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# A Construction of Certain Maximal Subgroups of the Algebraic Groups $E_6$ and $F_4$

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In this note we give explicit descriptions of certain maximal closed, connected subgroups of the exceptional algebraic groups $F_4$ and $E_6$, defined over an algebraically closed field of nonzero characteristic. Our original goal was to complete the work of [12], where we determined all possible closed subgroups of exceptional algebraic groups which act irreducibly on some nontrivial rational module for the overgroup. However, in three cases ($A_2 < E_6$, $G_2 < E_6$, and $G_2 < F_4$), we omitted the proof of the existence of an appropriate subgroup; this is contained in the proof of Theorem 1 below. As well, this work is part of the larger problem of describing (up to conjugacy) all maximal closed connected subgroups of the exceptional algebraic groups over a field of nonzero characteristic. Our main result is the following:

THEOREM 1. *Let k be an algebraically closed field of characteristic p.*

(a) *If $p \neq 2, 7$, the simply connected, simple algebraic group of type $E_6$ over k has exactly two conjugacy classes of closed, connected subgroups of type $G_2$ which act irreducibly on some nontrivial rational module for $E_6$. The subgroups are maximal among closed connected subgroups of $E_6$ and the two classes are conjugate in* Aut$(E_6)$.

(b) *If $p \neq 2, 5$, the simply connected, simple algebraic group of type $E_6$ over k has exactly two conjugacy classes of closed, connected subgroups of type $A_2$ (isomorphic to $PSL_3$) which act irreducibly on some nontrivial rational module for $E_6$. The subgroups are maximal among closed connected subgroups of $E_6$ if and only if $p \neq 3$. Moreover, the two classes are conjugate in* Aut$(E_6)$.

(c) *If $p = 7$, the simple algebraic group of type $F_4$ has exactly one conjugacy class of closed, connected subgroups of type $G_2$ which act irreducibly*



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*on some nontrivial rational module for* $F_4$. *The subgroups are maximal among closed connected subgroups of* $F_4$.

(d) *If* $p=5$, *the simply connected, simple algebraic group of type* $E_6$ *has two conjugacy classes of closed, connected subgroups of type* $A_2$ *which are maximal among closed connected subgroups of* $E_6$ *and which act reducibly on every nontrivial rational module for* $E_6$. *The two classes are conjugate in* $\mathrm{Aut}(E_6)$.

We obtain the results of (a), (b), and (c) by first working inside the overgroup $E_6$ or $F_4$, using the necessary action of the subgroup on a particular module (given by [12]) and the general theory of the structure of algebraic groups to describe (up to conjugacy) necessary conditions on the embedding. We then take a faithful representation of the overgroup and show that the necessary conditions are in fact sufficient to establish the conjugacy (in $GL_n$) of the subgroup to a known algebraic group. Noting that our embedding and the conjugating matrix are describable over a more general field $\mathscr{K}$, we obtain the following:

THEOREM 2. *Let* $\mathscr{K}$ *be an arbitrary field of characteristic* $p$ *and* $\mathfrak{K}$ *an algebraic closure of* $\mathscr{K}$. *Let* $E_6(\mathscr{K})$, $F_4(\mathscr{K})$, *and* $G_2(\mathscr{K})$ *denote universal Chevalley groups of type* $E_6$, $F_4$, *and* $G_2$, *respectively.*

(a) *If* $p\neq 2, 7$ *and* $(-7)^{1/2}\in\mathscr{K}$, $E_6(\mathscr{K})$ *has a subgroup isomorphic to* $G_2(\mathscr{K})$ *which acts irreducibly on the restricted, 27-dimensional rational modules for the group* $E_6(\mathfrak{K})$.

(b) *If* $p\neq 2, 5$ *and* $(-1)^{1/2}\in\mathscr{K}$, $E_6(\mathscr{K})$ *has a subgroup isomorphic to* $PSL_3(\mathscr{K})$ *which acts irreducibly on the restricted, 27-dimensional rational modules for the group* $E_6(\mathfrak{K})$.

(c) *If* $p=7$, $F_4(\mathscr{K})$ *has a subgroup isomorphic to* $G_2(\mathscr{K})$ *which acts irreducibly on the restricted, 26-dimensional rational module for the group* $F_4(\mathfrak{K})$.

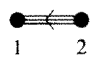
We note that a version of Theorems 1(a) and 2(a) has been proven with different methods by M. Aschbacher in [1] and that A. Ryba has communicated to the author a sketch of another proof of Theorem 2(a). Both Aschbacher and Ryba view $E_6(F)$, the universal Chevalley group of type $E_6$ over a field $F$, as the group of isometries of a symmetric trilinear form on a 27-dimensional module. Aschbacher shows that if $\mathrm{char}(F)\neq 2, 3, 7$, then $G_2(F)$, acting on a certain 27-dimensional module, preserves (up to scalar multiple) exactly two forms similar to the $E_6(F)$ form if and only if $F$ contains $(-7)^{1/2}$ and determines the conjugacy classes of such $G_2(F)$ in $E_6(F)$. Ryba uses the $E_6$ trilinear form and obtains precisely the result of Theorem 2(a). In concluding this introduction, the author wishes to thank

Ron Solomon for helpful conversations concerning the conjugacy questions in Theorem 1.

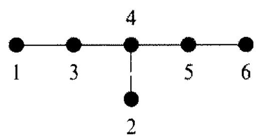
*Notation.* Throughout the paper, let $\mathscr{K}$ denote an arbitrary field of characteristic $p$ and $k$ an algebraically closed field of characteristic $p$. For $X$ a Chevalley group or a semisimple algebraic group, let $\Sigma(X)$ denote the root system associated with $X$, $\Pi(X)$ a base of $\Sigma(X)$, and $\Sigma^+(X)$ the corresponding set of positive roots. For a semisimple algebraic group defined over $k$, let $T_X$ denote a maximal torus of $X$, $U_\gamma$ the $T_X$-root subgroup associated with $\gamma \in \Sigma(X)$, $U_\gamma = \{x_\gamma(t) \mid t \in k\}$, $B_X = \langle U_\gamma \mid \gamma \in \Sigma^+(X) \rangle T_X$ (a Borel subgroup). If $\mu$ is a $T_X$ weight in a nontrivial rational $kX$ module $V$, let $V_{T_X}(\mu) = \{v \in V \mid tv = \mu(t)\, v, \text{ for all } t \in T_X\}$. Other notation will be standard as in [5] or [10]. We refer to [7] for a set of structure constants for the groups $E_6$ and $F_4$ and to [5] for the structure constants for $G_2$. In addition to the notation introduced thus far, we will use the following:

$A$: an algebraic group of type $A_2$ over $k$, $\Pi(A) = \{\alpha_1, \alpha_2\}$.

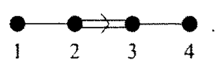
$G$: an algebraic group of type $G_2$ over $k$, $\Pi(G) = \{\gamma_1, \gamma_2\}$, with




$E$: a simply connected algebraic group of type $E_6$ over $k$, $\Pi(E) = \{\beta_i \mid 1 \leqslant i \leqslant 6\}$, with




$F$: an algebraic group of type $F_4$ over $k$, $\Pi(F) = \{\eta_1, \eta_2, \eta_3, \eta_4\}$, with




The proofs of Theorems 1 and 2 are contained in the proofs of the following seven results (the first of which is a straightforward, technical lemma). We use detailed information about certain rational representations of the universal Chevalley groups $E_6(\mathscr{K})$, $F_4(\mathscr{K})$, $G_2(\mathscr{K})$, and $A_2(\mathscr{K})$. For the sake of continuity, we have compiled this in an appendix (results in Appendixes E, A, G, and F) and refer to it when necessary.

LEMMA. *Let $SL_2(k) \cong X_1$ and $X_2 \cong SL_3(k)$. Let $\Pi(X_1) = \{\beta\}$, $\Pi(X_2) = \{\delta_1, \delta_2\}$ and let $W_i$ be the natural module for $X_i$, for $i = 1, 2$. Suppose $X$,* *a simple algebraic group of type $A_1$, is a closed subgroup of $X_i$, with $\Pi(X)=\{\alpha\}$, such that $W_i|X$ is a restricted irreducible rational kX module, $T_X \leqslant T_{X_i}$, and $U_\alpha \leqslant \langle U_\gamma \mid \gamma \in \Pi(X_i)\rangle$.*

(a) *If $i=1$, then $h_\alpha(c)=h_\beta(c)$ for all $c\in k^*$, and there exists $d\in k^*$ such that $x_\alpha(t)=x_\beta(dt)$ and $x_{-\alpha}(t)=x_{-\beta}((1/d)\,t)$, for all $t\in k$.*

(b) *If $i=2$, then $h_\alpha(c)=h_{\delta_1}(c^2)\,h_{\delta_2}(c^2)$ for all $c\in k^*$, and there exists $c_i\in k^*$ such that $x_\alpha(t)=x_{\delta_1}(c_1 t)\,x_{\delta_2}(c_2 t)\,x_{\delta_1+\delta_2}(-N\frac{1}{2}c_1 c_2 t^2)$ and $x_{-\alpha}(t)= x_{-\delta_1}(2t/c_1)\,x_{-\delta_2}(2t/c_2)\,x_{-\delta_1-\delta_2}(N2t^2/(c_1 c_2))$, for all $t\in k$, where $N$ is given by $[x_{\delta_1}(t), x_{\delta_2}(u)]=x_{\delta_1+\delta_2}(Ntu)$.*

*Proof.* Consider the case where $i=2$, so $X\leqslant X_2$. Since $T_X\leqslant T_{X_2}$, $h_\alpha(c)=h_{\delta_1}(c^k)\,h_{\delta_2}(c^l)$ for some $k, l\in\mathbb{Z}$. But $W_2|X$ a 3-dimensional restricted irreducible implies that $p\neq 2$, $W_2^*|X\cong W_2|X$, and $k=2=l$. Since $U_\alpha\leqslant\langle U_{\delta_i}\mid i=1,2\rangle$, $x_\alpha(t)=x_{\delta_1}(f_1(t))\,x_{\delta_2}(f_2(t))\,x_{\delta_1+\delta_2}(f_3(t))$, for some $f_i\in k[t]$. Moreover, $f_1(t)\neq 0\neq f_2(t)$, else $U_\alpha$ lies in the unipotent radical of a proper parabolic of $X_2$ and hence has a fixed point space on $W_2$ or $W_2^*$ of dimension greater than 1. Conjugating $x_\alpha(t)$ by $h_\alpha(c)$ we have $f_i(c^2t)=c^2(f_i(t))$ for $i=1,2$, and $f_3(c^2t)=c^4 f_3(t)$, for $c\in k^*$ and $t\in k$. Letting $t=1$, we find that $f_i(t)=c_i t$ for $i=1,2$ and $f_3(t)=c_3t^2$ for some $c_i\in k$, $c_1c_2\neq 0$. Also, $x_\alpha(t)\,x_\alpha(u)=x_\alpha(t+u)$ implies that $c_3=-\frac{1}{2}Nc_1c_2$, where $N$ is as in the statement of the result.

Let $\theta\colon X\to SL_2(k)$ be given by $\theta(x_\alpha(t))=\left[\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right]$ and $\theta(x_{-\alpha}(t))=\left[\begin{smallmatrix}1 & 0\\ t & 1\end{smallmatrix}\right]$. Let $\{x,y\}$ be a basis of the natural module for $SL_2(k)$ and let $W$ be the vector space of degree two homogeneous polynomials in $\{x,y\}$. Let $\pi\colon SL_2(k)\to SL_3(k)$ be the corresponding representation, where $SL_3(W)$ is identified with $SL_3(k)$ via the ordered basis $\{x^2, xy, y^2\}$. Let $\rho\colon SL_3(k)\to X_2$ be the isomorphism such that

$$\rho\left(\begin{bmatrix}1 & t & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)=x_{\delta_1}(t),\qquad \rho\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & t\\ 0 & 0 & 1\end{bmatrix}\right)=x_{\delta_2}(t),$$

$$\rho\left(\begin{bmatrix}1 & 0 & 0\\ t & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)=x_{-\delta_1}(t),\qquad\text{and}\qquad \rho\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & t & 1\end{bmatrix}\right)=x_{-\delta_2}(t).$$

Then let $\varphi=\rho\circ\pi\circ\theta\colon X\to X_2$. One checks that $\varphi(x_\alpha(t))=x_{\delta_1}(t)\,x_{\delta_2}(2t)\,x_{\delta_1+\delta_2}(-Nt^2)$, $\varphi(x_{-\alpha}(t))=x_{-\delta_1}(2t)\,x_{-\delta_2}(t)\,x_{-\delta_1-\delta_2}(Nt^2)$, and $\varphi(h_\alpha(c))= h_{\delta_1}(c^2)\,h_{\delta_2}(c^2)$. Let $\mathscr{I}_2\colon X\subsetneq X_2$ be the inclusion map. Then $\varphi$ and $\mathscr{I}_2$ are equivalent representations of $X$; so there exists $y\in X_2$ such that $\mathscr{I}_2(g)=y\varphi(g)\,y^{-1}$ for all $g\in X$. Now, $h_{\delta_1}(c^2)\,h_{\delta_2}(c^2)=\mathscr{I}_2(h_\alpha(c))= y\varphi(h_\alpha(c))\,y^{-1}=yh_{\delta_1}(c^2)\,h_{\delta_2}(c^2)\,y^{-1}$ implies that $y=h_{\delta_1}(e)\,h_{\delta_2}(f)$ for some $e$,

$f \in k^*$. Also, $\mathscr{I}_2(x_\alpha(t)) = y\varphi(x_\alpha(t))\, y^{-1}$ implies $e^2 f^{-1} = c_1$, $2e^{-1}f^2 = c_2$, and $ef = \frac{1}{2}c_1 c_2$. Thus, $\mathscr{I}_2(x_{-\alpha}(t)) = x_{-\delta_1}(2t/c_1)\, x_{-\delta_2}(2t/c_2)\, x_{-\delta_1-\delta_2}(N2t^2/(c_1c_2))$ and (b) holds.

We omit the proof of (a), which is similar to but easier than the above. ∎

We now begin our consideration of the irreducible $G_2$'s in $E_6$ and mention that the methods we use were developed in [8, 12].

