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Extending Morphisms from Finite to Algebraic Groups

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1. INTRODUCTION

In this paper we are concerned with the following problem. Given a homomorphism $\varphi: X \to G$ between finite groups of Lie type in characteristic p, when can this be lifted to a morphism of suitable algebraic groups? In the special case where G is a special linear group with φ an absolutely irreducible representation the answer is affirmative and a well-known result of Steinberg [16]. On the other hand there exist indecomposable representations for which there is no extension.

We show that if $\varphi(X)$ is not contained in a proper parabolic subgroup of G and if certain restrictions are put on p when G is an exceptional group, then φ can be extended. Results of this type are particularly important in the study of subgroups of finite groups of Lie type as they permit certain questions to be translated from the realm of finite group embeddings to the realm of embeddings of algebraic groups. As the maximal closed, connected subgroups of the classical algebraic groups are now fairly well understood and results are forthcoming on the exceptional groups (see [9] and [11] for surveys of the current situation), this can be an effective way to study the finite groups.

For the purposes of this paper we make the following definition. Let $\tilde{X}$ be a semisimple and simply connected algebraic group over the algebraic closure of F_p, and σ an endomorphism of $\tilde{X}$ normalizing each simple factor

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of $\tilde{X}$ and having finite fixed point group. Let $Z \leqslant Z(X_\sigma)$ and $\bar{X} = \tilde{X}/Z$. Then σ induces an endomorphism on $\bar{X}$, also denoted by σ. A *finite group of Lie type in characteristic p* is a group of the form $X = 0^{p'}(\bar{X}_\sigma)$. The group $\bar{X}$ is called the *ambient algebraic group* corresponding to X.

Throughout the paper X will denote a perfect finite group of Lie type in characteristic p, with ambient algebraic group $\bar{X}$ and morphism σ. Let $\bar{G}$ be a semisimple algebraic group over the closure of F_p and δ an endomorphism of $\bar{G}$ such that $G = 0^{p'}(\bar{G}_\delta)$ is finite.

THEOREM 1. *Assume $\varphi: X \to G$ is a homomorphism such that $\varphi(X)$ is contained in no proper parabolic subgroup of G. There is an integer N depending on the dimension of the largest simple factor of $\bar{G}$, such that if $p > N$, then φ can be extended to a homomorphism of algebraic groups $\bar{\varphi}: \bar{X} \to \bar{G}$. If each simple factor of $\bar{G}$ is of classical type, then no restriction on p is required.*

The theorem extends Theorem 4 of [10], which shows that if φ is an absolutely irreducible representation and if X preserves a nondegenerate bilinear or quadratic form on the underlying module, then the representation can be extended so that $\bar{X}$ preserves the extended form. It is necessary to require that $X = 0^{p'}(\bar{X}_\sigma)$ be perfect. For example, $X = PSL_2(2)$ preserves a quadratic form on the usual 2-dimensional module, although $\bar{X}$ preserves no quadratic form.

In the next section we prove the theorem, taking N as three times the dimension of the largest simple factor of $\bar{G}$. However, this bound is far too big and in Theorem 2 we give more manageable bounds in the case of $\bar{G}$ an exceptional group. It seems likely that only a very mild restriction on p is actually required, conceivably none at all, but we have been unable to establish this. Such a result would be an important tool for the study of subgroups of exceptional groups.

THEOREM 2. *Assume the hypothesis of Theorem 1 with $\bar{G}$ a simple exceptional group, and let $\bar{Y}$ be a simple factor of $\bar{X}$ of minimal dimension. Then $N = 7$ suffices unless $\bar{Y}$ and $N = N(\bar{G})$ are as in the following table. In particular, $N = 13$ suffices unless $\bar{Y} = A_1, A_2$, or B_2.*

$\bar{Y}$	$N(Eg)$	$N(E_7)$	$N(E_6)$	$N(F_4)$	$N(G_2)$
A_4, B_3, C_3	13				
G_2	13	13			
A_3	13	13	13		
B_2	23	19	13	13	
A_2	43	31	19	13	
A_1	113	67	43	43	19

The final theorem is an application to the study of maximal subgroups.

THEOREM 3. *Let $G = G(q) = 0^{p'}(\bar{G}_\delta)$ be a finite exceptional group over F_q and M a subgroup of G such that $X = F^*(M)$ is a commuting product of groups of Lie type in characteristic p. Assume $p > N$, where N is as in Theorem 2. Then one of the following holds*:

(i) *M is contained in a proper parabolic subgroup of G.*

(ii) *X is the derived group of the group of fixed points of a field or graph-field morphism of $\bar{G}$.*

(iii) *There is a reductive subgroup $\bar{D} = \bar{D}^\delta = \bar{D}^M < \bar{G}$ such that $X \leqslant 0^{p'}(\bar{D}_\delta)$. If M is maximal, then $X = 0^{p'}(\bar{D}_\delta)$.*

2. THE PROOF OF THEOREM 1

In this section we prove Theorem 1 by induction, taking $\bar{G}$ as a counterexample of minimal dimension. Take $N = 3k$, where k is the dimension of the largest simple factor of $\bar{G}$. It is immediate that the pair $(\bar{X}, X)$ can be replaced by the pair $(\tilde{X}, \tilde{X}_\sigma)$. Consequently, we now take $\bar{X}$ to be simply connected.

Suppose $\varphi(X)$ is contained in a δ-invariant connected subgroup $\bar{D}$ of $\bar{G}$. If $R_u(\bar{D}) \neq 1$, then $0_p(\bar{D}_\delta) \neq 1$, and so by the Borel–Tits Theorem [2] X is contained in a δ-invariant parabolic subgroup of $\bar{G}$, whence $\varphi(X)$ is in a proper parabolic of $G = \bar{G}_\delta$, a contradiction. If $\bar{D}$ is reductive, then we may apply induction (again using the Borel–Tits result to see that $\varphi(X)$ is not in a parabolic of $\bar{D}_\delta$). Thus, we may assume

There does not exist a δ-invariant connected subgroup $\bar{D}$ of $\bar{G}$ for which $\varphi(X) \leqslant \bar{D}$. (*)

Let $\tilde{G}$ be a simply connected cover of $\bar{G}$ and π the natural surjection from $\tilde{G}$ to $\bar{G}$. Since X is universal there is a homomorphism $\tilde{\varphi}: X \to \tilde{G}$ such that $\varphi = \pi \circ \tilde{\varphi}$ (see [8]). Let $\bar{G}_1, ..., \bar{G}_k$ be the simple factors of $\bar{G}$ and $\tilde{G}_1, ..., \tilde{G}_k$ the corresponding factors of $\tilde{G}$.

