EQUIVARIANT K-THEORY AND ALPERIN'S CONJECTURE

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Introduction

The purpose of this paper is to show that Alperin's conjecture in the modular representation theory of finite groups can be reinterpreted as a conjecture in equivariant K-theory, whose statement is surprisingly simple. Let G be a finite group and p a prime number. Brown's complex (for G and p) is the simplicial complex $\mathcal{S}_p(G)$ whose set of k-simplices consists of all chains

$$\sigma = (P_0 < P_1 < \ldots < P_k)$$

of non-trivial p-subgroups of G (the faces of σ being the shorter chains). This is an ordered simplicial complex which is finite, so that its geometric realization $|\mathcal{S}_p(G)|$ is a compact subset of some Euclidean space. In his work on Euler characteristic of discrete groups [11], K.S. Brown introduced this complex (allowing the group to be infinite), but the first systematic study of $\mathcal{S}_p(G)$ started with Quillen [19]. The complex was used more recently in the theory of p-modular representations and mod p cohomology of the group G by Webb [25, 26, 27], Knörr-Robinson [15], Bouc [10], and Thévenaz [21]. It appears to be of fundamental importance for handling p-local information in the representation theory of G. Quillen [19], Webb [26], and Aschbacher [3] have proposed conjectures about contractibility and simple connectivity of $\mathcal{S}_p(G)$, which have been solved in some special cases by Quillen [19], Aschbacher [3], Aschbacher-Kleidman [4], and Thévenaz [22].

The group G acts by conjugation on $\mathcal{S}_p(G)$ and some relevant information seems to lie in the G-homotopy type of $\mathcal{S}_p(G)$, rather than in its homotopy type. Thus we view $|\mathcal{S}_p(G)|$ as a compact G-space and we can consider G-equivariant $\mathbb{C}$-vector bundles on $|\mathcal{S}_p(G)|$. This leads to the equivariant K-theory groups $K^0_G(\mathcal{S}_p(G))$ and $K^1_G(\mathcal{S}_p(G))$ defined by Atiyah [5] and Segal [20]. (As usual any topological invariant of $\mathcal{S}_p(G)$, such as K^*_G, is defined to be the corresponding invariant of $|\mathcal{S}_p(G)|$). Equivariant K-theory is a cohomology theory on compact G-spaces and the Bott periodicity theorem implies that K^*_G is periodic of period 2, or in other words $\mathbb{Z}/2\mathbb{Z}$-graded. Thus a natural invariant of a compact G-space X is its "equivariant Euler characteristic"

$$\chi_G(X) = \dim\big(\mathbb{Q}\otimes K^0_G(X)\big) - \dim\big(\mathbb{Q}\otimes K^1_G(X)\big).$$

When G is the trivial group, then we are dealing with ordinary K-theory and it is well-known that this number is the ordinary Euler characteristic of X.

The present paper is concerned with a conjecture on the value of $\chi_G(\mathcal{S}_p(G))$, which has to do with the representation theory of G. Let $k(G)$ be the number of irreducible representations of G over $\mathbb{C}$ (i.e. the number of conjugacy classes of G) and let $z_p(G)$ be the number of those representations whose dimension is a multiple of $|G|_p$, the p-part of the order of the group. By elementary block theory, $z_p(G)$ is the number of p-blocks of G of defect zero, that is also, over an algebraically closed field k of characteristic p, the number of irreducible representations of G which are projective kG-modules.

Conjecture. $\chi_G(\mathcal{S}_p(G)) = k(G) - z_p(G)$.

The only cases where the conjecture is trivially satisfied are when $\mathcal{S}_p(G)$ is empty or G-contractible. In some cases where $\mathcal{S}_p(G)$ is just G-homotopically equivalent to a complex of dimension zero (e.g. if a Sylow p-subgroup is cyclic), the conjecture holds thanks to some highly non-trivial results in modular representation theory. The main result of the present paper asserts that the conjecture is equivalent (in a suitable sense) to the conjecture of Alperin [1] about the number of modular representations of G . Thus our approach provides an entirely different point of view on Alperin's conjecture and shows that the G-homotopy type of Brown's complex seems to be a relevant invariant in this subject. Note however that no significant progress is made towards the solution of Alperin's conjecture.

The fact that Alperin's conjecture holds for many families of finite groups provides the main evidence for the conjecture above. But as the equivalence between the two conjectures requires an inductive argument, we note that if Alperin's conjecture holds for some specific group G then it does not follow that the K-theoretic conjecture holds for G (and conversely). However the inductive argument works for instance for the family of soluble groups, and therefore the conjecture above holds if G is soluble since Alperin's conjecture has been proved in that case. For similar reasons, the conjecture also holds for finite groups of Lie type in characteristic p .

1. First examples

In this section we examine a few cases which are small enough to allow easy computation. But we start with some general remarks and fix some notation. First recall that the conjugation action of G on p-subgroups induces an action on $\mathcal{S}_p(G)$ such that if a simplex is fixed by $g \in G$, then it is fixed pointwise. Since equivariant K-theory only depends on the G-homotopy type of the G-space, we can replace $\mathcal{S}_p(G)$ by any G-homotopy equivalent complex, such as the subcomplex $\mathcal{A}_p(G)$ consisting of chains of elementary abelian p-subgroups, or the subcomplex $\mathcal{B}_p(G)$ consisting of chains of p-subgroups P satisfying $P = O_p(N_G(P))$. If G is a finite group of Lie type, then $\mathcal{S}_p(G)$ is also G-homotopy equivalent to the Tits building of G . The homotopy equivalences are due to Quillen (for $\mathcal{A}_p(G)$ and the building) and Bouc (for $\mathcal{B}_p(G)$), and the fact that they are all G-equivariant appears in [24]. Another complex which is G-homotopy equivalent to $\mathcal{S}_p(G)$ has been introduced by Alperin [2] (see also [3]). It is the complex $\mathcal{C}_p(G)$ whose vertices are the subgroups of order p and whose k-simplices consist of $k-1$ such subgroups centralizing each other.