PROPOSITION (G.1). *Suppose $G$ is isomorphic to a closed subgroup of $E$ and, identifying $G$ with the subgroup, suppose $V|G$ is irreducible for some nontrivial rational $kE$ module $V$. Then $p \neq 2, 7$ and up to conjugacy in $E$*

$$x_{\gamma_1}(t) = x_{\beta_1}(t)\, x_{\beta_3}(t)\, x_{\beta_1+\beta_3}(-\tfrac{1}{2}t^2)\, x_{\beta_2}(t)\, x_{\beta_5}(t)\, x_{\beta_6}(t)\, x_{\beta_5+\beta_6}(-\tfrac{1}{2}t^2),$$

$$x_{-\gamma_1}(t) = x_{-\beta_1}(2t)\, x_{-\beta_3}(2t)\, x_{-\beta_1-\beta_3}(2t^2)\, x_{-\beta_2}(t)\, x_{-\beta_5}(2t)\, x_{-\beta_6}(2t)\, x_{-\beta_5-\beta_6}(2t^2),$$

$$x_{\gamma_2}(t) = x_{\beta_3+\beta_4}(a_1 t)\, x_{\beta_2+\beta_4}(a_2 t)\, x_{\beta_4+\beta_5}((a_1 + \tfrac{1}{2}a_2)\, t),$$

*and*

$$x_{-\gamma_2}(t) = x_{-\beta_3-\beta_4}(t/a_1)\, x_{-\beta_2-\beta_4}(t/a_2)\, x_{-\beta_4-\beta_5}(t/(a_1 + \tfrac{1}{2}a_2)),$$

*for some $a_i \in k^*$ with $2a_1^2 + a_1 a_2 + a_2^2 = 0$.*

*Proof.* By the Main Theorem of [12], $p \neq 2, 7$ and $G$ acts irreducibly on $V(\lambda_1)$, the irreducible $kE$ module with high weight $\lambda_1$, where $\lambda_1$ is the fundamental dominant weight corresponding to $\beta_1$. Also, $V(\lambda_1)|G$ is the irreducible $kG$ module with high weight $2(2\gamma_1 + \gamma_2)$. Let $V = V(\lambda_1)$. Let $P \geqslant B_G$ be the parabolic subgroup of $G$ with Levi factor $L = \langle U_{\pm\gamma_1}\rangle T_G$ and unipotent radical $Q$. By the Borel–Tits theorem [2], there exists a parabolic $P_E$ of $E$ (with Levi factor $L_E$ and unipotent radical $Q_E$), such that $P \leqslant P_E$ and $Q \leqslant Q_E$. Up to conjugacy in $E$, we may assume $T_G \leqslant T_E \leqslant L_E$ and $B_G \leqslant B_E$. It follows from [9] that $V^Q = V^{Q_E}$ is a restricted 3-dimensional irreducible $kL$ module. But $V^{Q_E}$ is an irreducible $kL'_E$ module with high weight $\lambda_1 | (T_E \cap L'_E)$; thus $\langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$ is a simple component of $L'_E$. Considering the action of $G$ on $V^*$, we conclude as well that $\langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$ is a component of $L'_E$. However, $L'_E \neq \langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle \times \langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$ since otherwise, $h_{\gamma_1}(-1) = 1$. Hence, $L'_E = \langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle \times \langle U_{\pm\beta_2}\rangle \times \langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$. Moreover, the above argument shows that $P_E$ is minimal with respect to the conditions: $P_E$ a parabolic of $E$, $P \leqslant P_E$ and $Q \leqslant R_u(P_E)$. Hence, by (2.9) of [8], $Z(L)^\circ \leqslant Z(L_E)^\circ$. But $Z(L)^\circ$ and $Z(L_E)^\circ$ are 1-dimensional tori, so $Z(L)^\circ = Z(L_E)^\circ$. So $L \leqslant C_{P_E}(Z(L_E)^\circ) =$

$L_E$ and $L' \leqslant \langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle \times \langle U_{\pm\beta_2}\rangle \times \langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$. Moreover, the projection of $L'$ in $\langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$ (respectively, $\langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$) acting on $V^Q$ (resp., $(V^*)^Q$) satisfies the hypotheses of the lemma. Note that $0 \neq w \in V_{T_E}(\lambda_1 - \beta_1 - \beta_3 - \beta_4)$ is the maximal vector of an $L'_E$ composition factor of $V$ and hence of an $L'$ composition factor of $V$. One checks that (for all $p \neq 2$) $V|L'$ has no 6-dimensional tensor decomposable composition factor. The action of $T_G \cap L'$ on $w$ then implies that the natural module for $\langle U_{\pm\beta_2}\rangle$ is a restricted irreducible module for the projection of $L'$ in $\langle U_{\pm\beta_2}\rangle$. So by the lemma, there exist $d_i$, $e_i$, $r \in k^*$ such that

$$h_{\gamma_1}(c) = h_{\beta_1}(c^2)\, h_{\beta_3}(c^2)\, h_{\beta_2}(c)\, h_{\beta_5}(c^2)\, h_{\beta_6}(c^2),$$

$$x_{\gamma_1}(t) = x_{\beta_1}(d_1 t)\, x_{\beta_3}(d_2 t)\, x_{\beta_1+\beta_3}(-\tfrac{1}{2} d_1 d_2 t^2)\, x_{\beta_2}(rt)\, x_{\beta_5}(e_1 t)\, x_{\beta_6}(e_2 t) \cdot x_{\beta_5+\beta_6}(-\tfrac{1}{2} e_1 e_2 t^2),$$

and

$$x_{-\gamma_1}(t) = x_{-\beta_1}(2t/d_1)\, x_{-\beta_3}(2t/d_2)\, x_{-\beta_1-\beta_3}(2t^2/d_1 d_2)\, x_{-\beta_2}(t/r) \cdot x_{-\beta_5}(2t/e_1)\, x_{-\beta_6}(2t/e_2)\, x_{-\beta_5-\beta_6}(2t^2/e_1 e_2),$$

for all $t \in k$ and $c \in k^*$. Moreover, conjugating by an element of $T_E$ if necessary, we may assume $e_i = 1 = d_i = r$.

We must now consider the embedding of $\langle U_{\pm\gamma_2}\rangle$ in $E$. One checks that $Z(L)^\circ = \{z(c) = h_{\gamma_1}(c)\, h_{\gamma_2}(c^2) \mid c \in k^*\}$ and $Z(L_E)^\circ = \{h_{\beta_1}(d^2)\, h_{\beta_2}(d^3)\, h_{\beta_3}(d^4)\, h_{\beta_4}(d^6)\, h_{\beta_5}(d^4)\, h_{\beta_6}(d^2) = z_E(d) \mid d \in k^*\}$. Now, $z(c) = z_E(c^l)$ for some $l \in \mathbb{Z}$ and the action of $z(c)$ on $V_{T_E}(\lambda_1)$ implies that $l = 1$. Combining this with the known factorization of $h_{\gamma_1}(c)$, we have $h_{\gamma_2}(c) = h_{\beta_2}(c)\, h_{\beta_3}(c)\, h_{\beta_4}(c^3)\, h_{\beta_5}(c)$. Let $T_0 = \{h_{\gamma_1}(e^2)\, h_{\gamma_2}(e^3) \mid e \in k^*\}$. Then $\langle U_{\pm\gamma_2}\rangle \leqslant C_E(T_0)$, a connected reductive group containing $T_E$. One checks that $C_E(T_0)$ has root system $\{\pm(\beta_3+\beta_4),\ \pm(\beta_2+\beta_4),\ \pm(\beta_4+\beta_5)\}$. So $\langle U_{\pm\gamma_2}\rangle \leqslant \langle U_{\pm(\beta_3+\beta_4)}\rangle \times \langle U_{\pm(\beta_2+\beta_4)}\rangle \times \langle U_{\pm(\beta_4+\beta_5)}\rangle$. Since $U_{\gamma_2} \leqslant Q \leqslant Q_E$, $x_{\gamma_2}(t) = x_{\beta_3+\beta_4}(f_1(t))\, x_{\beta_2+\beta_4}(f_2(t))\, x_{\beta_4+\beta_5}(f_3(t))$ for some $f_i \in k[t]$. Now, $f_1 \neq 0$, else $0 \neq w \in V_{T_E}(\lambda_1 - \beta_1 - \beta_3 - \beta_4)$ is fixed by $B_G$, contradicting the irreducibility of $G$ on $V$. Arguing similarly with $V^*$, we see that $f_3 \neq 0$. So $\langle U_{\pm\gamma_2}\rangle$ projects nontrivially into $\langle U_{\pm(\beta_3+\beta_4)}\rangle$ and $\langle U_{\pm(\beta_4+\beta_5)}\rangle$. In fact, $\langle U_{\pm\gamma_2}\rangle$ projects nontrivially into $\langle U_{\pm(\beta_2+\beta_4)}\rangle$. For otherwise, $T_G \cap \langle U_{\pm\gamma_2}\rangle \leqslant \{h_{\beta_3+\beta_4}(c_1) \cdot h_{\beta_4+\beta_5}(c_2) \mid c_i \in k^*\}$, contradicting the given factorization of $h_{\gamma_2}(c)$. Finally, we note that the factorization of $h_{\gamma_2}(c)$ implies that the projection of $\langle U_{\pm\gamma_2}\rangle$ into each of the components $\langle U_{\pm(\beta_3+\beta_4)}\rangle$, $\langle U_{\pm(\beta_2+\beta_4)}\rangle$, and $\langle U_{\pm(\beta_4+\beta_5)}\rangle$ satisfies the hypotheses of the lemma. Thus $x_{\gamma_2}(t) = x_{\beta_3+\beta_4}(a_1 t) x_{\beta_2+\beta_4}(a_2 t)\, x_{\beta_4+\beta_5}(a_3 t)$, for some $a_i \in k^*$, and $x_{-\gamma_2}(t) = x_{-\beta_3-\beta_4}(t/a_1)\, x_{-\beta_2-\beta_4}(t/a_2)\, x_{-\beta_4-\beta_5}(t/a_3)$. The relation $[x_{-\gamma_1}(t), x_{\gamma_2}(u)] = 1$ implies that $a_3 = a_1 + \frac{1}{2} a_2$; the relation $[x_{\gamma_1}(t), x_{-\gamma_2}(t)] = 1$ implies that

$1/a_1 + 1/a_2 = 1/a_3$. So $2a_1^2 + a_1 a_2 + a_2^2 = 0$. This completes the proof of the proposition. ▮

*Notation.* Assume $\mathrm{char}(\mathscr{K}) \neq 2, 7$, $(-7)^{1/2} \in \mathscr{K}$, and let $\mathfrak{K}$ be an algebraic closure of $\mathscr{K}$. Let $a_1, a_2 \in \mathscr{K}^*$ such that $2a_1^2 + a_1 a_2 + a_2^2 = 0$ and fix $\varepsilon \in \mathscr{K}$ such that $\varepsilon^2 + 7 = 0$; so $a_2 = \frac{1}{2}(-1 \pm \varepsilon)\, a_1$. Let $G_{\mathscr{K}}(a_1, a_2) \leqslant E_6(\mathscr{K})$ be defined as follows: $G_{\mathscr{K}}(a_1, a_2) = \langle \bar{x}_{\gamma_1}(t), \bar{x}_{-\gamma_1}(t), \bar{x}_{\gamma_2}(t), \bar{x}_{-\gamma_2}(t) \mid t \in \mathscr{K} \rangle$, where

$$\begin{aligned}
\bar{x}_{\gamma_1}(t) &= x_{\beta_1}(t)\, x_{\beta_3}(t)\, x_{\beta_1+\beta_3}(-\tfrac{1}{2}t^2)\, x_{\beta_2}(t)\, x_{\beta_5}(t)\, x_{\beta_6}(t)\, x_{\beta_5+\beta_6}(-\tfrac{1}{2}t^2),\\
\bar{x}_{-\gamma_1}(t) &= x_{-\beta_1}(2t)\, x_{-\beta_3}(2t)\, x_{-\beta_1-\beta_3}(2t^2)\, x_{-\beta_2}(t)\, x_{-\beta_5}(2t)\, x_{-\beta_6}(2t)\\
&\quad \cdot x_{-\beta_5-\beta_6}(2t^2),\\
\bar{x}_{\gamma_2}(t) &= x_{\beta_3+\beta_4}(a_1 t)\, x_{\beta_2+\beta_4}(a_2 t)\, x_{\beta_4+\beta_5}((a_1 + \tfrac{1}{2}a_2)\, t),
\end{aligned}$$

and

$$\bar{x}_{-\gamma_2}(t) = x_{-\beta_3-\beta_4}(t/a_1)\, x_{-\beta_2-\beta_4}(t/a_2)\, x_{-\beta_4-\beta_5}(t/(a_1 + \tfrac{1}{2}a_2)).$$

The statements of Theorems 1(a) and 2(a) will follow immediately from Proposition (G.1) and the following:

THEOREM (G.2). *Let notation be as above. Then*

(a) $G_{\mathscr{K}}(a_1, a_2) \cong G_2(\mathscr{K})$.

(b) $G_{\mathscr{K}}(a_1, a_2)$ *acts irreducibly on the restricted, 27-dimensional rational modules for the group* $E_6(\mathfrak{K})$.

(c) $G_{\mathscr{K}}(a, \frac{1}{2}(-1+\varepsilon)\, a)$ *is conjugate in* $E_6(\mathscr{K})$ *to* $G_{\mathscr{K}}(b, \frac{1}{2}(-1+\varepsilon)\, b)$, *for any* $a, b \in \mathscr{K}^*$.

(d) *Let* $\tau$ *be the graph automorphism of* $E_6(\mathscr{K})$. *Then* $\tau(G_{\mathscr{K}}(a, \frac{1}{2}(-1+\varepsilon)\, a)) = G_{\mathscr{K}}(b, \frac{1}{2}(-1-\varepsilon)\, b)$, *where* $b = -\frac{1}{4}(3+\varepsilon)\, a$.

(e) *Over the algebraically closed field* $k$ *(characteristic* $k \neq 2, 7$*)* $G_k(1, \frac{1}{2}(-1+\varepsilon))$ *is not conjugate in* $E$ *to* $G_k(1, \frac{1}{2}(-1-\varepsilon))$.

(f) *With* $k$ *as in* (e), $G_k(a_1, a_2)$ *is maximal among closed, connected subgroups of* $E$.

*Proof.* In the Appendix, we explicitly describe faithful matrix representations of $E_6(\mathscr{K})$ and $G_2(\mathscr{K})$ in $SL_{27}(\mathscr{K})$. More precisely, $\varphi_E \colon E_6(\mathscr{K}) \to SL_{27}(\mathscr{K})$ corresponds to a representation on a 27-dimensional vector space $V$ with a fixed ordered basis $\mathscr{B}_E = \{v_i \mid 1 \leqslant i \leqslant 27\}$ and $\varphi_G \colon G_2(\mathscr{K}) \to SL_{27}(\mathscr{K})$ corresponds to a representation on a 27-dimensional vector space $W$ with a fixed ordered basis $\mathscr{B}_G = \{u_i \mid 1 \leqslant i \leqslant 27\}$.

