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Author(s): Donna M. Testerman
Source: *Proceedings of the American Mathematical Society*, Vol. 116, No. 3 (Nov., 1992), pp. 635-644
Published by: American Mathematical Society
Stable URL: https://www.jstor.org/stable/2159429
Accessed: 08-10-2019 06:19 UTC

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 3, November 1992

THE CONSTRUCTION OF THE MAXIMAL A_1'S IN THE EXCEPTIONAL ALGEBRAIC GROUPS

DONNA M. TESTERMAN

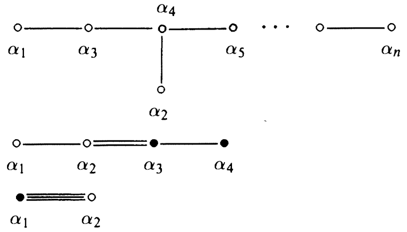
(Communicated by Warren J. Wong)

ABSTRACT. Let G be a simply connected simple algebraic group of exceptional type defined over an algebraically closed field of characteristic $p > 3, 3, 5, 7, 7$, for G of type G_2, F_4, E_6, E_7, E_8, respectively. We construct the maximal closed connected subgroups of G, that are simple of type A_1. This completes Seitz's classification (under the indicated prime restrictions) of the maximal closed connected subgroups of G.

Let G be a simply connected simple algebraic group of exceptional type defined over an algebraically closed field k of characteristic $p > 0$. In this paper we construct closed connected subgroups of G that are simple, of type A_1, and maximal among closed connected subgroups of G. Moreover, under certain weak prime restrictions, these are known to be the only maximal A_1's in the exceptional algebraic groups. (See [3].) To apply the results of [3], we construct subgroups of type A_1 with a maximal torus having a prescribed action on the Lie algebra of G. We state our main result in these terms, but first we introduce some notation.

Let $\Phi(G)$ denote the root system of G and take $\Pi(G) = \{\alpha_1, \alpha_2, \dots\}$ to be a fundamental system of $\Phi(G)$, with $\Phi^+(G)$ the associated set of positive roots. Let $\{x_\alpha, y_\alpha, t_\gamma \mid \alpha \in \Phi^+(G), \gamma \in \Pi(G)\}$ be a basis of $L(G)$, the Lie algebra of G, where $\langle t_\gamma \mid \gamma \in \Pi(G) \rangle$ is the Lie algebra of T, a maximal torus of G, and $\langle x_\alpha \rangle$, respectively $\langle y_\alpha \rangle$, is the T-root subspace corresponding to the root α, respectively $-\alpha$. (See [3, (1.1)].) We fix the following labelling of Dynkin diagrams, where the darkened nodes represent the short roots.

E_n:, F_4:, G_2:



We can now state our result:

Received by the editors November 26, 1990 and, in revised form, April 3, 1991; the results of this paper were presented at the 851st meeting of the AMS in Hoboken, NJ on October 22, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20E28, 20G05, 20G15.

Theorem 1. *For each of the groups G given below, and for each of the prime restrictions, there exists a closed, connected subgroup $X \leq G$ such that X has type A_1 and for some maximal torus $T_X = \{T_X(c) \mid c \in k^*\}$ of X, the action of T_X on $L(G)$ is given by $T_X(c)x_\alpha = c^{d(\alpha)}x_\alpha$, $T_X(c)y_\alpha = c^{-d(\alpha)}y_\alpha$, and $T_X(c)t_\gamma = t_\gamma$ for all $\alpha \in \Phi^+(G)$, $\gamma \in \Pi(G)$, where $d(\alpha + \beta) = d(\alpha) + d(\beta)$ and $\{d(\alpha) \mid \alpha \in \Pi(G)\}$ are as indicated.*

(i) $G = G_2$, $p \geq 7$, $d(\alpha) = 2$ *for all* $\alpha \in \Pi(G)$.
(ii) $G = F_4$, $p \geq 13$, $d(\alpha) = 2$ *for all* $\alpha \in \Pi(G)$.
(iii) $G = E_7$, $p \geq 19$, $d(\alpha) = 2$ *for all* $\alpha \in \Pi(G)$.
(iv) $G = E_7$, $p \geq 17$, $d(\alpha) = 2$ *for* $\alpha \in \Pi(G)$, $\alpha \neq \alpha_4$, *and* $d(\alpha_4) = 0$.
(v) $G = E_8$, $p \geq 31$, $d(\alpha) = 2$ *for all* $\alpha \in \Pi(G)$.
(vi) $G = E_8$, $p \geq 29$, $d(\alpha) = 2$ *for* $\alpha \in \Pi(G)$, $\alpha \neq \alpha_4$, *and* $d(\alpha_4) = 0$.
(vii) $G = E_8$, $p \geq 23$, $d(\alpha) = 2$ *for* $\alpha \in \Pi(G)$, $\alpha \neq \alpha_4, \alpha_6$, *and* $d(\alpha_4) = 0 = d(\alpha_6)$.