Let $\bar{A}$ be the product of a $\langle \delta \rangle$ orbit of $\bar{G}_i$'s and $\tilde{A}$ the corresponding orbit of $\tilde{G}_i$'s. If γ is the projection of $\tilde{G}$ to $\tilde{A}$ then $\pi \cdot \gamma \cdot \tilde{\varphi}$ is a homomorphism from X to the δ-invariant subgroup $\bar{A}$ of $\bar{G}$. If each such map can be extended to $\bar{X}$, then clearly φ can be extended. Therefore, we may assume $\langle \delta \rangle$ is transitive on the simple factors of $\bar{G}$.

Next, let $\bar{E}$ be the collection of all elements of $\bar{G}$ of the form $x \cdot x^\delta \cdot x^{\delta^2} \cdot ... \cdot x^{\delta^{k-1}}$ for x in $\bar{G}_1$. Then $\bar{E}$ is closed, connected, and δ^k-stable. We claim that $\varphi(X) \leqslant \bar{E}$. If $\bar{G}$ were a direct product, this would be obvious. Since $\bar{G}$ fails to be a direct product only by a (finite) center of semisimple elements and since X is generated by unipotent elements, we have the claim. Also, X is not contained in a proper parabolic subgroup of G_{δ^k}. So the pair $(\bar{G}, \delta)$ can be replaced by $(\bar{E}, \delta^k)$. Combining this with the above we may now assume $\bar{G}$ is simple. There is an endomorphism $\tilde{\delta}$ of $\tilde{G}$ such that $\pi \circ \tilde{\delta} = \delta \circ \pi$ on $\tilde{G}$. Consequently, it will suffice to extend $\tilde{\varphi}$. Therefore, we may take $\bar{G}$ to be simple and simply connected.

LEMMA 1. *φ can be extended if $\bar{G}$ is of classical type.*

Proof. Part of the argument here is similar to the proof of Proposition 3 of [10], but we include some details for completeness. Let θ be the representation of $\bar{G}$ on the standard classical module $\bar{V}$. Associated with $\bar{G}_\delta$ is a form of $\bar{V}$, say V, over a suitable finite field. The hypothesis together with (*) implies that $V|X$ is irreducible. Then $\bar{V}|X = V_1 \oplus \ldots \oplus V_k$, where the V_i are nonisomorphic and algebraic conjugates of V_1, an absolutely irreducible module for X.

By Steinberg [16] each V_i can be regarded as an irreducible module for $\bar{X}$, with high weight in the appropriate range. We obtain a representation μ of $\bar{X}$ into $SL(\bar{V})$ extending the representation $\theta \circ \varphi$ of X. So this completes the argument when $\bar{G} = SL(\bar{V})$.

Suppose G fixes a nondegenerate bilinear form on V. Then $\bar{G}$ fixes the extended form and for each i, either V_i is self dual or V_i is singular, and there exists a unique $j \neq i$ for which V_j is isomorphic to the dual of V_i (as the irreducibles are algebraic conjugates of V_1 this can occur only if some quasisimple factor of X is a twisted group). Write $\bar{V} = M_1 \perp \cdots \perp M_r$, where each M_k is either a single V_i or the sum of one and its dual.

Fix k and set $M = M_k$. Then $M \cong M^*$ as $\bar{X}$ modules (consider high weights, using (31.6) of [7]), hence $\bar{X}$ preserves a nondegenerate bilinear form on M_k. Choose a basis of $\bar{V}$ by taking the union of bases of the various V_i and let A_1 be the matrix of the form on $\bar{V}$ restricted to the nondegenerate space M and let A_2 denote the matrix of the form fixed by $\bar{X}$. Then $A_2 A_1^{-1}$ commutes with the action of X.

If $M = V_i$, then the irreducibility of X forces $A_2 A_1^{-1}$ to be a scalar matrix, whence $\bar{X}$ also fixes the original form on M. Otherwise, $M = V_i \oplus V_j$ and since these are nonequivalent irreducible modules, $A_2 A_1^{-1}$ is a diagonal matrix, scalar on each of V_i and V_j. It follows that $\bar{X}$ also fixes the bilinear form on M with matrix A_1.

If $p = 2$ and $\bar{G} = \bar{D}_n$, then we must show $\bar{X}$ also preserves the extended quadratic form. If $M = V_i \oplus V_j$, then the spaces V_i and V_j are totally singular and the assertion is immediate, since $\bar{X}$ preserves the bilinear form. Suppose M is irreducible. Say $\bar{X} = \bar{X}_1 \circ \circ \circ \bar{X}_s$, the product of quasisimple groups, with $s > 1$. Then M decomposes into a corresponding tensor product, $M = M^1 \otimes \ldots \otimes M^s$, and $\bar{X}_i$ preserves a nondegenerate bilinear form on M^i (another application of (31.6) of [7]). Hence, the assertion follows from the containment $\mathrm{Sp}(Z_1) \otimes \mathrm{Sp}(Z_2) \leqslant \Omega(Z_1 \otimes Z_2)$.

Finally, if $s=1$, the argument in Theorem 4 of [10] shows $\bar{X}$ preserves the extended quadratic form. We take this opportunity to fill in one detail of that proof. Using the notation of that argument, on p. 398, paragraph 2, we reached the situation where $\mu = -\alpha/2$ and it as asserted that as μ is the negative of a dominant weight, this is impossible. There is an exception if the group in question has type C_n with α the root of highest height. For this to occur $\alpha/2$ is subdominant to the high weight of the module, and since the module is assumed to be restricted and tensor indecomposable, the high weight necessarily has short support (see (1.6) of [12]). Consequently, L_s acts irreducibly, where L_s is the ideal of $L = L(\bar{X})$ generated by short roots. Thus all roots considered can be taken to be short, and w can be chosen as $f_\alpha z$ for α a short root.

The above remarks show that μ: $\bar{X} \to \theta(\bar{G})$. We must pull this back to $\bar{G}$, an issue only if $\bar{G}$ is of type D_n or B_n. For otherwise θ is an isomorphism. Temporarily exclude the B_n case if $p=2$.

If $\bar{X}$ were simple, then for each $i = 1, ..., r$, M_i is a Frobenius twist, by say F_i, of a representation σ_i such that σ_i is an irreducible representation or the sum of two irreducibles, each a tensor product of modules, one of which is restricted. If $\bar{X}$ is not simple then the same thing holds, except we must replace each of F_i and σ_i by products of maps, one for each of the simple factors of $\bar{X}$. Denote these products by $\bar{F}_i$ and $\bar{\sigma}_i$, respectively. Write

$$\mu\colon \bar{X} \to \bar{X} \oplus \dots \oplus \bar{X} \to I(M_1) \times \dots \times I(M_r),$$

where the first map is $(\bar{F}_1, ..., \bar{F}_r)$ and the second is $(\bar{\sigma}_1, ..., \bar{\sigma}_r)$.

