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CONTENTS

ABSTRACT

Let Y be a simply connected, simple algebraic group of exceptional type, defined over an algebraically closed field k of characteristic $p > 0$. The main result describes all semisimple, closed connected subgroups of Y which act irreducibly on some rational kY module V. This extends work of Dynkin who obtained a similar classification for algebraically closed fields of characteristic 0. The main result has been combined with work of G. Seitz to obtain a classification of the maximal closed connected subgroups of the classical algebraic groups defined over k.

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INTRODUCTION

Our purpose here is to study triples (A,Y,V), where Y is a simply connected, simple algebraic group of exceptional type, defined over an algebraically closed field k of characteristic $p > 0$, V is an irreducible rational kY module, and A is a semisimple, closed connected subgroup of Y such that $V|A$ is irreducible. (We refer to the above set of hypotheses as the "main problem.") In our main result, we obtain a precise description of the triples (A,Y,V).

Before stating our result, we introduce the following notation. Let T_A be a maximal torus of A, T_Y a maximal torus of Y, with $T_A \leq T_Y$. Let $\Pi(A) = \{\alpha_1, \alpha_2, \ldots\}$ and $\Pi(Y) = \{\beta_1, \beta_2, \ldots\}$ be bases of the root systems $\Sigma(A)$ and $\Sigma(Y)$, respectively, with μ_i the fundamental dominant weight corresponding to α_i and λ_i the fundamental dominant weight corresponding to β_i. Let λ be the high weight of V. (Our labelling of Dynkin diagrams is described on page 8.) Finally, we write $A = G_2$, for example, to mean that $\Sigma(A)$ has type G_2.

<u>Main Theorem.</u> If $V|Y$ is tensor indecomposable, one of the following holds:

(i) $A = A_1$, $Y = G_2$, $\lambda|T_A = 6\mu_1$, $\lambda|T_Y = \lambda_1$ and $p \geq 7$.

(ii) $Y = G_2$, $p=3$, $\Sigma(A)$ is a subsystem of $\Sigma(Y)$ containing all long (respectively, short) roots of $\Sigma(Y)$, and $\lambda|T_Y$ has long (short) support.

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(iii) $A = G_2$, $Y = F_4$, p=7, and $\lambda|T_A = 2\mu_1$ and $\lambda|T_Y = \lambda_4$.

(iv) $Y = F_4$, p=2, $\Sigma(A)$ is a subsystem of $\Sigma(Y)$ containing all long (respectively, short) roots, and $\lambda|T_Y$ has long (short) support.

(v) $A = A_2$, $Y = E_6$, $\lambda|T_A = 2\mu_1+2\mu_2$, $\lambda|T_Y = \lambda_1$ or λ_6, and $p\neq 2,5$.

(vi) $A = G_2$, $Y = E_6$, $\lambda|T_A = 2\mu_1$, $\lambda|T_Y = \lambda_1$ or λ_6, and $p\neq 2,7$.

(vii) $A = C_4$, $Y = E_6$, $\lambda|T_A = \mu_2$, $\lambda|T_Y = \lambda_1$ or λ_6, and $p\neq 2$.

(viii) $Y = E_6$, $A = F_4$ is the fixed point subgroup of the graph automorphism of Y and

(a) $\lambda|T_Y = \lambda_1 + (p-2)\lambda_3$ or $(p-2)\lambda_5 + \lambda_6$, for $p>2$, or

(b) $\lambda|T_Y = (p-3)\lambda_1$ or $(p-3)\lambda_6$, for $p>3$.

Moreover, if the pair (A,Y) is as in (ii), (iv) or (viii) V|A is irreducible. As well, if $p\geq 7$ (respectively, $p = 7$, $p \neq 2,5$, $p \neq 2,7$, $p \neq 2$) and Y has type G_2 (respectively, F_4, E_6, E_6, E_6), there exists a subgroup $B \leq Y$, of type A_1 (respectively, G_2, A_2, G_2, C_4) such that B acts irreducibly on $V(\lambda_1)$ (respectively, $V(\lambda_4)$, $V(\lambda_1)$, $V(\lambda_1)$, $V(\lambda_1)$) with the high weight described in (i) (respectively, (iii), (v), (vi), (vii)).□

The results of (i), (ii) and (iv) are proven in [12], where G. Seitz considered the main problem in case Y is a classical group. We establish (iii), (v), (vi) (vii) and (viii) and the existence of an irreducible A_1 in G_2 in this paper. The proof of the existence of an irreducible C_4 in E_6 was communicated to the author by Seitz and is also included here. The remaining existence proofs ($A_2 < E_6$, $G_2 < E_6$ and $G_2 < F_4$) are given in [16], where the conjugacy classes of the irreducible subgroups are also determined.

For an arbitrary irreducible rational kY module V, Steinberg's tensor product theorem ([15]) implies $V|Y = V_1^{q_1} \otimes \cdots \otimes V_k^{q_k}$, where each V_i is a nontrivial irreducible kY module with restricted high weight and $\{q_1, \ldots, q_k\}$ are distinct p-powers. (We refer to $V_i^{q_i}$ as a conjugate of V_i.) If V|A is irreducible, for some subgroup A, then $V_i|A$ is irreducible for each i and the triple (A,Y,V_i) is described in the above theorem. Hence, there is no loss of generality in assuming throughout that $V|Y$ is tensor indecomposable.

The consideration of triples (A,Y,V) in the case where char(k) = 0 was undertaken by E.B. Dynkin in [7]. Given A, a semisimple algebraic group and $\varphi:A \longrightarrow SL(V)$ an irreducible rational representation, Dynkin determined all overgroups of A in SL(V), Sp(V) or SO(V). In a straightforward way, this information yielded a classification of all maximal, proper, closed connected subgroups of the classical algebraic groups. In our situation, where char(k) = p, the Main Theorem has been combined with the results obtained by Seitz in [12] to obtain a similar classification of the maximal proper closed connected subgroups of the classical algebraic groups over k. (This is perhaps the most striking application of the results to date.)

Theorem (A). Let A be a simple algebraic group and $\varphi:A \longrightarrow SL(V)$ an irreducible, rational representaion which is tensor indecomposable. Then with specified exceptions, the image of A is maximal among proper, closed connected subgroups in one of SL(V), Sp(V) or SO(V). Moreover, any other maximal, proper closed connected subgroup of the isometry group of V arises naturally as the stabilizer of a subspace of V or the stabilizer of a tensor product decomposition of V.□

For a more precise statement and the proof, see Theorem (3) in [12]. By far, the major portion of the proof of Theorem (A) lies in describing the "specified exceptions." These fall into two categories, as follows:

Theorem (B). Let Y be a simple algebraic group and $\varphi:Y \rightarrow SL(V)$ an irreducible rational representation which is tensor indecomposable. If A is a proper, closed connected subgroup of Y with $V|\varphi(A)$ irreducible, then one of the following holds:

(i) $\varphi(Y) = SL(V)$, $Sp(V)$ or $SO(V)$.

(ii) $(\varphi(A),\varphi(Y),V)$ appears in Table 1.□

Table 1 contains the combined results of this paper and [12], and lists all embeddings $A < Y < SL(V)$ where A and Y are irreducible, $V|Y$ is tensor indecomposable and $Y \neq Sp(V)$ or $SO(V)$. For a complete explanation of the notation in Table 1, see the end of the introduction; we make a few remarks here. To describe the modules $V|A$ and $V|Y$, we give the high weights. To describe the embedding of A in Y, we indicate the action of a covering group of A on the irreducible kY module W, where W is the natural, classical module for Y, if Y is classical, and W is an irreducible, restricted kY module of minimal dimension, if Y is exceptional. Finally, we note that there are examples for arbitrarily large primes for which there are no counterparts in the characteristic zero result; e.g. $I_1{}'$ and T_1 in Table 3. Hence, interestingly enough, the philosophy that the answer to the sort of question studied here should be the same for large primes p as the answer to the analogous zero characteristic question fails to be justified.

The methods in [12] and this paper differ greatly from those of Dynkin, by necessity. Since in characteristic p, rational modules for simple groups need not be completely reducible nor tensor indecomposable, as in zero characteristic, some of Dynkin's key reductions do not carry over. Though we may assume that in the triple (A,Y,V), $V|Y$ is tensor indecomposable, it happens that $V|A$ can be tensor decomposable. One may notice that throughout the paper, case-by-case analysis is required whenever this possibility persists. If we desired only to prove Theorem (A) or to give a new proof of Dynkin's result, we could assume $V|A$ to be tensor indecomposable and shorten much of our work. As well, for a new proof of the zero characteristic result the small prime analysis of Chapter 9 and the difficulty created by the absence of formulae for the dimensions of and multiplicities of weights in irreducible modules could be avoided.

We now give a survey of the methods used in this paper. Let (A,Y,V) statisfy the hypotheses of the main problem. We obtain preliminary information about the triple (A,Y,V) via induction. Choose a maximal parabolic P_A of A, with unipotent radical Q_A and Levi factor L_A. By the Borel–Tits theorem [2], there exists a parabolic subgroup P_Y of Y with $P_A \le P_Y$ and $Q_A \le Q_Y = R_u(P_Y)$. If L_Y is a Levi factor of P_Y, a result of Smith ([13]) implies that L_A' and L_Y' act irreducibly on the fixed point space V_{Q_A}. Hence, considering the projection of L_A' into the quasisimple components of L_Y' which act nontrivially on V_{Q_A}, we obtain a smaller rank version of the original problem. Theorem (7.1) of [12] is a complete solution of the main problem in the case where rankA = 1. Working inductively, we may describe V_{Q_A} (so partially describe V) and partially describe the embedding of L_A' in L_Y'. Though we are inducting on the rank of A, we handle the case where rank(A) = 2 and $Y = E_n$ in Chapters 6 – 9. Hence, in Chapters 4 and 5, we assume the results of the later chapters.

In Chapter 2, we establish machinery for studying general parabolic embeddings. As well, we prove results applicable only in the context of irreducibility on some module. (Several of the results are proven in [12].) Through this work, we can see the influence of the inductive information on (1) the projections of L_A' in the components of L_Y' which act trivially on V_{Q_A} and (2) the embedding of Q_A in Q_Y. Our considerations are as follows. With L_Y acting on Q_Y via conjugation, certain quotients of Q_Y may be regarded as modules for L_Y', and hence as modules for L_A'. We consider the image of the L_A' module $Q_A/[Q_A,Q_A]$ in these quotients. Of course, in an arbitrary parabolic embedding, Q_A may appear in few L_Y'

composition factors of Q_Y. But another consequence of Smith's result is the equality of the commutator subspaces $[V,Q_A]$ and $[V,Q_Y]$. The existence of particular weight spaces in $[V,Q_Y]$ often forces $Q_A/[Q_A,Q_A]$ to appear in particular quotients of Q_Y. Moreover, in most cases $Q_A/[Q_A,Q_A]$ must appear as an L_A' submodule. This will place restrictions on the projection of L_A' in the quasisimple components of L_Y' which act nontrivially on particular composition factors of Q_Y. We compare this with the inductively given information and perhaps produce a contradiction, or at least broaden our knowledge of the embedding $P_A \leq P_Y$.

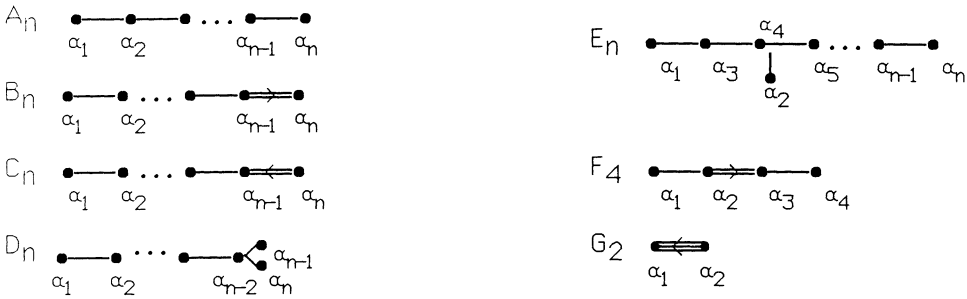
Throughout the paper, various numerical methods are employed as well. Since $[[V,Q_A],Q_A] \leq [V,Q_Y],Q_Y]$ and $[V,Q_A] = [V,Q_Y]$, $\dim([V,Q_Y]/[[V,Q_Y],Q_Y]) \leq \dim([V,Q_A]/[[V,Q_A],Q_A])$. Moreover, if $Z(L_A)^\circ \leq Z(L_Y)^\circ$ (which is usually implied by a suitable choice of P_Y), then the dimension of a $Z(L_Y)^\circ$ weight space of $[V,Q_Y]/[[V,Q_Y],Q_Y]$ is bounded by the dimensions of $Z(L_A)^\circ$ weight spaces of $[V,Q_A]/[[V,Q_A],Q_A]$. Seitz gives an explicit upper bound on the dimensions of the latter. This yields further restrictions on the high weight of $V|Y$. When the high weights of $V|A$ and $V|Y$ are almost explicitly determined, we attempt to show that $\dim V|Y$ exceeds the upper bound on $\dim V|A$ given by the Weyl degree formula. For this purpose, various methods for obtaining lower bounds on dimensions of kY modules are discussed in Chapter 1.

The absence of a "natural" module for the exceptional group Y gives rise to (expected) differences between Seitz's work in [12] and our work here. If Y is a classical group with natural (classical) module W, Seitz proves that most cases, $W|A$ is irreducible and tensor indecomposable. This provides information about the restriction of elements of $\Sigma(Y)$ to a maximal torus of A and, coupled with an inductive hypothesis, usually implies that $V|A$ is a conjugate of a restricted module. As mentioned before, the tensor indecomposable situation is much easier to handle.