Let $P: V \to W$ be given by

$$Pv_1 = 2u_1,\ Pv_2 = u_2,\ Pv_3 = u_3,\ Pv_4 = -a_1 u_4,$$
$$Pv_5 = \tfrac{1}{4}(2a_1 - a_2)\, u_5 - \tfrac{1}{2}a_2 u_6,$$
$$Pv_6 = -\tfrac{1}{4}(3a_1 + \tfrac{1}{2}a_2)\, u_5 - (\tfrac{1}{2}a_1 + \tfrac{1}{4}a_2)\, u_6,$$
$$Pv_7 = \tfrac{1}{8}(2a_1 - a_2)\, u_7 - \tfrac{1}{4}(a_1 + \tfrac{1}{2}a_2)\, u_8,$$
$$Pv_8 = -\tfrac{1}{4}(3a_1 + \tfrac{1}{2}a_2)\, u_7 + \tfrac{1}{8}(2a_1 - a_2)\, u_8,$$
$$Pv_9 = -\tfrac{1}{2}a_1 u_9,\ Pv_{10} = a_2(a_1 + \tfrac{1}{2}a_2)\, u_{10},$$
$$Pv_{11} = \tfrac{1}{8}a_2(2a_1 + a_2)\, u_{11} - \tfrac{1}{8}a_2(2a_1 - a_2)\, u_{12},$$
$$Pv_{12} = \tfrac{1}{8}a_1(2a_1 - a_2)\, u_{11} - \tfrac{1}{4}a_1(3a_1 + \tfrac{1}{2}a_2)\, u_{12},$$
$$Pv_{13} = -\tfrac{1}{7}a_2(a_1 + \tfrac{1}{4}a_2)\, u_{13} - \tfrac{1}{7}a_1(a_1 + \tfrac{1}{4}a_2)\, u_{14} + \tfrac{1}{14}(-5a_1^2 + \tfrac{1}{2}a_1 a_2)\, u_{15},$$
$$Pv_{14} = \tfrac{1}{28}a_1(5a_1 - \tfrac{1}{2}a_2)\, u_{13} + [\tfrac{1}{14}a_1^2 - \tfrac{9}{56}a_2^2]\, u_{14} + [\tfrac{3}{28}a_1^2 + \tfrac{5}{56}a_1 a_2]\, u_{15},$$
$$Pv_{15} = \tfrac{1}{28}(5a_1^2 - \tfrac{1}{2}a_1 a_2)\, u_{13} - [\tfrac{3}{28}a_1^2 + \tfrac{5}{56}a_1 a_2]\, u_{14} + [\tfrac{3}{28}a_1^2 + \tfrac{5}{56}a_1 a_2]\, u_{15},$$
$$Pv_{16} = -\tfrac{1}{8}a_1(3a_1 + \tfrac{1}{2}a_2)\, u_{16} + \tfrac{1}{2}a_1(\tfrac{1}{2}a_1 + \tfrac{1}{4}a_2)\, u_{17},$$
$$Pv_{17} = -\tfrac{1}{8}a_2(a_1 - \tfrac{1}{2}a_2)\, u_{16} + \tfrac{1}{2}a_1(\tfrac{1}{2}a_1 + \tfrac{1}{4}a_2)\, u_{17},$$
$$Pv_{18} = -\tfrac{1}{8}a_1(2a_1 - a_2)\, u_{18},\ Pv_{19} = -\tfrac{1}{2}a_1^2(a_1 + \tfrac{1}{2}a_2)\, u_{19},$$
$$Pv_{20} = \tfrac{1}{16}a_2^2(3a_1 + \tfrac{1}{2}a_2)\, u_{20} - \tfrac{1}{8}a_2^2(a_1 + \tfrac{1}{2}a_2)\, u_{21},$$
$$Pv_{21} = -\tfrac{1}{8}a_2^2(a_1 - \tfrac{1}{2}a_2)\, u_{20} + \tfrac{1}{4}a_1 a_2(a_1 + \tfrac{1}{2}a_2)\, u_{21},$$
$$Pv_{22} = \tfrac{1}{16}a_2^2(a_1 - \tfrac{1}{2}a_2)\, u_{22} + \tfrac{1}{8}a_2^2(a_1 + \tfrac{1}{2}a_2)\, u_{23},$$
$$Pv_{23} = -\tfrac{1}{16}a_2^2(a_1 + \tfrac{1}{2}a_2)\, u_{22} - \tfrac{1}{8}a_1 a_2(a_1 + \tfrac{1}{2}a_2)\, u_{23},$$
$$Pv_{24} = \tfrac{1}{4}a_1^2(a_1 + \tfrac{1}{2}a_2)\, u_{24},\ Pv_{25} = \tfrac{1}{2}a_1^2(a_1 + \tfrac{1}{2}a_2)^2\, u_{25},$$
$$Pv_{26} = -\tfrac{1}{4}a_1^2(a_1 + \tfrac{1}{2}a_2)^2\, u_{26},\ Pv_{27} = \tfrac{1}{4}a_1^2(a_1 + \tfrac{1}{2}a_2)^2\, u_{27}.$$

One checks that, for all $t \in \mathscr{K}$, for $1 \leqslant i \leqslant 27$, and for $j = 1, 2$, $\varphi_G(x_{\gamma_j}(t))\, Pv_i = P\varphi_E(\bar{x}_{\gamma_j}(t))\, v_i$ and $\varphi_G(x_{-\gamma_j}(t))\, Pv_i = P\varphi_E(\bar{x}_{-\gamma_j}(t))\, v_i$. Thus, conjugation by $P$ is an isomorphism between $\varphi_G(G_2(\mathscr{K}))$ and $\varphi_E(G_{\mathscr{K}}(a_1, a_2))$. So $G_{\mathscr{K}}(a_1, a_2) \cong G_2(\mathscr{K})$. The statement of (b) follows from Appendixes E and G. For (c), let $z = h_{\beta_1}(d^2)\, h_{\beta_2}(d^3)\, h_{\beta_3}(d^4)\, h_{\beta_4}(d^6) \cdot h_{\beta_5}(d^4)\, h_{\beta_6}(d^2)$, for $d = b/a$. Then one checks that $zG_{\mathscr{K}}(a, \tfrac{1}{2}(-1+\varepsilon)\, a)\, z^{-1} = G_{\mathscr{K}}(b, \tfrac{1}{2}(-1+\varepsilon)\, b)$. The statement of (d) is easily checked.

Now suppose there exists $y \in E$ such that $yG_k(1, \frac{1}{2}(-1+\varepsilon))\, y^{-1} = G_k(1, \frac{1}{2}(-1-\varepsilon))$. By (c) and (d), there exists $h \in E$ such that $h\tau(G_k(1, \frac{1}{2}(-1+\varepsilon)))\, h^{-1} = G_k(1, \frac{1}{2}(-1-\varepsilon))$. Set $G_k = G_k(1, \frac{1}{2}(-1+\varepsilon))$ (a simple algebraic group of type $G_2$). Then $y^{-1}h\tau(G_k)\, h^{-1}y = G_k$. We first note that $G_k$ is not pointwise fixed by $y^{-1}h\tau$ (viewed as an element of $\mathrm{Aut}(E)$). For there are two conjugacy classes of involutions in the coset $\tau E$ (in $\mathrm{Aut}(E)$), with fixed point subgroups of types $F_4$ and $C_4$. (See [6].) The $F_4$ (a conjugate of the fixed point subgroup of $\tau$) acts reducibly on the 27-dimensional $kE$ module $V(\lambda_1)$, so does not contain $G_k$. Since $p \neq 2$, the $C_4$ acts irreducibly on $V(\lambda_1)$. (See [12].) But the Main Theorem of [8] implies that no proper closed connected subgroup of $C_4$ acts irreducibly on a 27-dimensional $C_4$ module when $p \neq 2$. So $y^{-1}h\tau$ induces a nontrivial (algebraic group) automorphism of $G_k$. By Steinberg (see [10]), any such automorphism is induced by an inner automorphism. However, we then have $G_k$ pointwise fixed by an involution in $\tau E$, contradicting the above remarks. This completes the proof of (e). Finally, we note that (f) follows from (b) and the Main Theorems of [8, 12]. ∎

Note that if $\mathrm{char}(\mathscr{K}) = 7$, the definition of $G_{\mathscr{K}}(a_1, a_2) < E_6(\mathscr{K})$ makes sense, and in fact, we could argue that $G_{\mathscr{K}}(a_1, a_2) \cong G_2(\mathscr{K})$ in this case as well. Moreover, it is easy to see that when $p = 7$, $G_{\mathscr{K}}(a_1, a_2)$ is fixed by $\tau$, thus giving an embedding of $G_2(\mathscr{K})$ in $F_4(\mathscr{K})$. However, we will work inside $F_4(\mathscr{K})$ instead, in order to obtain the conjugacy statement in Theorem 1(c). The proofs of Theorems 1(c) and 2(c) closely parallel the above proofs, so we give an abbreviated version.

PROPOSITION (F1). *Suppose $G$ is isomorphic to a closed subgroup of $F$ and, identifying $G$ with the subgroup, suppose $V|G$ is irreducible for some nontrivial rational $kF$ module $V$. Then $p = 7$ and up to conjugacy in $F$,*

$$x_{\gamma_1}(t) = x_{\eta_1}(t)\, x_{\eta_3}(t)\, x_{\eta_4}(t)\, x_{\eta_3+\eta_4}(-\tfrac{1}{2}t^2),$$
$$x_{-\gamma_1}(t) = x_{-\eta_1}(t)\, x_{-\eta_3}(2t)\, x_{-\eta_4}(2t)\, x_{-\eta_3-\eta_4}(2t^2),$$
$$x_{\gamma_2}(t) = x_{\eta_1+\eta_2}(4bt)\, x_{\eta_2+\eta_3}(bt),$$

*and*

$$x_{-\gamma_2}(t) = x_{-\eta_1-\eta_2}(2t/b)\, x_{-\eta_2-\eta_3}(t/b),$$

*for some $b \in k^*$.*

*Proof.* By the Main Theorem of [12], $p = 7$ and $G$ acts irreducibly on the rational $kF$ module with high weight $\lambda = \eta_1 + 2\eta_2 + 3\eta_3 + 2\eta_4$, and for

$V = V(\lambda)$, $V|G$ is the irreducible $kG$ module with high weight $2(2\gamma_1 + \gamma_2)$. Let $P$ be as in the proof of (G.1). Arguing as in (G.1), we see that if $P_F \geqslant B_F$ is the parabolic subgroup of $F$ with Levi factor $L_F = (\langle U_{\pm\eta_1}\rangle \times \langle U_{\pm\eta_3}, U_{\pm\eta_4}\rangle)\, T_F$, then up to conjugacy in $F$, $P \leqslant P_F$, $Q \leqslant R_u(P_F) = Q_F$, $L' \leqslant L'_F$, $T_G \leqslant T_F$, and $Z(L)^\circ = Z(L_F)^\circ$. Moreover, again arguing as in (G.1), we see that the projection of $L'$ in each of the components of $L'_F$ satisfies the hypotheses of the lemma. Thus, there exist $d$, $e_i \in k^*$ such that

$$h_{\gamma_1}(c) = h_{\eta_1}(c)\, h_{\eta_3}(c^2)\, h_{\eta_4}(c^2),$$
$$x_{\gamma_1}(t) = x_{\eta_1}(dt)\, x_{\eta_3}(e_1 t)\, x_{\eta_4}(e_2 t)\, x_{\eta_3+\eta_4}(-\tfrac{1}{2}e_1 e_2 t^2),$$

and

$$x_{-\gamma_1}(t) = x_{-\eta_1}(t/d)\, x_{-\eta_3}(2t/e_1)\, x_{-\eta_4}(2t/e_2)\, x_{-\eta_3-\eta_4}(2t^2/e_1 e_2),$$

for all $t \in k$ and $c \in k^*$. Moreover, conjugating by an element of $T_F$ if necessary, we may assume $d = 1 = e_i$.

We now consider the embedding of $\langle U_{\pm\gamma_2}\rangle$ in $F$. Arguing as in (G.1), we find that $h_{\gamma_2}(c) = h_{\eta_1}(c)\, h_{\eta_2}(c^3)\, h_{\eta_3}(c)$. Let $T_0 = \{h_{\gamma_1}(e^2)\, h_{\gamma_2}(e^3) \mid e \in k^*\}$. Then $\langle U_{\pm\gamma_2}\rangle \leqslant C_F(T_0)$, a connected reductive group containing $T_F$. One checks that $C_F(T_0)$ has root system $\{\pm(\eta_1+\eta_2), \pm(\eta_2+\eta_3)\}$. So $\langle U_{\pm\gamma_2}\rangle \leqslant \langle U_{\pm(\eta_1+\eta_2)}\rangle \times \langle U_{\pm(\eta_2+\eta_3)}\rangle$. Since $U_{\gamma_2} \leqslant Q \leqslant Q_F$, $x_{\gamma_2}(t) = x_{\eta_1+\eta_2}(f_1(t))\; x_{\eta_2+\eta_3}(f_2(t))$ for some $f_i \in k[t]$. Now, $f_2 \neq 0$, else $0 \neq w \in V_{T_F}(\lambda - \eta_2 - \eta_3 - \eta_4)$ is fixed by $B_G$, contradicting the irreducibility of $G$ on $V$. So $\langle U_{\pm\gamma_2}\rangle$ projects nontrivially into $\langle U_{\pm(\eta_2+\eta_3)}\rangle$. In fact, $\langle U_{\pm\gamma_2}\rangle$ projects nontrivially into $\langle U_{\pm(\eta_1+\eta_2)}\rangle$. For otherwise, $T_G \cap \langle U_{\pm\gamma_2}\rangle \leqslant \{h_{\eta_2+\eta_3}(c) \mid c_1 \in k^*\}$, contradicting the given factorization of $h_{\gamma_2}(c)$. Finally, we note that the factorization of $h_{\gamma_2}(c)$ implies that the projection of $\langle U_{\pm\gamma_2}\rangle$ into each of the components $\langle U_{\pm(\eta_1+\eta_2)}\rangle$ and $\langle U_{\pm(\eta_2+\eta_3)}\rangle$ satisfies the hypotheses of the lemma. Thus $x_{\gamma_2}(t) = x_{\eta_1+\eta_2}(at)\, x_{\eta_2+\eta_3}(bt)$ and $x_{-\gamma_2}(t) = x_{-\eta_1-\eta_2}(t/a)\, x_{-\eta_2-\eta_3}(t/b)$ for some $a, b \in k^*$. The relation $[x_{-\gamma_1}(t), x_{\gamma_2}(u)] = 1$ implies that $a - 4b = 0$. This completes the proof of the proposition. ∎

*Notation.* Let $\mathscr{K}$ be an arbitrary field of characteristic 7 and let $\mathfrak{K}$ be an algebraic closure of $\mathscr{K}$. Let $b \in \mathscr{K}^*$ and let $G_{\mathscr{K}}(b) \leqslant F_4(\mathscr{K})$ be defined as follows: $G_{\mathscr{K}}(b) = \langle \bar{x}_{\gamma_1}(t), \bar{x}_{-\gamma_1}(t), \bar{x}_{\gamma_2}(t), \bar{x}_{-\gamma_2}(t) \mid t \in \mathscr{K}\rangle$, where

$$\bar{x}_{\gamma_1}(t) = x_{\eta_1}(t)\, x_{\eta_3}(t)\, x_{\eta_4}(t)\, x_{\eta_3+\eta_4}(-\tfrac{1}{2}t^2),$$
$$\bar{x}_{-\gamma_1}(t) = x_{-\eta_1}(t)\, x_{-\eta_3}(2t)\, x_{-\eta_4}(2t)\, x_{-\eta_3-\eta_4}(2t^2),$$
$$\bar{x}_{\gamma_2}(t) = x_{\eta_1+\eta_2}(4bt)\, x_{\eta_2+\eta_3}(bt),$$

and

$$\bar{x}_{-\gamma_2}(t) = x_{-\eta_1-\eta_2}(2t/b)\, x_{-\eta_2-\eta_3}(t/b).$$

The statements of Theorems 1(c) and 2(c) follow from Proposition (F.1) and the following:

THEOREM(F.2). *Let notation be as above. Then*

(a) $G_{\mathscr{K}}(b) \cong G_2(\mathscr{K})$.