For simplicity, we shall work with the category of ordered G-simplicial complexes, that is, ordered simplicial complexes endowed with a simplicial action of a finite group G . Thus throughout this paper, a *G-complex* Δ is an ordered G-simplicial complex. The interested reader can easily extend the results to the case of G-CW-complexes. Moreover we shall only work with finite complexes. For every simplex σ of Δ , we shall usually assume that the stabilizer G_σ of σ acts trivially on σ . This condition is satisfied by $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$ and the building (in the Lie type case), but in general it does not hold with $\mathcal{C}_p(G)$. In fact there is no real loss of generality in assuming this condition because if Δ does not satisfy the condition, then its barycentric subdivision does.

We write $R(G)$ for the Grothendieck ring of complex representations of G . Recall that the equivariant K-theory of a point is given by:

$$K^i_G(pt) = \begin{cases} R(G) & \text{if } i = 0, \\ 0 & \text{if } i = 1. \end{cases}$$

Moreover the *reduced* equivariant K-theory of a pointed G-complex Δ with base point x_0 (necessarily fixed by G) is by definition $\widetilde{K}_G^*(\Delta) = \mathrm{Ker}(K_G^*(\Delta) \to K_G^*(x_0))$, so that we have

$$K_G^i(\Delta) = \begin{cases} \widetilde{K}_G^0(\Delta) \oplus R(G) & \text{if } i = 0, \\ \widetilde{K}_G^1(\Delta) & \text{if } i = 1. \end{cases}$$

We shall often use the following basic result. If H is a subgroup of G and if Δ is a finite H-complex, we let $\mathrm{Ind}_H^G(\Delta) = G \times_H \Delta$, a disjoint union of copies of Δ permuted transitively by G. If Δ is a pointed H-complex with base point x_0 (fixed by H), we let $\widetilde{\mathrm{Ind}}_H^G(\Delta) = \mathrm{Ind}_H^G(\Delta)/\mathrm{Ind}_H^G(x_0)$, a wedge on x_0 of copies of Δ permuted transitively by G.

Lemma 1.1. *Let H be a subgroup of G and let Δ be a finite H- complex.*
(a) $K_G^*(\mathrm{Ind}_H^G(\Delta)) \cong K_H^*(\Delta)$.
(b) $\widetilde{K}_G^*(\widetilde{\mathrm{Ind}}_H^G(\Delta)) \cong \widetilde{K}_H^*(\Delta)$ *if Δ has a base point (fixed by H).*

Proof. The isomorphism $K_G^0(\mathrm{Ind}_H^G(\Delta)) \cong K_H^0(\Delta)$ appears on page 132 of [20]. For the convenience of the reader, we sketch the proof of the other statements. Since $\mathrm{Ind}_H^G(x_0)$ is a G-retract of $\mathrm{Ind}_H^G(\Delta)$ if Δ has a base point x_0, there is a split short exact sequence

$$0 \longrightarrow K_G^0(\mathrm{Ind}_H^G(\Delta), \mathrm{Ind}_H^G(x_0)) \longrightarrow K_G^0(\mathrm{Ind}_H^G(\Delta)) \longrightarrow K_G^0(\mathrm{Ind}_H^G(x_0)) \longrightarrow 0\,.$$

Since $K_G^0(\mathrm{Ind}_H^G(\Delta)) \cong K_H^0(\Delta)$ and $K_G^0(\mathrm{Ind}_H^G(x_0)) \cong K_H^0(x_0) \cong R(H)$, it follows that

$$\begin{aligned} \widetilde{K}_G^0(\widetilde{\mathrm{Ind}}_H^G(\Delta)) &= \widetilde{K}_G^0(\mathrm{Ind}_H^G(\Delta)/\mathrm{Ind}_H^G(x_0)) = K_G^0(\mathrm{Ind}_H^G(\Delta), \mathrm{Ind}_H^G(x_0)) \\ &\cong \mathrm{Ker}(K_H^0(\Delta) \to K_H^0(x_0)) = \widetilde{K}_H^0(\Delta)\,. \end{aligned}$$

Now K_G^1 is defined by adding an extra base point, suspending, and taking $\widetilde{K}_G^0$, or in other words $K_G^1(X) = \widetilde{K}_G^0(\overline{S}X)$, where $\overline{S}X$ denotes the suspension of X with the identification of the vertices of the two cones on X. Since $\overline{S}\,\mathrm{Ind}_H^G(\Delta) = \widetilde{\mathrm{Ind}}_H^G(\overline{S}\Delta)$, it follows that

$$K_G^1(\mathrm{Ind}_H^G(\Delta)) = \widetilde{K}_G^0(\overline{S}\,\mathrm{Ind}_H^G(\Delta)) = \widetilde{K}_G^0(\widetilde{\mathrm{Ind}}_H^G(\overline{S}\Delta)) \cong \widetilde{K}_H^0(\overline{S}\Delta) = K_H^1(\Delta)\,.$$

Finally if X has a base point (fixed by G), then $\widetilde{K}_G^1(X) = \widetilde{K}_G^0(\widetilde{S}X)$ where $\widetilde{S}X$ denotes the reduced suspension of X. We have $\widetilde{S}\,\widetilde{\mathrm{Ind}}_H^G(\Delta) = \widetilde{\mathrm{Ind}}_H^G(\widetilde{S}\Delta)$ and therefore

$$\widetilde{K}_G^1(\widetilde{\mathrm{Ind}}_H^G(\Delta)) \cong \widetilde{K}_G^0(\widetilde{S}\,\widetilde{\mathrm{Ind}}_H^G(\Delta)) \cong \widetilde{K}_G^0(\widetilde{\mathrm{Ind}}_H^G(\widetilde{S}\Delta)) \cong \widetilde{K}_H^0(\widetilde{S}\Delta) = \widetilde{K}_H^1(\Delta)\,. \ \square$$

Since we are only interested in Euler characteristics, we can tensor K_G^* with the field $\mathbb{Q}$, but for simplicity, as we shall use later roots of unity, we extend scalars to $\mathbb{C}$ and we define

$${}^{\mathbb{C}}K_G^i(X) = \mathbb{C} \otimes_{\mathbb{Z}} K_G^i(X)\,.$$

We also define similarly ${}^{\mathbb{C}}R(G) = \mathbb{C} \otimes_{\mathbb{Z}} R(G)$, and we have $k(G) = \dim {}^{\mathbb{C}}R(G)$.