Now let $Y_1, ..., Y_r$ be the connected components of the preimages under θ of the subgroups $\bar{\sigma}_1(\bar{X}), ..., \bar{\sigma}_r(\bar{X})$ of $I(M_1), ..., I(M_r)$.

Assume $\bar{X}$ does not contain a simple factor $\bar{Y}$ with $(\bar{Y}, p) = (B_n, 2)$, $(C_n, 2)$, $(F_4, 2)$, or $(G_2, 3)$. Then the differential of $\bar{\sigma}_i$ is an isomorphism when restricted to the Lie algebra of each maximal unipotent subgroup $\bar{U}$ of $\bar{X}$. Hence, $\bar{\sigma}_i$ restricts to an isomorphism of $\bar{U}$ (see (4.3.4)(ii) of [14]). As θ also restricts to an isomorphism of unipotent subgroups, each $\bar{\sigma}_i$ factors through Y_i (see the remarks on p. 193 of [15]). Putting things together we have a morphism γ: $\bar{X} \to \bar{G}$ such that $\theta \circ \gamma = \mu$. Then for $x \in X$, $\theta(\gamma(x)) = \mu(x) = \theta(\varphi(x))$. As θ is one-to-one on the set of unipotent elements of $\bar{G}$ and since X is generated by unipotent elements, it follows that $\varphi = \gamma | X$, completing the result.

The same argument works for the exceptional cases of $(\bar{Y}, p)$ unless the restricted part of $\bar{\sigma}_i$ has high weight with long support. For in the latter case the differential is not an isomorphism on the Lie algebra of a maximal unipotent group. However, in this case one can precede $\bar{\sigma}_i$ by a special isogeny to obtain a restricted module with short support (see (10.1) of [16]).

Finally, consider the case $\bar{G}=B_n$ and $p=2$. The above argument does not work since θ is not an isomorphism on maximal unipotent groups. However, the image of G under θ has type C_n and there is a special isogeny mapping C_n to B_n, say γ. Then $\gamma\circ\mu\colon \bar{X}\to\bar{G}$. Restricting to X we obtain $\gamma\circ\mu|X=\gamma\circ\theta\circ\varphi$. Also, $\gamma\circ\theta=F$ is a Frobenius morphism and we may arrange things so that $\delta=F^a$ for some $a\geqslant 1$. So, $F^{a-1}\circ\gamma\circ\mu\colon\bar{X}\to\bar{G}$ and restricts to φ on X. This completes the proof of Lemma 1.

LEMMA 2. *Assume $\bar{G}$ is an exceptional group with $p>N$. Then φ can be extended to a morphism $\bar{\varphi}\colon\bar{X}\to\bar{G}$.*

Proof. Let $\bar{G}$ be an exceptional group and let θ denote the adjoint representation of $\bar{G}$ on $L(\bar{G})$. If D is a subgroup of G with $O_p(D)\neq 1$, then by (3.12) of [2] D can be embedded in a parabolic subgroup P of G such that $O_p(D)\leqslant R_u(P)$. Repeating this with a series of nested parabolics of X we conclude $\operatorname{rank}(X)\leqslant\operatorname{rank}(G)$, where if X is twisted we refer to the rank of X as a group with a (BN)-pair.

Let $J\leqslant X$ with $J=A_1(p)$ with J projecting nontrivially to each quasi-simple factor of X and let H be a Cartan subgroup of J. As $p>N$ the J-composition factors on $L(\bar{G})$ have dimension less than $p/3$. The action of J on $L(\bar{G})$ can be realized as a representation of $SL_2(p)$ so each composition factor affords a weight string where the weights are of the form $c\to H(c)\to c^e$, for $c\in F_p^*$ and where $-(p-1)/3<e<(p-1)/3$. As H is cyclic we can embed H in a torus $\bar{T}$ of $\bar{G}$; so taking a corresponding set of root vectors of $L(\bar{G})$ we have $\theta(H(c)\,e_\alpha=c^{l_\alpha}e_\alpha$ for each root α, where $-(p-1)/3<l_\alpha<(p-1)/3$. Also, $\theta(H(c)\,h=h$ for $h\in L(\bar{T})$.

Now define a 1-dimensional torus $\bar{H}$ of $GL(L(\bar{G}))$ such that $\bar{H}(c)\,e_\alpha=c^{l_\alpha}e_\alpha$ for each root α and $\bar{H}(c)\,h=h$ for h in $L(\bar{T})$. Then $p>N$ implies that distinct H weights give distinct $\bar{H}$ weights. If $[e_\alpha,e_\beta]\neq 0$, then the restrictions on weights forces $l_\alpha+l_\beta=l_{\alpha+\beta}$ so that $\bar{H}$ preserves the Lie bracket on $L(\bar{G})$. Hence, $\bar{H}\leqslant\theta(\bar{G})$ and we set $\bar{R}=(\theta^{-1}(\bar{H}))^\circ$.

Write $\delta=q\tau$, where q denotes a field automorphism and τ a possibly trivial graph automorphism. Consideration of high weights shows that the representations $\theta\circ\delta$ and $F\circ\theta$ are equivalent, where F is a suitable field automorphism of $GL(L(\bar{G}))$. An application of Lang's theorem then yields

$$\theta(g^\delta)=\omega\theta(g)\,\omega^{-1}\qquad\text{for each}\quad g\in\bar{G},\tag{**}$$

where ω is the semilinear map $\sum c_iv_i\to\sum c_i{}^q\,v_i$ on $L(\bar{G})$ with respect to a suitable basis. Fix $t\in\bar{T}$. Then (**) shows that for each root α, $\theta(t)\,\omega e_\alpha=\omega\theta(t^{\delta^{-1}})\,e_\alpha=(\alpha(t^{\delta^{-1}})^q)\,\omega e_\alpha$. Now $\alpha(t^{\delta^{-1}})=(\tau(\alpha)(t))^{1/q}$, so $\theta(t)\,\omega e_\alpha=\tau(\alpha)(t)\,\omega e_\alpha$ and we conclude $\omega e_\alpha=de_{\tau(\alpha)}$ for some constant d.