Combining Theorem 1 with results in [3], we obtain

Theorem 2. *Let G be as above and assume $p > 3, 3, 5, 7, 7$, for G of type G_2, F_4, E_6, E_7, E_8, respectively. Then a simple closed connected subgroup Y of G, with Y of type A_1, is maximal among proper closed connected subgroups of G if and only if $G = G_2, F_4, E_7, E_7, E_8, E_8, E_8$, $p \geq 7, 13, 19, 17, 31, 29, 23$, respectively, and Y is conjugate in* Aut(G) *to X as described in Theorem* 1(i), (ii), (iii), (iv), (v), (vi), (vii), *respectively.*

Remarks. (1) Our construction of the A_1's in fact produces A_1's in Chevalley groups over arbitrary fields of suitable characteristic; we state this result (Theorem 3) after introducing further notation.

(2) The existence of a maximal A_1 (with the described action) in the algebraic group G_2 is established in [6]. Nevertheless, we include the proof here, since we establish as well the existence of the A_1 in the Chevalley groups over arbitrary fields of characteristic $p \geq 7$.

(3) In [3] under certain weak prime restrictions, Seitz establishes a list of the possible maximal (among closed connected subgroups) semisimple subgroups of the exceptional algebraic groups in nonzero characteristic. In every case, he establishes the maximality of the groups, assuming their existence; if such a subgroup has rank greater than 1, he establishes as well the existence. The existence of the rank 1 subgroups appearing on Seitz's list is provided by our Theorem 1.

(4) The method of construction is fairly general and should have further applications in the study of the subgroup structure of algebraic and finite groups. In particular, we describe a sufficient condition for exponentiating $\operatorname{ad} e$, for nilpotent elements e in a semisimple complex Lie algebra, to obtain automorphisms of Lie algebras over fields of characteristic p, for certain primes p for which $(\operatorname{ad} e)^p \neq 0$. (See Lemma 3.)

Before proceeding with the proof of the theorems, we wish to mention that the research for this paper was done while the author was in residence at the Institute for Advanced Study. We thank this institution for its hospitality and extend thanks as well to Professor Richard Lyons of Rutgers University for the helpful conversations we had concerning this project.

The A_1's are constructed by "exponentiating" suitable sl_2 subalgebras in a semisimple Lie algebra over $\mathbf{C}$. Indeed, the construction follows the construction of Chevalley groups as presented in [1, §§4.3, 4.4; 5, §3]. The essential difference is that we require the exponential to preserve a lattice over a localization of $\mathbf{Z}$ (p-local integers for some prime p) rather than over $\mathbf{Z}$ itself.

Let $L_G(\mathbf{C})$ be a simple Lie algebra over $\mathbf{C}$ with root system $\Phi(G)$ and Chevalley basis $\mathscr{B} = \{e_\alpha, f_\alpha, h_\gamma \mid \alpha \in \Phi^+(G), \gamma \in \Pi(G)\}$. Fix a prime p, and let $\mathbf{Z}_{(p)}$ be the localization of $\mathbf{Z}$ at the prime ideal $p\mathbf{Z}$. Let $L_G(\mathbf{Z}_{(p)})$ be the set of $\mathbf{Z}_{(p)}$ linear combinations of elements of the Chevalley basis. Then $L_G(\mathbf{Z}_{(p)}) \simeq \mathbf{Z}\mathscr{B} \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$. Let F be any field of characteristic p and $L_G(F) = L_G(\mathbf{Z}_{(p)}) \otimes_{\mathbf{Z}_{(p)}} F \simeq \mathbf{Z}\mathscr{B} \otimes_{\mathbf{Z}} F$. Then $L_G(F)$ is a Lie algebra over F with basis $\mathscr{B}' = \{v \otimes 1 \mid v \in \mathscr{B}\}$. For convenience, write $\bar{v}$ for $v \otimes 1$. Note that the multiplication constants of $L_G(F)$ with respect to the basis $\mathscr{B}'$ are those of $L_G(\mathbf{C})$ with respect to $\mathscr{B}$, interpreted as elements of the prime subfield of F. Finally, let $G(F)$ be the adjoint Chevalley group of type $\Phi(G)$ defined over F. So $G(F) \leq \operatorname{Aut}(L_G(F))$.

We can now state

Theorem 3. *Let G, p, and $d(\alpha)$ be as in Theorem* 1 *and F be an arbitrary field of characteristic p. Then there exists a homomorphism ϕ: $\mathrm{PSL}_2(F) \to \operatorname{Aut}(L_G(F))$ such that*

$$\phi\pi\begin{pmatrix} c & 0 \\ 0 & c^{-1}\end{pmatrix}\bar{e}_\alpha = c^{d(\alpha)}\bar{e}_\alpha,$$

$$\phi\pi\begin{pmatrix} c & 0 \\ 0 & c^{-1}\end{pmatrix}\bar{f}_\alpha = c^{-d(\alpha)}\bar{f}_\alpha,$$

$$\phi\pi\begin{pmatrix} c & 0 \\ 0 & c^{-1}\end{pmatrix}\bar{h}_\gamma = \bar{h}_\gamma$$

for all $\alpha \in \Phi^+(G)$, $\gamma \in \Pi(G)$, where π: $\mathrm{SL}_2(F) \to \mathrm{PSL}_2(F)$ is the natural surjection. Moreover, $\phi(\mathrm{PSL}_2(F)) \leq G(F)$.