(b) *$G_{\mathscr{K}}(b)$ acts irreducibly on the restricted, 26-dimensional rational module for the group $F_4(\mathfrak{K})$.*

(c) *$G_{\mathscr{K}}(b)$ is conjugate in $F_4(\mathscr{K})$ to $G_{\mathscr{K}}(a)$, for any $a \in \mathscr{K}^*$.*

(d) *Over the algebraically closed field k (of characterstic 7), $G_k(b)$ is maximal among closed, connected subgroups of F.*

*Proof.* In the Appendix, we explicitly describe a faithful matrix representation of $F_4(\mathscr{K})$, $\varphi_F: F_4(\mathscr{K}) \to S_{26}(\mathscr{K})$, where $\varphi_F$ corresponds to a representation on a 26-dimensional vector space $V$ with a fixed ordered basis $\mathscr{B}_F = \{y_i \mid 1 \leqslant i \leqslant 26\}$. As in (G.2), $\varphi_G: G_2(\mathscr{K}) \to SL_{27}(\mathscr{K})$ is a faithful representation of $G_2(\mathscr{K})$ on a 27-dimensional vector space $W$ with basis $\mathscr{B}_G = \{u_i \mid 1 \leqslant i \leqslant 27\}$. As noted in the Appendix, when $p=7$, $\varphi_G(G_2(\mathscr{K}))$ fixes a 1-space on $W$, namely $\langle u_{13} - 2u_{14} + 2u_{15} \rangle$. Let $\bar{W} = W/\langle u_{13} - 2u_{14} + 2u_{15} \rangle$ and let $\varphi'_G$ denote the corresponding faithful representation $\varphi'_G: G_2(\mathscr{K}) \to SL_{26}(\bar{W})$. Let $\mathscr{B}'_G = \{\bar{u}_i \mid 1 \leqslant i \leqslant 27,\ i \neq 15\}$, where $\bar{w}$ denotes the image of $w$ in $\bar{W}$, for $w \in W$. Then $\mathscr{B}'_G$ is a basis of $\bar{W}$. Let $P: V \to \bar{W}$ be defined as follows: $Py_1 = 2\bar{w}_1$, $Py_2 = \bar{w}_2$, $Py_3 = 6\bar{w}_3$, $Py_4 = b\bar{w}_4$, $Py_5 = 2b\bar{w}_5 + 3b\bar{w}_6$, $Py_6 = 2b\bar{w}_5 + 5b\bar{w}_6$, $Py_7 = 2b\bar{w}_7 + b\bar{w}_8$, $Py_8 = b\bar{w}_7 + 5b\bar{w}_8$, $Py_9 = 4b\bar{w}_9$, $Py_{10} = 3b^2\bar{w}_{10}$, $Py_{11} = 6b^2\bar{w}_{11} + 4b^2\bar{w}_{12}$, $Py_{12} = 6b^2\bar{w}_{11} + 5b^2\bar{w}_{12}$, $Py_{13} = 2b^2\bar{w}_{13} + 6b^2\bar{w}_{14}$, $Py_{14} = 3b^2\bar{w}_{14}$, $Py_{15} = 2b^2\bar{w}_{16} + 2b^2\bar{w}_{17}$, $Py_{16} = 6b^2\bar{w}_{16} + 5b^2\bar{w}_{17}$, $Py_{17} = 6b^2\bar{w}_{18}$, $Py_{18} = 3b^3\bar{w}_{19}$, $Py_{19} = b^3\bar{w}_{20} + b^3\bar{w}_{21}$, $Py_{20} = 6b^3\bar{w}_{20} + 5b^3\bar{w}_{21}$, $Py_{21} = 4b^3\bar{w}_{22} + 2b^3\bar{w}_{23}$, $Py_{22} = 6b^3\bar{w}_{22} + 4b^3\bar{w}_{23}$, $Py_{23} = 2b^3\bar{w}_{24}$, $Py_{24} = 3b^4\bar{w}_{25}$, $Py_{25} = 2b^4\bar{w}_{26}$, $Py_{26} = 5b^4\bar{w}_{27}$.

One checks that, for all $t \in \mathscr{K}$, for $1 \leqslant i \leqslant 26$, and for $j = 1, 2$, $\varphi'_G(x_{\gamma_j}(t))\, Py_i = P\varphi_F(\bar{x}_{\gamma_j}(t))\, y_i$ and $\varphi'_G(x_{-\gamma_j}(t))\, Py_i = P\varphi_F(\bar{x}_{-\gamma_j}(t))\, y_i$. Thus, conjugation by $P$ is an isomorphism between $\varphi'_G(G_2(\mathscr{K}))$ and $\varphi_F(G_{\mathscr{K}}(b))$. So $G_{\mathscr{K}}(b) \cong G_2(\mathscr{K})$. The statement of (b) follows from the remarks of Appendixes G and F. For (c), we note that if $h = h_{\eta_1}(d^3)\, h_{\eta_2}(d^6) \cdot h_{\eta_3}(d^4)\, h_{\eta_4}(d^2)$, for $d = a/b$, then $hG_{\mathscr{K}}(b)\, h^{-1} = G_{\mathscr{K}}(a)$. Finally, (d) follows from (b) and the Main Theorems of [8, 12]. ∎

We now begin our consideration of irreducible $A_2'$s in $E_6$.

PROPOSITION (A.1). *Suppose A is isomorphic to a closed subgroup of E and, identifying A with the subgroup, suppose $V \mid A$ is irreducible for some nontrivial kE module V. Then $p \neq 2, 5$ and up to conjugacy in E*

$$
\begin{aligned}
x_{\alpha_1}(t) &= x_{\beta_1}(t)\, x_{\beta_3}(t)\, x_{\beta_1+\beta_3}(-\tfrac{1}{2}t^2)\, x_{\beta_2}(t)\, x_{\beta_5}(t)\, x_{\beta_6}(t)\, x_{\beta_5+\beta_6}(-\tfrac{1}{2}t^2),\\
x_{-\alpha_1}(t) &= x_{-\beta_1}(2t)\, x_{-\beta_3}(2t)\, x_{-\beta_1-\beta_3}(2t^2)\, x_{-\beta_2}(t)\, x_{-\beta_5}(2t)\, x_{-\beta_6}(2t)\\
&\quad \cdot x_{-\beta_5-\beta_6}(2t^2),\\
x_{\alpha_2}(t) &= x_{\beta_1+\beta_3+\beta_4}(a_1 t)\, x_{\beta_2+\beta_4+\beta_5}(a_2 t)\\
&\quad \cdot x_{\beta_1+\beta_2+\beta_3+2\beta_4+\beta_5}(\tfrac{1}{2}a_1 a_2 t^2)\, x_{\beta_3+\beta_4+\beta_5}((a_1+\tfrac{1}{2}a_2)\, t)\\
&\quad \cdot x_{\beta_4+\beta_5+\beta_6}((a_1+a_2)\, t)\, x_{\beta_2+\beta_3+\beta_4}(a_2 t)\\
&\quad \cdot x_{\beta_2+\beta_3+2\beta_4+\beta_5+\beta_6}((\tfrac{1}{2}a_1 a_2+\tfrac{1}{2}a_2^2)\, t^2),\\
x_{-\alpha_2}(t) &= x_{-\beta_1-\beta_3-\beta_4}(2t/a_1)\, x_{-\beta_2-\beta_4-\beta_5}(2t/a_2)\\
&\quad \cdot x_{-\beta_1-\beta_2-\beta_3-2\beta_4-\beta_5}(-2t^2/(a_1 a_2))\\
&\quad \cdot x_{-\beta_3-\beta_4-\beta_5}((a_1+\tfrac{1}{2}a_2)^{-1}\, t)\\
&\quad \cdot x_{-\beta_4-\beta_5-\beta_6}(2t(a_1+a_2)^{-1}) \cdot x_{-\beta_2-\beta_3-\beta_4}(2t/a_2)\\
&\quad \cdot x_{-\beta_2-\beta_3-2\beta_4-\beta_5-\beta_6}(-2t^2(a_1 a_2+a_2^2)^{-1})
\end{aligned}
$$

*for some* $a_i \in k^*$ *such that* $2a_1^2+2a_1 a_2+a_2^2=0$.

*Proof.* By the Main Theorem of [12], $p \neq 2, 5$ and $A$ acts irreducibly on $V(\lambda_1)$, the irreducible $kE$ module with high weight $\lambda_1$. Also, $V(\lambda_1)|A$ is the irreducible $kA$ module with high weight $2(\alpha_1+\alpha_2)$. We argue as in Proposition (G.1) to see that up to conjugacy in $E$ the following hold:

$$
\begin{aligned}
x_{\alpha_1}(t) &= x_{\beta_1}(t)\, x_{\beta_3}(t)\, x_{\beta_1+\beta_3}(-\tfrac{1}{2}t^2)\, x_{\beta_2}(t)\, x_{\beta_5}(t)\, x_{\beta_6}(t)\, x_{\beta_5+\beta_6}(-\tfrac{1}{2}t^2),\\
x_{-\alpha_1}(t) &= x_{-\beta_1}(2t)\, x_{-\beta_3}(2t)\, x_{-\beta_1-\beta_3}(2t^2)\, x_{-\beta_2}(t)\, x_{-\beta_5}(2t)\, x_{-\beta_6}(2t)\\
&\quad \cdot x_{-\beta_5-\beta_6}(2t^2),\\
h_{\alpha_1}(c) &= h_{\beta_1}(c^2)\, h_{\beta_3}(c^2)\, h_{\beta_2}(c)\, h_{\beta_5}(c^2)\, h_{\beta_6}(c^2),
\end{aligned}
$$

and

$$U_{\alpha_2} \leqslant \langle U_\beta \mid \beta = \Sigma c_\gamma \gamma,\ c_\gamma \in \mathbb{Z}^+,\ c_{\beta_4} > 0 \rangle.$$

As well, if $L = \langle U_{\pm\alpha_1} \rangle T_A$, $Z(L)^\circ = \{h_{\alpha_1}(c)\, h_{\alpha_2}(c^2) = z(c) \mid c \in k^*\} = \{h_{\beta_1}(d^2) \cdot h_{\beta_2}(d^3)\, h_{\beta_3}(d^4)\, h_{\beta_4}(d^6)\, h_{\beta_5}(d^4)\, h_{\beta_6}(d^2) = z_E(d) \mid d \in k^*\}$. Now $z(c) = z_E(c^l)$ for some $l \in \mathbb{Z}$ and the action of $z(c)$ on $V_{T_E}(\lambda_1)$ implies that $l=3$. So $h_{\alpha_2}(c) = h_{\beta_1}(c^2)\, h_{\beta_2}(c^4)\, h_{\beta_3}(c^5)\, h_{\beta_4}(c^9)\, h_{\beta_5}(c^5)\, h_{\beta_6}(c^2)$. Let $T_0 = \{h_{\alpha_1}(e^2)\, h_{\alpha_2}(e) \mid e \in k^*\}$. Then $\langle U_{\pm\alpha_2} \rangle \leqslant C_E(T_0)$, a connected reductive group containing $T_E$. One checks that $C_E(T_0)$ has root system $\{\pm(\beta_1+\beta_3+\beta_4),\ \pm(\beta_2+\beta_4+\beta_5),\ \pm(\beta_1+\beta_2+\beta_3+2\beta_4+\beta_5),\ \pm(\beta_3+\beta_4+\beta_5),\ \pm(\beta_4+\beta_5+\beta_6),\ \pm(\beta_2+\beta_3+\beta_4),\ \pm(\beta_2+\beta_3+2\beta_4+\beta_5+\beta_6)\}$. So $\langle U_{\pm\alpha_2} \rangle \leqslant \langle U_{\pm(\beta_1+\beta_3+\beta_4)}, U_{\pm(\beta_2+\beta_4+\beta_5)} \rangle \times \langle U_{\pm(\beta_3+\beta_4+\beta_5)} \rangle \times \langle U_{\pm(\beta_4+\beta_5+\beta_6)}, U_{\pm(\beta_2+\beta_3+\beta_4)} \rangle$.

Now, $0 \neq w \in V_{T_E}(\lambda_1)$ affords a 3-dimensional (restricted) composition factor for $\langle U_{\pm\alpha_2}\rangle$. Thus, the projection of $\langle U_{\pm\alpha_2}\rangle$ in $\langle U_{\pm(\beta_1+\beta_3+\beta_4)}, U_{\pm(\beta_2+\beta_4+\beta_5)}\rangle$ satisfies the hypotheses of the lemma. Arguing similarly with $V^*$, we see that the same is true of the projection of $\langle U_{\pm\alpha_2}\rangle$ in $\langle U_{\pm(\beta_4+\beta_5+\beta_6)}, U_{\pm(\beta_2+\beta_3+\beta_4)}\rangle$. Finally, note that $\langle U_{\pm\alpha_2}\rangle$ projects non-trivially into $\langle U_{\pm(\beta_3+\beta_4+\beta_5)}\rangle$ else $h_{\alpha_2}(-1)=1$. In fact, one checks that $h_{\alpha_2}(c)=h_{\beta_1+\beta_3+\beta_4}(c^2)\, h_{\beta_2+\beta_4+\beta_5}(c^2)\, h_{\beta_3+\beta_4+\beta_5}(c)\; h_{\beta_4+\beta_5+\beta_6}(c^2).\; h_{\beta_2+\beta_3+\beta_4}(c^2)$, so the projection of $\langle U_{\pm\alpha_2}\rangle$ in $\langle U_{\pm(\beta_3+\beta_4+\beta_5)}\rangle$ also satisfies the hypotheses of the lemma. Thus,

$$\begin{aligned} x_{\alpha_2}(t) = {} & x_{\beta_1+\beta_3+\beta_4}(a_1 t)\, x_{\beta_2+\beta_4+\beta_5}(a_2 t) \\ & \cdot x_{\beta_1+\beta_2+\beta_3+2\beta_4+\beta_5}(\tfrac{1}{2}a_1 a_2 t^2) \\ & \cdot x_{\beta_3+\beta_4+\beta_5}(bt)\, x_{\beta_4+\beta_5+\beta_6}(c_1 t)\, x_{\beta_2+\beta_3+\beta_4}(c_2 t) \\ & \cdot x_{\beta_2+\beta_3+2\beta_4+\beta_5+\beta_6}(\tfrac{1}{2}c_1 c_2 t^2), \end{aligned}$$

and

$$\begin{aligned} x_{-\alpha_2}(t) = {} & x_{-\beta_1-\beta_3-\beta_4}(2t/a_1)\, x_{-\beta_2-\beta_4-\beta_5}(2t/a_2) \\ & \cdot x_{-\beta_1-\beta_2-\beta_3-2\beta_4-\beta_5}(-2t^2/(a_1 a_2))\, x_{-\beta_3-\beta_4-\beta_5}(t/b)) \\ & \cdot x_{-\beta_4-\beta_5-\beta_6}(2t/c_1)\, x_{-\beta_2-\beta_3-\beta_4}(2t/c_2) \\ & \cdot x_{-\beta_2-\beta_3-2\beta_4-\beta_5-\beta_6}(-2t^2/(c_1 c_2)), \end{aligned}$$

for some $a_i, b, c_i \in k^*$.