If B is a p-block of G, we write $k(B)$ for the number of irreducible complex representations of G lying in B, so that $k(G) = \sum_B k(B)$, where the sum runs over all p-blocks of G. In most cases where the K-theoretic conjecture is known to hold, the proof consists in the analysis of the structure of each block of G, followed by a sum of the results over the blocks. This is made explicit in some of the following examples. These examples also show that, in the present state of our knowledge, the conjecture is far from being obvious even in the case where $\mathcal{S}_p(G)$ is G-homotopy equivalenty to a complex of dimension zero. However we first start with two easy cases.

Example 1.2. Suppose that p does not divide the order of G. Then $\mathcal{S}_p(G)$ is empty and $\chi_G(\mathcal{S}_p(G)) = 0$. Since the dimension of every representation is a multiple of $|G|_p = 1$, we have $k(G) = z_p(G)$ and the conjecture holds.

Example 1.3. Suppose that G has a non-trivial normal p-subgroup P. Then $\mathcal{S}_p(G)$ is G-contractible (via the maps $Q \mapsto QP \mapsto P$, see [24, 1.2]). The equivariant K-theory of a G-contractible space is that of a point and therefore $\chi_G(\mathcal{S}_p(G)) = k(G)$. The conjecture holds in that case because it is well-known that $z_p(G) = 0$ when G has a non-trivial normal p-subgroup. One way to see this consists in the observation that the centre $Z(P)$ is a non-trivial normal abelian subgroup, so that any irreducible representation has a dimension dividing $|G : Z(P)|$ (Ito's theorem [13, 11.33]), hence cannot be divisible by $|G|_p$.

Notice conversely that if $\mathcal{S}_p(G)$ is G-contractible, then there is a G-fixed point, hence a G-fixed simplex, hence a G-fixed vertex, that is, a non-trivial normal p-subgroup. Quillen's conjecture asserts that if $\mathcal{S}_p(G)$ is contractible, then G has a non-trivial normal p-subgroup; the latter argument shows that the G-equivariant version of Quillen's conjecture holds trivially.

Example 1.4. Suppose that a Sylow p-subgroup P of G is cyclic and let Q be the unique subgroup of P of order p. Then $\mathcal{A}_p(G)$ is the zero dimensional complex whose set of points is the set of G-conjugates of Q (and in this case the G-homotopy equivalence with $\mathcal{S}_p(G)$ consists simply in mapping an arbitrary non-trivial p-subgroup to its unique subgroup of order p). Therefore $\mathcal{A}_p(G) \cong G/N_G(Q)$ and since $K_G^*(G/H) = K_H^*(pt)$ by Lemma 1.1, we obtain

$$\chi_G(\mathcal{S}_p(G)) = \chi_G(\mathcal{A}_p(G)) = \chi_{N_G(Q)}(pt) = \dim{}^{\mathbb{C}}R(N_G(Q)) = k(N_G(Q)) .$$

Thus the conjecture asserts that $k(G) - z_p(G) = k(N_G(Q))$. This equality is known to hold, thanks to the whole theory of blocks with cyclic defect group, which is one of the first non-trivial achievements of modular representation theory. More precisely for every block B of G of non-zero defect, some defect group D of B satisfies $Q \leq D \leq P$, so that $N_G(D) \leq N_G(Q)$, and the cyclic theory implies that $k(B) = k(b)$ where b is the block of $N_G(D)$ which is the Brauer correspondent of B (see [14, VII.2.12]). Similarly $k(B') = k(b)$ where B' is the block of $H = N_G(Q)$ which is the Brauer correspondent of b, and therefore $k(B) = k(B')$. Since $B \leftrightarrow b$ (respectively $B' \leftrightarrow b$) is a bijection between blocks of G (respectively of H) with defect group D and blocks of $N_G(D)$ with defect group D, it suffices to sum up over all such blocks and then over all possible non-trivial defect groups to get $k(G) - z_p(G) = k(H) - z_p(H)$. But $z_p(H) = 0$ because $H = N_G(Q)$ has a normal p-subgroup, proving the conjecture.

Example 1.5. The fact that $\mathcal{S}_p(G)$ is G-homotopy equivalenty to a complex of dimension zero also occurs when $p = 2$ and a Sylow 2-subgroup P of G is generalized quaternion. Indeed P has a unique subgroup Q of order 2 and the argument of Example 1.4 applies without change. Thus the conjecture asserts here that $k(G) - z_p(G) = k(N_G(Q))$. Again this is proved, using results of Olsson [18] on blocks with a quaternion defect group, and then summing over all blocks with non-trivial defect (which can be either quaternion or cyclic).

Example 1.6. Suppose that a Sylow p-subgroup P of G is T.I. ("trivial intersection"). This means that P intersects trivially each conjugate gPg^{-1} where $g \notin N_G(P)$. Equivalently this says that any non-trivial p-subgroup is contained in a unique Sylow p-subgroup. Then $\mathcal{B}_p(G)$ is the zero dimensional complex whose set of points is the set of G-conjugates of P (and in this case the G-homotopy equivalence with $\mathcal{S}_p(G)$ consists simply in mapping an arbitrary non-trivial p-subgroup to the unique Sylow p-subgroup containing it). For another way of seeing this, notice that $\mathcal{S}_p(G)$ is a disjoint union of cones, the vertex of each cone being a Sylow p-subgroup. As in the previous two examples we conclude that $\chi_G(\mathcal{S}_p(G)) = k(N_G(P))$ and the conjecture says that $k(G) - z_p(G) = k(N_G(P))$. Again this equality is known to hold, by a recent result of Blau and Michler [9]. Their proof uses the classification of all finite simple groups.

Example 1.7. Suppose that $\mathcal{S}_p(G)$ is disconnected. Then G acts transitively on the set of connected components and if H is the stabilizer of one component, then $\mathcal{S}_p(G) = \mathrm{Ind}_H^G(\mathcal{S}_p(H))$. Therefore $K_G^*(\mathcal{S}_p(G)) = K_H^*(\mathcal{S}_p(H))$ and so $\chi_G(\mathcal{S}_p(G)) = \chi_H(\mathcal{S}_p(H))$. The conjecture would imply that $k(G) - z_p(G) = k(H) - z_p(H)$. It is easy to see that the subgroup H is strongly p-embedded (see [19, 5.2]), and conversely the existence of a strongly p-embedded proper subgroup means that $\mathcal{S}_p(G)$ is disconnected. It has been verified in many cases that $k(G) - z_p(G) = k(H) - z_p(H)$ if H is strongly p-embedded (e.g. $H = N_G(Q)$ in Examples 1.4 and 1.5, and $H = N_G(P)$ in Example 1.6). As far as we know, the general case is still open, although it might be possible to settle it, in view of the classification of all finite groups with a strongly p-embedded proper subgroup (which is a consequence of the classification of all finite simple groups, see [3, 6.2]).