We now proceed with the basic lemmas that form the proofs of the theorems. We continue with the notation introduced thus far, and mention that any necessary restrictions on the prime p are indicated in the statements of the lemmas. We wish to define automorphisms of $L_G(F)$ associated with certain elements $e \in L_G(\mathbf{C})$. Let $e = \sum c_\alpha e_\alpha$, where α ranges over $\Phi^+(G)$ and $c_\alpha \in \mathbf{Z}$, such that $\operatorname{ad} e$ is a nilpotent transformation of $L_G(\mathbf{C})$ with $(\operatorname{ad} e)^k/k!$ preserving $L_G(\mathbf{Z}_{(p)})$ for all $k \geq 0$. That is, $[(\operatorname{ad} e)k/k!](L_G(\mathbf{Z}_{(p)})) \subseteq L_G(\mathbf{Z}_{(p)})$. Then for $\lambda \in \mathbf{C}$, if $\exp(\operatorname{ad} \lambda e)$ acting on $L_G(\mathbf{C})$ is represented by the matrix $A(\lambda)$ with respect to the basis $\mathscr{B}$, then the entries of $A(\lambda)$ lie in $\mathbf{Z}_{(p)}[\lambda]$. Now let $t \in F$ and $\overline{A}(t)$ be the matrix obtained from $A(\lambda)$ by replacing each entry $f(\lambda) \in \mathbf{Z}_{(p)}[\lambda]$ by $\bar{f}(t)$ where $\bar{f}$ is the image of f under the natural homomorphism $\mathbf{Z}_{(p)}[x] \to F[x]$. Define $x(t)$ to be the linear transformation of $L_G(F)$ represented by the matrix $\overline{A}(t)$ with respect to the basis $\mathscr{B}$. In Lemmas 1 and 2 we use the polynomial identities that hold for the entries of $A(\lambda)$ to establish identities for the entries of $\overline{A}(t)$, which then imply the stated results about $x(t)$.

Lemma 1. (1) *$x(t)$ is a Lie algebra automorphism of $L_G(F)$ for all $t \in F$.*

(2) *Assume $p > 3$, 2 for G of type E_6, E_7, respectively. Then $x(t) \in G(F) \leq \operatorname{Aut}(L_G(F))$.*

Proof. Since $A(\lambda)A(-\lambda) = I$, $x(t)x(-t)$ is the identity transformation of $L_G(F)$, so $x(t)$ is invertible.

Let $\{v_1, v_2, \dots\}$ be the Chevalley basis $\mathscr{B}$ of $L_G(\mathbf{C})$, hence $\mathscr{B}' = \{\bar{v}_1, \bar{v}_2, \dots\}$. Say

$$(\#) \qquad [v_i v_j] = \sum_k \gamma_{ijk} v_k \quad \text{for some } \gamma_{ijk} \in \mathbf{Z}.$$

Then, letting $\bar{\gamma}_{ijk}$ denote $\sum_{s=1}^{\gamma_{ijk}} 1_F$, we have $[\bar{v}_i \bar{v}_j] = \sum_k \bar{\gamma}_{ijk} \bar{v}_k$. For $\lambda \in \mathbf{C}$, $\exp(\operatorname{ad} \lambda e) v_i = \sum_j A(\lambda)_{ji} v_j$ and $x(t)\bar{v}_i = \sum_j \overline{A}(t)_{ji} \bar{v}_j$. Applying the algebra automorphism $\exp(\operatorname{ad} \lambda e)$ to both sides of (#), we obtain

$$\sum_{r,s} A(\lambda)_{ri} A(\lambda)_{sj} \gamma_{rsl} = \sum_k \gamma_{ijk} A(\lambda)_{lk} \quad \text{for all } i, j, l.$$

That is, the polynomial

$$\sum_k \gamma_{ijk} A(x)_{lk} - \sum_{r,s} A(x)_{ri} A(x)_{sj} \gamma_{rsl} \in \mathbf{Z}_{(p)}[x]$$

vanishes for all $\lambda \in \mathbf{C}$; so it is identically 0. Thus, the polynomial

$$\sum_k \bar{\gamma}_{ijk} \overline{A}(x)_{lk} - \sum_{r,s} \overline{A}(x)_{ri} \overline{A}(x)_{sj} \bar{\gamma}_{rsl} \in F[x]$$

is also identically 0, guaranteeing that

$$x(t)[\bar{v}_i \bar{v}_j] = [x(t)\bar{v}_i, x(t)\bar{v}_j] \quad \text{for all } i \text{ and } j \text{ and for all } t \in F.$$

Thus, $x(t)$ is a Lie algebra automorphism of $L_G(F)$.

By Steinberg (see [4, §4]) there exists a normal subgroup $A \le \operatorname{Aut}(L_G(F))$ with $G(F) \le A$ and such that $\operatorname{Aut}(L_G(F))/A$ is isomorphic to the group of graph automorphisms of $L_G(F)$ and $A/G(F)$ is isomorphic to the group $F^*/(F^*)^d$, where $d = 1$ for G of type G_2, F_4, E_8, $d = 3$ for G of type E_6 and $d = 2$ for G of type E_7. Since the order of $x(t)$ is p, the restrictions on p imply that $x(t) \in G(F)$, and the result holds.