Hence,

$$\omega\bar{H}(c)\,\omega^{-1}\,e_{\tau(\alpha)}=\omega\bar{H}(c)\,d^{-1/q}\,e_\alpha=\omega c^{l_\alpha}\,d^{-1/q}\,e_\alpha$$
$$=c^{ql_\alpha}\,d^{-1}\,\omega e_\alpha=c^{ql_\alpha}\,e_{\tau(\alpha)}. \qquad (***)$$

Taking c in the prime field, $\bar{H}(c)\in\theta(H)$, so that $\bar{H}(c)$ is centralized by ω. From (***) we conclude $l_{\tau(\alpha)}=l_\alpha$. Another application of (***), this time with c arbitrary, shows $\omega\bar{H}(c)\,\omega^{-1}=\bar{H}(c)^q$. By (**), $\theta^{-1}(\bar{H})$ is δ-invariant and hence $\bar{R}^\delta=\bar{R}$.

Suppose M is a proper X-invariant subspace of $L(\bar{G})$. Then M is spanned by weight vectors for H and each such weight vector is necessarily one for $\bar{H}$. So M is $\bar{H}$-invariant. Set $\bar{D}=\langle x\bar{R}x^{-1}:x\in X\rangle$. By the corollary in (7.5) of [7], $\bar{D}$ is connected. Also, $\bar{D}$ leaves M invariant. In view of the previous paragraph, $\bar{D}=\bar{D}^\delta$. Finally, $X=[X,H]\leqslant[X,\bar{R}Z(\bar{D})]=[X,\bar{R}]\leqslant\bar{D}$, contradicting (*).

Therefore, X acts irreducibly on $L(\bar{G})$. We claim this is only possible if $\bar{X}$ and $\bar{G}$ are of the same type. The irreducible representations of X occur by restriction from irreducible representations of $\bar{X}$ and we shall use the high weight theory in describing the latter representations. In addition we will use the following consequence of Smith's theorem [13]. Let P be a parabolic subgroup of X. Use [2] to embed P in a parabolic subgroup $\bar{P}$ of $\bar{G}$ such that $Q=0_p(P)<R_u(\bar{P})=\bar{Q}$. By [13] the fixed-point-space $V_{\bar{Q}}$ is irreducible under the action of $\bar{P}$ and by Theorem 1 of [10] the same holds for the action of P on the fixed-point-space V_Q. But P stabilizes $V_{\bar{Q}}$ and clearly $V_{\bar{Q}}\leqslant V_Q$. Hence, equality must hold and both P and $\bar{P}$ are irreducible on the common set of fixed points. The high weight of $V_{\bar{Q}}$ is given simply by restricting the high weight of the representation on $L(\bar{G})$ to the appropriate Levi factor of $\bar{P}$.

Suppose $X=X_1\circ X_2$, a commuting product of two nontrivial groups, and write $L(\bar{G})=L_1\otimes L_2$, a corresponding tensor product. From the dimension of $L(\bar{G})$ we see that one of the factors, say L_1, has dimension dividing 2, 4, 6, 7, 8, according to $\bar{G}$ of type G_2, F_4, E_6, E_7, or E_8. Choose a maximal parabolic subgroup P of X such that X_2 is contained in the Levi factor. An application of the argument in the previous paragraph yields a contradiction in every case. Consequently, X is quasisimple.

First assume X has type A_1. If we embed a Cartan subgroup of X in a maximal torus of $\bar{G}$, the Cartan group acts trivially on the Lie algebra of this maximal torus and hence has a fixed point space of dimension at least the rank of $\bar{G}$. On the other hand from the representation theory of X we see that the fixed point space on $L(\bar{G})$ has dimension 1 unless $L(\bar{G})|X$ is the Steinberg module. However, if the latter case held, then $\dim(\bar{G})$ would be a power of p, a contradiction. Thus $\operatorname{rank}(\bar{X})>1$.

Table 1 of Section 3 gives a list of certain restricted dominant weights λ. The list includes all λ for which the corresponding irreducible module is self-dual and of dimension at most that of $L(\bar{G})$. This list is obtained under much weaker prime restrictions than in the current situation. Moreover, under the current prime restriction we have $\langle \lambda + \rho, \alpha \rangle < p$ for each positive root α of $\bar{X}$, so from [1] we conclude that the dimension of the irreducible module of high weight λ is given by the Weyl degree formula.

Suppose $L(\bar{G})|X$ can be decomposed as a tensor product. As above, one of the tensor factors has small dimension. Moreover, from the theory of high weights and the fact that $L(\bar{G})$ is self-dual it follows that this factor must be self-dual (see 31.6 of [7]). In each case the other tensor factor must be the twist of a restricted module. A contradiction is now obtained using the list in Table I.

Thus, $L(\bar{G})|X$ is the twist of a restricted module and we again use the list of Table I to determine the possibilities. We find that either the claim holds or one of the following holds for the triple $(\bar{G}, X, \lambda)$, where λ is the high weight of the underlying restricted module: $(G_2, B_2, 2\lambda_1)$, (E_6, B_6, λ_2), $(E_6, C_6, 2\lambda_1)$. In each case let P be the maximal parabolic subgroup of X obtained by deleting the end node on the right (label Dynkin diagrams as in Section 3). At this point we embed P in a parabolic subgroup of $\bar{G}$, use the above argument, and compare dimensions to obtain an easy contradiction.

We now have $\bar{X}$ and $\bar{G}$ of the same type. By Steinberg [16] we can extend $\theta \circ \varphi$ to a representation μ of $\bar{X}$ into $GL(L(\bar{G}))$. Necessarily, μ is the twist of a restricted representation (the action is tensor indecomposable) and $L(\bar{G})$ is a twist of the adjoint module for $\bar{X}$. Consequently, X fixes two Lie products on $L(\bar{G})$, one, call it $[\ ,\ ]_{\bar{G}}$, coming from $L(\bar{G})$, and the other, $[\ ,\ ]_{\bar{X}}$, coming from $\bar{X}$ via the adjoint module.

One checks that in each case $L(\bar{G}) \wedge L(\bar{G}) = A \oplus L(\bar{G})$ as modules for X, where A is an irreducible module of high weight different from that of $L(\bar{G})$. Thus, $\mathrm{Hom}_X(L(\bar{G}) \wedge L(\bar{G}), L(\bar{G}))$ is a 1-space, $[\ ,\ ]_{\bar{G}}$ and $[\ ,\ ]_{\bar{X}}$ are scalar multiples of each other, and so $\mu(\bar{X}) = \theta(\bar{G})$ (as each group is the connected component of the full stabilizer in GL of the corresponding Lie product).