In our situation, where Y is an exceptional group, we may think of the "natural" module W as a restricted rational kY module of minimal dimension. However, there is no complete theory relating the subgroup structure of Y to its action on W. We do however use the module W whenever possible. We consider the action of L_A' on W, in particular the L_A' composition series of W. (This can be determined only when we have a fairly complete knowledge about the image of L_A' in L_Y') If dim(W) is relatively small (e.g., 26, 27 or 56) we can list all rational kA modules of this dimension, determine their L_A' composition series and compare with the given L_A' composition series of $W|L_A'$. Though fruitful in specific situations, this analysis does not serve the purpose that the natural module does for the classical groups. Rather, the bounded rank of the exceptional groups and our extensions of Seitz's results on parabolic embeddings enable us to restrict to the few possibilities of the Main Theorem.

For the convenience of the reader, many of the preliminary results from [12] are listed in this paper. It is useful to see that some of the results in Chapter 2 are natural extensions of the results on parabolic embeddings in [12]. A few essential theorems from [12], which we do not state, are often referenced. Theorem (7.1), mentioned already, is the solution of the main problem in case rank(A) = 1. Theorem (4.1) is a solution for the case where rank A = rank Y. (See Chapter 3 for a partial statement.) Theorem (8.1) gives the solution of the main problem for certain natural embeddings of classical groups. And finally, we refer to the list of all triples (A,Y,V), where Y has classical type, as the Main Theorem of [12].

Throughout the paper, we use the following labelling of Dynkin diagrams.



Let us make a few remarks about the notation in Table 1. The second column indicates the types of the groups A and Y, respectively When the symbol " $\longrightarrow'$ " occurs, $A \leq B < Y$, for a closed, connected subgroup B, which is a commuting product of quasisimple groups as indicated. The notation means that either A projects surjectively to each of the simple factors of Y or some factor is of type B_2 and the projection is an A_1 acting irreducibly on the spin module for B_2. Moreover, in order to make sure $V|A$ is irreducible, it may be necessary for the projections to involve distinct field twists.

The third column describes the action of a covering group of A on a particular irreducible kY module, W. If Y is classical, W is the natural module for Y; if Y has type G_2, F_4, or E_6 (E_7 and E_8 do not arise), W is a restricted module of dimension 7 (6 if p=2), 26 (25 if p=3) or 27, respectively.

In the fourth and fifth columns the actions of A and Y on the module V are described, and in the last column any prime restrictions are indicated. Column 1 associates with each example a number. In the cases where there is an analogous zero characteristic example, Dynkin's numbering has been used. So I_1–I_{12}, II_1–II_9, III_1, IV_1–IV_{10}, V_1 and VI_1–VI_3 appear in [7]. Notation such as VI_1' refers to a variant of

Dynkin's VI_1. Examples MR_i are those where rankA = rankY and examples labelled S_1–S_9 are special examples occuring only when p = 2 or 3; these were found by Seitz in [12]. Examples T_1 and T_2 are found in this paper.

In conclusion, the author would like to express thanks to Gary Seitz, who suggested the problem, read an earlier version of this paper and offered useful advice throughout. As well, special thanks are given to Mark Reeder for numerous mathematical insights.

CHAPTER 1: PRELIMINARY LEMMAS

Let V be a finite dimensional vector space over an algebraically closed field k of characteristic $p > 0$, and let X be a semisimple, closed, connected subgroup of $SL(V)$ with fixed maximal torus T. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a base for the root system $\Sigma(X)$ and let e_{α_i} and f_{α_i} denote the corresponding elements of the Lie algebra $L(X)$. Labelling Dynkin diagrams as in Table 1, let λ_i be the fundamental dominant weight corresponding to α_i. Assume $V|X$ is irreducible and let λ be the high weight of V. Then $\langle \lambda, \alpha_i \rangle \geq 0$, for each i and V is said to be *restricted* if $\langle \lambda, \alpha_i \rangle < p$, for $1 \leq i \leq n$. For a subgroup $N < X$, let V_N denote the space of fixed points of N on V and $[V,N]$ the commutator subspace $\langle v - nv \mid v \in V, n \in N \rangle$.

(1.1). (i) $V = V_1^{q_1} \otimes \cdots \otimes V_k^{q_k}$, where each V_i is an irreducible restricted module for X and $q_1, \ldots, q_k$ are distinct powers of p.

(ii) If V is restricted, then V is also irreducible when viewed as a module for $L(X)$.

Proof: (i) is Steinberg's tensor product theorem (see [15]). For (ii) see Section A of [1].□

(1.2). ([13]) Let P be a proper parabolic subgroup of X with unipotent radical Q and Levi factor L. Then $L \cong P/Q$ acts irreducibly on V_Q.□

(1.3). ((1.7) of [12]) Let P be a proper parabolic subgroup of X with unipotent radical Q and Levi factor L. Then $V/[V,Q]$ is irreducible for L. In fact, this quotient is L-isomorphic to $((V^*)_Q)^*$.□

(1.4). ([2]) Let $X \leq Y$, where Y is a closed, connected subgroup of $SL(V)$ and let P be a parabolic subgroup of X with unipotent radical Q. There is a parabolic subgroup P_Y of Y with unipotent radical Q_Y such that

$P \le P_Y$ and $Q \le Q_Y$.□

(1.5). Let X, Y, and P be as in (1.4), and choose P_Y as in (1.4) minimal such that $P \le P_Y$ and $Q \le Q_Y = R_u(P_Y)$. Suppose $L_Y' = L_1 \cdots L_r$, where L_i is a simple normal subgroup of L_Y, with root system of classical type for $1 \le i \le r$. Then, $Z(L)^\circ \le Z(L_Y)^\circ$.

Proof: This follows from the proof of (2.8) in [12].□

(1.6). ((1.4) in [12]) Let $X \le Y$, $P \le P_Y$, $Q \le Q_Y$ be as in (1.4). Then, $V_Q = V_{Q_Y}$. So L and L_Y are reductive groups both acting irreducibly on $M = V_Q$ and the image of L in SL(M) is contained in the image of L_Y.□

(1.7). ((1.6) of [12]) Suppose X is simple. Then V can be expressed as the tensor product, $V = V_1 \otimes V_2$, of two nontrivial restricted kX-modules if and only if V is restricted and the following conditions hold:

(i) X has type B_n, C_n, F_4, or G_2, with p = 2,2,2,3, respectively.

(ii) V_1, V_2 may be arranged such that each V_i has high weight λ_i, $\lambda = \lambda_1 + \lambda_2$, and λ_1 (respectively, λ_2) has support on those fundamental dominant weights corresponding to short (long) fundamental roots.□

(1.8) Definition: Suppose X is simple.

(A) We say V is *basic* (respectively, *p-basic*) if the following conditions hold.

(i) V is restricted.

(ii) If X has type B_n, C_n, F_4, or G_2 with p = 2,2,2,3, respectively, then λ has short (respectively, long) support.

(B) If X and p are as in (ii), we say the pair (X,p) is *special*.

(1.9). Let X, P, Y, and V be as in (1.4) and suppose V|X is basic. Also, if (X,p) is special (respectively, $(G_2,2)$), assume $\Pi(X) - \Pi(L)$ is $\{\alpha\}$ (respectively, long).Then there exists a parabolic subgroup P_Y, of Y, such that the following hold:

(i) $P \le P_Y$ and $Q \le Q_Y = R_u(P_Y)$.

(ii) $L \le C_Y(Z(L)^\circ) \le L_Y$, a Levi complement to Q_Y in P_Y.

(iii) If T_Y is any maximal torus of Y containing T, then $T_Y \leq L_Y$.

Proof: This follows from the first two paragraphs of the proof of (2.8) in [12].□

(1.10). ((2.16) of [12]) Let Y be a simple algebraic group and φ: Y $\longrightarrow$ SL(M) a basic representation. Suppose X is a simple, closed subgroup of Y and $\varphi|X$ is an algebraic conjugate of a restricted representation of X. Then $\varphi|X$ is restricted.□

(1.11). (i) V^* is irreducible with high weight $-w_0\lambda$, where w_0 is the long word in the fundamental reflections generating the Weyl group of X.

(ii) X leaves invariant a nondegenerate bilinear from on V if and only if $\lambda = -w_0\lambda$.

(iii) If X has type B_n, C_n, D_n for n even, E_7, E_8, F_4, or G_2, then X necessarily stabilizes a nondegenerate bilinear from on V.

(iv) If X has type A_n, D_n for n odd, or E_6, then X stabilizes a nondegenerate bilinear form on V if and only if $\lambda = \tau\lambda$, where τ is the graph automorphism of the Dynkin diagram of $\Sigma(X)$.

Proof: See Section 31 of [10].□

(1.12). ((1.14) of [12]) Let X = SL_{n+1}. For any integer p>c>0, the irreducible module V having high weight $c\lambda_1$ or $c\lambda_n$ is isomorphic to the space of homogeneous poylnomials of degree c in a basis of the usual module for X, or its dual. Thus, dimV = $(1/n!)(c+1)(c+2)\cdots(c+n)$.□

(1.13). ((1.13) of [12]) Suppose X = SL_2. Then the weight spaces of T on V are of dimension 1.□

(1.14). Suppose X = SL_2. Let $\Pi(X) = \{\alpha\}$ and let λ_α be the fundamental dominant weight corresponding to α. Then X fixes a symplectic (respectively, orthogonal) form on the restricted, irreducible kX-module with high weight $n\lambda_\alpha$, where n is odd (even).

Proof: This follows immediately from Lemma 79 of [14].□

(1.15). Suppose X = SL_2. Let $\Pi(X) = \{\alpha\}$ and λ_α the fundamental dominant weight corresponding to α. Let W be the rational kX-module

$V(a_1\lambda_\alpha) \otimes \cdots \otimes V(a_m\lambda_\alpha)$ where $V(a_i\lambda_\alpha)$ is an irreducible, rational kX-module with high weight $a_i\lambda_\alpha$, for $a_i = \Sigma c_{ij}p^j$, $0 \le j \le m_j$, $c_{ij} \in Z^+$ and $0 \le c_{ij} < p$. If $p>2$ and $\Sigma\Sigma c_{ij}$ is even (respectively, odd), then there is no submodule of W isomorphic to a conjugate of $V(\lambda_\alpha)$ or $V(3\lambda_\alpha)$ (respectively, $V(2\lambda_\alpha)$).

<u>Proof</u>: Suppose $\Sigma\Sigma c_{ij}$ is even. Then $W = \Sigma W_{2k\lambda_\alpha}$, $-r \le k \le r$, a sum of T weight spaces. Hence, since p is odd there is no weight vector with weight $q\lambda_\alpha$ or $3q\lambda_\alpha$, for any p-power q. Similarly, if $\Sigma\Sigma c_{ij}$ is odd, $W = \Sigma W_{(2k+1)\lambda_\alpha}$, $-r \le k \le r-1$, and there is no weight vector with weight $2q\lambda_\alpha$, for any p-power q.□

<u>(1.16).</u> Let X be simple and $\lambda = q_1\gamma_1 + \cdots + q_\ell\gamma_\ell$, where $\gamma_1, \ldots, \gamma_\ell$ are restricted dominant weights and $q_1, \ldots, q_\ell$ are distinct powers of p. Then X leaves invariant a nondegenerate bilinear form on $V = V(\lambda)$ if and only if X leaves such a form invariant on each of $V(\gamma_1), \ldots, V(\gamma_\ell)$.

<u>Proof</u>: This is immediate from (1.11).□

<u>(1.17).</u> ((1.12) of [12]) Let X be simple and $\lambda = \gamma_1 + \cdots + \gamma_\ell$, where $\gamma_1, \ldots, \gamma_\ell$ are arbitrary dominant weights. Suppose that X leaves invariant a nondegenerate symplectic form on $V(\gamma_1), \ldots, V(\gamma_k)$ and a nondegenerate orthogonal form on $V(\gamma_{k+1}), \ldots, V(\gamma_\ell)$. Then

(i) X leaves invariant a nondegenerate bilinear form on $V(\gamma_1) \otimes \cdots \otimes V(\gamma_\ell) = D$.

(ii) There is a singular subspace S of D such that V is X-isomorphic to a nondegenerate subspace of $S^\perp/S$.

(iii) X leaves invariant a symplectic form on V if k is odd and an orthogonal form if k is even.□

<u>(1.18).</u> ((1.9) of [12]) Let X be simple and $L = L(X)$ and suppose that $0 < I < L$ is an ideal of L not containing each e_α, f_α for $\alpha \in \Sigma^+(X)$. Then one of the following holds:

(i) $I \le Z(L) \le L(T)$, the Lie algebra of T.

(ii) L has Dynkin diagram of type B_n, C_n, F_4, or G_2, p = 2,2,2,3 respectively, and I contains all e_β for β a short root in $\Sigma(X)$.□

Fix a parabolic subgroup $P = QL$ of X, where $Q = R_u(P)$ and L is a Levi complement containing T. Choose P such that Q is the product of those T root subgroups corresponding to the roots $\Sigma^-(X) - \Sigma(L)$. Let U be the product of all T-root subgroups for roots in $\Sigma^+(X)$. Let $\Pi(L) = \Pi(X) \cap \Sigma(L)$ and $Z = Z(L)°$. We now establish notation and list certain useful results regarding the series $V > [V,Q] > [[V,Q],Q] > \cdots > 0$, where $[V,Q^0] = V$ and $[V,Q^i] = [[V,Q^{i-1}],Q]$. Note that $[V,Q^j]$ is T invariant for all j and hence has a decomposition into a sum of T weight spaces. For $d \geq 1$, set $V^d(Q) = [V,Q^{d-1}]/[V,Q^d]$.

<u>(1.19).</u> ((2.1) of [12]) $V/[V,Q]$ is irreducible as a module for L' of high weight $\lambda|T \cap L'$.□

<u>Definition</u>: Let μ be a weight of V, say $\mu = \lambda - \Sigma c_\ell \alpha_\ell$, with each $c_\ell \geq 0$. Then, the *Q-level* of μ is Σc_j, where the sum ranges over those j for which $\alpha_j \in \Pi(X) - \Pi(L)$. Let $V_T(\mu)$ denote the subspace of V consisting of T-weight vectors of weight μ.

