Homotopy Equivalence of Posets with a Group Action

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The purpose of the present paper is to provide proofs of some of the results announced in the survey paper [13]. These results deal with combinatorial topology, in particular with complexes associated with posets of subgroups of a finite group. For applications to group theory, the reader can refer to the above mentioned paper [13].

In his influential paper [11], Quillen proved that the poset $\mathcal{S}_p(G)$ consisting of the non-identity p-subgroups of a finite group G is homotopy equivalent to its subposet $\mathcal{A}_p(G)$ consisting of the non-identity elementary abelian p-subgroups. Subsequently Bouc [2] proved in a dual fashion that $\mathcal{S}_p(G)$ is also homotopy equivalent to the subposet

$$\mathcal{B}_p(G) = \{P \in \mathcal{S}_p(G) \mid P = O_p(N_G(P))\}.$$

In both cases the inclusion mappings are homotopy equivalences. Taking into account the conjugation action of G, we show that these mappings are in fact G-homotopy equivalences. This means that all mappings in the homotopy equivalence are G-equivariant.

Quillen proved the homotopy equivalence $\mathcal{A}_p(G) \simeq \mathcal{S}_p(G)$ by applying his 'Theorem A' [11, Prop. 1.6] to the inclusion mapping. Our first observation is that this theorem may be given an equivariant version, which works without finiteness assumptions.

Theorem 1. *Let G be a group, let $\mathcal{X}, \mathcal{Y}$ be G-posets and let $\phi : \mathcal{X} \to \mathcal{Y}$ be a mapping of G-posets. Suppose that either*

(i) for all $y \in \mathcal{Y}$, $\phi^{-1}(\mathcal{Y}_{\leq y})$ is G_y-contractible

or

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(ii) for all $y \in \mathcal{Y}$, $\phi^{-1}(\mathcal{Y}_{\geq y})$ is G_y-contractible.

Then ϕ is a G-homotopy equivalence.

Here a G-poset means a partially ordered set together with an order-preserving action of G, and G_y denotes the stabilizer of y. Moreover we define

$$\mathcal{Y}_{\leq y} = \{z \in \mathcal{Y} | z \leq y\}$$
$$\mathcal{Y}_{\geq y} = \{z \in \mathcal{Y} | z \geq y\}.$$

Finally a G_y-contractible poset is a G_y-poset which is G_y-homotopy equivalent to a point. When G is the trivial group, then Theorem 1 is precisely Quillen's Theorem A (for posets).

When G is a *finite* group, we use Theorem 1 to make equivariant versions of the theorems of Quillen and Bouc, and also to prove that another closely related simplicial complex is G-homotopy equivalent to the other complexes of p-subgroups. If $\mathcal{X}$ is a poset, we denote by $\Delta(\mathcal{X})$ the simplicial complex whose n-simplices are chains $x_0 < x_1 < \cdots < x_n$ in $\mathcal{X}$. We note here that by definition, all topological notions applied to a poset $\mathcal{X}$ are to be understood as being the corresponding notions for $\Delta(\mathcal{X})$. If $\mathcal{X}$ is some poset of subgroups of G, we denote by $\Delta_{\triangleleft}(\mathcal{X})$ the subcomplex of $\Delta(\mathcal{X})$ whose simplices are chains $P_0 < \ldots < P_n$ in $\mathcal{X}$ satisfying $P_i \triangleleft P_n$ for all i. The simplicial complex $\Delta_{\triangleleft}(\mathcal{S}_p(G))$ was first considered by G.R. Robinson in his reformulation of Alperin's conjecture (see [9] or [13]).

Theorem 2. *(i) Let $\mathcal{X}$ be a subposet of $\mathcal{S}_p(G)$ closed under conjugation satisfying either $\mathcal{X} \supseteq \mathcal{A}_p(G)$ or $\mathcal{X} \supseteq \mathcal{B}_p(G)$. Then the inclusion $\mathcal{X} \to \mathcal{S}_p(G)$ is a G-homotopy equivalence.*
(ii) $\Delta_{\triangleleft}(\mathcal{S}_p(G))$ is G-homotopy equivalent to $\Delta(\mathcal{S}_p(G))$.
In particular, $\Delta(\mathcal{A}_p(G)) \simeq_G \Delta(\mathcal{B}_p(G)) \simeq_G \Delta(\mathcal{S}_p(G)) \simeq_G \Delta_{\triangleleft}(\mathcal{S}_p(G))$.