The relations $[x_{\alpha_2}(t), x_{-\alpha_1}(u)]=1$ and $[x_{-\alpha_2}(t), x_{\alpha_1}(u)]=1$ force the following relations among the constants: $b=a_1+\frac{1}{2}a_2$, $c_1=a_1+a_2$, $c_2=a_2$, and $2a_1^2+2a_1a_2+a_2^2=0$. Thus the result holds. ∎

*Notation.* Assume $\operatorname{char}(\mathscr{K})\neq 2$, $(-1)^{1/2}\in\mathscr{K}$, and let $\mathfrak{K}$ be an algebraic closure of $\mathscr{K}$. Let $a_1, a_2 \in \mathscr{K}^*$ such that $2a_1^2+2a_1a_2+a_2^2=0$ and fix $\delta\in\mathscr{K}$ such that $\delta^2+1=0$; so $a_2=(-1\pm\delta)\,a_1$. Let $A_{\mathscr{K}}(a_1,a_2)\leqslant E_6(\mathscr{K})$ be defined as follows: $A_{\mathscr{K}}(a_1,a_2)=\langle \bar{x}_{\alpha_1}(t), \bar{x}_{-\alpha_1}(t), \bar{x}_{\alpha_2}(t), \bar{x}_{-\alpha_2}(t) \mid t\in\mathscr{K}\rangle$, where

$$\bar{x}_{\alpha_1}(t)=x_{\beta_1}(t)\,x_{\beta_3}(t)\,x_{\beta_1+\beta_3}(-\tfrac{1}{2}t^2)\,x_{\beta_2}(t)\,x_{\beta_5}(t)\,x_{\beta_6}(t)\,x_{\beta_5+\beta_6}(-\tfrac{1}{2}t^2),$$

$$\begin{aligned} \bar{x}_{-\alpha_1}(t) = {} & x_{-\beta_1}(2t)\,x_{-\beta_3}(2t)\,x_{-\beta_1-\beta_3}(2t^2)\,x_{-\beta_2}(t)\,x_{-\beta_5}(2t)\,x_{-\beta_6}(2t) \\ & \cdot x_{-\beta_5-\beta_6}(2t^2). \end{aligned}$$

$$\begin{aligned} \bar{x}_{\alpha_2}(t) = {} & x_{\beta_1+\beta_3+\beta_4}(a_1 t)\, x_{\beta_2+\beta_4+\beta_5}(a_2 t) \\ & \cdot x_{\beta_1+\beta_2+\beta_3+2\beta_4+\beta_5}(\tfrac{1}{2}a_1 a_2 t^2)\, x_{\beta_3+\beta_4+\beta_5}((a_1+\tfrac{1}{2}a_2)\,t) \\ & \cdot x_{\beta_4+\beta_5+\beta_6}((a_1+a_2)\,t)\, x_{\beta_2+\beta_3+\beta_4}(a_2 t) \\ & \cdot x_{\beta_2+\beta_3+2\beta_4+\beta_5+\beta_6}((\tfrac{1}{2}a_1 a_2+\tfrac{1}{2}a_2^2)\,t^2), \end{aligned}$$

$$\bar{x}_{-\alpha_2}(t) = x_{-\beta_1-\beta_3-\beta_4}(2t/a_1)\, x_{-\beta_2-\beta_4-\beta_5}(2t/a_2)$$
$$\cdot x_{-\beta_1-\beta_2-\beta_3-2\beta_4-\beta_5}(-2t^2/(a_1a_2))\, x_{-\beta_3-\beta_4-\beta_5}((a_1+\tfrac{1}{2}a_2)^{-1}t)$$
$$\cdot x_{-\beta_4-\beta_5-\beta_6}(2t(a_1+a_2)^{-1})\cdot x_{-\beta_2-\beta_3-\beta_4}(2t/a_2)$$
$$\cdot x_{-\beta_2-\beta_3-2\beta_4-\beta_5-\beta_6}(-2t^2(a_1a_2+a_2^2)^{-1}).$$

The statements of Theorems 1b, 1d, and 2b follow from Proposition (A.1) and the following:

THEOREM (A.2). *Let notation be as above. Then*

(a) $A_{\mathscr{K}}(a_1, a_2) \cong PSL_3(\mathscr{K})$.

(b) *If* $p \neq 5$, $A_{\mathscr{K}}(a_1, a_2)$ *acts irreducibly on the restricted, 27-dimensional rational modules for the group* $E_6(\mathfrak{K})$.

(c) $A_{\mathscr{K}}(a, (-1+\delta)\,a)$ is conjugate in $E_6(\mathscr{K})$ to $A_{\mathscr{K}}(b, (-1+\delta)\,b)$, *for any* $a, b \in \mathscr{K}^*$.

(d) *Let* $\tau$ *be the graph automorphism of* $E_6(\mathscr{K})$. *Then* $\tau(A_{\mathscr{K}}(a, (-1+\delta)\,a)) = A_{\mathscr{K}}(b, (-1-\delta)\,b)$, *where* $b = a\delta$.

(e) *Over the algebraically closed field* $k$ (char$(k) \neq 2$), $A_k(1, (-1+\delta))$ *is not conjugate in* $E$ *to* $A_k(1, (-1-\delta))$.

(f) *Over the algebraically closed field* $k$ (char$(k) \neq 2$), $A_k(a_1, a_2)$ *is maximal among closed connected subgroups of* $E$ *if and only if* char$(k) \neq 3$. *If* char$(k) = 5$, $A_k(a_1, a_2)$ *acts reducibly on every nontrivial rational* $kE$ *module.*

*Proof.* In the Appendix, we explicitly describe a faithful matrix representation of $PSL_3(\mathscr{K})$ in $SL_{27}(\mathscr{K})$. More precisely, $\varphi_A\colon PSL_3(\mathscr{K}) \to SL_{27}(\mathscr{K})$ corresponds to a representation on a 27-dimensional vector space $W$ with a fixed ordered basis $\mathscr{B}_A = \{w_i \mid 1 \leqslant i \leqslant 27\}$. As well, let $\varphi_E\colon E_6(\mathscr{K}) \to SL_{27}(\mathscr{K})$, $V$ and $\mathscr{B}_E$ be as in the proof of (G.2).

If char$(\mathscr{K}) \neq 5$, let $P\colon V \to W$ be given by

$$Pv_1 = 2w_1,\ Pv_2 = w_2,\qquad Pv_3 = w_3,\qquad Pv_4 = a_1 w_4,$$
$$Pv_5 = \tfrac{1}{5}(2a_1+3a_2)\, w_5 + \tfrac{1}{5}(a_2 - a_1)\, w_6,$$
$$Pv_6 = \tfrac{1}{10}(2a_1+3a_2)\, w_5 + \tfrac{1}{5}(2a_1+\tfrac{1}{2}a_2)\, w_6,$$
$$Pv_7 = \tfrac{1}{10}(2a_1+3a_2)\, w_7 + (\tfrac{3}{10}a_1+\tfrac{1}{5}a_2)\, w_8,$$
$$Pv_8 = \tfrac{1}{10}(2a_1+3a_2)\, w_7 + \tfrac{1}{5}(a_2 - a_1)\, w_8,$$
$$Pv_9 = \tfrac{1}{2}a_1 w_9,\qquad Pv_{10} = -a_2a_1 w_{10},$$

$$Pv_{11}=\tfrac{1}{10}(3a_1a_2+2a_2^2)\,w_{11}+\tfrac{1}{10}a_2(-a_1+a_2)\,w_{12},$$

$$Pv_{12}=\tfrac{1}{10}(3a_1a_2+2a_2^2)\,w_{11}+\tfrac{1}{5}a_2(2a_1+\tfrac{1}{2}a_2)\,w_{12},$$

$$Pv_{13}=[\tfrac{1}{5}a_1a_2+\tfrac{1}{20}a_2^2]\,w_{13}+\tfrac{1}{20}a_2(a_1+\tfrac{3}{2}a_2)\,w_{14}$$
$$+\tfrac{1}{10}a_2(a_1+\tfrac{3}{2}a_2)\,w_{15},$$

$$Pv_{14}=-[\tfrac{3}{20}a_2^2+\tfrac{1}{10}a_1a_2]\,w_{13}-[\tfrac{1}{5}a_2a_1+\tfrac{7}{40}a_2^2]\,w_{14}$$
$$-[\tfrac{1}{10}a_2a_1+\tfrac{3}{20}a_2^2]\,w_{15},$$

$$Pv_{15}=[\tfrac{1}{20}a_2^2+\tfrac{1}{5}a_1a_2]\,w_{13}+[\tfrac{1}{40}a_2^2+\tfrac{1}{10}a_1a_2]\,w_{14}$$
$$-[\tfrac{3}{20}a_2^2+\tfrac{1}{10}a_1a_2]\,w_{15},$$

$$Pv_{16}=-[\tfrac{7}{20}a_1a_2+\tfrac{3}{20}a_2^2]\,w_{16}-\tfrac{1}{5}a_2(a_1+\tfrac{1}{4}a_2)\,w_{17},$$

$$Pv_{17}=-[\tfrac{1}{10}a_2a_1+\tfrac{3}{20}a_2^2]\,w_{16}+\tfrac{1}{20}a_2(a_1-a_2)\,w_{17},$$

$$Pv_{18}=-\tfrac{1}{4}a_1a_2w_{18},\ Pv_{19}=\tfrac{1}{4}a_1a_2^2w_{19},$$

$$Pv_{20}=-\tfrac{1}{20}a_2^2(a_1+\tfrac{3}{2}a_2)\,w_{20}+\tfrac{1}{20}a_2^2(a_1-a_2)\,w_{21},$$

$$Pv_{21}=-\tfrac{1}{10}a_2^2(a_1+\tfrac{3}{2}a_2)\,w_{20}-\tfrac{1}{10}a_2^2(a_2+\tfrac{3}{2}a_1)\,w_{21},$$

$$Pv_{22}=a_2^2(\tfrac{3}{20}a_1+\tfrac{1}{10}a_2)\,w_{22}+\tfrac{1}{10}a_2^2(a_1+\tfrac{1}{4}a_2)\,w_{23},$$

$$Pv_{23}=\tfrac{1}{10}a_2^2(a_2+\tfrac{1}{4}a_1)\,w_{22}-\tfrac{1}{40}a_2^2(a_1-a_2)\,w_{23},$$

$$Pv_{24}=-\tfrac{1}{8}a_1a_2^2w_{24},\ Pv_{25}=-\tfrac{1}{4}a_1a_2^2(a_1+a_2)\,w_{25},$$

$$Pv_{26}=\tfrac{1}{8}a_1a_2^2(a_1+a_2)\,w_{26},\ Pv_{27}=-\tfrac{1}{8}a_1a_2^2(a_1+a_2)\,w_{27}.$$

If $\mathrm{char}(\mathscr{K})=5$, let $Q\colon V\to W$ be given by

$$Qv_1=2w_1,\ Qv_2=w_2,\ Qv_3=w_3,\ Qv_4=a_1w_4,\ Qv_5=4a_1w_5+a_1w_6,$$

$$Qv_6=2a_1w_5+a_1w_6,\ Qv_7=2a_1w_7+4a_1w_8,\ Qv_8=2a_1w_7+a_1w_8,$$

$$Qv_9=3a_1w_9,\ Qv_{10}=4a_1^2w_{10},\ Qv_{11}=4a_1^2w_{11}+3a_1^2w_{12},$$

$$Qv_{12}=4a_1^2w_{11}+a_1^2w_{12},\ Qv_{13}=3a_1^2w_{13}+3a_1^2w_{14}+a_1^2w_{15},$$

$$Qv_{14}=4a_1^2w_{13}+4a_1^2w_{15},\ Qv_{15}=3a_1^2w_{13}+4a_1^2w_{14}+4a_1^2w_{15},$$

$$Qv_{16}=2a_1^2w_{17},\ Qv_{17}=4a_1^2w_{16}+a_1^2w_{17},\ Qv_{18}=a_1^2w_{18},$$

$$Qv_{19}=-a_1^3w_{19},\ Qv_{20}=2a_1^3w_{20}+a_1^3w_{21},\ Qv_{21}=4a_1^3w_{20}+3a_1^3w_{21},$$

$$Qv_{22}=2a_1^3w_{22}+4a_1^3w_{23},\ Qv_{23}=2a_1^3w_{23},\ Qv_{24}=3a_1^3w_{24},$$

$$Qv_{25}=2a_1^4w_{25},\ Qv_{26}=4a_1^4w_{26},\ Qv_{27}=a_1^4w_{27}.$$

One checks that when $\mathrm{char}(\mathscr{K}) \neq 5$, for all $t \in \mathscr{K}$, for $1 \leqslant i \leqslant 27$, and for $j = 1, 2, \varphi_A(x_{\alpha_j}(t)) P v_i = P \varphi_E(\bar{x}_{\alpha_j}(t)) v_i$ and $\varphi_A(x_{-\alpha_j}(t)) P v_i = P \varphi_A(\bar{x}_{-\alpha_j}(t)) v_i$. Thus, conjugation by $P$ is an isomorphism between $\varphi_A(PSL_3(\mathscr{K}))$ and $\varphi_E(A_{\mathscr{K}}(a_1, a_2))$. So if $p \neq 5$, $A_{\mathscr{K}}(a_1, a_2) \cong PSL_3(\mathscr{K})$. If $\mathrm{char}(\mathscr{K}) = 5$, $a_2 = a_1$ or $a_2 = 2a_1$. If $a_2 = a_1$, then for all $t \in \mathscr{K}$, for $1 \leqslant i \leqslant 27$, and for $j = 1, 2, \varphi_A(x_{\alpha_j}(t)) Q v_i = Q \varphi_E(\bar{x}_{\alpha_j}(t)) v_i$ and $\varphi_A(x_{-\alpha_j}(t)) Q v_i = Q \varphi_A(\bar{x}_{-\alpha_j}(t)) v_i$. So conjugation by $Q$ is an isomorphism between $\varphi_A(PSL_3(\mathscr{K}))$ and $\varphi_E(A_{\mathscr{K}}(a_1, a_1))$ and when $p = 5$, $A_{\mathscr{K}}(a_1, a_1) \cong PSL_3(\mathscr{K})$.