<u>(1.20).</u> Suppose X is simple and V is basic, and let $d \geq 0$. If (X,p) is special, assume $\Pi(X) - \Pi(L) = \{\alpha\}$ for some $\alpha \in \Pi(X)$.

(i) If when $(X,p) = (G_2,2)$, $\Pi(L)$ is short, $[V,Q^d] = \oplus V_T(\mu)$, the sum ranging over those weights μ having Q-level at least d. Consequently, $V^{d+1}(Q)$ is isomorphic to the direct sum of those weights having Q-level d.

(ii) If when $(X,p) = (G_2,2)$, $\Pi(L)$ is short, $\dim([V,Q^d]/[V,Q^{d+1}]) \leq s \cdot \dim([V,Q^{d-1}]/[V,Q^d])$, where s is the number of positive roots β such that $U_{-\beta} \leq Q$ and $\beta = \alpha_j + \beta'$, for some $\alpha_j \in \Pi(X) - \Pi(L)$ and β' is 0 or a sum of roots in $\Pi(L)$.

(iii) If when $(X,p) = (G_2,2)$, $\Pi(L) = \{\alpha_2\}$ is long, $V^2(Q) = V^2(Q)_{\lambda-\alpha_1} \oplus V^2(Q)_{\lambda-2\alpha_1}$, a direct sum of Z weight spaces. Moreover, $\dim V^2(Q)_{\lambda-\alpha_1} \leq 2 \cdot \dim V^1(Q)$ and $\dim V^2(Q)_{\lambda-2\alpha_1} \leq \dim V^1(Q)$.

Proof: Statements (i) and (ii) follow from (2.3) of [12]. So consider the case where $(X,p) = (G_2,2)$ and $\Pi(L) = \{\alpha_2\}$. Since V is restricted, V is irreducible as a module for $L(X)$ and so is spanned by the weight vectors $f_{\gamma_1}\cdots f_{\gamma_m}v^+$, for $\gamma_i \in \Sigma^+(X)$. Order $\Sigma^+(X) = \{r_1,\ldots,r_N\}$ so that the roots $\beta \in \Sigma^+(X) - \Pi^+(L)$ occur first. Then, we may write $f_{\gamma_1}\cdots f_{\gamma_m}v^+$ as $f_{\gamma_1}\cdots f_{\gamma_j}w$, where $w \notin [V,Q]$ and $\gamma_i \in \Sigma^+(X) - \Sigma^+(L)$ for $1 \le i \le j$. Say $w \in V_T(\mu)$.

Claim: With $f_{\gamma_1}\cdots f_{\gamma_m}v^+$, w and j as above, $f_{\gamma_1}\cdots f_{\gamma_j}w \in [V,Q^j]$.

Reason: We use induction on j. If $j = 0$, there is nothing to show. So suppose $j > 0$ and $0 \ne f_{\gamma_1}\cdots f_{\gamma_j}w$. By induction, $f_{\gamma_2}\cdots f_{\gamma_j}w \in [V,Q^{j-1}]$. But $x_{-\gamma_1}(1)(f_{\gamma_2}\cdots f_{\gamma_j}w) = f_{\gamma_2}\cdots f_{\gamma_j}w + f_{\gamma_1}\cdots f_{\gamma_j}w + \Sigma w_\ell$, where the sum ranges over $\ell \ge 2$ and $w_\ell \in V_T(\mu-\gamma_2-\cdots-\gamma_j-\ell\gamma_1)$. In particular, $\{f_{\gamma_1}\cdots f_{\gamma_j}w,\ w_\ell \mid \ell \ge 2\}$ is a set of weight vectors for distinct T weights and so is a set of linearly independent vectors. Since $[x_{-\gamma_1}(1), f_{\gamma_2}\cdots f_{\gamma_j}w] \in [V,Q^j]$ and $[V,Q^j]$ is a sum of T weight spaces, we have $f_{\gamma_1}\cdots f_{\gamma_j}w \in [V,Q^j]$. Thus the claim holds.

In the situation where $(X,p) = (G_2,2)$ and $\Pi(L)$ is long, the above claim implies that $V^2(Q)$ is spanned by the images of $f_\gamma w$, for $w \notin [V,Q]$ and $\gamma \in \Sigma^+(Y) - \Sigma^+(L)$. But in fact, since $f_{3\alpha_1+2\alpha_2} = \pm(1/3)[f_{2\alpha_1+\alpha_2}, f_{\alpha_1+\alpha_2}]$ and $f_{3\alpha_1+\alpha_2} = \pm(1/3)[f_{2\alpha_1+\alpha_2}, f_{\alpha_1}]$, we obtain a spanning set for $V^2(Q)$ from the images of $f_\gamma w$ for $\gamma \in \{\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$ and $w \notin [V,Q]$. Hence, $V^2(Q)$ is spanned by weight vectors of Q-levels 1 and 2. So we may decompose $V^2(Q)$ as described in (iii). Moreover, the above remarks imply that the Z weight space $V^2(Q)_{\lambda-\alpha_1}$ (respectively, $V^2(Q)_{\lambda-2\alpha_1}$) is spanned by vectors of the form $f_\gamma w$ where $w \notin [V,Q]$ and $\gamma \in \{\alpha_1, \alpha_1+\alpha_2\}$ (respectively, $\gamma = 2\alpha_1+\alpha_2$). Thus, we have the given bounds on the dimensions of the Z weight spaces of $V^2(Q)$.□

(1.21). Suppose X is simple and $V = (V_1)^{q_1} \otimes \cdots \otimes (V_k)^{q_k}$, where each V_i is restricted and $q_1,\ldots,q_k$ are distinct powers of p. Then for each

$d \geq 0$, $[V,Q^d] = \Sigma[V_1,Q^{d_1}]^{q_1} \otimes \cdots \otimes [V_k,Q^{d_k}]^{q_k}$, the sum ranging over sets of nonnegative integers $d_1,\ldots,d_k$ with $\Sigma d_i = d$.

Proof: This follows from (2.5) of [12].

(1.22). Let V and X be as in (1.21), with $\Pi(X) - \Pi(L) = \{\alpha\}$. Set $W_i = V_i^{q_i}$, for $i = 1,\ldots,k$.

(i) As modules for L, $V^2(Q) \cong \oplus(W_1{}^1(Q) \otimes \cdots \otimes W_{i-1}{}^1(Q) \otimes W_i{}^2(Q) \otimes W_{i+1}{}^1(Q) \otimes \cdots \otimes W_k{}^1(Q))$, the sum over $i = 1,\ldots,k$.

(ii) Assume V_i is basic for each i and that α is long when $(X,p) = (G_2,2)$. The above summands of $V^2(Q)$ are the Z weight spaces for the respective weights $(\lambda-q_1\alpha)|Z,\ldots,(\lambda-q_k\alpha)|Z$. Each such weight space has dimension at most $s\cdot\dim V^1(Q)$, where s is as in (1.20).

(iii) Any Z weight space of $V^2(Q)$ has dimension at most $d\cdot\dim V^1(Q)$, where

$d = s$ as in (1.20), if one of the following holds: V_i is basic for all i; or (X,p) is special and $\Pi(X) - \Pi(L)$ is long; or $(X,p) = (G_2,2)$ and $\Pi(X) - \Pi(L)$ is long.

$d = \frac{1}{2}n(n+1)$, $2n-1$, 14 or 4, if $(X,p) = (B_n,2)$, $(C_n,2)$, $(F_4,2)$ or $(G_2,3)$, respectively, with $\Pi(X) - \Pi(L)$ short.

$d = 3$, if $(X,p) = (G_2,2)$ with $\Pi(X) - \Pi(L)$ short.

Proof: The proof of this result is found in the proofs of (2.12), (2.13) and (15.3) of [12], except for statement (iii) when $(X,p) = (G_2,2)$ and $\Pi(X) - \Pi(L) = \{\alpha_1\}$ is short. So we will consider this case. Let N_j be the j^{th} summand of the decomposition of $V^2(Q)$ given in (i). Then (1.20)(iii) implies that $N_j = N_j{}^1 \oplus N_j{}^2$, where $N_j{}^\ell$ lies in the Z weight space $V^2(Q)_{\lambda-\ell q_j\alpha_1}$, for $\ell = 1,2$. Hence, the Z weights in $V^2(Q)$ have the form $(\lambda-q_i\alpha_1)|Z$ or $(\lambda-2q_i\alpha_1)|Z$, for $1\leq i\leq k$. Suppose a Z weight space of $V^2(Q)$ intersects N_m and N_ℓ for some $m \neq \ell$. Then $q_i = 2q_j$ for some $1 \leq i\neq j \leq k$. Then $q_i \neq 2q_a$ for $a\neq j$ and $q_i \neq q_b$ for $b \neq i$. Thus, the Z weight space $V^2(Q)_{\lambda-q_i\alpha_1} = N_i{}^1 \oplus N_j{}^2$, and by (1.20)(iii), $\dim(N_i{}^1) \leq 2\cdot\dim V^1(Q)$ and $\dim(N_j{}^2) \leq \dim V^1(Q)$. So the result of (iii) follows.□

Let $X \le Y$, for Y a closed subgroup of $SL(V)$ and let $P_Y = Q_Y L_Y$ be a parabolic subgroup of Y such that $P \le P_Y$, $Q \le Q_Y = R_u(P_Y)$, and $T \le T_Y$, for T_Y a maximal torus of L_Y. Set $Z_Y = Z(L_Y)^\circ$. Let $\Sigma(Y)$ be the root system of Y and $\Pi(Y)$ a fundamental system of $\Sigma(Y)$. We choose $\Pi(Y)$ of $\Sigma(Y)$ such that $U \cap L \le Q_Y(U_Y \cap L_Y)$, where U_Y is the product of all T_Y root subgroups for roots in $\Sigma^+(Y)$ and Q_Y is the product of T_Y root subgroups for roots in $\Sigma^-(Y) - \Sigma(L_Y)$. Set $\Pi(L_Y) = \Pi(Y) \cap \Sigma(L_Y)$.

<u>(1.23).</u> (i) $[V,Q] = [V,Q_Y]$.

(ii) $V^1(Q) = V^1(Q_Y)$ is an irreducible module for L and L_Y.

(iii) $V^2(Q_Y)$ is an L-invariant quotient of $V^2(Q)$.

(iv) If w_0 (respectively, s_0) is the long word in the Weyl group of Y (respectively, X), then $-w_0(\lambda)|T \cap L' = -s_0(\lambda)|T \cap L'$.

<u>Proof</u>: (i) – (iii) follow from (2.10) in [12].

Let $W = V^*$. Then, by (1.11), W has high weight $-w_0(\lambda)$ (respectively, $-s_0(\lambda)$) as a Y (respectively, X) module. By (ii) and (1.19), $W^1(Q) = W^1(Q_Y)$ is an irreducible L' (respectively, L_Y') module with high weight $-s_0(\lambda)|T \cap L'$ ($-w_0(\lambda)|T_Y \cap L_Y'$). Since, $U \cap L \le U_Y \cap L_Y$, if $\langle w^+ + [W,Q_Y]\rangle$ is the unique 1-space of $W^1(Q_Y)$ invariant under $U_Y \cap L_Y$, then $\langle w^+ + [W,Q_Y]\rangle$ is also the unique 1-space of $W^1(Q)$ invariant under $U \cap L$. Recalling that $T \le T_Y$, the result follows.□

<u>Definition</u>: For $\gamma \in \Pi(Y) - \Pi(L_Y)$, set $V^\gamma(T_Y) = \Sigma V_{T_Y}(\mu)$, the sum ranging over those μ for which $\lambda - \mu - \gamma$ is a sum of roots in $\Pi(L_Y)$. Since the T_Y weights in $V^1(Q_Y)$ all differ from λ by a sum of roots in $\Pi(L_Y)$, it follows that $V^\gamma(T_Y) \le [V,Q_Y]$ and we let

$$V_\gamma(Q_Y) = (V^\gamma(T_Y) + [V,Q_Y{}^2])/[V,Q_Y{}^2].$$

<u>(1.24).</u> ((2.15) in [12]) Assume $V|Y$ is basic and $\gamma \in \Pi(Y) - \Pi(L_Y)$.

(i) If $\langle \lambda,\gamma\rangle \ne 0$, then some L_Y' composition factor of $V_\gamma(Q_Y)$ has high weight $\lambda - \gamma$.

(ii) Suppose $\langle \lambda,\gamma\rangle = 0$, but $(\Sigma L_0,\gamma) \ne 0$ for some simple factor L_0 of L_Y' satisfying $\langle \lambda,\Sigma L_0\rangle \ne 0$. Then there exist distinct roots

$\beta_1, \ldots, \beta_\ell \in \Pi(L_Y) \cap \Sigma L_0$ such that some L_Y composition factor of $V_\gamma(Q_Y)$ has high weight $\lambda - \beta_1 - \cdots - \beta_k - \gamma$.□

(1.25). Assume $V|Y$ is basic, X is simple, and that $V|X = V_1^{q_1} \otimes \cdots \otimes V_k^{q_k}$ with each V_i restricted and $q_1, \ldots, q_k$ distinct powers of p. Then

(i) $V^2(Q_Y) = \oplus V_\gamma(Q_Y)$, the sum ranging over $\gamma \in \Pi(Y) - \Pi(L_Y)$.

(ii) For each $\gamma \in \Pi(Y) - \Pi(L_Y)$, $V_\gamma(Q_Y) \cong V^\gamma(T_Y)$ (as vector spaces).

(iii) For each $\gamma \in \Pi(Y) - \Pi(L_Y)$, $V_\gamma(Q_Y)$ is a weight space for Z_Y of weight $(\lambda - \gamma)|Z_Y$. The decomposition in (i) is the decomposition of $V^2(Q_Y)$ into distinct weight spaces for Z_Y.