2. Equivariant Euler characteristic

In this section we prove a general result about $\chi_G(\Delta)$, where Δ is a finite G-complex. This result is a consequence of Segal's work [20] and can probably be better understood by introducing Segal's spectral sequence, but for the convenience of the reader, we give here a direct proof. We simply use standard facts from algebraic topology, which just need to be made explicit in the case of equivariant K-theory. For a short proof of Proposition 2.1 below using Segal's spectral sequence and for a deeper understanding of the theory, we refer the interested reader to the Appendix, where we have gathered some general facts about equivariant K-theory.

Let Δ be a finite G-complex. Using the character ring $R(G_\sigma)$ of each stabilizer G_σ, we describe how to construct a coefficient system $\mathcal{R}$ on Δ. To each simplex σ is associated the abelian group $R(G_\sigma)$; if τ is a face of σ, there is the restriction map $R(G_\tau) \to R(G_\sigma)$ (which satisfies the obvious transitivity condition); finally if $g \in G$, there is the conjugation map $R(G_\sigma) \to R(gG_\sigma g^{-1}) = R(G_{g\sigma})$ (which satisfies the obvious condition for an action, and commutes with restriction in the obvious way). These data form a G-equivariant coefficient system $\mathcal{R}$ on Δ and we write ${}^{\mathbb{C}}\mathcal{R}$ for the coefficient system of $\mathbb{C}$-vector spaces obtained from $\mathcal{R}$ by extension of scalars.

One can build from this a cochain complex of vector spaces $C^*(\Delta/G, {}^{\mathbb{C}}\mathcal{R})$ as follows. First define

$$C^k(\Delta, \mathcal{R}) = \bigoplus_{\sigma \in \Delta_k} R(G_\sigma),$$

where Δ_k denotes the set of k-simplices of Δ, and define a coboundary map

$$\delta^k : C^k(\Delta, \mathcal{R}) \to C^{k+1}(\Delta, \mathcal{R})$$

by using the alternating sum of the restriction maps to faces of a simplex. This is a cochain complex of abelian groups and since G acts on the whole situation, we can consider the subcomplex of G-fixed points $C^*(\Delta, \mathcal{R})^G$. We define

$$C^k(\Delta/G, \mathcal{R}) = C^k(\Delta, \mathcal{R})^G = \bigoplus_{\sigma \in [\Delta_k/G]} R(G_\sigma)\,,$$

where $[\Delta_k/G]$ denotes an arbitrary set of representatives of the G-orbits of k-simplices. Finally $C^*(\Delta/G, {}^{\mathbb{C}}\mathcal{R})$ is obtained by extending scalars to $\mathbb{C}$:

$$C^k(\Delta/G, {}^{\mathbb{C}}\mathcal{R}) = \mathbb{C} \otimes C^k(\Delta/G, \mathcal{R}) = \bigoplus_{\sigma \in [\Delta_k/G]} {}^{\mathbb{C}}R(G_\sigma)\,.$$

Proposition 2.1. *Let Δ be a finite G-complex and assume that the stabilizer of every simplex σ fixes σ pointwise. Then the Euler characteristic $\chi_G(\Delta)$ of the equivariant K-theory of Δ is equal to the Euler characteristic of the cochain complex $C^*(\Delta/G, {}^{\mathbb{C}}\mathcal{R})$. In other words*

$$\chi_G(\Delta) = \sum_{\sigma \in [\Delta/G]} (-1)^{\dim(\sigma)} \dim {}^{\mathbb{C}}R(G_\sigma)\,.$$

Proof. Let $|\Delta|^k$ denote the k-th skeleton of $|\Delta|$ and write S^k for the sphere of dimension k . Then

$$|\Delta|^k/|\Delta|^{k-1} \cong \bigvee_{\sigma \in \Delta_k} S^k$$

for $k \geq 1$, while $|\Delta|^0 = \Delta_0$ is a finite set of points. The action of G on $|\Delta|^k/|\Delta|^{k-1}$ permutes the spheres of the wedge, and since G_σ fixes σ pointwise, G_σ acts trivially on the sphere S^k indexed by σ . As reduced K-theory $\widetilde{K}^*_G$ behaves additively with respect to the wedge of G-complexes, we can decompose into orbits, and then we obtain by Lemma 1.1

$$\begin{aligned} K^*_G(|\Delta|^k, |\Delta|^{k-1}) = \widetilde{K}^*_G(|\Delta|^k/|\Delta|^{k-1}) &= \bigoplus_{\sigma \in [\Delta_k/G]} \widetilde{K}^*_G(\widetilde{\mathrm{Ind}}_{G_\sigma}^G(S^k)) \\ &\cong \bigoplus_{\sigma \in [\Delta_k/G]} \widetilde{K}^*_{G_\sigma}(S^k) \cong \bigoplus_{\sigma \in [\Delta_k/G]} \widetilde{K}^*(S^k) \otimes R(G_\sigma)\,, \end{aligned}$$

using also the fact that, for an H-complex X with trivial H-action, there is an isomorphism $K^*_H(X) \cong K^*(X) \otimes R(H)$ (see [20, 2.2]). Now the (non-equivariant) K-theory of spheres is well-known [5, 2.5]:

$$K^i(S^k) = 0 \quad \text{if} \quad i \neq k \pmod 2\,, \text{ and } \quad K^i(S^k) = \mathbb{Z} \quad \text{if} \quad i = k \pmod 2\,.$$

Therefore

$$K^i_G(|\Delta|^k, |\Delta|^{k-1}) \cong \begin{cases} \bigoplus_{\sigma \in [\Delta_k/G]} R(G_\sigma) & \text{if } \; i = k \pmod 2, \\ 0 & \text{if } \; i \neq k \pmod 2. \end{cases}$$