The statement of (b) follows from the remarks of Appendixes E and A. Let $z$ be as in the proof of (G.2)(c). Then $z A_{\mathscr{K}}(a, (-1+\delta) a) z^{-1} = A_{\mathscr{K}}(b, (-1+\delta) b)$. So (c) holds. The statement of (d) is easily checked. In particular, since (d) holds for all $p \neq 2$, we now have $A_{\mathscr{K}}(a_1, a_2) \cong PSL_3(\mathscr{K})$ when $p = 5$, for any choice of $a_1$ and $a_2$.

Now suppose $p \neq 5$ and there exists $y \in E$ such that $y A_k(1, -1+\delta) y^{-1} = A_k(1, -1-\delta)$. Then by (c) and (d), there exists $h \in E$ such that $h\tau(A_k(1, -1+\delta)) h^{-1} = A_k(1, -1-\delta)$. Set $A_k = A_k(1, -1+\delta)$, a simple algebraic group of type $A_2$. Then $y^{-1} h\tau(A_k) h^{-1} y = A_k$. Now argue as in the proof of Theorem (G.2) to see that $A_k$ is not pointwise fixed by $y^{-1} h\tau$ (viewed as an element of $\mathrm{Aut}(Y)$). So $y^{-1} h\tau$ induces a nontrivial (algebraic group) automorphism of $A_k$. Let $\rho$ be the graph automorphism of $A_k$. Then by Steinberg (see [10]), there exists $a \in A_k$ such that $(y^{-1} h\tau)|_{A_k} = (\rho a)|_{A_k}$ (where we view $a$ as an element of $\mathrm{Aut}(E)$).

CLAIM. *There exists $w \in N_E(T_E)$ such that $w A_k w^{-1} = A_k$ and conjugation by $w$ induces $\rho$.*

*Proof of Claim.* For $r \in \Sigma^+(E)$, let $n_r \in N_E(T_E)$ be as defined in Section 7 of [5]. (For $r = \beta_i$, denote $n_r$ by $n_i$; for $r = \Sigma a_i \beta_i \notin \Pi(E)$, denote $n_r$ by $n_{a_1 a_2 a_3 a_4 a_5 a_6}$.) Set $w' = n_{122321} n_4 n_{001110} n_{101111} n_1 n_3 n_1 n_5 n_6 n_5 n_2 n_{101100} \cdot n_{010110} n_{101100} n_{000111} n_{011100} n_{000111} n_{001110} n_1 n_3 n_1 n_5 n_6 n_5 n_2$; $w'$ is the product of the long word of the Weyl group of $E$ and the long word of the Weyl group of $A_k$. Let $h = h_{\beta_1}(-1)\, h_{\beta_2}(-\delta-1)\, h_{\beta_4}(1-\delta)\, h_{\beta_5}(-\delta)\, h_{\beta_6}(-1)$. Then one checks that $w = hw'$ satisfies the claim.

Thus, $(y^{-1} h\tau)|_{A_k} = (wa)|_{A_k}$. But now we have $A_k$ pointwise fixed by an element of $\tau E$ in $(\mathrm{Aut}(E))$ so we may argue as in (G.2) to produce a contradiction. Thus, (e) holds when $\mathrm{char}(k) \neq 5$.

Now, let $k$ be an algebraically closed field of characteristic 5 and set $A_k = A_k(a_1, a_2)$. By Proposition (A.1) and (A.2)(a), $A_k$ acts reducibly on every nontrivial rational $kE$ module. Moreover, by the remarks of Appendix A and the previous work of this result, $V | A_k$ has composition factors of dimensions 8 and 19. (Recall $V$ is the rational $kE$ module with high weight $\lambda_1$.) It is a straightforward check to see that $A_k(a_1, a_1)$ leaves invariant an

8-space on $V$ while $A_k(a_1, 2a_1)$ does not. Thus, (e) holds when $\text{char}(k)=5$ also.

Now, with $A_k = A_k(a_1, a_2)$, $k$ an algebraically closed field, $\text{char}(k) \neq 2, 3$, suppose $A_k < X < E$, for $X$ a closed connected subgroup of $E$. Then $X$ is reductive, since [9] and the action of $A_k$ on $V$ imply that $A_k$ does not lie in a proper parabolic of $E$. So $A_k \leqslant X' = [X, X]$, a semisimple algebraic group. Now, suppose $X'$ acts irreducibly on $V$, which must be the case if $\text{char}(k) \neq 5$. Then by the Main Theorem of [12], $X' = G_2$ or $C_4$. If $X' = C_4$, $A_k$ must lie in a proper parabolic of $C_4$. Now, $V(\lambda_1)|C_4$ is the 27-dimensional irreducible occurring in the wedge product of the natural $C_4$ module with itself. But [9] implies that every proper parabolic of $C_4$ stabilizes a nontrivial subspace of this 27-dimensional irreducible of dimension less than 8. So $X' = G_2$ and since $\text{char}(k) \neq 3$, $A_k$ is generated by the long root subgroups of $G_2$. But then $A_k$ has a 6-dimensional composition factor on the 27-dimensional module for $G_2$, contradicting the above remarks. Hence, $V|X'$ is reducible, $\text{char}(k)=5$, and $V|X'$ has an 8- and a 19-dimensional composition factor. But now one checks, using Table 1 of [4] and (1.10) of [8], that there is no semisimple group, other than $A_2$, having both 8- and 19-dimensional irreducible representations when $p=5$. So $X' = A_k$ and $X = A_k T_0$, where $T_0$ is a torus lying in $C_E(A_k)$. But the above remarks about the action of $A_k(a_1, 2a_1)$ on $V$ imply that $V|A_k$ is indecomposable. Hence, any semisimple element in $SL_{27}(k)$ centralizing $\varphi_E(A_k)$ must be a scalar and therefore $T_0 \leqslant Z(E)^\circ = \{1\}$. Thus, if $\text{char}(k) \neq 3$, $A_k$ is maximal among closed connected subgroups of $E$.

Now suppose $\text{char}(k)=3$. Then Propositions (G.1), (G.2), and (A.1) and Theorem (4.1) of [8] imply that $A_k < B < E$, where $B$ is a closed connected simple subgroup of $E$ of type $G_2$. Hence, $A_k$ is not maximal and (f) holds. ∎

## APPENDIX.

In this section, we describe certain matrix representations of Chevalley groups associated with finite-dimensional Lie algebras of types $E_6$, $F_4$, $A_2$, and $G_2$, over arbitrary fields. We include enough information to check the statements made in the proofs of the previous results.

Let $\mathscr{L}(E)$ be a finite-dimensional complex simple Lie algebra of type $E_6$. Fix a Chevalley basis $\{e_\beta, f_\beta, h_{\beta_i} \mid \beta \in \Sigma^+(E)\}$ of $\mathscr{L}(E)$. If $\beta = \Sigma a_i \beta_i$, we will write $f_{a_1a_2a_3a_4a_5a_6}$ for $f_\beta$. Let $V$ be the irreducible $\mathscr{L}(E)$ module with high weight $\lambda_1$, where $\lambda_1$ is the fundamental dominant weight corresponding to $\beta_1$. Choose $0 \neq v^+ \in V$ such that $e_\alpha v^+ = 0$ for all $\alpha \in \Sigma^+(E)$. Fix the following (Kostant) basis of $V$.

| | | |
|---|---|---|
| $v_1 = v^+,$ | $v_{10} = f_{111210} v^+,$ | $v_{19} = f_{112321} v^+$ |
| $v_2 = f_{100000} v^+$ | $v_{11} = f_{111211} v^+$ | $v_{20} = f_{100000} f_{112321} v^+$ |
| $v_3 = f_{101000} v^+$ | $v_{12} = f_{112210} v^+$ | $v_{21} = f_{122321} v^+$ |
| $v_4 = f_{101100} v^+$ | $v_{13} = f_{112211} v^+$ | $v_{22} = f_{100000} f_{122321} v^+$ |
| $v_5 = f_{111100} v^+$ | $v_{14} = f_{111221} v^+$ | $v_{23} = f_{101000} f_{112321} v^+$ |
| $v_6 = f_{101110} v^+$ | $v_{15} = f_{100000} f_{112210} v^+$ | $v_{24} = f_{101000} f_{122321} v^+$ |
| $v_7 = f_{101111} v^+$ | $v_{16} = f_{100000} f_{112211} v^+$ | $v_{25} = f_{101100} f_{122321} v^+$ |
| $v_8 = f_{111110} v^+$ | $v_{17} = f_{112221} v^+$ | $v_{26} = f_{101110} f_{122321} v^+$ |
| $v_9 = f_{111111} v^+$ | $v_{18} = f_{100000} f_{112221} v^+$ | $v_{27} = f_{101111} f_{122321} v^+.$ |

*Appendix* E

Let $\mathscr{B}_E = \{v_i \mid 1 \leqslant i \leqslant 27\}$ (an ordered basis) and set $M = \Sigma \mathbb{Z} v_i$. It is well known that $M$ is invariant under $\{(e_\alpha^n)/n!, (f_\alpha^n)/n! \mid \alpha \in \Sigma^+(E), n \in \mathbb{Z}^+\}$ and that $e_\alpha^n$ and $f_\alpha^n$ act as zero on $V$ for sufficiently large values of $n$. Set $V(\mathscr{K}) = M \otimes_{\mathbb{Z}} \mathscr{K}$. Then, for $t \in \mathscr{K}$, we have an action of $\exp(te_\alpha) = 1 + \Sigma_1^\infty (te_\alpha)^n/n!$ and $\exp(tf_\alpha)$ on $V(\mathscr{K})$. We may then define a faithful rational representation $\varphi_E\colon E_6(\mathscr{K}) \to SL_{27}(\mathscr{K})$ on the generators of $E_6(\mathscr{K})$ by $\varphi_E(x_\beta(t)) = \exp(te_\beta)$ and $\varphi_E(x_{-\beta}(t)) = \exp(tf_\beta)$. Note that if $\mathfrak{K}$ is an algebraic closure of $\mathscr{K}$, then $\varphi_E(E_6(\mathscr{K}))$ acts irreducibly on $V(\mathfrak{K})$, the irreducible rational $E_6(\mathfrak{K})$ module with high weight $\lambda_1$. (See [11] and Section 12 of [10].) We identify $SL(V(\mathscr{K}))$ with $SL_{27}(\mathscr{K})$ via the ordered basis $\mathscr{B}_E$. A description of $\varphi_E$ is given below, with $E_{i,j}$ denoting the $27 \times 27$ matrix whose $(k, l)$ entry is $\delta_{ik}\delta_{jl}$ and $I$ denoting the $27 \times 27$ identity matrix.

$$\varphi_E(x_{\beta_2}(t)) = I + t(-E_{4,5} - E_{6,8} - E_{7,9} - E_{19,21} - E_{20,22} - E_{23,24}),$$
$$\varphi_E(x_{\beta_1}(t)) = I + t(E_{1,2} + E_{12,15} + E_{13,16} + E_{17,18} + E_{19,20} + E_{21,22}),$$
$$\varphi_E(x_{\beta_3}(t)) = I + t(E_{2,3} - E_{10,12} - E_{11,13} - E_{14,17} + E_{20,23} + E_{22,24}),$$
$$\varphi_E(x_{\beta_1+\beta_3}(t)) = I + t(E_{1,3} + E_{10,15} + E_{11,16} + E_{14,18} + E_{19,23} + E_{21,24}),$$
$$\varphi_E(x_{\beta_5}(t)) = I + t(E_{4,6} + E_{5,8} - E_{11,14} - E_{13,17} - E_{16,18} + E_{25,26}),$$
$$\varphi_E(x_{\beta_6}(t)) = I + t(E_{6,7} + E_{8,9} + E_{10,11} + E_{12,13} + E_{15,16} + E_{26,27}),$$
$$\varphi_E(x_{\beta_5+\beta_6}(t)) = I + t(E_{4,7} + E_{5,9} + E_{10,14} + E_{12,17} + E_{15,18} + E_{25,27}),$$
$$\varphi_E(x_{\beta_2+\beta_4}(t)) = I + t(E_{3,5} + E_{6,10} + E_{7,11} - E_{17,21} - E_{18,22} - E_{23,25}),$$
$$\varphi_E(x_{\beta_3+\beta_4}(t)) = I + t(E_{2,4} - E_{8,12} - E_{9,13} + E_{14,19} + E_{18,23} + E_{22,25}),$$
$$\varphi_E(x_{\beta_4+\beta_5}(t)) = I + t(E_{3,6} + E_{5,10} + E_{9,14} - E_{13,19} - E_{16,20} + E_{24,26}),$$
$$\varphi_E(x_{\beta_1+\beta_3+\beta_4}(t)) = I + t(E_{1,4} + E_{8,15} + E_{9,16} - E_{14,20} + E_{17,23} + E_{21,25}),$$