(iv) If, in addition, $\Pi(X) - \Pi(L) = \{\alpha\}$ and $Z \le Z_Y$, then for each $\gamma \in \Pi(Y) - \Pi(L_Y)$, $\dim V_\gamma(Q_Y) \le \dim V^1(Q_A) \cdot d$, where d is as in (1.22)(iii).

Proof: This follows from (2.11) and (2.14) in [12] and (1.22) (iii) above.□

In the remainder of this section, we list and establish certain results which will be used in obtaining upper and lower bounds on the dimensions of modules. Beyond the notation established in the next paragraph, this material will be explicitly referenced when necessary and is not required for the reading of the next chapter.

Assume, for the remainder of this chapter, that X is simple. Let $W(\lambda)$ be the Weyl module corresponding to the dominant weight λ. Let δ be the half sum of the positive roots in $\Sigma(X)$. Write $\mathbb{Z}\Sigma(X)$ for the $\mathbb{Z}$-span of $\Sigma(X)$ and normalize the inner product on $\mathbb{Z}\Sigma(X)$ so that long roots have length 1. Write $\mu \preceq \lambda$ if $\mu = \lambda - \Sigma c_i \alpha_i$, for $c_i \in \mathbb{Z}^+$. Let $P(\lambda - \mu)$ denote the number of distinct ways of writing $\lambda - \mu$ as an integral linear combination of elements of $\Sigma^+(X)$ with nonnegative coefficients. Also, let $m(\mu)$ denote the multiplicitly of the weight μ in $W(\lambda)$. Finally, let $W(X)$ denote the Weyl group of X.

(1.26). (Weyl degree formula): $\dim W(\lambda) = (\prod(\lambda+\delta,\alpha))/(\prod(\delta,\alpha))$, where each product is taken over $\alpha \in \Sigma^+(X)$.

Proof: See Section 24 of [9].□

(1.27). Suppose rank $X = 2$ and $\lambda = m_1\lambda_1 + m_2\lambda_2$.

(a) If X has type A_2, $\dim W(\lambda) = (1/2)(m_1+1)(m_2+1)(m_1+m_2+2)$.

(b) If X has type B_2, $\dim W(\lambda) =$
$(1/6)(m_1+1)(m_2+1)(m_1+m_2+2)(2m_1+m_2+3)$.

(c) If X has type G_2, $\dim W(\lambda) =$
$(1/5!)(m_1+1)(m_2+1)(m_1+m_2+2)(m_1+2m_2+3)(m_1+3m_2+4)(2m_1+3m_2+5)$.

Proof: This follows directly from (1.26).□

(1.28). (Freudenthal): $m(\mu) = (2\Sigma m(\mu+i\alpha)(\mu+i\alpha,\alpha))/((\lambda+\delta,\lambda+\delta) - (\mu+\delta,\mu+\delta))$, where the sum is taken over $\alpha \in \Sigma^+(X)$ and $i \geq 1$.

Proof: See Section 22 of [9].□

(1.29). Let $\langle v^+\rangle$ be the unique 1-space of V invariant under $U = \langle U_r | r \in \Sigma^+(X)\rangle$. Assume λ is restricted and $\mu \leq \lambda$. Let $N = |\Sigma^+(X)|$ and let $\{s_1, s_2, \ldots, s_N\}$ denote any sequence of nonnegative integers. Given a fixed ordering in $\Sigma^+(X) = \{\beta_1, \beta_2, \ldots, \beta_N\}$, $V_T(\mu) = \langle (f_{\beta_1}{}^{s_1} \cdots f_{\beta_N}{}^{s_N})v^+ \mid \lambda - \mu = \Sigma s_i\beta_i\rangle$.

Proof: By (1.1), V is irreducible as a module for L(X). As L(U) leaves $\langle v^+\rangle$ invariant, $V_T(\mu) = \langle f_{\gamma_1} \cdots f_{\gamma_t} v^+ \mid \lambda - \mu = \Sigma \gamma_i\rangle$. The result then follows from the Poincare-Birkhoff-Witt Theorem. (See Section 17 of [9].)□

(1.30). Suppose V is a restricted kX-module, $\alpha \in \Sigma^+(X)$ and $\mu \leq \lambda$, such that $V_T(\mu) \neq 0$. Assume $0 < \langle \mu,\alpha\rangle < p$.

(i) $V_T(\mu - d\alpha) \neq 0$, for each $0 \leq d \leq \langle \mu,\alpha\rangle$.

(ii) If $\alpha \in \Pi(X)$, $\dim V_T(\lambda - d\alpha) = 1$ for $0 \leq d \leq \langle \lambda,\alpha\rangle$.

Proof: View V as an irreducible module for L(X). (See (1.1).) Let $\mathcal{H}$ be a Cartan subalgebra of L(X) and let $\mathfrak{N} = \langle e_\alpha, f_\alpha, \mathcal{H}\rangle$. Then $\mathfrak{N} = \mathfrak{N}_0 \oplus \mathcal{H}_0$ (direct sum of Lie subalgebras), where $\mathfrak{N}_0 = \langle e_\alpha, f_\alpha\rangle$ and $\mathcal{H}_0 = C_{\mathcal{H}}(\mathfrak{N}_0)$. Also, $\mathfrak{N}_0 \cong sl_2$.

Now, let $0 \neq v \in V_T(\mu)$ and take a composition series of V under the action of $\mathfrak{N}$, such that $v \in V_{i+1} - V_i$. Then, $\mathcal{H}_0$ induces scalars on V_{i+1}/V_i, so $\mathfrak{N}_0$ acts irreducibly on V_{i+1}/V_i. The irreducible modules for $\mathfrak{N}_0$ are all restricted (see Section A of [1]); in particular, the weights form a chain. Since $\langle \mu,\alpha \rangle < p$, (i) holds.

The result of (ii) follows from (i) and (1.29).□

(1.31). For $\alpha \in \Pi(X)$, $\dim V_T(\lambda - k\alpha) \leq 1$, for any $k \in \mathbb{Z}^+$.

Proof: Write $V = V_1{}^{q_1} \otimes \cdots \otimes V_k{}^{q_k}$, where V_i is a restricted irreducible kX-module with high weight ν_i, and $q_1, \ldots, q_k$ are distinct powers of p. Suppose $\lambda - k\alpha = \Sigma(q_i\nu_i - n_iq_i\alpha) = \Sigma(q_i\nu_i - m_iq_i\alpha)$, for some integers $0 \leq m_i, n_i < p$. Then $k = \Sigma n_iq_i = \Sigma m_iq_i$ is the p-adic expansion of the integer k. So $m_i = n_i$ for all i and $V_T(\lambda - k\alpha) = (V_1{}^{q_1})_T(q_1(\nu_1 - n_1\alpha)) \otimes \cdots \otimes (V_k{}^{q_k})_T(q_k(\nu_k - n_k\alpha))$. So $\dim V_T(\lambda - k\alpha) = \prod \dim((V_i{}^{q_i})_T(\nu_i - n_i\alpha)) \leq 1$ by (1.29) and (1.30).□

(1.32). ((1.10) in [12]) Let μ be a dominant weight of T and $W_0 \leq W(X)$ be generated by those fundamental reflections corresponding to simple roots $\alpha \in \Pi(X)$ with $\langle \mu,\alpha \rangle = 0$. Then W_0 is the stabilizer of μ in $W(X)$; so there are $|W(X):W_0|$ distinct conjugates of $V_T(\mu)$ in V.□

(1.33). (i) (Linkage principle) Assume X is simply connected and let X(T) be the group of rational characters of T. If μ and ν are high weights of composition factors of an indecomposable kX module, then

(a) $w(\mu+\delta) - (\nu+\delta) \in pX(T)$, for some $w \in W(X)$, and

(b) μ and ν lie in the same coset of $X(T)/X_r(T)$, where $X_r(T)$ is the sublattice generated by $\Sigma(X)$.

(ii) Suppose that μ is a dominant weight and that $W(\lambda)$ contains an X-composition factor of high weight μ. Assume that $p > 2$, and that $p > 3$ when A has type G_2. Write $\mu = \lambda - \Sigma c_i\alpha_i$, where each $c_i \geq 0$. Then

(i) $2(\lambda+\delta, \Sigma c_i\alpha_i) - (\Sigma c_i\alpha_i, \Sigma c_i\alpha_i) \in (p/2)Z$, if $X \neq G_2$.

(ii) $2(\lambda+\delta, \Sigma c_i\alpha_i) - (\Sigma c_i\alpha_i, \Sigma c_i\alpha_i) \in (p/6)Z$ if $X = G_2$.

Proof: The statement of (i)(a) is Theorem 3 of [11]. Statement (ii) is (6.2) of [12]. Part (b) of (i) follows from the fact that $U_r(V_T(\eta)) \subset \Sigma V_T(\eta+ir)$, for $i \geq 0$, η a weight in a rational kX module V, and $r \in \Sigma(X)$. See Lemma 72 of [14].□

(1.34). ((8.6) of [12]) Let $X = SL_{n+1}$ and let V be a restricted irreducible kX-module with high weight λ. Suppose $1 \leq r \leq i < j \leq s \leq n$ and $\lambda = a\lambda_i + b\lambda_j$ for $a \neq 0 \neq b$. Then $V_T(\lambda - (\alpha_r + \cdots + \alpha_s))$ is spanned by vectors $v_t = f_{\alpha_r + \cdots + \alpha_{t-1}} f_{\alpha_t + \cdots + \alpha_s} v^+$, for $i \leq t \leq j$. Moreover, $v_i, \ldots, v_j$ are linearly independent unless $a+b+j-i \equiv 0 \pmod p$, in which case they span a $j-i$ space and $bv_i + v_{i+1} + \cdots + v_j = 0$.□

(1.35). Let V be a restricted kX-module. Let $\alpha, \beta \in \Pi(X)$, with $(\alpha,\beta) < 0$, such that $\langle \lambda,\alpha \rangle = c$, $\langle \lambda,\beta \rangle = d$, for $0 < c,d$. Then $2 \geq \dim V_T(\lambda-\alpha-\beta) > 0$ and

(i) if $(\alpha,\alpha) = (\beta,\beta)$, $\dim V_T(\lambda-\alpha-\beta) = 1$ if and only if $c+d = p-1$;

(ii) if $(\alpha,\alpha) = 2(\beta,\beta)$, $\dim V_T(\lambda-\alpha-\beta) = 1$ if and only if $2c+d+2 \equiv 0 \pmod p$; and

(iii) if $(\alpha,\alpha) = 3(\beta,\beta)$, $\dim V_T(\lambda-\alpha-\beta) = 1$ if and only if $3c+d+3 \equiv 0 \pmod p$.

Proof: This follows from (1.28) and the final proposition of [4].□

(1.36). Let Y be a simple closed subgroup of SL(V). Let $P_Y = Q_Y L_Y$, T_Y, U_Y be as in results (1.23) – (1.25). Let $\gamma \in \Pi(Y) - \Pi(L_Y)$ and $\beta \in \Pi(L_Y)$ be such that $(\gamma,\gamma) = (\beta,\beta)$, $(\gamma,\beta) < 0$, $\langle U_{\pm\beta} \rangle$ is a simple normal subgroup of L_Y', $\langle \lambda,\gamma \rangle \neq 0$ and $\langle \lambda,\beta \rangle = p-1$. Then, there exists $0 \neq w \in V_{T_Y}(\lambda-\gamma-\beta)$ such that $f_\gamma v^+$ and w afford distinct L_Y' composition factors of $V_\gamma(Q_Y)$.

Proof: Note first that for $0 \neq v \in V_{T_Y}(\lambda-\gamma-\beta)$, either v is a maximal vector for $L_Y' \cap U_Y$ or v lies in an L_Y composition factor of $V_\gamma(Q_Y)$ with high weight $\lambda-\gamma$. Clearly, $f_\gamma v^+$ is a maximal vector for $L_Y' \cap U_Y$, and so affords an L_Y' composition factor of $V_\gamma(Q_Y)$. Moreover, by (1.31), there exists a unique L_Y composition factor of $V_\gamma(Q_Y)$ with high weight $\lambda-\gamma$. But $\langle \lambda,\beta \rangle = p-1$ and (1.35) imply that $\dim V_{T_Y}(\lambda-\gamma-\beta) = 2$.

Hence, there exists $0 \neq w \in V_{T_Y}(\lambda-\gamma-\beta)$ as claimed.□

<u>(1.37).</u> Let rank $X = 2$, with $\Pi(X) = \{\alpha,\beta\}$. Let P, Q, and L be as in (1.19) with $L' = \langle U_{\pm\beta}\rangle$. Suppose $V = V_1{}^{q_1} \otimes \cdots \otimes V_k{}^{q_k}$, for each V_i is a basic module for X and $q_1,\ldots,q_k$ are distinct powers of p. Then $\dim((V_T(\lambda - q_i\beta - q_i\alpha) + [V,Q^2])/[V,Q^2]) \leq 2$, for each i.