The map μ is a combination of a suitable Frobenius map F and the adjoint representation Ad of $\bar{X}$. We have two groups $\bar{X}$ and $\bar{G}$ mapping to the same group $\theta(\bar{G})$. In each case the image arises as the adjoint representation, which is an isomorphism when restricted to a maximal unipotent subgroup. As in Lemma 1, Ad can be factored through $\bar{G}$. That is, there exists a homomorphism $\zeta\colon \bar{X} \to \bar{G}$ such that $Ad = \theta \circ \zeta$. Now for $x \in X$ we have $\theta(\varphi(x)) = Ad(F(x)) = \theta(\zeta(F(x)))$. As before this yields $\varphi = \zeta \circ F|X$, and we have the required extension.

3. THEOREM 2

In this section we prove Theorem 2. We follow the proof of Lemma 2 of Theorem 1, being more careful with the definition of the relevant 1-dimensional torus. Let $\bar{G}$ be an exceptional group, take $\bar{Y}$ to be a simple factor of $\bar{X}$ of minimal dimension, and let N be as given in Theorem 2. With one exception, the number N given for the pair $(\bar{Y}, \bar{G})$ is at least that obtained by considering any other simple factor of $\bar{X}$. The exception is (G_2, E_6), with $\bar{X}$ containing a simple factor of type A_3. However, an argument using [2] shows that E_6 cannot contain such a subgroup. Consequently, the number N will suffice for each simple factor of $\bar{X}$.

The prime restrictions in Theorem 1 arose at two points of the proof. The first and most important was in the course of defining a certain 1-dimensional torus. Later, prime restrictions were used in order to show X acts irreducibly on $L(\bar{G})$ and that $\bar{X}$ and $\bar{G}$ are of the same type.

The definition of the torus depended on the composition factors of a subgroup J of X isomorphic to $SL_2(p)$ or $PSL_2(p)$. If J can be chosen so that all composition factors are small, then it is possible to improve on the bound. We consider an appropriate group J in $\bar{Y}$ and obtain a 1-dimensional torus as before. We repeat this for each simple factor of $\bar{X}$. At this point we either can use the product of the tori obtained or can take a diagonal 1-dimensional torus of the product.

If $\operatorname{rank}(\bar{Y}) > 1$, we will proceed by first obtaining restrictions on the possible composition factors of X on $L(\bar{G})$, assuming relatively mild restrictions on p. This yields Table I. Then we choose $J \leqslant Y$ to be contained in a fundamental SL_2 corresponding to a long root and argue that all composition factors of J on $L(\bar{G})$ must be of low dimension. If $\bar{Y} = A_1$, the bounds are obtained using information on unipotent classes.

Once again assume we are in the inductive proof of the theorem. We first argue that $\varphi(Y)$ acts on $L(\bar{G})$ as a simple group. For suppose $Z = Z(\varphi(Y)) \nleqslant Z(\bar{G})$. Then Z is either cyclic or $Z_2 \times Z_2$. In either case II.5.1 of [15] implies Z is contained in a maximal torus, so the preimage under θ of $C_{\bar{G}}(Z)$ is a δ-invariant subgroup of positive dimension containing $\varphi(X)$, contradicting (*). Hence, $\varphi(Y)$ is simple (as $p > N$).

We also claim X acts without fixed points on $L(\bar{G})$. First note that $\bar{G}$ acts irreducibly on $L(\bar{G})$ (see [6]). In particular, $Z(L(\bar{G})) = 0$. Suppose $S = C_{L(\bar{G})}(X) \neq 0$. Then S is ω-invariant, so there exists $0 \neq v \in S$ fixed by ω (see the proof of Proposition AG(14.2) of [3]). But as $\bar{D} = C_{\bar{G}}(v)$ has positive dimension, this contradicts (*). Thus, $C_{L(\bar{G})}(X) = 0$.

Case 1. $\bar{Y} = A_1$.

The choice of N yields $p > N > 3(h_{\bar{G}} - 1)$, where $h_{\bar{G}}$ is the Coxeter number of $\bar{G}$. Consequently, we may apply the results of Chapter 5 of

Carter [5]. Let $B = UT$ be a Borel subgroup of $J = SL_2(p) \leqslant Y$, with $U = \langle u \rangle$ and $T = \langle t \rangle$.

If $e \neq 0$ is a nilpotent element of $L(\bar{G})$, then e can be embedded in a subalgebra E of type A_1 ((5.3.1) of [5]) which can be exponentiated ((5.5.5) of [5]) to yield a subgroup $\bar{D}$ of $\theta(\bar{G})$ of type A_1. By (5.4.7) and (5.5.5) of [5], it follows that E and $\bar{D}$ are completely reducible on $L(\bar{G})$ with irreducibles of the same dimension. Let $\bar{T}$ be a Cartan subgroup of $\bar{D}$ normalizing the unipotent subgroup $\bar{U} \leqslant \bar{D}$, where $L(\bar{U}) = \langle e \rangle$. Embedding $\bar{T}$ in a maximal torus of $\bar{G}$ we obtain a weighted Dynkin diagram (see (5.6.6) of [5]) which in turn yields all weights of $\bar{T}$ on $L(\bar{G})$. The highest such weight possible is twice the height of the highest root. As this number is less than p it follows that all composition factors of $\bar{D}$ on $L(\bar{G})$ are restricted.

For $g \in \bar{G}$, $\exp(ad(Ad(g)\, e)) = Ad(g) \cdot \exp(ad(e)) \cdot Ad(g)^{-1}$, so that the map $\xi \colon e \to u(e)$ is a map of $\bar{G}$-classes, where $u(e)$ is the unipotent part of $\theta^{-1}(\exp(ad(e)))$. We claim ξ is a bijection. The $\bar{G}$-class of e is determined by the above labelling of the Dynkin diagram of $\bar{G}$, which in turn is determined by the dimensions and multiplicities of the simple E-submodules (see the proof of (5.6.7) of [5]). But these are also simple modules for $\bar{D}$ and therefore Jordan blocks for $\exp(ad(e))$. It follows that ξ is injective. The Springer correspondence gives a $\bar{G}$-equivariant map between the set of unipotent elements of $\bar{G}$ and the set of nilpotent elements of $L(\bar{G})$. So the number of classes are equal and the claim follows. The surjectivity of ξ implies there is a subgroup $\bar{D} \leqslant \bar{G}$ of type A_1 and containing $u = u(e)$ for suitable e.

Each irreducible summand of $L(\bar{G})|\bar{D}$ is a Jordan block of u, and these summands are determined by the weighted diagram. Since u also acts as a single Jordan block on each J-composition factor of $L(\bar{G})$, the largest such factor is bounded by the dimension of the largest composition factor of $\bar{D}$.