Note that this also holds for $k = 0$, with the convention that $|\Delta|^{-1} = \emptyset$. It follows that the "long" exact sequence for the pair $(|\Delta|^k, |\Delta|^{k-1})$ (which is in fact a hexagon because of periodicity) has only 5 terms. If k is even, we get

$$0 \longrightarrow K^1_G(|\Delta|^k) \longrightarrow K^1_G(|\Delta|^{k-1}) \longrightarrow \bigoplus_{\sigma \in [\Delta_k/G]} R(G_\sigma) \longrightarrow K^0_G(|\Delta|^k) \longrightarrow K^0_G(|\Delta|^{k-1}) \longrightarrow 0\,,$$

and this works also for $k=0$ because the empty set has zero K-theory. If k is odd, the sequence is

$$0 \longrightarrow K^0_G(|\Delta|^k) \longrightarrow K^0_G(|\Delta|^{k-1}) \longrightarrow \bigoplus_{\sigma\in[\Delta_k/G]} R(G_\sigma) \longrightarrow K^1_G(|\Delta|^k) \longrightarrow K^1_G(|\Delta|^{k-1}) \longrightarrow 0\,.$$

Tensoring with $\mathbb{C}$ and taking the alternating sum of dimensions, we obtain

$$\chi_G(|\Delta|^k) = \chi_G(|\Delta|^{k-1}) + (-1)^k \dim\big(\bigoplus_{\sigma\in[\Delta_k/G]} {}^{\mathbb{C}}R(G_\sigma)\big)\,.$$

Since $|\Delta|^n = |\Delta|$ if Δ has dimension n, we deduce

$$\chi_G(\Delta) = \sum_{k=0}^{n} (-1)^k \dim\big(\bigoplus_{\sigma\in[\Delta_k/G]} {}^{\mathbb{C}}R(G_\sigma)\big) = \sum_{\sigma\in[\Delta/G]} (-1)^{\dim(\sigma)} \dim {}^{\mathbb{C}}R(G_\sigma)\,,$$

and the proof is complete. □

We now deduce a formula for $\chi_G(\Delta)$ which appears in the recent work of Baum-Connes [8], Atiyah-Segal [7], and Kuhn [16].

Corollary 2.2. *Let Δ be a finite G-complex. Then*

$$\chi_G(\Delta) = \sum_{[g]} \chi(\Delta^g/C_G(g))\,,$$

where Δ^g is the subcomplex of g-fixed points, $C_G(g)$ is the centralizer of g in G, $\Delta^g/C_G(g)$ is the quotient complex, and the sum runs over all conjugacy classes $[g]$ of G.

Proof. Replacing Δ by its barycentric subdivision, we can assume that the stabilizer of every simplex σ fixes σ pointwise. Since the number of conjugacy classes of H is

$$\dim {}^{\mathbb{C}}R(H) = \sum_{h\in H} \frac{1}{|H : C_H(h)|}$$

we have

$$\begin{aligned}
\chi_G(\Delta) &= \sum_{\sigma\in[\Delta/G]} (-1)^{\dim(\sigma)} \dim {}^{\mathbb{C}}R(G_\sigma) = \sum_{\sigma\in\Delta} (-1)^{\dim(\sigma)} \frac{1}{|G:G_\sigma|} \sum_{g\in G_\sigma} \frac{1}{|G_\sigma : C_{G_\sigma}(g)|} \\
&= \frac{1}{|G|} \sum_{\sigma\in\Delta} (-1)^{\dim(\sigma)} \sum_{g\in G_\sigma} |C_G(g)_\sigma| \\
&= \frac{1}{|G|} \sum_{g\in G} |C_G(g)| \sum_{\sigma\in\Delta^g} (-1)^{\dim(\sigma)} \frac{1}{|C_G(g) : C_G(g)_\sigma|} \\
&= \sum_{g\in G} \frac{1}{|G : C_G(g)|} \sum_{\sigma\in[\Delta^g/C_G(g)]} (-1)^{\dim(\sigma)} = \sum_{[g]} \chi(\Delta^g/C_G(g))\,. \ \square
\end{aligned}$$

There is a technical detail: $\Delta^g/C_G(g)$ need not be a simplicial complex, unless one replaces Δ by its barycentric subdivision. Thus the last equality of the proof actually makes sense for the barycentric subdivision. But the proof shows that the Euler characteristic of the quotient space $|\Delta^g|/C_G(g)$ can in fact be computed by the last formula of the proof without bothering about putting a structure of simplicial complex on the quotient.

3. Alperin's conjecture

Let k be an algebraically closed field of characteristic p and let $\ell_p(G)$ be the number of irreducible representations of G over k (which is also the number of p-regular conjugacy classes of G by a well-known result of Brauer). Only a few simple kG-modules are projective (unless p does not divide $|G|$) and we let $z_p(G)$ be the number of projective simple kG-modules. The first standard result of block theory [13, 18.28] implies that this number $z_p(G)$ coincides with the number defined in the introduction.

Alperin's conjecture [1]. *$\ell_p(G) = \sum_P z_p(N_G(P)/P)$, where P runs over a set of representatives of conjugacy classes of p-subgroups of G .*

Apart from some special cases, no bijection is expected to exist, but only a mere equality of numbers. The conjecture is proved for soluble groups, groups of Lie type in characteristic p , symmetric groups, $GL_n(\mathbb{F}_q)$ in arbitrary characteristic, groups with a cyclic or generalized quaternion Sylow p-subgroup, and a few other cases. For each p-block of G , there is also a version of the conjecture which we do not discuss here; the version above is obtained by summing over the blocks the equalities for each block.

Now Knörr and Robinson have found a new formulation of the conjecture, with two equivalent versions. The equivalence between the following two statements is proved in [15] (see also [21] for another approach).

Conjecture (Knörr-Robinson [15]). *The following two conjectures are equivalent.*
(a) $\ell_p(G) - z_p(G) = \sum_{\sigma\in[\mathcal{S}_p(G)/G]} (-1)^{\dim(\sigma)}\, \ell_p(G_\sigma)$.
(b) $k(G) - z_p(G) = \sum_{\sigma\in[\mathcal{S}_p(G)/G]} (-1)^{\dim(\sigma)}\, k(G_\sigma)$.

Now the equivalence between Alperin's conjecture and the Knörr-Robinson conjecture requires an inductive argument. Here is the precise statement (Knörr-Robinson [15]). A slightly different proof can be found in [26].