<u>Proof</u>: By (1.22), $0 \neq w \in V_T(\lambda - q_i\beta - q_i\alpha)$ corresponds to a nonzero vector in $W_1{}^1(Q) \otimes \cdots \otimes W_i{}^2(Q) \otimes \cdots \otimes W_k{}^1(Q)$, where $W_i = V_i{}^{q_i}$. Let μ_ℓ be the high weight of W_ℓ, for $1 \leq \ell \leq k$. Suppose $w \notin (W_1)_T(\mu_1) \otimes \cdots \otimes (W_i)_T(\mu_i - q_i\beta - q_i\alpha) \otimes \cdots \otimes (W_k)_T(\mu_k)$. Then w projects nontrivially into some weight space of the form $(W_1)_T(\mu_1 - n_1q_1\beta) \otimes \cdots \otimes (W_i)_T(\mu_i - q_i\alpha) \otimes \cdots \otimes (W_k)_T(\mu_k - n_kq_k\beta)$, for $0 \leq n_\ell < p$. Hence, $\lambda - (\Sigma n_\ell q_\ell \beta) - q_i\alpha = \lambda - q_i\beta - q_i\alpha$. So $\Sigma n_\ell q_\ell = q_i$. Dividing this equation by the highest power of p which occurs, and taking congruences modulo p, we obtain $n \equiv 0 \pmod p$ for some $0 < n < p$. Contradiction. Thus, $w \in (W_1)_T(\mu_1) \otimes \cdots \otimes (W_i)_T(\mu_i - q_i\beta - q_i\alpha) \otimes \cdots \otimes (W_k)_T(\mu_k)$, which has dimension equal to $\dim((W_i)_T(\mu_i - q_i\beta - q_i\alpha))$. But this is at most 2, by (1.29). Thus, the result holds.□

Using the methods of (1.26), (1.30), (1.32), (1.34) and referring to [8], we obtain the following lower bounds on dimensions of irreducible kX modules.

<u>(1.38).</u> For V a restricted kX-module with high weight λ, let $d(\lambda,p) = \dim V$, where p is the characteristic of k. Then

(i) $X = F_4$, $d(2\lambda_2,p) \geq 2^6\cdot 3\cdot 5$

(ii) $X = E_6$, 1. $d(\lambda_1+\lambda_3+\lambda_5+\lambda_6,2) \geq 2^6\cdot 3^3\cdot 5$

2. $d(\lambda_1+\lambda_3+\lambda_6,2) \geq 2^4\cdot 3^3\cdot 19$ 3. $d(\lambda_1+\lambda_2+\lambda_6,2) \geq 2^4\cdot 3^3\cdot 13$

4. $d(\lambda_3+\lambda_4+\lambda_5,3) \geq 2^4\cdot 3^3\cdot 5\cdot 13$ 5. $d(2\lambda_1+2\lambda_6,p) \geq 2\cdot 3^3\cdot 5\cdot 11$

(iii) $X = E_7$, 1. $p \neq 3$, $d(\lambda_6+\lambda_7,p) \geq 2^3 \cdot 3^2 \cdot 7 \cdot 19$

2. $d(2\lambda_1,p) \geq 2 \cdot 3^2 \cdot 7 \cdot 17$ 3. $x \neq 0$, $d(\lambda_4+x\lambda_7,3) \geq 2^6 \cdot 3^2 \cdot 7 \cdot 19$

4. $d(3\lambda_7,p) \geq 2^5 \cdot 5^2 \cdot 7$ 5. $d(2\lambda_2,p) \geq 2^5 \cdot 3^2 \cdot 37$

6. $d(\lambda_1+\lambda_5,2) \geq 2^6 \cdot 3^2 \cdot 7 \cdot 17$ 7. $d(\lambda_2+\lambda_5,2) \geq 2^8 \cdot 3^2 \cdot 5 \cdot 7$

8. $d(\lambda_2+\lambda_7,2) \geq 2^4 \cdot 3 \cdot 7 \cdot 37$ 9. $d(\lambda_1+\lambda_2,2) \geq 2^6 \cdot 3^2 \cdot 7 \cdot 11$

10. $d(2\lambda_2+\lambda_7,3) \geq 2^6 \cdot 3^2 \cdot 7 \cdot 13$ 11. $d(\lambda_4,3) \geq 2^5 \cdot 3^2 \cdot 5 \cdot 7$

12. $d(2\lambda_7,p) \geq 2^2 \cdot 7 \cdot 29$ 13. $d(\lambda_1+2\lambda_7,3) \geq 2^4 \cdot 3^4 \cdot 7$

(iv) $X = E_8$, 1. $\{a,b\} = \{1,2\}$, $d(a\lambda_1+b\lambda_8,3) \geq 2^5 \cdot 3^3 \cdot 5 \cdot 241$

2. $d(\lambda_2+\lambda_3+\lambda_8,2) \geq 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$

3. $d(2\lambda_8,p) \geq 2^4 \cdot 3 \cdot 5 \cdot 37$ 4. $d(\lambda_1+\lambda_2+\lambda_7,2) \geq 2^{10} \cdot 3^5 \cdot 5 \cdot 7$

5. $d(4\lambda_8,p) \geq 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ 6. $d(\lambda_1+\lambda_2+\lambda_8,2) \geq 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7$

7. $p>2$, $d(\lambda_2,p) \geq 2^6 \cdot 3 \cdot 5^3$ 8. $d(2\lambda_1,p) \geq 2^4 \cdot 3^4 \cdot 5 \cdot 11$

9. $d(\lambda_2+\lambda_6+\lambda_8,2) \geq 2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$

10. $d(3\lambda_8,p) \geq 2^5 \cdot 3^3 \cdot 5 \cdot 17$ 11. $d(\lambda_5+\lambda_8,2) \geq 2^{10} \cdot 3^3 \cdot 5 \cdot 7$

12. $d(\lambda_2,2) \geq 2^8 \cdot 3^2 \cdot 5^2$ 13. $d(\lambda_2+\lambda_8,2) \geq 2^8 \cdot 3^2 \cdot 5^2 \cdot 11$

14. $d(\lambda_2+\lambda_8,3) \geq 2^{12} \cdot 3^3 \cdot 5$ 15. $d(\lambda_1+\lambda_8,3) \geq 2^5 \cdot 3^3 \cdot 5^2 \cdot 7$

16. $d(2\lambda_2+\lambda_8,3) \geq 2^9 \cdot 5 \cdot 37$ 17. $d(\lambda_1+\lambda_6+\lambda_8,2) \geq 2^7 \cdot 3^3 \cdot 5^2 \cdot 7^2$

CHAPTER 2: PARABOLIC EMBEDDINGS

In this chapter, we establish certain results concerning the embeddings of parabolic subgroups, and in particular, embeddings of unipotent radicals. We will adopt the following notation for this entire chapter.

Notation and Hypothesis (2.0). Let Y be a simply connected, simple algebraic group over an algebraically closed field k of characteristic $p > 0$. Let $\theta: Y \longrightarrow SL(V)$ be a nontrivial, finite dimensional, irreducible rational representation. Suppose $A = A^\circ$ is a simple closed subgroup of Y such that $V|A$ is irreducible.

Let $\Sigma(A)$, $\Sigma(Y)$ denote the root systems of A, Y respectively, and take $\Pi(A) = \{\alpha_1, \alpha_2, \ldots\}$ to be a fundamental system of $\Sigma(A)$, with μ_i the fundamental dominant weight corresponding to α_i. Let $B_A = U_A T_A$ be a Borel subgroup of A with maximal torus T_A and unipotent radical U_A, chosen so that U_A is the product of T_A-root subgroups corresponding to roots in $\Sigma^+(A)$. Write B_A^- for $(U_A^-)T_A$, where U_A^- is the opposite unipotent radical. If $\Sigma(A)$ or $\Sigma(Y)$ has only one root length, we will refer to all roots as being "long." Assume $\Sigma(Y)$ has type G_2, F_4 or E_n.

Fix a maximal parabolic subgroup $P_A = Q_A L_A$ of A, where $Q_A = R_u(P_A)$ and L_A is a Levi factor containing T_A. Set $\Pi(L_A) = \Pi(A) \cap \Sigma(L_A)$ and $\Pi(A) - \Pi(L_A) = \{\alpha\}$. We will choose P_A such that α corresponds to an end node of the Dynkin diagram and Q_A is the product of T_A-root subgroups corresponding to the roots in $\Sigma^-(A) - \Sigma(L_A)$. Let $T(L_A') = T_A \cap L_A'$ and set $Z_A = Z(L_A)^\circ$. We will abuse notation and write μ_i for $\mu_i|T(L_A')$.

Let $P_Y = Q_Y L_Y$ be a parabolic subgroup of Y such that $P_A \leq P_Y$,

$Q_A \le Q_Y = R_u(P_Y)$. (The existence of such a parabolic P_Y is given by the Borel–Tits theorem.) Choose P_Y minimal with these properties. Let $T_A \le T_Y$ for T_Y a maximal torus of L_Y. We choose an ordering of $\Sigma(Y)$ and a corresponding base $\Pi(Y) = \{\beta_1, \ldots \beta_n\}$, such that $U_A \cap L_A \le Q_Y(U_Y \cap L_Y)$, where U_Y is the product of those T_Y root subgroups corresponding to the roots in $\Sigma^+(Y)$ and Q_Y is the product of T_Y-root subgroups for roots in $\Sigma^-(Y) - \Sigma(L_Y)$. Write $\Pi(L_Y)$ for $\Pi(Y) \cap \Sigma(L_Y)$ and set $Z_Y = Z(L_Y)^\circ$.

We will write U_r for the T_A (respectively, T_Y) root subgroup corresponding to the root $r \in \Sigma(A)$ (respectively, $\Sigma(Y)$). Also, let $x_r(t)$ denote elements of U_r, for $t \in k$ and $h_\gamma(c)$ denote the element of T_A, or T_Y, corresponding to the root $\gamma \in \Pi(A)$, or $\Pi(Y)$, for $c \in k^*$. As well, let e_r and f_r denote the corresponding elements of the Lie algebra $L(Y)$ or $L(A)$. For Y of type E_n, we will sometimes abbreviate the above notation in the following manner: For $r \in \Sigma^+(Y)$ such that $r = \beta_{i_1} + \cdots + \beta_{i_t}$, $\{\beta_{i_1}, \ldots, \beta_{i_t}\} \subset \Pi(Y)$ with $i_1 < i_2 < \cdots < i_t$, we will write $U_{\pm i_1 i_2 \cdots i_t}$ for $U_{\pm r}$, $e_{i_1 i_2 \cdots i_t}$ for e_r and similarly for f_r, $x_{\pm r}(t)$ and $h_r(c)$. For $r \in \Sigma^+(Y)$, $r = \Sigma a_i \beta_i$, $a_i \in \mathbb{Z}^+$ with some $a_i > 1$, we will write $U_{\pm(a_1, a_2, \ldots, a_n)}$ for $U_{\pm r}$, etc.

Write $L_Y' = L_1 \times \cdots \times L_r$, a direct product of simple algebraic groups. We will refer to L_i as a *component* of L_Y'. By (1.23), L_A' and L_Y' are irreducible on $V^1(Q_Y)$. Then $V^1(Q_Y) = M_1 \otimes \cdots \otimes M_r$, where each M_i is an irreducible L_i module. The embedding $\rho: L_A \to P_Y/Q_Y \cong L_Y$ gives an embedding of L_A' in $L_1 \times \cdots \times L_r$ and we let $\rho_i: L_A' \to L_i$ be the corresponding projection. Then any module for L_i, in particular M_i, can be regarded as a module for L_A'.

Remark: If L_i is of classical type, with natural module W_i, the proper parabolic subgroups of L_i correspond to stabilizers of flags of totally singular subspaces of W_i. Thus, P_Y minimal implies $W_i|\rho_i(L_A')$ is either irreducible or $\rho_i(L_A')$ stabilizes a nonsingular subspace of W_i. Hereafter, we will use this fact without reference to this remark.

Write $V = V(\lambda)$, where λ is a dominant weight of T_Y. Let λ_i denote the fundamental dominant weight corresponding to the root β_i. Let $\langle v^+\rangle$ be the unique 1-space of $V|Y$ invariant under $U_Y T_Y = B_Y$. We may assume, as discussed in the introduction,

(i) $V|Y$ is a restricted module, and

(ii) $V|A = V_1^{q_1} \otimes \cdots \otimes V_k^{q_k}$, where $V_1, \ldots, V_k$ are nontrivial restricted kA-modules and $q_1, \ldots, q_k$ are distinct powers of p.

As well, note that

(iii) $V|Y \not\cong L(Y)$, the Lie algebra of Y, as L(X), the Lie algebra of X, is always a proper invariant kX-submodule of L(Y). We will use this fact frequently without reference.

For each $\gamma \in \Pi(Y) - \Pi(L_Y)$, we define a certain normal subgroup K_γ of P_Y, which in most cases is just the largest normal subgroup of P_Y that is contained in Q_Y and does not contain the T_Y root subgroup corresponding to $-\gamma$. Let $\Sigma_\gamma(Y)$ denote the set of roots in $\Sigma(Y)$ having coefficient of γ equal to -1 and zero coefficient for other roots in $\Pi(Y) - \Pi(L_Y)$. Then let K_γ be the product of those T_Y root subgroups U_β for which $\beta \in \Sigma^-(Y) - \Sigma^-(L_Y) - \Sigma_\gamma(Y)$. From the commutator relations it follows that $K_\gamma \trianglelefteq P_Y$.

(2.1). ((3.1) of [12])

(i) Q_Y/K_γ is isomorphic to the direct product of those T_Y root subgroups for roots $\beta \in \Sigma_\gamma(Y)$.

(ii) There is an L_Y-module structure on Q_Y/K_γ such that Z_Y acts by scalars and such that there is a maximal vector of weight $-\gamma$.

(iii) Q_Y/K_γ is an irreducible L_Y-module, unless γ is a long root with $(\gamma, \Sigma L_Y) \neq 0$ and $\Sigma(Y) = G_2$ or F_4, with $p = 3$ or 2, respectively.□

The above considerations apply to the parabolic subgroup P_A. Here we have only $\alpha \in \Pi(A) - \Pi(L_A)$ and we write $Q_A/K_\alpha = Q_A{}^\alpha$. If (A,p) is not special and if α is long when (A,p) has type $(G_2,2)$, then $Q_A{}^{\alpha} = Q_A/[Q_A,Q_A]$.

(2.2). ((3.2) of [12])

(i) $Q_A{}^{\alpha}$ has an L_A-module structure and $-\alpha$ is the high weight of a composition factor.

(ii) If (A,p) is not special, $Q_A{}^{\alpha}$ is an irreducible L_A module.