In addition to the above, let $f = \sum d_\alpha f_\alpha$, $\alpha \in \Phi^+(G)$, $d_\alpha \in \mathbf{Z}$, such that $\operatorname{ad} f$ is a nilpotent transformation of $L_G(\mathbf{C})$ with $(\operatorname{ad} f)^k/k!$ preserving $L_G(\mathbf{Z}_{(p)})$ for $k \ge 0$. For $t \in F$, let $y(t)$ be the automorphism of $L_G(F)$, represented by the matrix $\overline{B}(t)$ with respect to the basis $\mathscr{B}'$, where $\overline{B}(t)$ is obtained (as with $\overline{A}(t)$) from the matrix $B(\lambda)$ representing $\exp(\operatorname{ad} \lambda f)$ with respect to the basis $\mathscr{B}$. Moreover, assume e and f canonically generate an $sl_2(\mathbf{C})$ subalgebra of $L_G(\mathbf{C})$ with $[e, f] \in \sum\{\mathbf{Z} h_\gamma \mid \gamma \in \Pi(G)\}$. That is, $[[e, f]e] = 2e$ and $[[e, f]f] = -2f$, and if $h = [e, f]$ then $\mathscr{B}$ is a basis of eigenvectors for $\operatorname{ad} h$, with $[h, e_\alpha] = \alpha(h) e_\alpha$, $[h, f_\alpha] = -\alpha(h) f_\alpha$, and $[h, h_\gamma] = 0$ for $\alpha \in \Phi^+(G)$, $\gamma \in \Pi(G)$. Moreover, $\alpha(h) \in \mathbf{Z}$ for all $\alpha \in \Phi^+(G)$.

Lemma 2. (i) *$X = \langle x(t), y(t) \mid t \in F \rangle$ is isomorphic to* $(\mathrm{P})\mathrm{SL}_2(F)$.

(ii) *Taking $T_X = \{h(c) \mid c \in F^*\}$ where $h(c) = x(c)y(-c^{-1})x(c)x(-1) \times y(1)x(-1)$, T_X is isomorphic to the multiplicative group of F and its action on $L_G(F)$ is a diagonal action with respect to $\mathscr{B}'$ given by $h(c)\bar{e}_\alpha = c^{\alpha(h)} \bar{e}_\alpha$, $h(c)\bar{f}_\alpha = c^{-\alpha(h)} \bar{f}_\alpha$, and $h(c)\bar{h}_\gamma = \bar{h}_\gamma$ for $\alpha \in \Phi^+(G)$, $\gamma \in \Pi(G)$, $c \in F^*$.*

Proof. To see that $X \cong \mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$, we will check the following relations of Steinberg (from [5, §6]):

(a) $x(t)$ is additive in t.

(b) $w(t)x(u)w(-t) = y(-t^{-2}u)$ for $t \in F^*$, $u \in F$, where $w(t) = x(t)y(-t^{-1})x(t)$.
(c) $h(t)$ is multiplicative in t, where $h(t) = w(t)w(-1)$.

Recall the following basic

Lemma [1, 5.1.1]. *Let L be a simple Lie algebra over $\mathbf{C}$. Let $y \in L$ such that $\operatorname{ad} y$ is nilpotent and let $\theta \in \operatorname{Aut}(L)$. Then $\theta \exp(\operatorname{ad} y)\theta^{-1} = \exp(\operatorname{ad} \theta y)$.*

This lemma implies the following identities, where $A(\lambda)$ and $B(\mu)$ are the matrices corresponding to the automorphisms $\exp(\operatorname{ad} \lambda e)$ and $\exp(\operatorname{ad} \mu f)$, respectively, and

$$W(\lambda) = A(\lambda)B(-\lambda^{-1})A(\lambda) \quad \text{for } \lambda \in \mathbf{C},\ \lambda \neq 0.$$

(A) $A(\lambda)A(\mu) = A(\lambda + \mu)$ for all $\lambda, \mu \in \mathbf{C}$ and
(B) $W(\lambda)A(\mu)W(-\lambda) = B(-\lambda^{-2}\mu)$ for all $\lambda, \mu \in \mathbf{C}$, $\lambda \neq 0$.
As well, by Lemma 19 of [5],
(C) $W(\lambda)W(-1)W(\mu)W(-1) = W(\mu)W(-1)$ for all $\lambda, \mu \in \mathbf{C}^*$.

Now, the identity (A) produces polynomial identities in $\mathbf{Z}_{(p)}[x, x^{-1}, y]$, which then imply the polynomial identities in $F[x, x^{-1}, y]$ necessary to establish the identity $\overline{A}(t)\overline{A}(u) = \overline{A}(t+u)$ for $t, u \in F$. But this last equation is the matrix form of (a). Argue similarly, using (B) and (C), to obtain (b) and (c).

For (ii) we again refer to Lemma 19 of [5] to see that, for $\lambda \in \mathbf{C}^*$, $\alpha \in \Phi^+(G)$, $\gamma \in \Pi(G)$,

$$W(\lambda)W(-1)e_\alpha = \lambda^{\alpha(h)}e_\alpha,$$
$$W(\lambda)W(-1)f_\alpha = \lambda^{-\alpha(h)}f_\alpha,$$

and

$$W(\lambda)W(-1))h_\gamma = h_\gamma.$$

These equalitites produce the necessary matrix identities to conclude that $h(c)\bar{e}_\alpha = c^{\alpha(h)}\bar{e}_\alpha$, $h(c)\bar{f}_\alpha = c^{-\alpha(h)}\bar{f}_\alpha$, and $h(c)\bar{h}_\gamma = \bar{h}_\gamma$ for $\alpha \in \Phi^+(G)$, $\gamma \in \Pi(G)$, $c \in F^*$.