Theorem 2 has been announced in [13, 2.3], but no proof of the result has yet appeared in print and the present paper fills this gap. We refer the reader to [13] for applications of the theorem. Let us simply mention that in particular all four complexes have homology groups which are isomorphic as G-modules and that their Lefschetz invariants are equal.

1. Equivariant homotopy equivalences

Throughout this section, G denotes an arbitrary group. For later use, we first deal with the easiest case of G-homotopy equivalence of posets.

(1.1) Proposition. *Let $\mathcal{X}$ be a G-poset and let $\phi, \psi : \mathcal{X} \to \mathcal{X}$ be two G-maps such that $\phi(x) \geq \psi(x)$ for all $x \in \mathcal{X}$. Then ϕ and ψ are G-homotopic.*

Proof Consider the poset $I = \{0, 1\}$ with $0 < 1$ and trivial G-action. Then the map $H : I \times \mathcal{X} \to \mathcal{X}$ defined by $H(0, x) = \psi(x)$, $H(1, x) = \phi(x)$ is order preserving and induces a G-homotopy from ϕ to ψ (see [11, 1.3] for details).

(1.2) Corollary. *Let X be a G-poset and let $\phi : \mathcal{X} \to \mathcal{X}$ be a G-map such that $\phi(x) \geq x$ for all $x \in \mathcal{X}$. Then $\phi : \mathcal{X} \to Im(\phi)$ is a G-homotopy equivalence.*

Proof Let $i : \mathrm{Im}(\phi) \to \mathcal{X}$ be the inclusion. Then by (1.1), $i\phi$ is G-homotopy equivalent to id_X. Similarly $\phi i \simeq_G \mathrm{id}_{\mathrm{Im}(\phi)}$.

Now Quillen's Theorem A and its equivariant version Theorem 1 are more powerful ways of obtaining homotopy equivlences. In the proof of Theorem 1, our main ingredient for dealing with G-homotopy equivalences is the following result which reduces the question to the non-equivariant case.

(1.3) Proposition (Bredon [3, §II], see also [7, II.2.7]). *Let $\mathcal{X}$, $\mathcal{Y}$ be G-CW-complexes and let $\phi : \mathcal{X} \to \mathcal{Y}$ be a G-equivariant cellular map. Then ϕ is a G homotopy equivalence if and only if $\phi^H : \mathcal{X}^H \to \mathcal{Y}^H$ is a homotopy equivalence for each subgroup H of G.*

Here $\mathcal{X}^H$ denotes the H-fixed point subcomplex and ϕ^H the restriction of ϕ.

(1.4) Proof of Theorem 1. We will assume condition (i) in Theorem 1 holds. If we assume condition (ii) instead, the result follows by considering the opposite posets. By (1.3), we have to show that the mapping $\phi^H : \mathcal{X}^H \to \mathcal{Y}^H$ is a homotopy equivalence. We apply Quillen's Theorem A (that is, Theorem 1 in case G is the trivial group). Suppose

$y \in \mathcal{Y}^H$, that is, $H \leq G_y$. Then $(\phi^H)^{-1}({\mathcal{Y}^H}_{\leq y}) = (\phi^{-1}(\mathcal{Y}_{\leq y}))^H$ and by condition (i) this latter poset is contractible since the G_y-contraction of $\phi^{-1}(\mathcal{Y}_{\leq y})$ restricts to a contraction of fixed point sets. Hence by Quillen's Theorem A, ϕ^H is a homotopy equivalence. This completes the proof of Theorem 1.

Quillen's Theorem A was first proved in the general context of the classifying space of a category. In the special case of posets, Walker [12, 2.2] gave an elementary proof. It can be checked that each step of Walker's proof can be given an equivariant version, and we now sketch how this may be done. We assume familiarity with Walker's argument and his terminology, and simply indicate how the notions must be modified to cope with a G-action.

Let K be an *admissible* simplicial G-complex, that is a simplicial complex on which G acts simplicially so that the stabilizer G_σ of every simplex σ fixes σ pointwise. We define a *contractible G-carrier* C from K to X to be a contractible carrier (satisfying conditions (i) and (ii) of [12, p.374]) such that

(iii) $C({}^g\sigma) = {}^gC(\sigma)$ for all $g \in G$, for all simplices σ of K

(iv) G_σ acts trivially on $C(\sigma)$, for all simplices σ of K.