(iii) $Q_A{}^{\alpha}$ has a unique maximal L_A-invariant subgroup, M_{α}, which is a submodule having quotient module with high weight $-\alpha$.□

Notation: We will write I_{α} for the irreducible quotient, $(Q_A{}^{\alpha})/M_{\alpha}$, described in (iii) of the above result.

(2.3). ((3.3) of [12]) Assume $Z_A \le Z_Y$ and let $\gamma \in \Pi(Y) - \Pi(L_Y)$.

(i) Q_AK_{γ}/K_{γ} is an L_A invariant submodule of Q_Y/K_{γ}.

(ii) If $V_{\gamma}(Q_Y) \ne 0$, then $Q_A \nleq K_{\gamma}$.

(iii) Suppose rank$(\Sigma(A)) > 1$ and $\langle \lambda, \gamma \rangle \ne 0$. Then $(\gamma, \Sigma(L_Y)) \ne 0$.□

(2.4). Assume $Z_A \le Z_Y$ and let $\gamma \in \Pi(Y) - \Pi(L_Y)$. Suppose that $\delta \in \Sigma(A)$, δ has α-coefficient equal to $-e$ and $U_{\delta} \nleq K_{\gamma}$. Then

(i) $\gamma|Z_A = r\alpha|Z_A$, where r is a positive integral power of p. Moreover, $r = q_i$, for some i, in case each V_j is basic and $V_{\gamma}(Q_Y) \ne 0$.

(ii) There exist unique roots $\beta_1,\ldots,\beta_s$ in $\Sigma_{\gamma}(Y)$ and $c_1,\cdots,c_s$ in k^* such that for $t \in k$, $x_{\delta}(t)K_{\gamma} = x_{\beta_1}(c_1t^{r/e})\cdots x_{\beta_s}(c_st^{r/e})K_{\gamma}$.

(iii) $\beta_j|T_A = (r/e)\delta$, for $j = 1,\ldots,s$ as in (ii).

(iv) If $D = Q_A \cap K_{\gamma}$, then $U_{-\alpha} \nleq D$, so Q_AK_{γ}/K_{γ} has an L_A composition factor isomorphic to $(I_{\alpha})^r$.

(v) If (A,p) is not special, $Q_AK_{\gamma} \le \langle U_{-s} \mid U_{-s} \nleq K_{\gamma}, s|T(L_A') = r\eta$, for some $\eta \in \Sigma^+(A)$ with $U_{-\eta} \nleq Q_A')K_{\gamma}$. If (A,p) is special and $\eta \in \Sigma^+(A)$ with $U_{-\eta} \nleq M_{\alpha}$ (see 2.2(iii)), then

$U_{-\eta}K_{\gamma} \le \langle U_{-s} \mid U_{-s} \nleq K_{\gamma}, s|T(L_A') = r\eta\rangle K_{\gamma}$.

Proof: Statements (i) - (iii) are contained in (3.4) of [12]. As well, if when $(A,p) = (G_2,2)$ we take α to be long, (iv) follows from the proof of (3.4). Suppose $(A,p) = (G_2,2)$ and α is short. Let $\Pi(L_A) = \{\beta\}$. Then, $Q_A' = \langle U_{-3\alpha-\beta}, U_{-3\alpha-2\beta}\rangle \le D$. If $U_{-\alpha} \le D$, then $(U_{-\alpha})^{s_\beta} = U_{-\alpha-\beta} \le D$, where s_β is the reflection corresponding to the root β. But as well, $(U_{-\alpha})^{x_{-\beta}(1)} \le D$; and a nonidentity element from $U_{-2\alpha-\beta}$ occurs in the factorization of this last expression. Hence, if $U_{-\alpha} \le D$, then $Q_A \le D$. Contradiction. Thus, (iv) holds. Finally, since $Q_A K_\gamma / K_\gamma$ is an L_A invariant submodule of Q_Y/K_γ, it is a sum of $T(L_A')$ weight spaces. The result of (v) then follows from (iv).□

For the following result, we will need additional notation. Recall that $V|A = V_1^{q_1} \otimes \cdots \otimes V_k^{q_k}$, where V_i is a restricted irreducible kA module and $q_1, \ldots, q_k$ are distinct p-powers. Write $V_i = V_i^s \otimes V_i^\ell$, where $V_i = V_i^s$ unless (X,p) is special. If (X,p) is special, V_i^s and V_i^ℓ are the short and long parts of V_i, as in (1.7). We will write $V_i^\sim$ to indicate one of V_i, V_i^s, V_i^ℓ.

<u>(2.5).</u> ((3.5) of [12])

(i) For $i = 1, \ldots, r$, M_i is restricted.

(ii) For $i = 1, \ldots, r$, M_i is irreducible under the action of L_A' and there is a uniquely determined subset $\{q_{i_1}, \ldots, q_{i_d}\}$ of $\{q_1, \ldots, q_r\}$ such that $M_i|L_A' \cong (V_{i_1}{}^{\sim 1}(Q_A))^{q_{i_1}} \otimes \cdots \otimes (V_{i_d}{}^{\sim 1}(Q_A))^{q_{i_d}}$.□

<u>Definition</u>: Suppose one of the following holds:

(i) L_i is a classical group with natural module W_i and $W_i|L_A'$ is an algebraic conjugate, by a p-power q, of a nontrivial restricted module.

(ii) $\rho_i(L_A') \le L_i$ is the natural embedding of a group of type B_m in a group of type D_{m+1} and taking $\Pi(L_A) = \{\gamma_1, \ldots, \gamma_m\}$, $\Pi(L_i) = \{\tau_1, \ldots, \tau_{m+1}\}$, we have $\rho_i(x_{\pm\gamma_\ell}(t)) = x_{\pm\tau_\ell}(t^q)$ for $1 \le \ell < m$ and $\rho_i(x_{\pm\gamma_m}(t)) = x_{\pm\tau_m}(t^q) x_{\pm\tau_{m+1}}(t^q)$, for all $c \in k^*$.

(iii) $L_A' \cong L_i$ and if $\Pi(L_A) = \{\gamma_1, \ldots, \gamma_m\}$ and $\Pi(L_i) = \{\tau_1, \ldots, \tau_m\}$,

$\rho_i(x_{\pm\gamma_j}(t)) = x_{\pm\tau_j}(t^q)$, for $1 \le j \le m$, for all $t \in k$.
Then we call q the *field twist on the embedding of L_A' in L_i.*

(2.6). Suppose L_i is a classical group and $M_i|L_A'$ is the q_j twist of a nontrivial basic or p-basic module. Then, q_j is the field twist on the embedding of L_A' in L_i.

Proof: If $\rho_i(L_A') \le L_i$ is not the usual embedding of a group of type B_m in a group of type D_{m+1}, this follows from (9.1) of [12]. Suppose $\rho_i(L_A') \le L_i$ is of type $B_m \le D_{m+1}$. Let $\Pi(L_A)$ and $\Pi(L_i)$ be as in (ii) above. Then if $\langle\lambda,\tau_i\rangle = c_i$ for $1 \le i < m$, then $\langle\lambda,\gamma_i\rangle = c_i q$, where q is the field twist on the embedding of L_A' in L_i. And by (8.1) of [12], $\langle\lambda,\tau_m\rangle = 0$ or $\langle\lambda,\tau_{m+1}\rangle = 0$. So $\langle\lambda,\gamma_m\rangle = dq$, where $d = \langle\lambda,\tau_m\rangle + \langle\lambda,\tau_{m+1}\rangle$. By (2.5), $M_i|L_i$ is restricted so $c_i, d < p$ and $q_j = q$.□

Definition: For $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \ne 0$, suppose Q_Y/K_γ is an irreducible L_Y' module. Then, for $1 \le i \le r$ with $(\gamma,\Sigma L_i) \ne 0$, let $V_{L_i}(-\gamma)$ denote the irreducible L_i module with high weight $-\gamma$. Suppose $V_{L_i}(-\gamma)|L_A' \cong D_1^{r_1} \otimes \cdots \otimes D_d^{r_{d_i}}$, for $D_1,\ldots,D_d$ nontrivial, restricted irreducible L_A' modules and $r_1,\ldots,r_{d_i}$ distinct p-powers. Let $S_i(\gamma,L_A) = \{r_1,\ldots,r_{d_i}\}$.

(2.7). Assume $Z_A \le Z_Y$. Let $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $Q_A \nleq K_\gamma$. Suppose one of the following holds:

(i) There exists a unique pair $1 \le i,j \le r$ such that $(\Sigma L_i,\gamma) \ne 0 \ne (\Sigma L_j,\gamma)$ and $V_{L_m}(-\gamma)|L_A'$ is irreducible for $m = i,j$.

(ii) There exist distinct $1 \le i,j,\ell \le r$ such that $(\Sigma L_m,\gamma) \ne 0$ for $m = i,j,\ell$ and $V_{L_m}(-\gamma)|L_A'$ is irreducible for $m = i,j,\ell$.

If (i) holds, either $(A,p) = (G_2,2)$ with α long and $\mathrm{rank}(L_i) = 1 = \mathrm{rank}(L_j)$, or $S_i(\gamma,L_A) \cap S_j(\gamma,L_A) \ne \varnothing$. If (ii) holds, $S_i(\gamma,L_A) \cap (S_j(\gamma,L_A) \cup S_\ell(\gamma,L_A)) \ne \varnothing$.

Proof: Since $Q_A \nleq K_\gamma$, (2.4) implies that there exists an L_A' composition factor of Q_Y/K_γ isomorphic to a twist of I_α. Now, as L_A'

modules $Q_Y/K_\gamma \cong V_{L_i}(-\gamma)|L_A' \otimes V_{L_j}(-\gamma)|L_A'$, if (i) holds, and $V_{L_i}(-\gamma)|L_A' \otimes V_{L_j}(-\gamma)|L_A' \otimes V_{L_k}(-\gamma)|L_A'$, if (ii) holds. If A has type G_2, p=2, and α is long, $I_\alpha \cong W \otimes W^2$, where W is the restricted 2-dimensional L_A' irreducible. In every other case, I_α is a tensor indecomposable L_A' module. The result then follows from the Steinberg tensor product theorem. (See [15].) □

Hypothesis. For the remainder of Chapter 2, assume $Z_A \leq Z_Y$.

(2.8). Assume (Y,p) is not special and let $\gamma,\delta \in \Pi(Y) - \Pi(L_Y)$.

(i) Suppose there exist $r,s \in \Sigma^+(Y) - \Sigma^+(L_Y)$ such that $r+s \in \Sigma^+(Y)$, $U_{-r} \nleq K_\gamma$, $U_{-s} \nleq K_\delta$ and $x_{-\alpha}(t) = x_{-r}(c_1t^q)x_{-s}(c_2t^{q_0})w$, for $c_i \in k^*$, q and q_0 positive integral powers of p and $w \in \langle U_{-\beta} \mid \beta \in \Sigma^+(Y) - \Sigma^+(L_Y) - \{r,s\}\rangle$. If $q \neq q_0$, there exists a pair of roots $\{r_0,s_0\} \subset \Sigma^+(Y) - \Sigma^+(L_Y) - \{r,s\}$ such that $r_0+s_0 = r+s$ and a nonidentity element from each of U_{-r_0} and U_{-s_0} occurs in the factorization of $x_{-\alpha}(t)$.

(ii) Let $1 \leq i,j \leq r$ such that $(\Sigma L_i,\gamma) \neq 0 \neq (\Sigma L_j,\delta)$ and $(\gamma,\delta) < 0$. If $Q_A \nleq K_\gamma$ and $Q_A \nleq K_\delta$, there exists a p-power q such that $\gamma|Z_A = q\alpha = \delta|Z_A$.

Proof: Let r, s, q and q_0 be as in (i). If there does not exist a pair $\{r_0,s_0\}$ as described, then in the expression for $1 = [x_{-\alpha}(t), x_{-\alpha}(u)]$, the contribution to the root group U_{-r-s} is $c_1c_2(at^qu^{q_0} - bu^qt^{q_0})$, for some $a,b \in k^*$. (Here we have used the fact that (Y,p) is not a special pair.) Since $c_i \neq 0$, $a = -b$ and $q = q_0$. Thus (i) holds.

For (ii), let $r,s \in \Sigma^+(Y) - \Sigma^+(L_Y)$ such that $U_{-r} \nleq K_\gamma$ and $U_{-s} \nleq K_\delta$, and $x_{-\alpha}(t) = x_{-r}(c_1t^{q_1})x_{-s}(c_2t^{q_2})w$, for $c_i \in k^*$, $w \in Q_Y$, q_i a positive integral power of p. Also, we may choose w so that no nonidentity element from the set $U_{-r}U_{-s}$ occurs in its factorization. Note that {r,s} is the unique pair of roots in $\Sigma^+(Y) - \Sigma^+(L_Y)$ whose sum is r+s. Thus, by part (i), $q_1 = q_2$ and (ii) then follows from (2.4).□

(2.9). Assume (Y,p) is not special, and let $\gamma,\delta \in \Pi(Y) - \Pi(L_Y)$, as in (2.8)(ii). Suppose $Q_A' = \{1\}$. Then, $Q_A \le K_\gamma$ or $Q_A \le K_\delta$.

Proof: Suppose false; i.e., suppose $Q_A \nleq K_\gamma$ and $Q_A \nleq K_\delta$. Then, by (2.8), $\delta|Z_A = q\alpha = \gamma|Z_A$, for some p-power q. Let $r \in \Sigma^+(Y) - \Sigma^+(L_Y)$ and $c_r \in k^*$ be such that $U_{-r} \nleq K_\gamma$ and $x_{-r}(c_r t^q)$ occurs in the factorization of $x_{-\alpha}(t)$. Let $\beta \in \Sigma^+(L_A)$ such that $U_{-\alpha-\beta} \nleq M_\alpha$. (See 2.2(iii).) Let $s \in \Sigma^+(Y) - \Sigma^+(L_Y)$ and $c_s \in k^*$ be such that $U_{-s} \nleq K_\delta$ and $x_{-s}(c_s t^q)$ occurs in the factorization of $x_{-\beta-\alpha}(t)$. (We have used (2.4) here.)