The following lemma provides a criterion for establishing the condition

$$[(\operatorname{ad} e)^k/k!]\big(L_G(\mathbf{Z}_{(p)})\big) \subseteq L_G(\mathbf{Z}_{(p)}).$$

The proof is based on a variation of the arguments given in §5.7 of [2] and allows one to "exponentiate" certain nilpotent elements e for some primes p for which $(\operatorname{ad} e)^p \neq 0$. Before stating the lemma, we need additional notation.

For a subset $J \subseteq \Pi(G)$, let $\Phi(J) = \Phi(G) \cap \sum\{\mathbf{Z}\alpha \mid \alpha \in J\}$ and $\Phi^+(J) = \Phi(J) \cap \Phi^+(G)$. Recall the height function (relative to $\Pi(G)$) defined on Φ by $\operatorname{ht}(\sum_{\gamma\in\Pi(G)} k_\gamma\gamma) = \sum_{\gamma\in\Pi(G)} k_\gamma$. Recall as well, the partial ordering induced by $\Pi(G)$ on the Euclidean space E spanned by the roots Φ: $\mu \prec \lambda$ if and only if $\lambda - \mu$ is a sum of positive roots. We define a new height function corresponding to a subset $J \subseteq \Pi(G)$,

$$\operatorname{ht}_J\left(\sum_{\gamma\in\Pi(G)} k_\gamma\gamma\right) = \sum_{\gamma\notin J} k_\gamma.$$

So $\mathrm{ht}_\varnothing(r)$ is $\mathrm{ht}(r)$ in the usual sense, and for $\alpha \prec r$, $\mathrm{ht}_J(\alpha) \le \mathrm{ht}_J(r)$. For $\alpha \in \Phi^+(G)$, let $L_\alpha = \langle e_\alpha \rangle$, $L_{-\alpha} = \langle f_\alpha \rangle$, and $L_0 = \langle h_\gamma \mid \gamma \in \Pi(G)\rangle$ in $L_G(\mathbf{C})$. For $n = \sum_{\alpha\in\Phi^+(G)} c_\alpha e_\alpha$, $c_\alpha \in \mathbf{C}$ and for $J \subseteq \Pi(G)$, write $n_J = \sum_{\alpha\in\Phi^+(G)-\Phi^+(J)} c_\alpha e_\alpha$. Finally, let r_o denote the highest root in $\Phi(G)$.

Lemma 3. *Let* $e \in \sum_{\alpha\in\Phi^+(G)} \mathbf{Z}e_\alpha$.

(i) *If* $p > \mathrm{ht}(r_o)$, *then* $(\mathrm{ad}\, e)^k/k!$ *preserves* $L_G(\mathbf{Z}_{(p)})$ *for all* $k \ge 0$.

(ii) *Assume* $e = e_J = e_1 + e_2$, *where* $e_i \in \sum_{\alpha\in\Phi^+(G)} \mathbf{Z}e_\alpha$ *and* $e_i = (e_i)_{J_i}$ *for some* J, $J_i \subseteq \Pi(G)$ *with* $p > \mathrm{ht}_J(r_o)$, $p > 2\,\mathrm{ht}_{J_i}(r_o)$. *Then* $(\mathrm{ad}\, e)^k/k!$ *preserves* $L_G(\mathbf{Z}_{(p)})$ *for all* $k \ge 0$.

Proof. We first note some identities in the polynomial ring $\mathbf{Z}[x, y]$. For a prime p, $(x-y)^p = x^p - y^p + p\sum_{i=1}^{p-1} m_i x^i y^{p-i}$, $m_i \in \mathbf{Z}$, and for any integer k, $x^k - y^k = (x-y)\sum_{i=0}^{k-1} x^{k-i-1}y^i$. So the irreducible polynomial $(x-y)$ divides $p\sum_{i=1}^{p-1} m_i x^i y^{p-i}$; so $(x-y)$ divides $\sum_{i=1}^{p-1} m_i x^i y^{p-i}$ in $\mathbf{Z}[x, y]$. That is, $\sum_{i=1}^{p-1} m_i x^i y^{p-i} = (x-y)g(x, y)$ for some $g(x, y) \in \mathbf{Z}[x, y]$. (It is not difficult to express $g(x, y)$ explicitly, however, it is not necessary for our purposes.) Combining these statements, we have

$$(x-y)^p = (x-y)\sum_{i=0}^{p-1} x^{p-i-1}y^i + p(x-y)g(x, y).$$

So

$$(1) \qquad (x-y)^{p-1} = \sum_{i=0}^{p-1} x^{p-i-1}y^i + pg(x, y) \quad \text{for some } g(x, y) \in \mathbf{Z}[x, y].$$

Now following Jacobson [2, 5.7], we use (1) to obtain relations in any associative algebra $\mathscr{A}$. Let $A \in \mathscr{A}$. Then in the above we may take $x = A_L$, $y = A_R$, the left and right multiplications determined by A. Doing so, we have

$$(A_L - A_R)^{p-1} = \sum_{i=0}^{p-1} A_L^{p-i-1}A_R^i + pg(A_L, A_R);$$

so

$$(2) \qquad (\mathrm{ad}\, A)^{p-1}(a) = \sum_{i=0}^{p-1} A^{p-i-1}aA^i + pg(A_L, A_R)(a) \quad \text{for any } a \in \mathscr{A}.$$

Now let $a, b \in \mathscr{A}$ and λ be an indeterminate. Set

$$(3) \qquad (\lambda a + b)^p = \lambda^p a^p + b^p + \sum_{i=1}^{p-1} s_i(a, b)\lambda^i,$$

where $s_i(a, b)$ is a polynomial in a and b of total degree p.