Theorem 3.1 (Knörr-Robinson [15]). *Let G be a finite group and let $\mathcal{P}$ be a set of representatives of conjugacy classes of p-subgroups of G . The following conditions on G are equivalent.*
(a) Alperin's conjecture holds for G and for every group $N_G(P)/P$ where $P\in\mathcal{P}$.
(b) The Knörr-Robinson conjecture holds for G and for every group $N_G(P)/P$ where $P\in\mathcal{P}$.

By Proposition 2.1, the right hand side of the second form of the Knörr-Robinson conjecture is equal to $\chi_G(\mathcal{S}_p(G))$. Therefore the Knörr-Robinson conjecture can be restated as:

$$k(G) - z_p(G) = \chi_G(\mathcal{S}_p(G))\,.$$

This is precisely the conjecture of the introduction. Thus our K-theoretic conjecture is equivalent to Alperin's conjecture (using the precise formulation of the theorem above). This provides the main evidence for the K-theoretic conjecture. In particular it holds if G is soluble, because each group $N_G(P)/P$ is soluble and Alperin's conjecture holds for soluble groups (using results of Okuyama [17]).

The conjecture also holds for a group of Lie type in characteristic p , but this requires a slight improvement of the statement of the theorem above. It turns out that the inductive argument only uses unipotent radicals P of parabolic subgroups, in which case $N_G(P)$ is precisely a parabolic subgroup. For this improvement one uses the complex $\mathcal{B}_p(G)$ described in Section 1, which is G-homotopy equivalent to $\mathcal{S}_p(G)$, and which coincides here with the complex of unipotent radicals of parabolic subgroups (see [24] for details). The use of $\mathcal{B}_p(G)$ is made quite explicit in [26], and appears also in [15]. When P is a unipotent radical of a parabolic subgroup, then $N_G(P)/P$ is again of Lie type and Cabanes [12] has proved that Alperin's conjecture holds for these groups in defining characteristic p. Therefore the version of Theorem 3.1 using $\mathcal{B}_p(G)$ instead of all p-subgroups implies that the K-theoretic conjecture holds for G.

By Example 1.3, the K-theoretic conjecture holds if $\mathcal{S}_p(G)$ is G-contractible, that is, if G has a normal p-subgroup P. We note however that Alperin's conjecture is not proved in this case. In fact Alperin's conjecture for G and for G/P yield the same formula, so that the proof for a group with a normal p-subgroup would be a proof for all groups. Thus Alperin's conjecture is not directly linked with the topology of $\mathcal{S}_p(G)$, whereas the K-theoretic conjecture or the Knörr-Robinson conjecture are.

4. Orbit complexes

It follows Proposition 2.1 that the cochain complex $C^*(\Delta/G, \mathcal{R})$ is a relevant object to consider. Moreover an efficient way for analyzing this cochain complex lies in the decomposition over conjugacy classes $[g]$ as in Corollary 2.2 (see also Proposition A5 of the Appendix). Therefore in order to obtain information on the equivariant K-theory, it would be helpful to understand better each complex $\Delta^g/C_G(g)$. This includes the orbit complex Δ/G for $g = 1$.

In the special case of Brown's complex $\Delta = \mathcal{S}_p(G)$, some more information is available which says that $\mathcal{S}_p(G)^g/C_G(g)$ is very often contractible, or at least has trivial Euler characteristic. But we are going to see with a few examples that there is still a large variety of possibilities. For simplicity we write $\mathcal{S}_p(G)^g/C_G(g)$, but it should be emphasized that one needs in fact to pass to the barycentric subdivision of $\mathcal{S}_p(G)$ in order to have a simplicial structure on orbit complexes. Recall that g is called *p-regular* if p does not divide the order of g, and *p-singular* otherwise.

Lemma 4.1. *If g is p-singular, then $\mathcal{S}_p(G)^g/C_G(g)$ is contractible.*

Proof. Let P be the Sylow p-subgroup of the cyclic group $< g >$. Then P is non-trivial by assumption and by Lemma 2.1.2 in [27], $\mathcal{S}_p(G)^g$ is contractible via the contractions $K \mapsto KP \mapsto P$ for every p-subgroup K fixed by g. Now this contraction is $C_G(g)$-equivariant and by Corollary 1.2 in [24], this implies that $\mathcal{S}_p(G)^g$ is $C_G(g)$-contractible. Therefore the orbit space is contractible. □

For $g = 1$, there is a weaker result which is sufficient for our purposes.

Lemma 4.2 (Webb [25, 8.2])**.** $\chi(\mathcal{S}_p(G)/G) = 1$ *if p divides $|G|$.*

Another proof of this appears in [21, 4.4]. In fact Webb has proved that $\mathcal{S}_p(G)/G$ is mod p acyclic and he conjectures that $\mathcal{S}_p(G)/G$ is always contractible [26]. The conjecture is proved for p-soluble groups, groups of Lie type in characteristic p, and a few other cases [22].

By Corollary 2.2, our K-theoretic conjecture takes the form

$$k(G) - z_p(G) = \sum_{[g]} \chi(\mathcal{S}_p(G)^g/C_G(g)) \,.$$

Substracting the number of conjugacy classes $k(G)$ and considering the reduced Euler characteristic

$\widetilde{\chi}(\mathcal{S}_p(G)^g/C_G(g)) = \chi(\mathcal{S}_p(G)^g/C_G(g)) - 1$, one obtains the equivalent formulation

$$-z_p(G) = \sum_{[g]} \widetilde{\chi}(\mathcal{S}_p(G)^g/C_G(g)) \,. \tag{4.3}$$

This form of the conjecture was first observed by Bouc and appears explicitly in [21, 6.3]. The advantage is that the sum actually runs over non-trivial p-regular elements, since all p-singular elements as well as $g = 1$ have a zero contribution. Note however that an empty complex has a contribution -1; this occurs when g does not normalize any non-trivial p-subgroup of G.

The problem is now to analyze $\mathcal{S}_p(G)^g/C_G(g)$ when g is p-regular. The following examples show the type of phenomenon which one may expect, but no deep understanding seems to be at hand.