First suppose u is a semiregular unipotent element; i.e., $C_{\bar{G}}(u) = Q \times Z$, with Q a connected unipotent group and $Z = Z(\bar{G})$. We have $u = u(e)$ with $\langle e \rangle = L(\bar{U})$ and $C_{\bar{G}}(e)$ is $\bar{T}$-invariant. Now $C_{L(\bar{G})}(e) = C_{L(\bar{G})}(u)$ (check the representations of the corresponding A_1's) so (1.14) of [5] implies $C_{\bar{G}}(e)$ and $C_{\bar{G}}(u)$ have the same dimension. A quick check of pages 401–406 of [5] shows that there are unique nilpotent and unipotent classes with centralizers of this dimension. It follows that the $\bar{G}$-class of e corresponds under the Springer map to the $\bar{G}$-class of u. Clearly, $C_{\bar{G}}(e) \leqslant C_{\bar{G}}(\exp(ad(e))) = C_{\bar{G}}(u)$, so it follows that $C_{\bar{G}}(e) = C_{\bar{G}}(u)$ and thus $C_{\bar{G}}(u)$ is $\bar{T}$-invariant.

For $t \in T$, $u^t = u^k$ for some $k \in \bar{T}$, whence $t \in C_{\bar{G}}(u)\, \bar{T}$. An argument with (9.3) of [3] shows every semisimple element of $C_{\bar{G}}(u)\, \bar{T}$ is conjugate to an element of $Z \times \bar{T}$, so conjugating if necessary we may assume $t \in Z \times \bar{T}$. As $\theta(T) \leqslant \theta(\bar{T})$, we can use $\bar{T}$ in place of the torus $\bar{R}$ in the proof of

Theorem 1. It is still necessary to assume that the J-composition factors have dimension at most $(p-1)/2$ in order to avoid congruences among the weights. This is consistent with the bounds in Theorem 2.

Now suppose u is not a semiregular unipotent element. Here we use the list of labelled diagrams on pages 401–406 of [5] in order to determine the highest weight of $\bar{D}$ on $L(\bar{G})$. This weight is $m = \sum b_i l_i$, where the numbers l_i are the labels given and the b_i are the coefficients of the highest root. As indicated above, this yields an upper bound on the largest composition factor of J. We check that our assumption on p guarantees that $(p-1)/3 > m$, so we are in a position to define the torus $\bar{H}$. This completes the argument for $\bar{Y} = A_1$.

Case 2. *$\bar{Y}$ has rank $\geqslant 2$.*

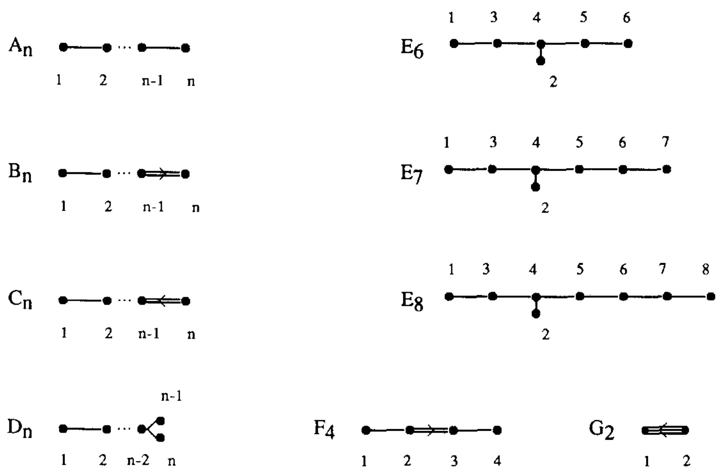
As mentioned at the start of his section our goal here is to show that the composition factors of a subgroup $J = SL_2(p)$ contained in a fundamental SL_2 in Y are of small dimension. The group J is contained in the subgroup C of Y defined over the prime field. All composition factors of C on $L(\bar{G})$ can be regarded as restrictions to C of restricted modules for $\bar{Y}$. Let λ be the high weight of such a composition factor, $V = V(\lambda)$. Since $L(\bar{G})$ is self-dual (see 31.6 of [7]) either V is self-dual or V^* is also a composition factor. Consequently we begin with Table I which gives conditions on the high weights of all the restricted self-dual representations for $\bar{Y}$ having dimension at most $\dim(L(\bar{G}))$ and the restricted non-self-dual modules of dimension at most $(\frac{1}{2})\dim(L(\bar{G}))$. The results are obtained under the assumption $p > N$. In some cases, we give separate restrictions according to the various possibilities for $\bar{G}$.

In the following we describe our methods for obtaining Table I although the actual details are left to the reader and are somewhat tedious. Let $V = V(\lambda)$ be as above and let ρ denote the half-sum of positive roots. Express λ as nonnegative combination of the fundamental dominant weights, $\lambda = \sum c_i \lambda_i$, where the fundamental dominant weight λ_i corresponds to the fundamental root α_i.

If $\langle \lambda + \rho, \alpha \rangle \leqslant p$ for each positive root α, then the Weyl module of high weight λ is irreducible (see [1], so $\dim(V)$ is given by the Weyl degree formula. If $\sum c_i$ is reasonably small, then these dimensions are listed in [4]. On the other hand, if $\sum c_i$ is large, then the degree formula gives a contradiction.

Now suppose there exists a positive root α such that $\langle \lambda + \rho, \alpha \rangle > p$. We obtain a lower bound on $\dim(V)$ by counting conjugates of certain dominant weights appearing in V. Namely, $\dim(V)$ is bounded below by the sum of orbit lengths of all subdominant weights to λ for which the corresponding weight space of V is nontrivial. To get subdominant weights with nontrivial weight space, proceed as follows. If $c_i > 0$, then V contains a composition factor for the fundamental A_1 corresponding to α_i with high weight $c_i\lambda_i$. Therefore each of the weights λ-$b\alpha_i$ occurs with nonzero multiplicity for $b \leqslant c_i$. Moreover, for $b \leqslant c_i/2$, these weights are dominant. Given such a b and α_j adjacent to α_i, one checks that the A_1 corresponding to α_j has a composition factor of high weight $d\lambda_j$, where $d = c_j - \langle b\alpha_i, \alpha_j\rangle$. Hence, λ-$b\lambda_i$-$c\lambda_j$ is also a dominant weight of V for all $0 \leqslant d/2$. One can repeat this process to get further dominant weights.

We make use of Table I, which we base on the labelling of Dynkin diagrams below. In some cases symmetries of the Dynkin diagram can be applied to yield further entries. In all cases we assume $p > N$, where N is as given in Theorem 2.