Recall that a and b do not necessarily commute. Then, differentiating both sides of (3) with respect to λ gives

$$\sum_{i=0}^{p-1}(\lambda a + b)^{p-i-1}a(\lambda a + b)^i = p\lambda^{p-1}a^p + \sum_{\iota=0}^{p-1} \iota s_\iota(a, b)\lambda^{\iota-1}.$$

Then by (2),

$$(\mathrm{ad}(\lambda a+b))^{p-1}(a)-pg((\lambda a+b)_L,(\lambda a+b)_R)(a)=p\lambda^{p-1}a^p+\sum_{i=1}^{p-1} is_i(a,b)\lambda^{i-1};$$

so

(4) for $i=1,\dots,p-1$, $is_i(a,b)$ is the coefficient of λ^{i-1} in $\left[(\mathrm{ad}(\lambda a+b))^{p-1}(a)-pg((\lambda a+b)_L,(\lambda a+b)_R)(a)\right]$.

Now let $n\in\sum_{\alpha\in\Phi^+(G)}\mathbf{C}e_\alpha$ such that $n=n_J$. We consider the action of $(\mathrm{ad}\,n)^k$ on $\mathscr{B}$, the Chevalley basis of $L_G(\mathbf{C})$. First note that since $(\mathrm{ad}\,n)^k(e_\gamma)\subseteq\sum\{L_s\mid \mathrm{ht}_J(s)\geq\mathrm{ht}_J(\gamma)+k\}$,

(5) if $k\geq\mathrm{ht}_J(r_o)$ then $(\mathrm{ad}\,n)^k(e_\gamma)=0$ for all $\gamma\in\Phi^+(G)-\Phi(J)$.

Also, since $(\mathrm{ad}\,n)^j(L_r)\subseteq\sum\{L_s\mid\mathrm{ht}_J(s)\geq\mathrm{ht}_J(r)+j\}$ for all $r\in\Phi(G)$,

$$(\mathrm{ad}\,n)^j(L_G(\mathbf{C}))\subseteq\sum\{L_s\mid\mathrm{ht}_J(s)\geq\mathrm{ht}_J(-r_o)+2\,\mathrm{ht}_J(r_o)+1\}.$$

In particular,

(6) if $j>2\,\mathrm{ht}_J(r_o)$ then $(\mathrm{ad}\,n)^j=0$.

Therefore, $(\mathrm{ad}\,n)^{2p}=0$. Thus, the only possible p-divisible denominators in a kth power arise from the case $k=p$ itself. So to show that $(\mathrm{ad}\,n)^k/k!$ preserves $L_G(\mathbf{Z}_{(p)})$ for all $k\geq 0$, it suffices to show that $(\mathrm{ad}\,n)^p/p!$ preserves $L_G(\mathbf{Z}_{(p)})$.

Let $\mathscr{A}=\mathrm{gl}(\mathscr{U})$, where $\mathscr{U}$ is the universal enveloping algebra of $L_G(\mathbf{C})$. Recalling that ad is a multiplicative homomorphism of $\mathscr{U}$ into $\mathscr{A}$ and using induction on k, one checks the following identity in operators in $\mathscr{A}$:

(7) $$(\mathrm{ad}_{\mathscr{A}}(\lambda\,\mathrm{ad}\,x+\mathrm{ad}\,y))^k(\mathrm{ad}\,u)=\mathrm{ad}((\mathrm{ad}(\lambda x+y))^k(u)),$$ for all $k\geq 0$ and for $x,y,u\in L_G(\mathbf{C})$.

Now apply (3) and (4) with $\lambda=1$, $a=\mathrm{ad}\,e_1$, and $b=\mathrm{ad}\,e_2$ to get

$$(\mathrm{ad}\,e)^p=(\mathrm{ad}\,e_1)^p+(\mathrm{ad}\,e_2)^p+\sum_{i=1}^{p-1}s_i(\mathrm{ad}\,e_1,\mathrm{ad}\,e_2),$$

where $s_i(\mathrm{ad}\,e_1,\mathrm{ad}\,e_2)$ is $(1/i)$ times the coefficient of λ^{i-1} in

$$\left(\mathrm{ad}_{\mathscr{A}}(\lambda\,\mathrm{ad}\,e_1+\mathrm{ad}\,e_2)\right)^{p-1}(\mathrm{ad}\,e_1)-pg((\lambda\,\mathrm{ad}\,e_1+\mathrm{ad}\,e_2)_L,(\lambda\,\mathrm{ad}\,e_1+\mathrm{ad}\,e_2)_R)(\mathrm{ad}\,e_1).$$