$$\varphi_E(x_{\beta_2+\beta_4+\beta_5}(t)) = I + t(E_{3,8} - E_{4,10} - E_{7,14} - E_{13,21} - E_{16,22} - E_{23,26}),$$
$$\varphi_E(x_{\beta_3+\beta_4+\beta_5}(t)) = I + t(E_{2,6} + E_{5,12} + E_{9,17} + E_{11,19} + E_{16,23} + E_{22,26}),$$
$$\varphi_E(x_{\beta_4+\beta_5+\beta_6}(t)) = I + t(E_{3,7} - E_{8,14} + E_{5,11} + E_{12,19} + E_{15,20} + E_{24,27}),$$
$$\varphi_E(x_{\beta_2+\beta_3+\beta_4}(t)) = I + t(E_{2,5} + E_{6,12} + E_{7,13} + E_{14,21} - E_{20,25} + E_{18,24}),$$
$$\varphi_E(x_{\beta_1+\beta_2+\beta_3+2\beta_4+\beta_5}(t)) = I + t(E_{3,15} + E_{1,10} + E_{7,20} - E_{13,25} + E_{9,22} + E_{17,26}),$$
$$\varphi_E(x_{\beta_2+\beta_3+2\beta_4+\beta_5+\beta_6}(t)) = I + t(E_{2,11} + E_{6,19} - E_{3,13} + E_{8,21} + E_{15,25} + E_{18,27}),$$
$$\varphi_E(x_{-\beta_1}(t)) = I + t(E_{2,1} + E_{15,12} + E_{16,13} + E_{18,17} + E_{20,19} + E_{22,21}),$$
$$\varphi_E(x_{-\beta_3}(t)) = I + t(E_{3,2} - E_{12,10} - E_{13,11} - E_{17,14} + E_{23,20} + E_{24,22}),$$
$$\varphi_E(x_{-\beta_1-\beta_3}(t)) = I + t(E_{3,1} + E_{15,10} + E_{16,11} + E_{18,14} + E_{23,19} + E_{24,21}),$$
$$\varphi_E(x_{-\beta_2}(t)) = I + t(-E_{5,4} - E_{8,6} - E_{9,7} - E_{21,19} - E_{22,20} - E_{24,23}),$$
$$\varphi_E(x_{-\beta_5}(t)) = I + t(E_{6,4} + E_{8,5} - E_{14,11} - E_{17,13} - E_{18,16} + E_{26,25}),$$
$$\varphi_E(x_{-\beta_6}(t)) = I + t(E_{7,6} + E_{9,8} + E_{11,10} + E_{13,12} + E_{16,15} + E_{27,26}),$$
$$\varphi_E(x_{-\beta_5-\beta_6}(t)) = I + t(E_{7,4} + E_{9,5} + E_{14,10} + E_{17,12} + E_{18,15} + E_{27,25}),$$
$$\varphi_E(x_{-\beta_2-\beta_4}(t)) = I + t(E_{5,3} + E_{10,6} + E_{11,7} - E_{21,17} - E_{22,18} - E_{25,23}),$$
$$\varphi_E(x_{-\beta_3-\beta_4}(t)) = I + t(E_{4,2} - E_{12,8} - E_{13,9} + E_{19,14} + E_{23,18} + E_{25,22}),$$
$$\varphi_E(x_{-\beta_4-\beta_5}(t)) = I + t(E_{6,3} + E_{10,5} + E_{14,9} - E_{19,13} - E_{20,16} + E_{26,24}),$$
$$\varphi_E(x_{-\beta_1-\beta_3-\beta_4}(t)) = I + t(E_{4,1} + E_{15,8} + E_{16,9} + E_{23,17} - E_{20,14} + E_{25,21}),$$
$$\varphi_E(x_{-\beta_2-\beta_4-\beta_5}(t)) = I + t(E_{8,3} - E_{10,4} - E_{14,7} - E_{21,13} - E_{22,16} - E_{26,23}),$$
$$\varphi_E(x_{-\beta_1-\beta_2-\beta_3-2\beta_4-\beta_5}(t)) = I + t(E_{10,1} + E_{15,3} + E_{20,7} + E_{22,9} - E_{25,13} + E_{26,17}).$$
$$\varphi_E(x_{-\beta_3-\beta_4-\beta_5}(t)) = I + t(E_{6,2} + E_{12,5} + E_{19,11} + E_{17,9} + E_{23,16} + E_{26,22}),$$
$$\varphi_E(x_{-\beta_4-\beta_5-\beta_6}(t)) = I + t(E_{7,3} + E_{11,5} - E_{14,8} + E_{19,12} + E_{20,15} + E_{27,24}),$$
$$\varphi_E(x_{-\beta_2-\beta_3-\beta_4}(t)) = I + t(E_{5,2} + E_{12,6} + E_{13,7} + E_{21,14} + E_{24,18} - E_{25,20}),$$
$$\varphi_E(x_{-\beta_2-\beta_3-2\beta_4-\beta_5-\beta_6}(t)) = I + t(E_{11,2} - E_{13,3} + E_{19,6} + E_{21,8} + E_{25,15} + E_{27,18}).$$

Let $\mathscr{L}(A)$ be a simple, finite-dimensional complex Lie algebra of type $A_2$. Let $\{e_{\alpha_1}, e_{\alpha_2}, f_{\alpha_1}, f_{\alpha_2}, e_{\alpha_1+\alpha_2}, f_{\alpha_1+\alpha_2}, h_{\alpha_1}, h_{\alpha_2}\}$ be a Chevalley basis of $\mathscr{L}(A)$. We take the structure constants to be determined by $[e_{\alpha_1}, e_{\alpha_2}] = e_{\alpha_1+\alpha_2}$ and relations (4.1.2) of [5]. Let $W$ be the 27-dimensional irreducible $\mathscr{L}(A)$ module with high weight $2(\alpha_1+\alpha_2)$. Choose $0 \neq w^+ \in W$ such that $e_\alpha w^+ = 0$ for all $\alpha \in \Sigma^+(A)$. Fix the following (Kostant) basis of $W$.

$w_1 = w^+,$ $w_2 = f_{\alpha_1} w^+,$ $w_3 = \frac{1}{2}(f_{\alpha_1})^2 w^+,$ $w_4 = f_{\alpha_2} w^+,$ $w_5 = f_{\alpha_1+\alpha_2} w^+,$ $w_6 = f_{\alpha_1} f_{\alpha_2} w^+,$ $w_7 = f_{\alpha_1+\alpha_2} f_{\alpha_1} w^+,$ $w_8 = \frac{1}{2}(f_{\alpha_1})^2 f_{\alpha_2} w^+,$ $w_9 = \frac{1}{2} f_{\alpha_1+\alpha_2}(f_{\alpha_1})^2 w^+,$

$w_{10} = \frac{1}{2}(f_{\alpha_2})^2 w^+,$ $w_{11} = f_{\alpha_1+\alpha_2} f_{\alpha_2} w^+,$ $w_{12} = \frac{1}{2} f_{\alpha_1}(f_{\alpha_2})^2 w^+,$ $w_{13} = \frac{1}{4}(f_{\alpha_1})^2 (f_{\alpha_2})^2 w^+,$ $w_{14} = f_{\alpha_1+\alpha_2} f_{\alpha_1} f_{\alpha_2} w^+,$ $w_{15} = \frac{1}{2}(f_{\alpha_1+\alpha_2})^2 w^+,$ $w_{16} = \frac{1}{2}(f_{\alpha_1+\alpha_2})^2 f_{\alpha_1} w^+,$ $w_{17} = \frac{1}{2} f_{\alpha_1+\alpha_2}(f_{\alpha_1})^2 f_{\alpha_2} w^+,$ $w_{18} = \frac{1}{4}(f_{\alpha_1+\alpha_2})^2 (f_{\alpha_1})^2 w^+,$

$w_{19} = \frac{1}{2} f_{\alpha_1+\alpha_2}(f_{\alpha_2})^2 w^+$ $w_{20} = \frac{1}{2}(f_{\alpha_1+\alpha_2})^2 f_{\alpha_2} w^+$ $w_{21} = \frac{1}{2} f_{\alpha_1+\alpha_2} f_{\alpha_1}(f_{\alpha_2})^2 w^+$ $w_{22} = \frac{1}{6}(f_{\alpha_1+\alpha_2})^3 w^+$ $w_{23} = \frac{1}{2}(f_{\alpha_1+\alpha_2})^2 f_{\alpha_1} f_{\alpha_2} w^+$ $w_{24} = \frac{1}{6}(f_{\alpha_1+\alpha_2})^3 f_{\alpha_1} w^+$ $w_{25} = \frac{1}{4}(f_{\alpha_1+\alpha_2})^2 (f_{\alpha_2})^2 w^+$ $w_{26} = \frac{1}{6}(f_{\alpha_1+\alpha_2})^3 f_{\alpha_2} w^+$ $w_{27} = \frac{1}{24}(f_{\alpha_1+\alpha_2})^4 w^+.$

*Appendix* A

As in Appendix E we obtain a rational representation $\varphi_A: A_2(\mathscr{K}) \to SL_{27}(\mathscr{K})$, where we have now identified $SL(W(\mathscr{K}))$ with $SL_{27}(\mathscr{K})$ via the ordered basis $\mathscr{B}_A = \{w_i \mid 1 \leqslant i \leqslant 27\}$. However, the center of $A_2(\mathscr{K})$ acts trivially, so we have $\varphi_A: PSL_3(\mathscr{K}) \to SL_{27}(\mathscr{K})$, a faithful representation. Let $\mathfrak{K}$ be an algebraic closure of $\mathscr{K}$. Then, [3] implies that if $\text{char}(\mathscr{K}) \neq 2, 5$, $\varphi_A: PSL_3(\mathfrak{K}) \to SL(W(\mathfrak{K}))$ is the irreducible representation with high weight $2(\alpha_1 + \alpha_2)$. Moreover, if $\text{char}(\mathscr{K}) = 5$, $\varphi_A(PSL_3(\mathfrak{K}))$ acts on $W(\mathfrak{K})$ with two composition factors, of dimensions 19 and 8. As in Appendix E, if $\text{char}(\mathscr{K}) \neq 5$, $\varphi_A(PSL_3(\mathscr{K}))$ acts irreducibly on $W(\mathfrak{K})$. A description of $\varphi_A$ is given below:

$$\begin{aligned}\varphi_A(x_{\alpha_1}(t)) = I &+ t(2E_{1,2} + E_{2,3} - E_{4,5} + 3E_{4,6} + E_{5,7} - E_{6,7} + 2E_{6,8} - E_{8,9} \\ &- 2E_{10,11} + 4E_{10,12} + 3E_{12,13} + 2E_{11,14} - 2E_{12,14} - E_{11,15} \\ &- E_{14,16} + E_{14,17} - 2E_{13,17} - E_{17,18} - E_{16,18} - 2E_{19,20} \\ &+ 3E_{19,21} - E_{20,22} + E_{20,23} - 2E_{21,23} - E_{22,24} - E_{23,24} \\ &- 2E_{25,26} - E_{26,27}) + t^2(E_{1,3} - 2E_{4,7} + 3E_{4,8} - E_{6,9} - 6E_{10,14} \\ &+ 6E_{10,13} + E_{10,15} - E_{11,16} + E_{11,17} + E_{12,16} - 4E_{12,17} \\ &+ E_{13,18} + E_{19,22} - 4E_{19,23} + E_{21,24} + E_{25,27}) \\ &+ t^3(-E_{4,9} + 2E_{10,16} - 6E_{10,17} + E_{12,18} + E_{19,24}) + t^4(E_{10,18}).\end{aligned}$$

$$\begin{aligned}\varphi_A(x_{-\alpha_1}(t)) = I &+ t(E_{2,1} + 2E_{3,2} + E_{6,4} + E_{7,5} + 2E_{8,6} + 2E_{9,7} - 3E_{9,8} \\ &+ E_{12,10} + E_{14,11} + 2E_{13,12} - 3E_{17,13} - 3E_{16,13} + 2E_{17,14} \\ &+ E_{16,15} + 2E_{18,16} - 6E_{18,17} + E_{21,19} + E_{23,20} - 2E_{22,21} \\ &- 2E_{23,21} + E_{24,22} - 4E_{24,23} - E_{26,25} - 2E_{27,26}) + t^2(E_{3,1} + E_{8,4} \\ &+ E_{9,5} - 3E_{9,6} + E_{13,10} - 3E_{17,12} - 3E_{16,12} + E_{17,11} \\ &+ 6E_{18,13} + E_{18,15} - 6E_{18,14} - E_{22,19} - E_{23,19} + 3E_{24,21} \\ &- 2E_{24,20} + E_{27,25}) + t^3(-E_{9,4} - E_{17,10} - E_{16,10} \\ &+ 4E_{18,12} - 2E_{18,11} + E_{24,19}) + t^4(E_{18,10}).\end{aligned}$$

$$\begin{aligned}\varphi_A(x_{\alpha_2}(t)) = I + t(&2E_{1,4} + E_{4,10} + E_{2,5} + 2E_{2,6} + 2E_{5,11} + E_{6,11} + E_{6,12}\\ &+ E_{11,19} + E_{12,19} + 2E_{3,7} + 2E_{3,8} + E_{8,13} + 2E_{7,14} + 2E_{8,14}\\ &+ E_{7,15} + 2E_{15,20} + E_{14,20} + E_{14,21} + 2E_{13,21} + E_{20,25} + E_{21,25}\\ &+ 2E_{9,16} - E_{9,17} + E_{16,22} + 2E_{16,23} + 2E_{17,23} + 2E_{22,26}\\ &+ E_{23,26} + 2E_{18,24} + E_{24,27}) + t^2(E_{1,10} + 2E_{2,11} + E_{2,12} + E_{5,19}\\ &+ E_{6,19} + 4E_{3,14} + E_{3,15} + E_{3,13} + 2E_{8,21} + 2E_{7,20} + E_{7,21}\\ &+ E_{8,20} + E_{15,25} + E_{14,25} + E_{13,25} + E_{9,22} + E_{9,23} + 2E_{16,26}\\ &+ E_{17,26} + E_{18,27}) + t^3(E_{2,19} + 2E_{3,20} + 2E_{3,21} + E_{8,25}\\ &+ E_{7,25} + E_{9,26}) + t^4(E_{3,25}).\end{aligned}$$

$$\begin{aligned}\varphi_A(x_{-\alpha_2}(t)) = I + t(&E_{4,1} + 2E_{10,4} + E_{5,2} + E_{6,2} + E_{11,5} + 2E_{12,6} + E_{11,6}\\ &+ 2E_{19,11} + E_{19,12} + E_{8,3} + E_{7,3} + E_{14,7} + 2E_{15,7} + 2E_{13,8}\\ &+ E_{14,8} + E_{21,13} + 2E_{21,14} + 2E_{20,14} + E_{20,15} + 2E_{25,20}\\ &+ 2E_{25,21} + E_{17,9} + 2E_{16,9} + 3E_{22,16} + E_{23,16} - 2E_{22,17}\\ &+ E_{26,22} + E_{26,23} + E_{24,18} + 2E_{27,24}) + t^2(E_{10,1} + E_{11,2} + E_{12,2}\\ &+ E_{19,5} + 2E_{19,6} + E_{14,3} + E_{13,3} + E_{15,3} + 2E_{20,7} + E_{20,8}\\ &+ E_{21,7} + 2E_{21,8} + E_{25,13} + E_{25,15} + 4E_{25,14} + 2E_{22,9} + E_{23,9}\\ &+ 2E_{26,16} - E_{26,17} + E_{27,18}) + t^3(E_{19,2} + E_{21,3} + E_{20,3}\\ &+ 2E_{25,7} + 2E_{25,8} + E_{26,9}) + t^4(E_{25,3}).\end{aligned}$$