Example 4.4. Suppose that G is p-nilpotent, that is, G has a normal subgroup N of order prime to p such that G/N is a p-group. Then for any $g \in N$, we have $\mathcal{S}_p(G)^g = \mathcal{S}_p(C_G(g))$, because any p-subgroup P normalized by g is centralized by g (since the commutators $[g,P]$ lie in $P \cap N = 1$). By Lemma 4.2, we have $\widetilde{\chi}(\mathcal{S}_p(C_G(g))/C_G(g)) = 0$ if p divides $|C_G(g)|$. In fact $\mathcal{S}_p(C_G(g))/C_G(g)$ is contractible because Webb's conjecture above holds for p-soluble groups. If p does not divide $|C_G(g)|$, which by definition means that g has defect zero, then we get the empty complex, with reduced Euler characteristic -1. It follows that the whole sum 4.3 counts the number of classes of defect zero and that the total result is $z_p(G)$. This is indeed the case because Alperin's conjecture is proved for p-nilpotent groups, so that the formula 4.3 holds. The fact that $z_p(G)$ is the number of classes of defect zero for p-nilpotent groups can also be proved directly.

Example 4.5. It is rather special to have $\mathcal{S}_p(G)^g/C_G(g)$ either empty or contractible, as in the previous example. When $G = GL_n(\mathbb{F}_q)$ and q is a power of p, then a complete information on $\mathcal{S}_p(G)^g/C_G(g)$ appears in [23, 3.3]. Recall that $\mathcal{S}_p(G)$ is G-homotopy equivalent to the building of G, that is, the simplicial complex Δ of chains of non-zero proper subspaces of $V = \mathbb{F}_q^n$. The complex Δ^g of g-invariant subspaces depends on the action of g on V. If g is p-regular (i.e. semi-simple) and if some isotypical component of g has multiplicity ≥ 2, then $\Delta^g/C_G(g)$ is contractible. If g is semi-simple and has k isotypical components, each with multiplicity one, then $\Delta^g/C_G(g)$ is the boundary of a $(k-1)$-simplex, hence is homeomorphic to a sphere S^{k-2} (with the convention $S^{-1} = \emptyset$ when $k = 1$). Thus we see that $\mathcal{S}_p(G)^g/C_G(g)$ can be of arbitrary large dimension. Each Euler characteristic is ± 1 and the formula 4.3 (which is proved for groups of Lie type in characterisic p) yields a curious polynomial identity for partitions of the integer n (see [23] for details). Note that the condition that all isotypical components of g have multiplicity one is equivalent to the requirement that $C_G(g)$ has order prime to p (i.e. g has defect zero).

Example 4.6. It is rather special to have $\mathcal{S}_p(G)^g/C_G(g)$ non-contractible only when g has defect zero, as in the previous two examples. In fact the following example shows that we may have $\widetilde{\chi}(\mathcal{S}_p(G)^g/C_G(g)) \neq 0$ when g has non-zero defect. Let $G = S_5$ be the symmetric group and take $p = 2$. The 3-cycle $g = (123)$ is not of defect zero because (45) centralizes g and in fact $C_G(g) = <(123)> \times <(45)>$. Now $\mathcal{S}_2(G)^g$ consists of 3 points $\{P, Q, R\}$, where P is the Klein four-group which is normal in the symmetric group on the letters $\{1,2,3,4\}$, Q is the Klein four-group which is normal in the symmetric group on the letters $\{1,2,3,5\}$, and R is generated by (45). The action of $C_G(g)$ permutes P and Q, so that $\mathcal{S}_2(G)^g/C_G(g)$ consists of 2 points, with reduced Euler characteristic 1. Note that the only other non-zero contribution to the sum 4.3 appears for a 5-cycle, with an empty complex and reduced Euler characteristic -1; thus we get $z_p(G) = 0$, which is indeed the case for S_5 in characteristic 2.

Appendix: More about equivariant K-theory

We wish to put the results of Section 2 in a more conceptual framework, using the work of G. Segal [20], P. Baum and A. Connes [8], M. Atiyah and G. Segal [7], and N. Kuhn [16]. The whole discussion holds for a compact G-space, but for simplicity we stick to the case of a finite G-complex Δ. We have already noticed at the end of Section 2 that Δ/G may not be a simplicial complex. Thus in order to obtain a simplicial structure on quotients, we assume that Δ is *regular*, in the following sense. Let

$$\sigma = (x_0 < x_1 < \ldots < x_n) \qquad \text{and} \qquad \tau = (y_0 < y_1 < \ldots < y_n)$$

be two (ordered) simplices, and assume that for each i, the vertices x_i and y_i are in the same G-orbit; then σ and τ are in the same G-orbit. When this regularity condition is satisfied, the set Δ/G of orbits is again an ordered simplicial complex, and $|\Delta/G| = |\Delta|/G$, the orbit space. It is no loss of generality to assume this condition since the second barycentric subdivision of an arbitrary G-complex is always regular.

The Atiyah-Hirzebruch spectral sequence for ordinary K-theory has a generalization in the equivariant case which is due to Segal [20, 5.3]. In the special case of finite G-complexes, filtered by skeletons, the spectral sequence can be described as

$$\text{(A1)} \qquad E_2^{pq} = H^p(\Delta/G, \mathcal{R}^q) \implies K_G^*(\Delta)\,,$$

where $\mathcal{R}^q = \mathcal{R}$ is the coefficient system defined in Section 2 if q is even, and $\mathcal{R}^q = 0$ if q is odd. In other words, the coefficients are just the equivariant K-theory of a point made into a coefficient system on Δ/G. When G is the trivial group, we recover the Atiyah-Hirzebruch spectral sequence. We have $E_1^{pq} = C^p(\Delta/G, \mathcal{R}^q)$, the cochain complex of Section 2, and therefore Proposition 2.1 is in fact just saying that the Euler characteristic goes through the spectral sequence, which is essentially obvious since E_{r+1}^{pq} is obtained from E_r^{pq} by taking homology, and the Euler characteristic of a chain complex is equal to the Euler characteristic of its homology.

We are going to see that the spectral sequence collapses when tensored with $\mathbb{C}$. This is a consequence of the following result, due independently to Baum-Connes [8], Kuhn [16], and Atiyah-Segal [7].

Theorem A2. *Let Δ be a finite G-complex. Then there is an isomorphism*

$$\theta : {}^{\mathbb{C}}K_G^*(\Delta) \xrightarrow{\sim} \bigoplus_{[g]} {}^{\mathbb{C}}K^*(\Delta^g/C_G(g))\,,$$

where Δ^g denotes the subcomplex of g-fixed points, $\Delta^g/C_G(g)$ is the quotient complex, and the sum runs over all conjugacy classes $[g]$ of G.