Using the information in Table I we can improve the bounds. We have $J \simeq SL_2(p)$ with $J < C \leqslant Y$. Each composition factor of C on $L(\bar{G})$ arises from an irreducible module $V = V(\lambda)$ for $\bar{Y}$ with high weight $\lambda = \sum c_i\lambda_i$ satisfying the conditions in Table I. In some cases the list can be further shortened. For example, as mentioned earlier, $\varphi(Y)$ acts as a simple group. So, for example, this would rule out the case $\bar{Y} = C_3$ with $\lambda = \lambda_1$ or $3\lambda_1$.

We must rule out a few special configurations. If $\bar{Y} = G_2$ and $\bar{G} = E_6$, then $\lambda \neq 2\lambda_2$. For suppose otherwise. Then by [6], $\dim(V(\lambda)) = 77$ and as $L(\bar{G})$ is self-dual and of dimension 78 we conclude that X has fixed points on $L(\bar{G})$, contrary to earlier remarks. Similarly, we show it is not the case that $\bar{X} = C_3$, $\bar{G} = E_7$, and $\lambda = 4\lambda_1$. For otherwise, $\dim(V(\lambda)) = 126$ (by (8.1) and (1.14) of [12]). As $Z(\bar{Y})$ acts trivially on $L(\bar{G})$, there is but one nontrivial composition factor of Y on $L(\bar{G})$. Hence Y fixes a nonzero element $l \in L(G)$ which can be taken to be either nilpotent or semisimple.

TABLE I

$\bar{Y}$		$\bar{G}$	Conditions on $\lambda = \sum c_i\lambda_i$
A_n	$4 < n$		$\sum c_i \leqslant 3$ and $\sum c_i \leqslant 2$ if $n = 8$
A_4		E_8	$\sum c_i \leqslant 4$
		$\neq E_8$	$\sum c_i \leqslant 3$
A_3		E_6, E_7, E_8	$\sum c_i \leqslant 5$ or $\lambda = 6\lambda_1, 7\lambda_1$
		F_4, G_2	$\sum c_i \leqslant 3$
A_2		E_8	$\sum c_i \leqslant 10$ or $\lambda = c\lambda_1, c \leqslant 14$
		E_7	$\sum c_i \leqslant 8$ or $\lambda = 9\lambda_1, 10\lambda_1$
		E_6	$\sum c_i \leqslant 6$ or $\lambda = 7\lambda_1$
		F_4	$\sum c_i \leqslant 4$ or $\lambda = 5\lambda_1$
		G_2	$\sum c_i \leqslant 2$
B_n, C_n	$4 \leqslant n \leqslant 8$	E_8	At most 2 c_i's are $\neq 0$ and $3 \geqslant \sum c_i \leqslant 1 + c_1 + c_n$.
		$\neq E_8$	At most 2 c_i's are $\neq 0$. Either $n = 4$ with $\lambda = 3\lambda_1$ or $2 \geqslant \sum c_i \leqslant 1 + c_1 + c_n$.
B_3		E_8	$\sum c_i \leqslant 4$. If $c_2 \neq 0$, then $\lambda = 2\lambda_2$ or $c_2 = 1$ and $\sum c_i \leqslant 3$.
		$\neq E_8$	$\sum c_i \leqslant 3$. If $c_2 \neq 0$, then $c_2 = 1$ and $\sum c_i \leqslant 2$.
C_3		E_8	$\sum c_i \leqslant 4$
		$\neq E_8$	$\sum c_i \leqslant 3$ or $\lambda = 4\lambda_1$
B_2		E_8	$\sum c_i \leqslant 9$
		E_7	$\sum c_i \leqslant 7$
		E_6, F_4	$\sum c_i \leqslant 5$
		G_2	$\sum c_i \leqslant 2$
D_n	$5 < n$		$\lambda = \lambda_i$ or $2\lambda_1$
	$n = 4,5$		$\sum c_i \leqslant 3$. If $c_j > 0$ for $j \neq 1, n, n-1$, then j is unique, $c_j = 1$, and $\sum c_i \leqslant 2$
G_2		E_7, E_8	$c_1 + 2c_2 \leqslant 4$
		F_4, E_6	$c_1 + 2c_2 \leqslant 3$
E_6			λ_1, λ_2
E_7			λ_1, λ_7
E_8			λ_8
F_4			λ_1, λ_4

Then $Y \leqslant C_G(l)$, so Y is contained in either a proper parabolic subgroup or a reductive subgroup of maximal rank. In either case Y must act nontrivially on the Lie algebra of $C_G(l)$. But no proper subgroup of $\bar{G}$ has dimension at least 126.

Lastly, consider the case $\bar{Y} = A_3$ with $\lambda = 6\lambda_1$ or $7\lambda_1$. The latter case is trivial for otherwise Y would not act as a simple group. Assume $\lambda = 6\lambda_1$, so that $\dim(V) = 84$ by (1.14) of [12]. As V is not self-dual, V^* must also appear as a composition factor, forcing $\bar{G} = E_8$. Hence, $p > N = 13$. If $p > 19$, this case presents no difficulties for the arguments to follow. So assume $p = 19$ or 17. Since Y must act as a simple group we have $C = SL(4, p)$ or $SU(4, p)$, respectively. Let K be a maximal torus of a fundamental $SL(2, p)$ of order 18, let t be the involution in this torus, and let w have order 3, with w central in a subgroup F of C of type $SL(3, p)$ or $SU(3, p)$. Choose w to commute with K. Then $C_{\bar{G}}(t) = D_8$ or A_1E_7, of dimension 120 or 136, respectively, and $C_{\bar{G}}(w) = A_2E_6$ of dimension 86 (here we use the fact that $w \in F = F'$). In this way we have the possible dimensions of the centralizers of t and w on $L(\bar{G})$. It is then possible to use Table I and the action of t and w on various modules to determine all possibilities for the composition factors of $L(\bar{G})|Y$. They are as follows, where we use the notation $(abc)^d$ to indicate the irreducible module of high weight $a\lambda_1 + b\lambda_2 + c\lambda_3$ occurs with multiplicity d: $(600)^1/(006)^1/(200)^4/(002)^4$, $(600)^1/(006)^1/(010)^{12}/(000)^8$, $(600)^1/(006)^1/(020)^1/(010)^{10}$. We next calculate $\dim(C_{\bar{G}}(K)) = 64, 64, 58$, respectively. As K is cyclic it can be embedded in a maximal torus of $\bar{G}$, so that $C_{\bar{G}}(K)$ is a group of maximal rank with the given dimension. One checks that the only possibility is $C_{\bar{G}}(K) = T_1A_7$ of dimension 64. However, another check shows $C_{\bar{G}}(K) \cap C_{\bar{G}}(w)$ has dimension 18, whereas viewing $w \in T_1A_7$, we see that this is not possible.