Consider the commutator $[x_{-\alpha}(t), x_{-\beta-\alpha}(t)]$. There is a nontrivial contribution to the root group U_{-r-s} from $[x_{-r}(c_r t^q), x_{-s}(c_s t^q)]$. Thus, as the commutator is 1, there must be another contribution to this root group. Now, $\{r,s\}$ is the unique pair of roots in $\Sigma^+(Y) - \Sigma^+(L_Y)$ whose sum is r+s. Thus, a nonidentity element from the root group U_{-r} must occur in the factorization of $x_{-\beta-\alpha}(t)$, and a nonidentity element from the root group U_{-s} must occur in the factorization of $x_{-\alpha}(t)$. But this contradicts (2.4). Thus, the result holds.□

For the following general lemmas we will need additional notation. Let $\gamma \in \Pi(Y) - \Pi(L_Y)$ and suppose that one of the following holds:

(a) There exists $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\gamma) < 0$.

(b) There exist $1 \le j \le r$ and $\delta \in \Pi(Y) - \Pi(L_Y)$ with $\gamma \ne \delta$ and $(\Sigma L_j,\gamma) \ne 0 \ne (\Sigma L_j,\delta)$.

Let $K \le K_\delta$ be defined as follows: $K = \langle U_{-r} \mid r = \Sigma n_\beta \beta,\ \beta \in \Pi(Y),\ n_\gamma > 1 \text{ or } n_\delta > 1 \text{ or } n_\tau > 0 \text{ for some } \tau \in \Pi(Y) - \Pi(L_Y) \text{ with } \tau \ne \gamma,\delta \rangle$. Then, $K \trianglelefteq P_Y$ and K_δ/K is an abelian group with an L_Y module structure, where L_Y acts by conjugation and the scalar action is defined as follows: for $c,d \in k$ and $s \in \Sigma^+(L_Y)$ such that $U_{-s} \nleq K$, $cx_{-s}(d)K = x_{-s}(cd)K$. Then T_Y preserves this scalar action and the image of U_{-s} in K_δ/K is a T_Y weight space of weight $-s$. As L_Y modules, $K_\delta/K \cong K_1/K \times K_2/K$, where K_1/K is the irreducible L_Y module with high weight $-\gamma$. If (a) holds, K_2/K is the irreducible L_Y module with high weight $-\gamma-\delta$. If (b) holds, let $r \in \Sigma^+(L_j)$ such that $\delta+r+\gamma \in \Sigma^+(Y)$ and if $s \in \Sigma^+(L_j)$ with $\delta+s+\gamma \in \Sigma^+(Y)$, then $ht(r) < ht(s)$. Then K_2/K is the irreducible L_Y module with high weight $-\gamma-r-\delta$. Let $Q_Y(\gamma,\delta)$ denote the L_Y composition factor, K_2/K, of Q_Y.

We wish to study the action of L_A on K_δ/K. We note that the commutator relations imply that K_2/K is an L_A invariant subspace of K_δ/K. However, K_1/K is not necessarily L_A invariant; in particular, under the conditions of the following

Definition: Let $\delta \in \Pi(Y) - \Pi(L_Y)$. We say *$-\delta$ is involved in L_A'* if there exists $r \in \Sigma(L_A)$ and $s \in \Sigma^+(Y) - \Sigma(L_Y)$ such that $U_{-s} \nleq K_\delta$ and a nonidentity element from the root group U_{-s} occurs in the factorization of $x_r(t)$.

Consider the following example to see how we may insure, in a particular case, that $-\delta$ is not involved in L_A', when $Q_A \leq K_\delta$. Suppose $(\delta,\Sigma L_j) \neq 0$ for a unique component L_j and L_j has type A_k for some k. Suppose, in addition, that $V_{L_j}(-\delta) \cong W_j$, the natural module for L_j, and hence is an irreducible $\rho_j(L_A')$ module. Say, $W_j|\rho_j(L_A)$ has high weight ν_j. Let $P_Y\hat{} \geq B_Y^-$ be the parabolic subgroup with Levi factor $L_Y\hat{} = \langle L_Y, U_{\pm\delta}\rangle$; so $P_A \leq P_Y\hat{}$ and $Q_A \leq Q_Y\hat{} = R_u(P_Y\hat{}) = K_\delta$. Let $\rho\hat{}: L_A \to L_Y\hat{}$ be the natural homomorphism and $\rho_j\hat{}$ be $\rho\hat{}|L_A'$ followed by the projection of $L_Y\hat{}$ onto the component $L_j\hat{} = \langle L_j, U_{\pm\delta}\rangle$. Then $\rho_j\hat{}$ is a rational morphism of L_A' into a group of type A_{k+1}. Moreover, $W_j\hat{}$, the natural module for $L_j\hat{}$, has two $\rho_j(L_A')$ composition factors – a factor isomorphic to W_j (or W_j^*) and a one-dimensional factor. Hence, if ν_j is not linked to the 0 weight, in the sense of (1.33), then $\rho_j(L_A')$ acts completely reducibly on $W_j\hat{}$ and we may assume, up to conjugacy by $L_j\hat{}$, that $-\delta$ is not involved in L_A'.

We give one additional criterion for $-\delta$ to be involved in L_A'.

(2.10). If $-\delta$ is involved in L_A', then $\delta|Z_A = 0$. Moreover, if r and s are as in the above definition, $-s(h_r(c)) = c^k$ for some $k \in 2\mathbb{Z}^+ - \{0\}$.

Proof: Let r and s be as given. Then there exists $0 \neq f(t) \in k[t]$ such that $x_r(t) = \ell x_{-s}(f(t))u$, for some $\ell \in L_Y'$ and some $u \in Q_Y$ such that u has no nonidentity element from U_{-s} in its factorization. Conjugating by $z \in Z_A$ and using uniqueness of factorization in U_Y, we have $-s(z) = 0$. But, if $\beta \in \Pi(L_Y)$, $\beta(z) = 0$, so the first statement holds. Also, conjugating by $h_r(c)$, we have $f(c^2t) = -s(h_r(c))f(t)$. Let $t = 1$ and the result follows.□

(2.11). Let $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $V_\gamma(Q_Y) \neq 0$. Suppose one of the following holds:

(a) There exists $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\delta) < 0$ and $Q_A \leq K_\delta$. If $p=2$, assume $(\gamma,\gamma) = (\delta,\delta)$ or δ is long.

(b) There exist $1 \leq j \leq r$ and $\delta \in \Pi(Y) - \Pi(L_Y)$ with $\gamma \neq \delta$, $(\Sigma L_j,\gamma) \neq 0 \neq (\delta,\Sigma L_j)$ and $Q_A \leq K_\delta$. If $p=2$, assume δ is long.

Then,

(i) if $-\delta$ is not involved in L_A', $Q_Y(\gamma,\delta)$ has an L_A' composition factor isomorphic to $(I_\alpha)^r$, for some p-power r, and $(\gamma+\delta)|Z_A = r\alpha$. Moreover, if $V|A = V_1{}^{q_1} \otimes \cdots V_k{}^{q_k}$, where each V_j is basic, then $r = q_i$, for some $1 \leq i \leq k$.

(ii) Suppose in addition that if (b) holds with Y of type E_n and $\delta=\beta_4$, then $\{\beta_2,\beta_3,\beta_5\} \not\subseteq \Pi(L_Y)$. Then there exists a parabolic subgroup $P_Y\hat{} \geq B_Y{}^-$, the opposite Borel subgroup of Y, with Levi factor $L_Y\hat{} = \langle L_Y, U_{\pm\delta}\rangle$ and such that $P_A \leq P_Y\hat{}$, $Q_A \leq R_u(P_Y\hat{}) = Q_Y\hat{}$.

(iii) Let $P_Y\hat{}$ be as in (ii). If $-\delta$ is involved in L_A' or if $\gamma|Z_A = r\alpha$, where r is as in (i), then $Z_A \leq Z(L_Y\hat{})°$.

Proof: Let $K \leq K_\delta$, K_1 and K_2 be as defined prior to (2.10). Note that if $-\delta$ is not involved in L_A', the commutator relations imply that K_1/K is an L_A invariant subgroup of K_δ/K. Moreover, since $Z_A \leq Z_Y$, L_A preserves the given scalar action on K_δ/K. So K_i/K has an L_A module structure for each i. Consider the image of Q_A in K_δ/K. Now, $Q_AK/K \nleq K_1/K$, else $[V,K_1] = [V,Q_Y]$. But there exists $b \in \Sigma^+(L_Y) \cup \{0\}$, with $V_{T_Y}(\lambda-b-\gamma-\delta) \neq 0$.

And $V_{T_Y}(\lambda-b-\gamma-\delta) \leq [V,Q_Y] - [V,K_1]$. Thus, Q_AK/K projects nontrivially into the L_A submodule $K_2/K = Q_Y(\gamma,\delta)$. Moreover, Z_A either acts trivially on $Q_Y(\gamma,\delta)$ or induces a full set of scalars on $Q_Y(\gamma,\delta)$; so in fact, Q_AK/K is an L_A submodule of $Q_Y(\gamma,\delta)$. One may now argue as in our proof of (2.4) and the proof of (3.4) of [12] that $U_{-\alpha}K/K \nleq K_1/K$. So the image of $x_{-\alpha}(1)K/K$ in $Q_Y(\gamma,\delta)$ affords the high weight space of an L_A' composition factor.

Let $s \in \Sigma^+(L_Y) \cup \{0\}$ and $0 \neq f(x) \in k[x]$ such that $x_{-s-\gamma-\delta}(f(t))$ occurs in the factorization of $x_{-\alpha}(t)$ and such that s is of minimal height with this property. (Say ht(0) = 0.) Conjugating $x_{-\alpha}(t)$ by an element of T_A which does not centralize $x_{-\alpha}(t)$, we obtain $f(c^kt) = c^\ell f(t)$, for some integers k and ℓ, and for all $c \in k^*$, $t \in k$. Letting t=1, we have $f(x) = a_1x^{\ell/k}$, for some $a_1 \in k$. Moreover, having chosen s of minimal height, $x_{-\alpha}(t)x_{-\alpha}(u) = x_{-\alpha}(t+u)$ implies that ℓ/k is a positive integral power of p. Let $\ell/k = r$. Then, by (2.4), $(s+\gamma+\delta)|T_A = r\alpha$. Thus, $(\gamma+\delta)|Z_A = r\alpha$ and the L_A' composition factor of $Q_Y(\gamma,\delta)$ afforded by the image of $U_{-\alpha}K/K$ is isomorphic to $(I_\alpha)^r$.

We must now show that if $V|A$ is a tensor product of basic modules, then $r = q_i$, for some i. Let $b \in \Sigma^+(L_Y) \cup \{0\}$ be as in the first paragraph. So $V_0 = V_{T_Y}(\lambda-b-\gamma-\delta) \neq 0$. Moreover, $V_0 \nleq [V,Q_A{}^2]$, since $Q_A \leq K_\delta$ implies that $[V,Q_A{}^2] \leq [V,K_\delta{}^2]$. Since $V_0 \nleq [V,Q_A{}^2]$, $(V_0 + [V,Q_A{}^2])/[V,Q_A{}^2] \leq V^2(Q_A)_{\lambda - q_i\alpha}$, a Z_A weight space, for some $1 \leq i \leq k$. See (1.22) for the description of Z_A weight spaces of $V^2(Q_A)$. Thus, $(\gamma+\delta)|Z_A = q_i\alpha$ and $r = q_i$ as desired. The statement of (ii) is clear and (iii) follows from (2.10).□

<u>(2.12).</u> Suppose there exist distinct $\tau_0,\ldots,\tau_m \in \Pi(Y) - \Pi(L_Y)$, with $(\tau_\ell,\tau_{\ell+1}) < 0$ for $0 \leq \ell < m$, $(\tau_\ell,\Sigma L_Y) = 0$ for $0 < \ell \leq m$, $(\tau_0,\Sigma L_Y) \neq 0$, and $V_{\tau_0}(Q_Y) \neq 0$. If m = 1 and p = 2, assume $(\tau_0,\tau_0) = (\tau_1,\tau_1)$; if m > 1 and Y has type F_4, assume p > 2. Suppose there exists a unique p-power q such

that $(I_\alpha)^q$ is an L_A' composition factor of Q_Y/K_{τ_0}.

Then $\tau_i|Z_A = 0$ for $i > 0$. Also, if $P_Y\hat{} \geq B_Y^-$ is the parabolic subgroup of Y with Levi factor $L_Y\hat{} = \langle L_Y, U_{\pm\tau_1},\ldots,U_{\pm\tau_m}\rangle$, then $P_A \leq P_Y\hat{}$, $Q_A \leq R_u(P_Y\hat{}) = Q_Y\hat{}$ and $Z_A \leq Z(L_Y\hat{})^\circ$.

If, in addition, there exists a unique L_A' composition factor of Q_Y/K_{τ_0} isomorphic to $(I_\alpha)^q$, then $\tau_i|T_A = 0$ for $i > 0$.

<u>Proof</u>: We use induction on m. Let m = 1 and note that since $(\tau_1,\Sigma L_Y) = 0$, $\tau_1|T(L_A') = 0$ and (2.10) implies that $-\tau_1$ is not involved in L_A'. By (2.4), $\tau_0|Z_A = q\alpha$ and since $Q_A \leq K_{\tau_1}$, (2.11) implies that there is a nontrivial image of I_α in $Q_Y(\tau_0,\tau_1)$. But $Q_Y(\tau_0,\tau_1) \cong Q_Y/K_{\tau_0}$, as L_Y' modules, so all L_A' composition factors of $Q_Y(\tau_0,\tau_1)$ isomorphic to a twist of I_α are isomorphic to $(I_\alpha)^q$. Thus, $(\tau_0+\tau_1)|Z_A = q\alpha$. Hence, $\tau_1|Z_A = 0$. The statements about the parabolic $P_Y\hat{}$ are clear.