By (7),

$$\left(\mathrm{ad}_{\mathscr{A}}(\lambda\,\mathrm{ad}\,e_1+\mathrm{ad}\,e_2)\right)^{p-1}(\mathrm{ad}\,e_1)=\mathrm{ad}((\mathrm{ad}(\lambda e_1+e_2))^{p-1}(e_1)).$$

Now applying (5) to λe_1+e_2 in place of n, and recalling the fact that $p>\mathrm{ht}_J(r_o)$, we have $\mathrm{ad}(\lambda e_1+e_2)^{p-1}(e_1)=0$. So for $1\leq i\leq p-1$,

$$s_i(\mathrm{ad}\,e_1,\mathrm{ad}\,e_2)$$
$$=-\frac{p}{i}(\text{coeff. of }\lambda^{i-1})\text{ in }g((\lambda\,\mathrm{ad}\,e_1+\mathrm{ad}\,e_2)_L,(\lambda\,\mathrm{ad}\,e_1+\mathrm{ad}\,e_2)_R)(\mathrm{ad}\,e_1).$$

Then dividing by $p!$ gives

$$s_i(\operatorname{ad} e_1, \operatorname{ad} e_2)$$
$$= -\frac{1}{i(p-1)!}(\text{coeff. of } \lambda^{i-1}) \text{ in } g((\lambda \operatorname{ad} e_1 + \operatorname{ad} e_2)_L, (\lambda \operatorname{ad} e_1 + \operatorname{ad} e_2)_R)(\operatorname{ad} e_1),$$

which clearly preserves $L_G(\mathbf{Z}_{(p)})$. Finally, we note that since $e_i = (e_i)_{J_i}$ and $p > 2\,\mathrm{ht}_{J_i}(r_o)$, (6) implies that $(\operatorname{ad} e_i)^p = 0$. So $(\operatorname{ad} e_1)^p/p! + (\operatorname{ad} e_2)^p/p!$ preserves $L_G(\mathbf{Z}_{(p)})$ as well, and (ii) holds. For (i), we take $J = \varnothing$ and for each root system $\Phi(G)$, we indicate below J_1, J_2 such that $\Pi(G) = J_1 \cup J_2$ and such that $p > \mathrm{ht}_J(r_o)$ implies $p > 2\,\mathrm{ht}_{J_i}(r_o)$. Then (i) follows directly from (ii).

G	J_1	J_2	r_o
G_2	$\{\alpha_1\}$	$\{\alpha_2\}$	$3\alpha_1 + 2\alpha_2$
F_4	$\{\alpha_1, \alpha_2\}$	$\{\alpha_3, \alpha_4\}$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
E_6	$\{\alpha_1, \alpha_2, \alpha_3\}$	$\{\alpha_4, \alpha_5, \alpha_6\}$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$\{\alpha_1, \alpha_3, \alpha_4\}$	$\{\alpha_2, \alpha_5, \alpha_6, \alpha_7\}$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$	$\{\alpha_5, \alpha_6, \alpha_7, \alpha_8\}$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$

Remark. Lemma 3 holds for G of classical type as well. For (i) one checks that in each case there exists a decompostion $\Pi(G) = J_1 \cup J_2$ such that $p > \mathrm{ht}(r_o)$ implies $p > 2\,\mathrm{ht}_{J_i}(r_o)$.

In the following lemma, we list the specific sl_2 subalgebras in $L_G(\mathbf{C})$ to which we will apply Lemmas 1 and 2. For the purposes of this lemma, we simplify our notation for certain elements of $\mathscr{B}$ as follows: if $\gamma = \alpha_i \in \Pi(G)$, we write e_i, f_i, h_i for e_γ, f_γ, h_γ, respectively.

Lemma 4. *In each of the following, $\{e, f, h\}$ is the standard basis of an* sl_2 *subalgebra in $L_G(\mathbf{C})$, for G as indicated, with the action of $\Pi(G)$ on h as given. That is, for e, f, and h as given, e and f are* ad-*nilpotent, $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, and the $\alpha_i(h)$ are as indicated.*

(1) *$G = G_2$ with Cartan matrix*

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix};$$

$e = e_1 + e_2$,
$f = 6f_1 + 10f_2$,
$h = 6h_1 + 10h_2$, *and*
$\alpha_i(h) = 2$ *for* $i = 1, 2$.

(2) *$G = F_4$ with Cartan matrix*

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix};$$

$e = e_1 + e_2 + e_3 + e_4$,
$f = 22f_1 + 42f_2 + 30f_3 + 16f_4$,
$h = 22h_1 + 42h_2 + 30h_3 + 16h_4$, *and*
$\alpha_i(h) = 2$ *for* $1 \le i \le 4$.

(3) $G = E_7$ *with Cartan matrix*

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix};$$

(a) $e = \sum_{i=1}^{7} e_i$,
$f = 34f_1 + 49f_2 + 66f_3 + 96f_4 + 75f_5 + 52f_6 + 27f_7$,
$h = 34h_1 + 49h_2 + 66h_3 + 96h_4 + 75h_5 + 52h_6 + 27h_7$, *and*
$\alpha_i(h) = 2$ *for* $1 \leq i \leq 7$.

