q_i = P \cap {}^xP$.

Recall that Q is a *tame intersection* $Q = P \cap S$ where S is some Sylow p-subgroup if $N_P(Q)$ is a Sylow p-subgroup of $N_G(Q)$ and similarly $N_S(Q)$ is a Sylow p-subgroup of $N_G(Q)$.

COROLLARY. *Let* H *be a subgroup of* G *containing* P. *The following conditions are equivalent.*
(a) H *controls* p*-fusion in* G.
(b) $N_G(Q) = N_H(Q)C_G(Q)$ *for every tame intersection* $Q = P \cap {}^xP$.

Proof. (a) implies (b) by definition, and (b) implies (a) by Alperin's theorem. □

The proof of the main theorem of this note breaks up into three steps. The first one is a generalization of the fact (due to Burnside) that the normalizer of an *abelian* Sylow p-subgroup controls p-fusion.

LEMMA 1. *Let* A *be a strongly characteristic central subgroup of* P. *Then* $N_G(A)$ *controls* p*-fusion in* G.

Proof. We first observe that if A is contained in some p-subgroup R, then A is central in R and strongly characteristic in R. Indeed we have $A \leq R \leq {}^gP$ for some $g \in G$ (because all Sylow subgroups are conjugate), so that ${}^{g^{-1}}A \leq P$ and ${}^{g^{-1}}A = A$ since A is strongly characteristic in P. Now by conjugation $A = {}^gA$ is strongly characteristic and central in ${}^gP$, hence in R.

Since A is characteristic in P, we have $P \leq N_G(A)$, which is the first condition for control of fusion. Suppose that $Q, {}^gQ \leq N_G(A)$ for some $g \in G$. Then both AQ and $A\,{}^gQ$ are p-subgroups and by the observation above, A is central in both of them. Thus $A, {}^{g^{-1}}A \leq C_G(Q)$ and so if S is a Sylow p-subgroup of $C_G(Q)$ containing A, there exists $c \in C_G(Q)$ such that ${}^{cg^{-1}}A \leq S$. Since A is strongly characteristic in S by the observation above, we have $A = {}^{cg^{-1}}A$, that is, $cg^{-1} \in N_G(A)$, or in other words $g \in N_G(A)C_G(Q)$. □

Lemma 1 allows to replace G by $N_G(A)$, that is, we can assume that A is normal.

LEMMA 2. *Let* A *be a normal* p*-subgroup of* G *which is central in* P. *Let* H *be a subgroup of* G *containing* $N_G(P)$. *If* $C_H(A) = H \cap C_G(A)$ *controls* p*-fusion in* $C_G(A)$, *then* H *controls* p*-fusion in* G *(the converse being obvious).*

Proof. First note that since A is central in P, we have $P \leq C_G(A)$. But since $C_G(A)$ is a normal subgroup of G, all the Sylow p-subgroups are contained in $C_G(A)$ and therefore every p-subgroup centralizes A. Moreover since all Sylow p-subgroups are conjugate under $C_G(A)$, we have $G = N_G(P)C_G(A)$ (Frattini argument). Now let $Q, {}^gQ \leq H$ for some $g \in G$, and write $g = nc$ with $n \in N_G(P)$ and $c \in C_G(A)$. By assumption $n \in H$, so we only have to deal with c. Thus we have $Q, {}^cQ \leq H$, hence $Q, {}^cQ \leq C_H(A)$ since p-subgroups centralize A. By assumption it follows that $c = hd$ with $h \in C_H(A)$ and d centralizing Q. Finally $g = nhd$ with $nh \in H$ and $d \in C_G(Q)$, as was to be shown. □

Lemma 2 allows to replace G by $C_G(A)$, that is, we can assume that A is central.

LEMMA 3. *Let* A *be a central* p*-subgroup of* G, *let* $\overline{G} = G/A$ *and write* $\overline{K} = K/A$ *for every* $A \leq K \leq G$. *Let* H *be a subgroup of* G *containing* A. *If* $\overline{H}$ *controls* p*-fusion in* $\overline{G}$, *then* H *controls* p*-fusion in* G *(the converse being obvious).*

Proof. Since $\overline{H}$ contains a Sylow p-subgroup $\overline{P}$ of $\overline{G}$ and since $A \leq H$, we have $P \leq H$. By the corollary of Alperin's fusion theorem, it suffices to show that $N_G(Q) = N_H(Q)C_G(Q)$ for every tame intersection $Q = P \cap {}^xP$. Note that $A \leq Q$ since A is a normal subgroup. By assumption we have

$$N_{\overline{G}}(\overline{Q}) = N_{\overline{H}}(\overline{Q})C_{\overline{G}}(\overline{Q})$$

and we take the inverse image of this equation in G. There is no surprise with normalizers and we obtain

$$N_G(Q) = N_H(Q)C_G(\overline{Q}),$$

where $C_G(\overline{Q})$ is the centralizer of $\overline{Q}$ in G. Thus it suffices to prove that

$$C_G(\overline{Q}) \leq N_H(Q)C_G(Q).$$

For every $g \in C_G(\overline{Q})$, consider the map

$$\phi_g : Q \longrightarrow A, \qquad u \mapsto [g, u] = gug^{-1}u^{-1},$$

which has image in A by assumption on g, and let

$$\phi : C_G(\overline{Q}) \longrightarrow \mathrm{Hom}(Q, A), \qquad g \mapsto \phi_g.$$

Since $[g, u]$ is central, we have $[g, uu'] = [g, u][g, u']$ so that $\phi_g \in \mathrm{Hom}(Q, A)$, and also $[gg', u] = [g, u][g', u]$ so that ϕ is in turn a group homomorphism (relative to the group structure on $\mathrm{Hom}(Q, A)$ induced by the product in A). By construction $\mathrm{Ker}(\phi) = C_G(Q)$ and therefore $C_G(\overline{Q})/C_G(Q)$ is a p-group since it embeds in the p-group $\mathrm{Hom}(Q, A)$.