The isomorphism is a generalization of the isomorphism ${}^{\mathbb{C}}R(G) \cong \bigoplus_{[g]} \mathbb{C}$ given by evaluation of characters, which in fact corresponds to the case $\Delta = pt$. The map θ is obtained by restricting an equivariant vector bundle to each subcomplex Δ^g, writing it as a direct sum of (non-equivariant) vector bundles consisting of the eigenspaces of g in every fibre, and putting the corresponding eigenvalue as a scalar coefficient in $\mathbb{C}$. This is where one needs roots of unity and this is why we have tensored K-theory with $\mathbb{C}$ (although $\mathbb{Q}(\zeta)$ would do as well, where ζ is a primitive $|G|$-th root of unity). The above procedure defines an element of ${}^{\mathbb{C}}K^*(\Delta^g)^{C_G(g)}$ for each $[g]$, and the last step in the definition of the map θ uses the isomorphism $K^*(\Delta^g)^{C_G(g)} \cong K^*(\Delta^g/C_G(g))$.

In fact the isomorphism θ also holds for relative K-theory, and since the construction of θ is so "natural", it is indeed a natural transformation of functors (from pairs of finite G-complexes to $\mathbb{C}$-vector spaces). Therefore, for a pair (Δ, Γ) of finite G-complexes, the long exact sequence of equivariant K-theory breaks up as the direct sum over conjugacy classes $[g]$ of the long exact sequences of ordinary K-theory for $(\Delta^g/C_G(g), \Gamma^g/C_G(g))$.

Now the Segal spectral sequence A1 above is constructed from the skeletal filtration of Δ and the corresponding various long exact sequences of equivariant K-theory. Therefore over $\mathbb{C}$, the whole spectral sequence for Δ decomposes as the direct sum over conjugacy classes $[g]$ of the Atiyah-Hirzebruch spectral sequences for the ordinary K-theory of $\Delta^g/C_G(g)$. But it is well-known that the Atiyah-Hirzebruch spectral sequence over $\mathbb{C}$ stops at the E_2-page, that is, $E_2^{pq} = E_\infty^{pq}$ (see [6]). Therefore the same holds for the spectral sequence A1.

The fact that the Atiyah-Hirzebruch spectral sequence collapses tells us that

$$\text{(A3)} \qquad {}^{\mathbb{C}}K^i(\Delta^g/C_G(g)) \cong \begin{cases} H^{even}(\Delta^g/C_G(g), \mathbb{C}) & \text{if } i = 0, \\ H^{odd}(\Delta^g/C_G(g), \mathbb{C}) & \text{if } i = 1. \end{cases}$$

The isomorphism is in fact given by the Chern character. Since the spectral sequence A1 also collapses, we have the following analogous result, which can be viewed as a much more precise version of Proposition 2.1. This is the equivariant Chern character isomorphism of Baum and Connes [8].

Theorem A4. *Let Δ be a finite G-complex. Then*

$$ {}^{\mathbb{C}}K_G^i(\Delta) \cong \begin{cases} H^{even}(\Delta/G, {}^{\mathbb{C}}\mathcal{R}) & \text{if } i = 0, \\ H^{odd}(\Delta/G, {}^{\mathbb{C}}\mathcal{R}) & \text{if } i = 1. \end{cases}$$

The cohomology appearing in this theorem also breaks up as a direct sum over conjugacy classes of G. Indeed we have the decomposition of the cochain complex

$$\begin{aligned} C^k(\Delta/G, {}^{\mathbb{C}}\mathcal{R}) &= \bigoplus_{\sigma\in[\Delta_k/G]} {}^{\mathbb{C}}R(G_\sigma) \cong \bigoplus_{\sigma\in[\Delta_k/G]} \bigoplus_{\{[g] \mid g\in G_\sigma\}} \mathbb{C} \\ &= \bigoplus_{\{[g] \mid g\in G\}} \bigoplus_{\sigma\in[(\Delta^g)_k/C_G(g)]} \mathbb{C} = \bigoplus_{\{[g] \mid g\in G\}} C^k(\Delta^g/C_G(g), \mathbb{C}), \end{aligned}$$

and it is easy to see that the coboundary of $C^*(\Delta/G, {}^{\mathbb{C}}\mathcal{R})$ is the direct sum over $[g]$ of the coboundaries of $C^*(\Delta^g/C_G(g), \mathbb{C})$. Therefore we have the following proposition.

Proposition A5. $H^*(\Delta/G, {}^{\mathbb{C}}\mathcal{R}) \cong \bigoplus_{[g]} H^*(\Delta^g/C_G(g), \mathbb{C})$.

In fact Baum and Connes view the right hand side as the cohomology of a single space, namely $\widehat{\Delta}/G$, where $\widehat{\Delta} = \{ (x,g) \in \Delta \times G \mid g\cdot x = x \}$, using the given action of G on Δ and the conjugation action on the second factor G. It is easy to see that $\widehat{\Delta}/G$ is the disjoint union of the complexes $\Delta^g/C_G(g)$.

Putting together A2, A3, A4 and A5, we obtain two commutative diagrams of isomorphisms:

$$\text{(A6)} \qquad \begin{array}{ccc} {}^{\mathbb{C}}K_G^0(\Delta) & \xrightarrow{\sim} & H^{even}(\Delta/G, {}^{\mathbb{C}}\mathcal{R}) \\ \downarrow\wr & & \downarrow\wr \\ \bigoplus_{[g]} {}^{\mathbb{C}}K^0(\Delta^g/C_G(g)) & \xrightarrow{\sim} & \bigoplus_{[g]} H^{even}(\Delta^g/C_G(g), \mathbb{C}), \end{array}$$

and similarly

$$\text{(A7)}\qquad \begin{array}{ccc} {}^{\mathbb{C}}K^1_G(\Delta) & \xrightarrow{\sim} & H^{odd}(\Delta/G, {}^{\mathbb{C}}\mathcal{R}) \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus\limits_{[g]} {}^{\mathbb{C}}K^1(\Delta^g/C_G(g)) & \xrightarrow{\sim} & \bigoplus\limits_{[g]} H^{odd}(\Delta^g/C_G(g), \mathbb{C})\,. \end{array}$$

These two diagrams summarize the whole discussion and imply in particular the formulae for the Euler characteristic $\chi_G(\Delta)$ given in Proposition 2.1 and Corollary 2.2.

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