(b) $e = e_1 + e_3 + [e_3, e_4] + [e_2, e_4] + e_5 + e_6 + e_7$,
$f = 26f_1 - 15f_2 - 37[f_2, f_4] + 15f_3 - 35[f_3, f_4] + 57f_5 - 35[f_4, f_5] + 40f_6 + 21f_7$,
$h = 26h_1 + 37h_2 + 50h_3 + 72h_4 + 57h_5 + 40h_6 + 21h_7$, *and*
$\alpha_i(h) = 2$ *for* $i \neq 4$, $\alpha_4(h) = 0$.

(4) $G = E_8$ *with Cartan matrix*

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix};$$

(a) $e = \sum_{i=1}^{8} e_i$,
$f = 92f_1 + 136f_2 + 182f_3 + 270f_4 + 220f_5 + 168f_6 + 114f_7 + 58f_8$,
$h = 92h_1 + 136h_2 + 182h_3 + 270h_4 + 220h_5 + 168h_6 + 114h_7 + 58h_8$, *and*
$\alpha_i(h) = 2$ *for* $1 \leq i \leq 8$.

(b) $e = e_1 + e_2 + [e_2, e_4] + [e_3, e_4] + e_5 + e_6 + e_7 + e_8$,
$f = 72f_1 + 38f_2 - 68[f_2, f_4] - 38f_3 - 142[f_3, f_4] + 172f_5 - 68[f_4, f_5] + 132f_6 + 90f_7 + 46f_8$,
$h = 72h_1 + 106h_2 + 142h_3 + 210h_4 + 172h_5 + 132h_6 + 90h_7 + 46h_8$, *and*
$\alpha_i(h) = 2$ *for* $i \neq 4$, $\alpha_4(h) = 0$.

(c) $e = e_1 + e_2 + e_3 + [e_2, e_4] + [e_4, e_5] + [e_5, e_6] + [e_6, e_7] + e_8$,
$f = 60f_1 + 22f_2 - 66[f_2, f_4] + 118f_3 + 66[f_3, f_4] + 22f_5 - 108[f_4, f_5] - 34[f_5, f_6] + 22f_7 - 74[f_6, f_7] + 38f_8$,
$h = 60h_1 + 88h_2 + 118h_3 + 174h_4 + 142h_5 + 108h_6 + 74h_7 + 38h_8$, *and*
$\alpha_i(h) = 2$ *for* $i \neq 4, 6$ *and* $\alpha_4(h) = 0 = \alpha_6(h)$.

Proof. The proof consists of a straightforward check.

Lemma 5. *Let* G, e, f, h *be as in Lemma* 4 *and* $J = \{\gamma \in \Pi(G) \mid \gamma(h) = 0\}$, *and assume* $p > \mathrm{ht}_J(r_o)$. *Then* $(\mathrm{ad}\, e)^k/k!$ *and* $(\mathrm{ad}\, f)^k/k!$ *preserve* $L_G(\mathbf{Z}_{(p)})$ *for all* $k \geq 0$.

Proof. Since we may take $-\Pi(G) = \{-\gamma \mid \gamma \in \Pi(G)\}$ as a base of $\Phi(G)$, we may apply Lemma 3 to f as well as e. For e, f as in 1, 2, 3(a), and

4(a) of Lemma 4, $\alpha_i(h) \neq 0$ for all i, so $J = \varnothing$ and Lemma 3(i) implies that $(\operatorname{ad} e)^k/k!$ and $(\operatorname{ad} f)^k/k!$ preserve $L_G(\mathbf{Z}_{(p)})$ for all $k \geq 0$. For e, f as in 3(b) of Lemma 4, where $J = \{\alpha_4\}$, take $J_1 = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ and $J_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then J_i (respectively $-J_i$) satisfy the hypotheses of Lemma 3 for e (respectively f). For e, f as in 4(b) of Lemma 4, where $J = \{\alpha_4\}$, take $J_1 = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ and $J_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Finally, for e, f as in 4(c) of Lemma 4, where $J = \{\alpha_4, \alpha_6\}$, take $J_1 = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ and $J_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6\}$. Then in each case, Lemma 3(ii) implies $(\operatorname{ad} e)^k/k!$ and $(\operatorname{ad} f)^k/k!$ preserve $L_G(\mathbf{Z}_{(p)})$ for all $k \geq 0$.

Proof of Theorems 1 *and* 3. Theorems 1 and 3 follow directly from Lemmas 2–5.

Proof of Theorem 2. Assume $p > 3, 3, 5, 7, 7$ for G of type G_2, F_4, E_6, E_7, E_8, respectively. Under these prime restrictions, Seitz establishes a list of the possible subgroups $Y \leq G$, Y of type A_1, such that Y is maximal among proper closed connected subgroups of G. (See [3, Theorem (4.2)].) Each possibility is determined up to conjugacy in $\operatorname{Aut}(G)$ by the integers $\{d(\alpha) \mid \alpha \in \Pi(G)\}$ given in Theorem 1. Moreover, (17.2) of [3] proves that if G has a closed connected subgroup A of type A_1 with a maximal torus whose action on $L(G)$ is given by the integers $\{d(\alpha) \mid \alpha \in \Pi(G)\}$ for any of the cases (i)–(vii) of Theorem 1, then A is maximal among proper closed connected subgroups of G.

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DEPARTMENT OF MATHEMATICS, WESLEYAN UNIVERSITY, MIDDLETOWN, CONNECTICUT 06457