Let $\mathscr{L}(G)$ be a simple, finite-dimensional Lie algebra of type $G_2$. Let $\{e_{\gamma_i}, f_{\gamma_i}, e_{k\gamma_1+\gamma_2}, e_{3\gamma_1+2\gamma_2}, f_{k\gamma_1+\gamma_2}, f_{3\gamma_1+2\gamma_2}, h_{\gamma_i} \mid i=1, 2, k=1, 2, 3\}$ be a Chevalley basis for $\mathscr{L}(G)$. Let $W$ be the irreducible $\mathscr{L}(G)$ module with high weight $2(2\gamma_1+\gamma_2)$. Choose $0 \neq u^+ \in W$ such that $e_\gamma u^+ = 0$ for all $\gamma \in \Sigma^+(G)$. Fix the following (Kostant) basis of $W$:

| | | |
|---|---|---|
| $u_1 = u^+$, | $u_{10} = \frac{1}{2}(f_{\gamma_1+\gamma_2})^2 u^+$ | $u_{19} = f_{3\gamma_1+2\gamma_2} f_{\gamma_1+\gamma_2} u^+$ |
| $u_2 = f_{\gamma_1} u^+$, | $u_{11} = f_{2\gamma_1+\gamma_2} f_{\gamma_1+\gamma_2} u^+$, | $u_{20} = f_{3\gamma_1+2\gamma_2} f_{2\gamma_1+\gamma_2} u^+$ |
| $u_3 = \frac{1}{2}(f_{\gamma_1})^2 u^+$, | $u_{12} = f_{3\gamma_1+2\gamma_2} u^+$, | $u_{21} = \frac{1}{2}(f_{2\gamma_1+\gamma_2})^2 f_{\gamma_1+\gamma_2} u^+$ |
| $u_4 = f_{\gamma_1+\gamma_2} u^+$, | $u_{13} = \frac{1}{2}(f_{2\gamma_1+\gamma_2})^2 u^+$, | $u_{22} = \frac{1}{6}(f_{2\gamma_1+\gamma_2})^3 u^+$ |
| $u_5 = f_{2\gamma_1+\gamma_2} u^+$, | $u_{14} = f_{3\gamma_1+2\gamma_2} f_{\gamma_1} u^+$, | $u_{23} = f_{3\gamma_1+2\gamma_2} f_{3\gamma_1+\gamma_2} u^+$ |
| $u_6 = f_{\gamma_1+\gamma_2} f_{\gamma_1} u^+$, | $u_{15} = f_{3\gamma_1+\gamma_2} f_{\gamma_1+\gamma_2} u^+$, | $u_{24} = \frac{1}{2}(f_{3\gamma_1+\gamma_2})^2 f_{\gamma_1+\gamma_2} u^+$ |
| $u_7 = f_{3\gamma_1+\gamma_2} u^+$, | $u_{16} = f_{3\gamma_1+\gamma_2} f_{2\gamma_1+\gamma_2} u^+$, | $u_{25} = \frac{1}{2}(f_{3\gamma_1+2\gamma_2})^2 u^+$ |
| $u_8 = f_{2\gamma_1+\gamma_2} f_{\gamma_1} u^+$, | $u_{17} = \frac{1}{2}(f_{2\gamma_1+\gamma_2})^2 f_{\gamma_1} u^+$, | $u_{26} = \frac{1}{6}(f_{2\gamma_1+\gamma_2})^3 f_{\gamma_1+\gamma_2} u^+$ |
| $u_9 = f_{3\gamma_1+\gamma_2} f_{\gamma_1} u^+$, | $u_{18} = \frac{1}{2}(f_{3\gamma_1+\gamma_2})^2 u^+$, | $u_{27} = \frac{1}{24}(f_{2\gamma_1+\gamma_2})^4 u^+$. |

*Appendix* G

As in Appendix E we obtain a faithful rational representation, $\varphi_G: G_2(\mathscr{K}) \to SL_{27}(\mathscr{K})$, where we have identified $SL(W(\mathscr{K}))$ with $SL_{27}(\mathscr{K})$ via the ordered basis $\mathscr{B}_G = \{u_i \mid 1 \leqslant i \leqslant 27\}$. Let $\mathfrak{K}$ be an algebraic closure of $\mathscr{K}$. Then Table 1 of [4] implies that if $\mathrm{char}(\mathscr{K}) \neq 2, 7$, $\varphi_G: G_2(\mathfrak{K}) \to SL(W(\mathfrak{K}))$ is the irreducible rational representation with high weight $2(2\gamma_1 + \gamma_2)$. If $\mathrm{char}(\mathscr{K}) = 7$, one checks that $\varphi_G(G_2(\mathscr{K}))$ fixes the 1-space $\langle u_{13} - 2u_{14} + 2u_{15} \rangle$. The quotient is again the irreducible $G_2(\mathfrak{K})$ module with high weight $2(2\gamma_1 + \gamma_2)$. Thus, as in Appendix E if $\mathrm{char}(\mathscr{K}) \neq 7$, $\varphi_G(G_2(\mathscr{K}))$ acts irreducibly on $W(\mathfrak{K})$ and if $\mathrm{char}(\mathscr{K}) = 7$, $\varphi_G(G_2(\mathscr{K}))$ acts irreducibly on the quotient of $W(\mathfrak{K})$ by $\langle u_{13} - 2u_{14} + 2u_{15} \rangle$. A description of $\varphi_G$ is given below:

$$\begin{aligned}
\varphi_G(x_{\gamma_1}(t)) = I &+ t(2E_{1,2} + E_{2,3} + 2E_{4,5} - E_{4,6} - E_{5,7} + 2E_{5,8} + 2E_{6,8} + 2E_{7,9} \\
&- E_{8,9} + 4E_{10,11} + 2E_{11,13} - 3E_{12,13} + 2E_{12,14} - E_{11,15} \\
&+ 2E_{15,16} - 2E_{13,16} - E_{14,17} - 2E_{13,17} + 2E_{15,17} - E_{16,18} \\
&+ 2E_{19,20} + E_{19,21} - 3E_{20,22} - E_{20,23} + 2E_{21,22} - E_{22,24} \\
&- E_{23,24} - 2E_{25,26} - E_{26,27}) + t^2(E_{1,3} - E_{4,7} + E_{4,8} - E_{6,9} \\
&- 2E_{5,9} - 2E_{10,15} + 4E_{10,13} - 3E_{11,16} - 3E_{11,17} + 3E_{12,16} \\
&+ 2E_{12,17} + E_{13,18} - E_{15,18} - 2E_{19,22} - E_{19,23} + 2E_{20,24} \\
&- E_{21,24} + E_{25,27}) + t^3(-E_{4,9} - 4E_{10,16} - 4E_{10,17} + E_{11,18} \\
&- E_{12,18} + E_{19,24}) + t^4(E_{10,18}).
\end{aligned}$$

$$\begin{aligned}
\varphi_G(x_{-\gamma_1}(t)) = I &+ t(E_{2,1} + 2E_{3,2} + 2E_{5,4} + E_{6,4} - 3E_{7,5} + 2E_{7,6} + E_{8,5} + E_{9,7} \\
&- E_{9,8} + E_{11,10} - E_{12,10} + E_{14,11} + 2E_{13,11} - 2E_{15,11} \\
&+ E_{14,12} + E_{17,13} - 3E_{16,13} + 2E_{16,14} - 2E_{17,14} + E_{17,15} \\
&- 4E_{18,16} - 4E_{18,17} + 2E_{20,19} - E_{21,19} - 2E_{23,20} - E_{22,20} \\
&- 2E_{23,21} - 2E_{24,22} - E_{24,23} - E_{26,25} - 2E_{27,26}) + t^2(E_{3,1} + E_{8,4} \\
&- 2E_{7,4} - 2E_{9,5} + E_{9,6} + E_{13,10} - E_{15,10} - 2E_{16,11} \\
&- E_{17,12} + E_{16,12} - E_{17,11} + 4E_{18,13} - 2E_{18,15} - E_{22,19} \\
&- E_{23,19} + E_{24,21} + 2E_{24,20} + E_{27,25}) + t^3(-E_{9,4} - E_{16,10} \\
&+ 4E_{18,11} + E_{24,19}) + t^4(E_{18,10}).
\end{aligned}$$

$$\begin{aligned}
\varphi_G(x_{\gamma_2}(t)) = I &+ t(-E_{2,4} - 2E_{3,6} - E_{6,10} - E_{5,10} - E_{8,11} - E_{7,12} - E_{9,14} \\
&- E_{9,15} - E_{14,19} - E_{15,19} - E_{16,20} - E_{17,21} - 2E_{18,23} - E_{23,25} \\
&- E_{24,26}) + t^2(E_{3,10} + E_{9,19} + E_{18,25}).
\end{aligned}$$

$$\begin{aligned}
\varphi_G(x_{-\gamma_2}(t)) = I &+ t(-E_{4,2} - E_{5,3} - E_{6,3} - 2E_{10,6} - E_{12,7} - E_{11,8} - E_{14,9} \\
&- E_{15,9} - E_{19,14} - E_{19,15} - E_{20,16} - E_{21,17} - E_{23,18} - 2E_{25,23} \\
&- E_{26,24}) + t^2(E_{10,3} + E_{19,9} + E_{25,18}).
\end{aligned}$$

Let $\mathscr{L}(F)$ be a finite-dimensional complex simple Lie algebra of type $F_4$. Fix a Chevalley basis $\{e_\beta, f_\beta, h_{\beta_i} \mid \beta \in \Sigma^+(F)\}$ of $\mathscr{L}(F)$ and write $f_{a_1a_2a_3a_4}$ for $f_\beta$, where $\beta = \Sigma a_i \beta_i$. Let $V$ be the irreducible $\mathscr{L}(F)$ module with high weight $\lambda = \eta_1 + 2\eta_2 + 3\eta_3 + 2\eta_4$, the fundamental dominant weight corresponding to $\eta_4$. Choose $0 \neq y^+ \in V$ such that $e_\alpha y^+ = 0$ for all $\alpha \in \Sigma^+(F)$. Fix the following (Kostant) basis of $V$.

| | | |
|---|---|---|
| $y_1 = y^+$, | $y_9 = f_{1122}\, y^+$, | $y_{18} = f_{1342}\, y^+$ |
| $y_2 = f_{0001}\, y^+$, | $y_{10} = f_{1221}\, y^+$, | $y_{19} = f_{2342}\, y^+$ |
| $y_3 = f_{0011}\, y^+$, | $y_{11} = f_{1222}\, y^+$, | $y_{20} = f_{1222} f_{0121}\, y^+$ |
| $y_4 = f_{0111}\, y^+$, | $y_{12} = f_{1231}\, y^+$, | $y_{21} = f_{1222} f_{1121}\, y^+$ |
| $y_5 = f_{0121}\, y^+$, | $y_{13} = f_{1232}\, y^+$, | $y_{22} = f_{1232} f_{0121}\, y^+$ |
| $y_6 = f_{1111}\, y^+$, | $y_{14} = f_{1111} f_{0121}\, y^+$, | $y_{23} = f_{1232} f_{1121}\, y^+$ |
| $y_7 = f_{1121}\, y^+$, | $y_{15} = f_{1242}\, y^+$, | $y_{24} = f_{2342} f_{0111}\, y^+$ |
| $y_8 = f_{0122}\, y^+$, | $y_{16} = f_{1122} f_{0111}\, y^+$, | $y_{25} = f_{2342} f_{0121}\, y^+$ |
| | $y_{17} = f_{1122} f_{0121}\, y^+$ | $y_{26} = f_{2342} f_{0122}\, y^+$. |

*Appendix* F

As in Appendix E, we obtain a faithful rational representation $\varphi_F$: $F_4(\mathscr{K}) \to SL_{26}(\mathscr{K})$, where we have now identified $SL_{26}(W(\mathscr{K}))$ with $SL_{26}(\mathscr{K})$ via the ordered basis $\mathscr{B}_F = \{y_i \mid 1 \leqslant i \leqslant 26\}$. Let $\mathfrak{K}$ be an algebraic closure of $\mathscr{K}$. Then Table 1 of [4] implies that if $\mathrm{char}(\mathscr{K}) \neq 3$, $\varphi_F$: $F_4(\mathfrak{K}) \to SL_{26}(\mathfrak{K})$ is the irreducible rational representation with high weight $\lambda$. So as in Appendix E, $\varphi_F(F_4(\mathscr{K}))$ acts irreducibly on $W(\mathfrak{K})$. A description of $\varphi_F$ is given below:

$$\begin{aligned}
\varphi_F(x_{\eta_1}(t)) &= I + t(-E_{4,6} - E_{5,7} - E_{8,9} + E_{18,19} - E_{20,21} - E_{22,23}),\\
\varphi_F(x_{\eta_3}(t)) &= I + t(-E_{2,3} + E_{4,5} + E_{6,7} + E_{10,12} + 2E_{11,13} + E_{11,14}\\
&\quad + E_{13,15} + E_{16,17} + E_{20,22} + E_{21,23} + E_{24,25}) + t^2(E_{11,15}),\\
\varphi_F(x_{\eta_4}(t)) &= I + t(E_{5,8} + E_{7,9} + E_{10,11} + E_{12,13} - E_{12,14} + E_{13,16}\\
&\quad - E_{14,16} + E_{15,17} + E_{18,10} - E_{19,21} + E_{25,26}) + t^2(E_{12,16}),\\
\varphi_F(x_{\eta_3+\eta_4}(t)) &= I + t(E_{1,3} + E_{4,8} + E_{6,9} - E_{10,13} - 2E_{10,14} + E_{11,16}\\
&\quad - E_{12,15} + E_{14,17} - E_{18,22} + E_{19,23} + E_{24,26}) + t^2(-E_{10,17}),\\
\varphi_F(x_{\eta_1+\eta_2}(t)) &= I + t(-E_{3,6} - E_{5,10} - E_{8,11} - E_{15,19} + E_{17,21} + E_{22,24}),\\
\varphi_F(x_{\eta_2+\eta_3}(t)) &= I + t(-E_{2,4} - E_{3,5} - E_{6,10} + E_{7,12} + 2E_{9,13} + E_{9,14}\\
&\quad - E_{13,18} - E_{16,20} + E_{17,22} + E_{21,24} - E_{23,25}) + t^2(-E_{9,18}),\\
\varphi_F(x_{-\eta_1}(t)) &= I + t(-E_{6,4} - E_{7,5} - E_{9,8} + E_{19,18} - E_{21,20} - E_{23,22}),
\end{aligned}$$

$$\varphi_F(x_{-\eta_3}(t)) = I + t(-E_{3,2} + E_{5,4} + E_{7,6} + E_{12,10} + E_{13,11} + 2E_{15,13} + E_{15,14} + E_{17,16} + E_{22,20} + E_{23,21} + E_{25,24}) + t^2(E_{15,11}),$$

$$\varphi_F(x_{-\eta_4}(t)) = I + t(E_{2,1} + E_{8,5} + E_{9,7} + E_{11,10} + E_{13,12} - E_{14,12} + E_{16,13} - E_{16,14} + E_{17,15} + E_{20,18} - E_{21,19} + E_{26,25}) + t^2(E_{16,12}),$$

$$\varphi_F(x_{-\eta_3-\eta_4}(t)) = I + t(E_{3,1} + E_{8,4} + E_{9,6} - E_{14,10} - E_{15,12} + E_{16,11} + E_{17,13} + 2E_{17,14} - E_{22,18} + E_{23,19} + E_{26,24}) + t^2(-E_{17,10}),$$

$$\varphi_F(x_{-\eta_1-\eta_2}(t)) = I + t(-E_{6,3} - E_{10,5} - E_{11,8} - E_{19,15} + E_{21,17} + E_{24,22}),$$

$$\varphi_F(x_{-\eta_2-\eta_3}(t)) = I + t(-E_{4,2} - E_{5,3} - E_{10,6} + E_{12,7} + E_{13,9} - 2E_{18,13} - E_{18,14} - E_{20,16} + E_{22,17} + E_{24,21} - E_{25,23}) + t^2(-E_{18,9}).$$

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