Since Q is a tame intersection, $N_P(Q)$ is a Sylow p-subgroup of $N_G(Q)$. As $C_G(\overline{Q})$ is a normal subgroup of $N_G(Q)$, we obtain a Sylow p-subgroup of $C_G(\overline{Q})$ by mere intersection; thus $C_P(\overline{Q})$ is a Sylow p-subgroup of $C_G(\overline{Q})$. It follows that $C_P(\overline{Q})$ covers the quotient group $C_G(\overline{Q})/C_G(Q)$, which is a p-group. Therefore

$$C_G(\overline{Q}) = C_P(\overline{Q})C_G(Q) \leq N_H(Q)C_G(Q),$$

since $P \leq H$. This completes the proof. □

Proof of the main theorem. We proceed by induction on the length of the given series of P. If we let $A = P_1$, lemma 1 implies that $N_G(A)$ controls p-fusion. Since $N_G(P) \leq N_G(A)$, we can replace G by $N_G(A)$ and assume that A is normal in G. By lemma 2, it suffices to prove that $N_G(P) \cap C_G(A)$ controls p-fusion in $C_G(A)$. Thus replacing G by $C_G(A)$, we can now assume that A is central in G. Let $\overline{G} = G/A$. Then $\overline{P} = P/A$ has a strongly characteristic central series of shorter length and so by induction $N_{\overline{G}}(\overline{P})$ controls p-fusion in $\overline{G}$. By lemma 3 it follows that $N_G(P)$ controls p-fusion in G. □

COROLLARY 1. *Let G be a finite group with a Sylow p-subgroup P having a strongly characteristic central series and such that $\mathrm{Out}(P)$ is a p-group. Then P controls p-fusion in G and so G is p-nilpotent.*

Proof. By the theorem, $N_G(P)$ controls p-fusion in G. But since $\mathrm{Out}(P)$ is a p-group and $N_G(P)/PC_G(P)$ has order prime to p, we have $N_G(P) = PC_G(P)$. Since $C_G(P)$ does not play any role in the fusion of subgroups of P, it follows that P controls p-fusion in G. In particular $N_G(Q) = N_P(Q)C_G(Q)$ for every $Q \leq P$ so that $N_G(Q)/C_G(Q)$ is a p-group for all Q. (In fact by Alperin's theorem this last condition is equivalent to the property that P controls p-fusion). Now by a classical theorem of Frobenius [Go, Section 7.4], this implies that G is p-nilpotent. □

The computation of stable elements in mod p cohomology for both G and the subgroup controling p-fusion gives the same answer (see [Be, 3.8.4] for details). Thus the above results immediately yield the following corollary, which establishes the connection with the approach of Henn and Priddy.

COROLLARY 2. *Let G be a finite group with a Sylow p-subgroup P.*

(a) If P has a strongly characteristic central series, then the restriction map

$$H^*(G, \mathbb{F}_p) \longrightarrow H^*(N_G(P), \mathbb{F}_p)$$

in mod p cohomology is an isomorphism.

(b) If P has a strongly characteristic central series and if $\mathrm{Out}(P)$ is a p-group, then the restriction map

$$H^*(G, \mathbb{F}_p) \longrightarrow H^*(P, \mathbb{F}_p)$$

in mod p cohomology is an isomorphism.

In fact by a result of Mislin [Mi], the property that the restriction map in mod p cohomology is an isomorphism is equivalent to the fact that the subgroup controls p-fusion (in the special case of a Sylow p-subgroup, this is due to Tate [Ta]).

Another consequence of the main theorem has to do with a conjecture of Webb. Let $S_p(G)$ be the poset of non-trivial p-subgroups of G, let $\Delta(S_p(G))$ be the associated simplicial complex of chains of non-trivial p-subgroups (called Brown's complex), and let $|\Delta(S_p(G))|$ be the geometric realization of $\Delta(S_p(G))$. The group G acts by conjugation on $S_p(G)$, hence also on $\Delta(S_p(G))$ and $|\Delta(S_p(G))|$, and Webb [We] conjectured that the orbit space $|\Delta(S_p(G))|/G$ is contractible. The next result shows that Webb's conjecture holds for most finite groups, in view of the first result of Henn and Priddy mentioned at the beginning of this paper.

COROLLARY 3. *If a Sylow p-subgroup P of G has a strongly characteristic central series, then $|\Delta(S_p(G))|/G$ is contractible.*

Proof. In the remark following Proposition 2.3 of [Th], it is shown that Webb's conjecture holds if $N_G(P)$ controls p-fusion. □

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