Now V occurs as a composition factor of the tensor product of the fundamental modules $V(\lambda_i)$, each repeated c_i times. It is fairly easy to obtain the action of J on fundamental modules. For example, say $\bar{X} = C_n$. Then each of the fundamental modules occurs within an appropriate wedge of the usual module. Since J is generated by a pair of opposite root groups acting as groups of transvections on the usual module it follows that composition factors of J on a wedge module are of dimension 1 or 2. So if, for example, $\lambda = \lambda_1 + \lambda_2$, all composition factors of J on V have dimension 1, 2, or 3. In some of the other cases a few additional comments are required to obtain the action of J on fundamental modules. If $\bar{X} = D_n$ or B_n, then J has composition factors on the spin modules of dimensions 1 and 2. This is proved by induction, noting that a Levi subgroup of type D_{n-1} or B_{n-1} decomposes the space into the sum of two spin modules. For the exceptional groups there are at most two fundamental modules to consider. One is the adjoint module where J has composition factors of dimension 1, 2, or 3. The other one (for $\bar{X} \neq E_8$) is a module on which J has composition factors of dimensions 1 or 2. This is seen by realizing the appropriate module within a parabolic subgroup of some larger group and then using the commutator relations to see that unipotent elements of J have quadratic minimal polynomial.

At this point we carry out the analysis in each case to obtain the possible sizes of composition factors of J. If the largest possible dimension is $b + 1$, then the required inequality is $(p-1)/3 > b$. For this allows us to define the required 1-dimensional torus.

Having defined the torus we next argue that X acts irreducibly on $L(\bar{G})$. For otherwise, the argument of Lemma 2 produces a connected,

δ-invariant subgroup $\bar{D}$ containing X and the torus. We recall some of the arguments at the start of the proof of Theorem 1. An argument with the Borel–Tits Theorem shows $\bar{D}$ is reductive. Next argue that we may assume $\langle\delta\rangle$ is transitive on the simple factors of $\bar{D}$. Now reduce to $X \leqslant \bar{E}$, where $\bar{E}$ is a simple diagonal subgroup of $\bar{D}$ invariant under δ^k, where k is the number of simple factors of $\bar{D}$. If $\bar{E}$ is classical then by Lemma 1 $\varphi\colon X \to \bar{E}$ can be extended to $\bar{X}$, so X leaves invariant the subspace $L(\bar{\varphi}(\bar{X}))$ and we can repeat the argument unless $\varphi(\bar{X}) = \bar{E}$. At this point we have the assertion. If $\bar{E}$ is exceptional replace $\bar{G}$ with $\bar{E}$ and repeat the above arguments. Eventually we either obtain the result, or reach a situation where X acts irreducibly on $L(\bar{G})$ and $\bar{G}$ is an exceptional group.

Therefore we now assume X acts irreducibly on $L(\bar{G})$. It will suffice to show $\bar{X}$ and $\bar{G}$ are of the same type, for given this we complete the proof as in Lemma 2. As before, we show X is quasisimple and not of type A_1. In the proof of Theorem 1 we then used Table 1, the Weyl degree formula, and [4] to determine $\dim(V)$. However, in view of the weakened restrictions on p, we can no longer assert that all Weyl modules are irreducible for weights λ in Table I. Consequently some changes are required. From [1] we argue that all relevant Weyl modules are irreducible unless $\bar{X} = D_n$ for $n \geqslant 6$ and $\lambda = 2\lambda_1$ or $\bar{X} = B_n$, C_n for $n \geqslant 3$. In the following we indicate the sorts of arguments required to complete the proof for these cases.

As before, we see that $L(\bar{G})|X$ is tensor indecomposable, so there is an irreducible restricted module for $\bar{X}$ of dimension equal to that of $L(\bar{G})$. Let λ be the high weight. We give the details for the case $\bar{X} = C_6$, leaving the remaining cases, which are based on the same sorts of arguments, to the reader. If $p > 13$, then by [1] all relevant Weyl modules are irreducible and we have the same contradiction as before. Choose a maximal parabolic subgroup P of X with Levi factor L of type A_5 and use the argument described in the middle of the proof of Lemma 2. Namely, embed P in a parabolic subgroup $\bar{P}$ of $\bar{G}$ having Levi factor $\bar{L}$ such that $Q = O_p(P) < R_u(\bar{P}) = \bar{Q}$. Then $V_Q = V_{\bar{Q}}$, this module is irreducible for both L and $\bar{L}$, and the latter module is of known type. The prime restrictions force the Weyl module corresponding to $V_Q|L$ to be irreducible and we conclude $\lambda = a\lambda_1 + b\lambda_5 + c\lambda_6$, with $a + b \leqslant 1$. Now replace P by the parabolic subgroup with Levi factor of type C_5 and repeat the argument. We conclude that $b = c = 0$, so $\lambda = \lambda_1$, contradicting the irreducibility of X.

4. Theorem 3

Here we sketch the proof of Theorem 3, which is a slight variation of the arguments of the previous sections. Let $\bar{G}$ be an exceptional group, let

$N = N(\bar{G})$ be as in Theorem 2, and fix notation as in the proofs of Theorems 1 and 2.

First argue as in the proof of Theorem 2 that if X acts irreducibly on $L(\bar{G})$, then $\bar{X}$ and $\bar{G}$ are of the same type and there is a surjective morphism $\bar{\varphi}$ of $\bar{X}$ to $\bar{G}$. Applying (**) to the representation $\theta \circ \bar{\varphi}$ we see that (ii) of Theorem 3 holds. Now suppose S is a proper X-invariant subspace. Using (**) we check that M and δ normalize the group

$$\bar{D} = \bigcap N_{\bar{G}}(\omega^{-i}\theta(m)\, S),$$

the intersection taken over all $m \in M$ and all $i \geqslant 0$. Each of the spaces $\theta(m)\, S$ is X-invariant and hence invariant under the constructed 1-dimensional torus. Since ω normalizes this torus (see the comments following (***)), the torus is contained in $\bar{D}$ so that $\bar{D}$ has positive dimension. If $\bar{D}$ is not reductive, then by [2] it can be embedded in a canonical parabolic subgroup $\bar{P}$, which is then invariant under M and δ. But then $M \leqslant N_G(\bar{P}_\delta) = \bar{P}_\delta$, a parabolic subgroup of G. If $\bar{D}$ is reductive, we have the first assertion of the theorem as $X \leqslant \bar{D}$. In fact $p > N$ implies $X \leqslant \bar{D}^\circ$. If M is maximal, then necessarily $X = F^*(M) = 0^{p'}(\bar{D}_\delta)$.

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