We say that a G-map $f : |K| \to X$ is *carried by C* if $f(|\sigma|) \subset C(\sigma)$ for all simplices σ of K. We now have the equivariant analogue of [12, 2.1]:

(1.5) Lemma. *If C is a contractible G-carrier from K to X then*

(a) there exists a continuous G-map $f : |K| \to X$ carried by C,

(b) any two continuous G-maps carried by C are G-homotopic.

Proof. Proceed by induction on the skeletons of K as in [12]. One works with a chosen simplex in each G-orbit and defines f on the rest of the orbit by requiring G-equivariance.

(1.6) Proof of Theorem 1 in the manner of [12, 2.2].

The function $\sigma \mapsto |(\phi^{-1}(\mathcal{Y}_{\leq \max \sigma}))^{G_\sigma}|$ is a contractible G-carrier from $\Delta(\mathcal{Y})$ to $|\Delta(\mathcal{X})|$ (note that since $G_\sigma \leq G_{\max \sigma}$, the fixed point set is contractible). By Lemma 1.5 there exists a G-map $\theta : |\Delta(\mathcal{Y})| \to |\Delta(\mathcal{X})|$ satisfying $\theta|\sigma| \subseteq |(\phi^{-1}(\mathcal{Y}_{\leq \max\sigma}))^{G_\sigma}|$. We show that θ is a homotopy inverse for ϕ.

The map $\sigma \mapsto |(\mathcal{Y}_{\leq \max\sigma})^{G_\sigma}|$ is a contractible G-carrier from $\Delta(\mathcal{Y})$ to $|\Delta(\mathcal{Y})|$ which carries both $\phi \circ \theta$ and $\mathrm{id}_{|\Delta(\mathcal{Y})|}$. Therefore $\phi \circ \theta \simeq_G \mathrm{id}_{|\Delta(\mathcal{Y})|}$.

The map $\tau \mapsto |(\phi^{-1}(\mathcal{Y}_{\leq \max\phi(\tau)}))^{G_\tau}|$ is a contractible G-carrier from $\Delta(\mathcal{X})$ to $|\Delta(\mathcal{X})|$ which carries both $\theta \circ \phi$ and $\mathrm{id}_{|\Delta(\mathcal{X})|}$. Therefore $\theta \circ \phi \simeq_G \mathrm{id}_{|\Delta(\mathcal{X})|}$.

Although the above proof of Theorem 1 appears to be notationally complex, it has the merit of being self-contained and does not use Proposition 1.3 or Quillen's Theorem A. For this reason it would give a more streamlined approach in an exposition in which nothing was assumed. Moreover it turns out from this proof that the assumption of Theorem 1 can be slightly relaxed: instead of requiring $\phi^{-1}(\mathcal{Y}_{\leq y})$ to be G_y-contractible, it suffices to assume that $\phi^{-1}(\mathcal{Y}_{\leq y})^{G_\sigma}$ is contractible for every simplex σ in $\Delta(\mathcal{Y})$ with $\max(\sigma) = y$, and that $\phi^{-1}(\mathcal{Y}_{\leq y})^{G_\tau}$ is contractible for every simplex τ in $\Delta(\mathcal{X})$ with $\max(\phi(\tau)) = y$. In other words the fixed points $\phi^{-1}(\mathcal{Y}_{\leq y})^H$ do not play any rôle when $H \leq G_y$ is not the stabilizer of a simplex (either in $\Delta(\mathcal{Y})$ or in $\Delta(\mathcal{X})$).

We come at last to the applications of Theorem 1. It has been observed by several people [2, Prop. 4], [10, Prop. 1.6], [12, Prop. 6.1] that if we remove from a poset $\mathcal{Y}$ any element y such that $\mathcal{Y}_{<y}$ (respectively $\mathcal{Y}_{>y}$) is contractible, then the inclusion $(\mathcal{Y} - \{y\}) \to \mathcal{Y}$ is a homotopy equivalence. For instance, when G is a finite group, starting from $\mathcal{S}_p(G)$, this process may be iterated so that we are left with $\mathcal{A}_p(G)$ (respectively $\mathcal{B}_p(G)$), and hence we deduce the homotopy equivalence of these posets. We modify this argument to cater for G-equivariance.

(1.7) Proposition. *Let $\mathcal{Y}$ be a G-poset of finite length and $\mathcal{X}$ a G-invariant subposet of $\mathcal{Y}$ such that for each $y \in \mathcal{Y} - \mathcal{X}$, $\mathcal{Y}_{<y}$ is G_y-contractible. Then the inclusion $\mathcal{X} \to \mathcal{Y}$ is a G-homotopy equivalence.*

Proof. To say that $\mathcal{Y}$ has finite length means there exists a number N so that no chain in $\mathcal{Y}$ has length greater than N. We construct a chain of posets

$$\mathcal{Y} = \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \cdots \supset \mathcal{Y}_n = \mathcal{X}$$

by at each stage removing from $\mathcal{Y}_i$ those elements which are maximal subject to not being in $\mathcal{X}$ and calling the resulting subposet $\mathcal{Y}_{i+1}$. Note that $\mathcal{Y}_{i+1}$ is G-invariant, and that we reach $\mathcal{X}$ after finitely many steps because of the finite length condition. Let $\phi_i : \mathcal{Y}_{i+1} \to \mathcal{Y}_i$ be the inclusion map, and consider the preimage $\phi_i^{-1}((\mathcal{Y}_i)_{\leq y})$ for $y \in \mathcal{Y}_i$. If $y \in \mathcal{Y}_{i+1}$, then $\phi_i^{-1}((\mathcal{Y}_i)_{\leq y}) = (\mathcal{Y}_{i+1})_{\leq y}$ has a maximal element. So $\Delta((\mathcal{Y}_{i+1})_{\leq y})$ is a cone on the G_y-fixed element y, hence is G_y-contractible. The other situation is when $y \notin \mathcal{Y}_{i+1}$, in which case $\phi_i^{-1}((\mathcal{Y}_i)_{\leq y}) = \mathcal{Y}_{<y}$ because y is removed in forming $\mathcal{Y}_{i+1}$, but no element below y has been removed yet. By assumption, $\mathcal{Y}_{<y}$ is G_y-contractible. It follows from Theorem 1 that ϕ_i is a G-homotopy equivalence. Therefore so is the composite inclusion $\mathcal{X} \to \mathcal{Y}$.

Recall that if $\bar{\mathcal{L}}$ is a bounded lattice (with meet $\wedge$ and join $\vee$), then $\bar{\mathcal{L}}$ has a unique maximal element $\hat{1}$ and a unique minimal element $\hat{0}$. The poset $\mathcal{L} = \bar{\mathcal{L}} - \{\hat{0}, \hat{1}\}$ is called the *proper part* of $\bar{\mathcal{L}}$. If $x \in \mathcal{L}$, an *upper semicomplement* of x is an element $c \in \mathcal{L}$ such that $c \vee x = \hat{1}$, while a *complement* of x is an element $c \in \mathcal{L}$ such that $c \vee x = \hat{1}$ and $c \wedge x = \hat{0}$. The following result is the G-equivariant version of a theorem of Walker [12, 8.1]; see also [10, Prop. 1.8]. We omit the proof for it is again an easy modification of the argument used in the non-equivariant case.

(1.8) Proposition. *Let $\mathcal{L}$ be the proper part of a bounded G-lattice and let $x \in \mathcal{L}$ be fixed under G. If B is a G-invariant set of upper semi-complements of x, including all of the complements of x, then the G-poset $\mathcal{L} - B$ is G-contractible.*

(1.9) Corollary. *If x has no complement, then $\mathcal{L}$ is G-contractible.*

2. Posets of subgroups

Throughout this section, G denotes a finite group. The following lemma expresses the duality which exists between $\mathcal{A}_p(G)$ and $\mathcal{B}_p(G)$. We remind the reader that $\mathcal{S}_p(G)_{<P}$ is the set of proper non-identity subgroups of the p-group P, and $\mathcal{S}_p(G)_{>P}$ is the set of p-subgroups of G properly containing P.

(2.1) Lemma. *Let $P \in \mathcal{S}_p(G)$.*

(i) $\mathcal{S}_p(G)_{<P}$ is $N_G(P)$-contractible if and only if $P \notin \mathcal{A}_p(G)$.

(ii) $\mathcal{S}_p(G)_{>P}$ is $N_G(P)$-contractible if and only if $P \notin \mathcal{B}_p(G)$.

Proof. (i) If $P \in \mathcal{A}_p(G)$ then $\mathcal{S}_p(G)_{<P}$ is the lattice of subspaces of the vector space P, with the top and bottom elements removed, and this has the homotopy type of a bouquet of spheres (as in the Solomon-Tits theorem), or is empty. Thus $\mathcal{S}_p(G)_{<P}$ is not contractible, and in particular it is not $N_G(P)$-contractible.

Conversely, if $P \notin \mathcal{A}_p(G)$, the Frattini subgroup $\Phi(P)$ is non-trivial. If $Q \in \mathcal{S}_p(G)_{<P}$, then $Q \leq Q \cdot \Phi(P) \geq \Phi(P)$. By Corollary 1.2, the map $Q \to Q \cdot \Phi(P)$ followed by the constant map on $\Phi(P)$ is an $N_G(P)$-homotopy equivalence from $\mathcal{S}_p(G)_{<P}$ to a point.

(ii) Assume first that $\mathcal{S}_p(G)_{>P}$ is $N_G(P)$-contractible. Such a contraction restricts to a contraction of fixed points, so $(\mathcal{S}_p(G)_{>P})^{N_G(P)}$ is contractible and in particular is non-empty. Thus there exists a p-subgroup $Q > P$ normalized by $N_G(P)$. Now $N_Q(P) = Q \cap N_G(P)$ is a p-group strictly containing P which is normal in $N_G(P)$, so $P \neq O_p N_G(P)$. Hence $P \notin \mathcal{B}_p(G)$.

Conversely, if $P \notin \mathcal{B}_p(G)$, then $\bar{P} = O_p(N_G(P))$ is strictly larger than P. If $Q \in \mathcal{S}_p(G)_{>P}$, then $Q \geq Q \cap N_G(P) \leq (Q \cap N_G(P)) \cdot \bar{P} \geq \bar{P}$. By Corollary 1.2, we obtain a three-step $N_G(P)$-homotopy equivalence from $\mathcal{S}_p(G)_{>P}$ to a point.

Proof of Theorem 2. Part (i) is a direct application of Proposition 1.7 (or its analogue with opposite posets). The assumptions of (1.7) are satisfied thanks to the above lemma.

The proof of part (ii) is more delicate. Let $\mathcal{X}$ be the poset of simplices of $\Delta_{\triangleleft}(\mathcal{S}_p(G))$. Then $\Delta(\mathcal{X})$ is the barycentric subdivision of $\Delta_{\triangleleft}(\mathcal{S}_p(G))$, hence is clearly G-homotopy equivalent to it. Let $\mathcal{Y}$ be the poset of abelian p-subgroups of G. Since $\mathcal{A}_p(G) \subseteq \mathcal{Y} \subseteq \mathcal{S}_p(G)$, $\mathcal{Y}$ is G-homotopy equivalent to $\mathcal{S}_p(G)$ by part (i). We will show that $\mathcal{X}$ and $\mathcal{Y}$ are G-homotopy equivalent, and this will complete the proof. Consider the map $\phi : \mathcal{X} \to \mathcal{Y}$ defined by $\phi(P_0 < \cdots < P_n) = \bigcap_{i=0}^n Z(P_i)$, where $Z(P_i)$ denotes the centre of P_i. Note that $\bigcap_{i=0}^n Z(P_i) = P_0 \cap Z(P_n)$ and since $P_0 \triangleleft P_n$ by definition of $\mathcal{X}$, $P_0 \cap Z(P_n)$ is indeed a *non-trivial* abelian p-subgroup (see [8, III.2.6] for this well-known property of the centre of a p-group).

Clearly ϕ is an order-reversing G-equivariant map, that is, a map of G-posets from $\mathcal{X}^{\mathrm{op}}$ to $\mathcal{Y}$. Since $\Delta(\mathcal{X}) = \Delta(\mathcal{X}^{\mathrm{op}})$, this change of order has no effect for our purposes. We wish to apply Theorem 1 to $\phi : \mathcal{X}^{\mathrm{op}} \to \mathcal{Y}$.

Let $A \in \mathcal{Y}$ and $(P_0 < \cdots < P_n) \in \mathcal{X}$. Then $(P_0 < \cdots < P_n) \in \phi^{-1}(\mathcal{Y}_{\geq A})$ if and only if $A \leq P_0 \leq P_n \leq C_G(A)$. In particular $A \triangleleft P_n$ and $(P_0 < \cdots < P_n)$ is a face of the simplex $A \leq P_0 < \cdots < P_n$, which lies in $\mathcal{X}$. Here we intend the notation $A \leq P_0 < \cdots < P_n$ to mean $A < P_0 < \cdots < P_n$ if $A \neq P_0$ and $P_0 < \cdots < P_n$ if $A = P_0$. Therefore in the poset $\mathcal{X}$, we have

$$(P_0 < \cdots < P_n) \leq (A \leq P_0 < \cdots < P_n) \geq A.$$

By Corollary 1.2, we obtain an $N_G(A)$-homotopy equivalence from $\phi^{-1}(\mathcal{Y}_{\geq A})$ to the point A. Thus the assumptions of Theorem 1 are satisfied and the result follows.

(2.3) Remarks. (i) The G-homotopy equivalence $\mathcal{A}_p(G) \simeq_G \mathcal{S}_p(G)$ also holds for infinite groups. In that case $\mathcal{S}_p(G)$ denotes the poset of finite p-subgroups of G. One cannot use the step-by-step argument of Proposition 1.7, but Quillen's original approach [11, 2.1] goes through. One proves in one go that the inclusion $\mathcal{A}_p(G) \to \mathcal{S}_p(G)$ is a G-homotopy equivalence, using Theorem 1. In order to check the assumptions, one proves that if P is a finite p-group, then $\mathcal{A}_p(P)$ is P-contractible. This follows from the same contraction as in [11, 2.2], using (1.2).

(ii) Proposition 1.7 can be applied to other posets of subgroups in a similar way. If $\mathcal{Y}$ is a poset of non-trivial subgroups of G closed under taking non-trivial subgroups and under G-conjugation, then one gets a G-homotopy equivalent subposet by removing from $\mathcal{Y}$ all the subgroups H such that H has a characteristic subgroup without complement in H. Indeed $\mathcal{Y}_{<H}$ is $N_G(H)$-contractible, by (1.9).

(iii) If $\mathcal{Y}$ is a poset of subgroups, it is natural to ask whether $\Delta_{\triangleleft}(\mathcal{Y})$ is homotopy equivalent to $\Delta(\mathcal{Y})$. This is the case for $\mathcal{Y} = \mathcal{S}_p(G)$ by Theorem 2, but such a result does not hold for $\mathcal{Y} = \mathcal{B}_p(G)$. This corrects a mistake on page 352 of [13]. We are grateful to L.G. Griffiths for pointing out this fact and for providing an example. We wish to thank him for allowing us to include it here. One takes $G = S_7$, the symmetric group on 7 letters, and $p = 2$. Then it turns out that the Euler characteristics of $\Delta_{\triangleleft}(\mathcal{B}_2(G))$ and of $\Delta(\mathcal{B}_2(G))$ do not coincide. In other words, while counting the number of chains in $\mathcal{B}_2(G)$, there are chains of subgroups not lying in $\Delta_{\triangleleft}(\mathcal{B}_2(G))$ which do not cancel out in the computation of the Euler characteristic.

(iv) Whenever G is a finite simple group of Lie type in defining charcteristic p then $\Delta(\mathcal{B}_p(G))$ is the barycentric subdivision of the building of G. It is this remark which underlies much of our philosophy for studying $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$ and $\mathcal{S}_p(G)$ in general, because apart from the fact that there is already a literature of theorems concerning these posets with applications in cohomology and representation theory, it allows us to regard the study of these posets as a generalisation to all finite groups of the notion of a building. We wish to give references for this identification of $\Delta(\mathcal{B}_p(G))$ with the barycentric subdivision of the building. It follows from

(2.4) Theorem. *Let G be a finite simple group of Lie type in defining characteristic p. Then a subgroup U lies in $\mathcal{B}_p(G)$ if and only if U is the unipotent radical of a parabolic subgroup of G.*

The implication that unipotent radicals lie in $\mathcal{B}_p(G)$ is in [6, (69.10)]. One needs to know that groups of Lie type have split BN-pairs [5]. The converse implication is due to Borel and Tits [1]; a more elementary argument for finite groups may be found in [4].

The building of G has as its simplices the proper parabolic subgroups, and P_1 is a face of P_2 whenever $P_1 \geq P_2$. Now by Theorem 2.4 above, $\mathcal{B}_p(G)$ is isomorphic to the opposite of the poset of proper parabolic subgroups of G, since for parabolic subgroups P_1 and P_2, $P_1 \geq P_2$ if and only if $O_p(P_1) \leq O_p(P_2)$ [6, Sec. 69]. Since the barycentric subdivision of the building is the complex of chains of proper parabolics, it is isomorphic to $\Delta(\mathcal{B}_p(G))$.

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