Now, suppose there exists a unique L_A' composition factor of Q_Y/K_{τ_0} isomorphic to $(I_\alpha)^q$. Let $r_1,\ldots,r_j \in \Sigma^+(Y)$ be such that $x_{-r_1}(c_1)\cdots x_{-r_j}(c_j)K_{\tau_0}$ spans the high weight space of this composition factor. So $r_\ell|T_A = q\alpha$, for $1\leq\ell\leq j$. Then, there exists an L_A' composition series of $Q_Y(\tau_0,\tau_1)$ such that $x_{-r_1-\delta}(c_1)\cdots x_{-r_j-\delta}(c_j)$ affords the high weight space of the unique composition factor of $Q_Y(\tau_0,\tau_1)$ isomorphic to $(I_\alpha)^q$. Thus, $(r_\ell+\tau_1)|T_A = q\alpha$, for $1\leq\ell\leq j$. Hence, $\tau_1|T_A = 0$.

Now suppose m>1. By induction, $\tau_1|T_A = 0 = \cdots = 0 = \tau_{m-1}|T_A$. Also, the parabolic subgroup $D_Y \geq B_Y^-$ with Levi factor $M_Y = \langle L_Y, U_{\pm\tau_1},\ldots,U_{\pm\tau_{m-1}}\rangle$ has the properties: $P_A \leq D_Y$, $Q_A \leq R_u(D_Y)$, and $Z_A \leq Z(M_Y)^\circ$. Let $K_{\tau_m} \leq R_u(D_Y)$ be as usual. Then $Q_A \leq K_{\tau_m}$, as all L_A' composition factors of $R_u(D_Y)/K_{\tau_m}$ are trivial. Also, since $(\tau_m,\Sigma L_Y) = 0$, $\tau_m|T(L_A') = 0$ and (2.10) implies that $-\tau_m$ is not involved in L_A'. Note that the given hypotheses on Q_Y/K_{τ_0} carry over to $R_u(D_Y)/K_{\tau_m}$; i.e., if Q_Y/K_{τ_0} has a unique L_A' composition factor isomorphic to a twist of I_α, then so does $R_u(D_Y)/K_{\tau_m}$. So we may argue as in the case m = 1 to obtain the result.□

In (2.11) and (2.12), the purpose of constructing the parabolic $P_Y\hat{}$ is to point out that since $Q_A \le Q_Y\hat{}$, $[V,Q_A{}^2] \le [V,(Q_Y\hat{})^2]$, and so there are many T_Y weight vectors of Q_Y level greater than 1, which are not contained in $[V,Q_A{}^2]$. Often, this construction will produce a $V_\gamma(Q_Y\hat{})$ which exceeds the bound in (1.25).

<u>Hypothesis (G)</u>: (i) There exists $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $Q_AK_\gamma/K_\gamma = Q_Y/K_\gamma$ and $\dim(Q_AK_\gamma/K_\gamma) = \dim(I_\alpha)$.

(ii) There exists $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\delta) < 0$, $(\delta,\Sigma L_Y) = 0$, and $(\gamma,\gamma) = (\delta,\delta)$ when p=2.

(iii) $V_\gamma(Q_Y) \neq 0$.

(iv) There exists a unique $1 \le j \le r$ with $(\gamma,\Sigma L_j) \neq 0$.

(v) $L_A' \cong L_j$, and q is the field twist on the embedding of L_A' in L_j.

(vi) $V|A = V_1{}^{q_1} \otimes \cdots \otimes V_k{}^{q_k}$, where each V_m is basic.

<u>(2.13).</u> (a) If Hypothesis (G) (i) holds, there exists a p-power r with $\gamma|T_A = r\alpha$. If (i), (iii) and (vi) hold, $r = q_i$ for some $1 \le i \le k$. If (i), (iv), and (v) hold, q = r.

(b) If Hypothesis (G) (i) – (iii) hold, $\delta|T_A = 0$ and $\langle\lambda,\gamma\rangle = 0$.

<u>Proof</u>: Condition (i), together with (2.4), implies $Q_Y/K_\gamma \cong (I_\alpha)^r$ as L_A' modules, for some p-power r. Comparing high weight vectors in the two modules, we have $\gamma|T_A = r\alpha$. If (iii) and (vi) hold as well, then (2.4) implies that $r = q_i$ for some i. If $L_A' \cong L_j$, let q be as in Hypothesis (G) (v). So if $\Pi(L_A) = \{\eta_1,\ldots\eta_m\}$ we may take $\Pi(L_j) = \{\tau_1,\ldots\tau_m\}$, such that $\rho_j(h_{\eta_\ell}(c)) = h_{\tau_\ell}(c^q)$, for $1 \le \ell \le m$ and for all $c \in k^*$. Moreover, $(\alpha,\eta_\ell) = (\gamma,\tau_\ell)$ for all ℓ, else Q_AK_γ/K_γ and Q_Y/K_γ are non-isomorphic $\rho(L_A')$ modules. Thus, the high weight space, $U_{-\gamma}K_\gamma/K_\gamma$, of Q_Y/K_γ affords $T(L_A')$ weight $-q\alpha$, and $Q_Y/K_\gamma \cong (I_\alpha)^q$, as L_A' modules. Thus, q = r. Hence, (a) holds.

Let δ be as in Hypothesis (G) (ii) and r as in (a). Then (2.12) implies

$\delta|T_A = 0$. Thus $\langle\lambda,\gamma\rangle = 0$, else $f_\gamma v^+$ and $f_{\gamma+\delta}v^+$ are two linearly independent vectors in $V_{T_A}(\lambda - r\alpha)$, contradicting (1.31). Thus, (b) holds.□

<u>(2.14).</u> Assume Hypothesis (G). In the p-adic expansion of $\langle\lambda,\alpha\rangle$, $q = q_i$ has nonzero coefficient. Moreover, if $p>2$ when Y has type F_4, L_Y' is not a simple algebraic group.

<u>Proof</u>: Note that M_j is nontrivial since $V_\gamma(Q_Y) \neq 0$, $(\gamma,\Sigma L_\ell) = 0$, for all $\ell\neq j$, and $\langle\lambda,\gamma\rangle = 0$, by (2.13). Choose $\beta_0 \in \Pi L_j$ of minimal distance from γ (on the Dynkin diagram) such that $\langle\lambda,\beta_0\rangle \neq 0$. Then there exist distinct $\beta_1,\ldots,\beta_t \in \Pi L_j$ with $(\beta_\ell,\beta_{\ell+1}) \neq 0$ for $0\leq\ell<t$ and $(\beta_t,\gamma) \neq 0$. Also, there exist distinct $\alpha_1,\ldots,\alpha_t \in \Pi(L_A)$ with $\beta_\ell|T_A = q_i\alpha_\ell$, for $0\leq\ell\leq t$, $(\alpha_\ell,\alpha_{\ell+1}) \neq 0$ for $0\leq\ell<t$, $(\alpha_t,\alpha) \neq 0$, $\langle\lambda,\alpha_0\rangle \neq 0$, and $\langle\lambda,\alpha_\ell\rangle = 0$ for $0<\ell\leq t$. Let $s = \beta_0+\cdots+\beta_t$ and $r = \alpha_0+\cdots+\alpha_t$. Then $f_{s+\gamma}v^+$ and $f_{s+\gamma+\delta}v^+$ are two linearly independent vectors in $(V_{T_A}(\lambda - q_ir - q_i\alpha) + [V,Q_A{}^2])/[V,Q_A{}^2]$. But if q_i has zero coefficient in the p-adic expansion of $\langle\lambda,\alpha\rangle$, the indicated T_A weight space in $V^2(Q_A)$ has dimension at most 1. (Here we use (1.22) and (1.29).) Thus, the first statement of the result holds.

Now, suppose L_Y' is a simple algebraic group; then $L_Y' = L_j$. Also, assume $p > 2$ when Y has type F_4. If $\tau \in \Pi(Y) - \Pi(L_Y)$ with $(\tau,\Sigma L_Y) \neq 0$, then $V_\tau(Q_Y) \neq 0$ and $Q_A \nleq K_\tau$. Also, Q_Y/K_τ is an irreducible L_Y' module and so an irreducible L_A' module. Thus, $Q_Y/K_\tau = Q_AK_\tau/K_\tau$. Comparing high weights, we have $(\tau,\beta_t) \neq 0$ and $\tau|T_A = q_i\alpha$. Now, if $\tau \in \Pi(Y) - \Pi(L_Y)$ with $(\tau,\Sigma L_Y) = 0$, (2.12) implies $\tau|T_A = 0$. Thus, we have completely determined the action of $\Pi(Y)$ on T_A.

The work of the first paragraph implies $V_{T_A}(\lambda - q_i\alpha) \neq 0$. Thus, there exists $\tau_0 \in \Pi(Y) - \Pi(L_Y)$ with $(\tau_0,\Sigma L_Y) \neq 0$ and $\langle\lambda,\tau_0\rangle \neq 0$. Moreover, there exists a unique τ_0 with these properties, else $\dim V_{T_A}(\lambda - q_i\alpha) > 1$, contradicting (1.31). Note that there does not exist a nonzero vector with T_A weight $\lambda - q_0\alpha$ for $q_0 \neq q_i$. Thus, $V|A$ is a conjugate of a basic module, and so by (1.10), $q_i = 1$. Also, note that if Y has type F_4, the above work implies $L_Y' = \langle U_{\pm\beta_2}\rangle$.

We now claim that $\langle\lambda,\alpha\rangle = \langle\lambda,\tau_0\rangle$. Certainly, $x = \langle\lambda,\tau_0\rangle \leq \langle\lambda,\alpha\rangle = y$, as $0 \neq ((f_{\tau_0})^x)v^+ \in V_{T_A}(\lambda - x\alpha)$. But, in fact, $x \geq y$, else there does not exist a vector in $V|Y$ with T_A weight $\lambda - y\alpha$. By (1.29), $\dim V_{T_A}(\lambda-r-\alpha) \leq t+2$. By (1.34), $\dim V_{T_Y}(\lambda-s-\tau_0) \geq t+1$. So $\dim(V_{T_Y}(\lambda-s-\tau_0) \oplus V_{T_Y}(\lambda-s-\gamma) \oplus V_{T_Y}(\lambda-s-\gamma-\delta)) \geq t+3$. But each of these T_Y weight spaces lies in $V_{T_A}(\lambda-r-\alpha)$, contradicting the given bound.

This completes the proof of (2.14).□

<u>(2.15).</u> Assume Hypothesis (G) and let rank A = 2, with $L_A' = \langle U_{\pm\beta}\rangle$. Assume also that (A,p) is not special and not of type $(G_2,2)$.

(i) There does not exist $\tau \in \Pi(Y) - \Pi(L_Y)$, with $\tau \neq \gamma$, such that $(\tau,\Sigma L_j) \neq 0$, $(\tau,\Sigma L_\ell) = 0$ for $\ell \neq j$.

(ii) If Y has type F_4, assume $p>2$. Then, there does not exist $\tau \in \Pi(Y) - \Pi(L_Y)$ such that $(\tau,\delta) < 0$, $(\tau,\Sigma L_Y') = 0$.

<u>Proof</u>: Let $L_j = \langle U_{\pm\beta_j}\rangle$, for some $\beta_j \in \Pi(L_Y)$. Then $\langle\lambda,\beta_j\rangle \neq 0$, as $V_\gamma(Q_Y) \neq 0$, $\langle\lambda,\gamma\rangle = 0$ and $(\Sigma L_\ell,\gamma) = 0$ for all $\ell \neq j$. Suppose there exists τ as in (i). Then, Hypothesis (G) implies that $V_\tau(Q_Y) \neq 0$. So by (2.13), $\tau|T_A = q_i\alpha$, for some i. However, $f_{\beta_j+\gamma}v^+$, $f_{\beta_j+\gamma+\delta}v^+$ and $f_{\beta_j+\tau}v^+$ are three linearly independent vectors in $(V_{T_A}(\lambda - q_i\beta - q_i\alpha) + [V,Q_A{}^2])/[V,Q_A{}^2]$, contradicting (1.37). Thus, (i) holds. If there exists τ as in (ii), (2.12) implies $\tau|T_A = 0$. But then $f_{\beta_j+\gamma}v^+$, $f_{\beta_j+\gamma+\delta}v^+$ and $f_{\beta_j+\gamma+\delta+\tau}v^+$ are again three linearly independent vectors in $(V_{T_A}(\lambda - q_i\beta - q_i\alpha) + [V,Q_A{}^2])/[V,Q_A{}^2]$. Again, this produces a contradiction. Thus, (ii) holds.□

We close this chapter with three technical results which apply only when rank A = 2. Hypothesis (G) is no longer necessary.

For (2.16) – (2.18), our only assumptions are that $\Pi(A) = \{\alpha,\beta\}$, so $L_A' = \langle U_{\pm\beta}\rangle$ and μ_β is the fundamental dominant weight corresponding to β.

<u>(2.16).</u> Suppose $\dim(Q_A/[Q_A,Q_A]) = 2$. Let $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $Q_A \nleq K_\gamma$ and L_i has type A_{k_i}, for some $k_i \geq 1$, for all i such that $(\Sigma L_i,\gamma) \neq 0$. Let W_i denote the natural module for L_i, and suppose $W_i|L_A'$ is tensor indecomposable, for all such i. If $\dim(Q_Y/K_\gamma) > 2$ and $-\gamma$ affords $T(L_A')$ weight $q_0\mu_\beta$, then $\gamma|Z_A \neq q_0\alpha$.

<u>Proof</u>: Suppose false. Let r_i be the field twist on the embedding of L_A' in L_i. Since, for each i with $(\Sigma L_i,\gamma) \neq 0$, $W_i|L_A'$ is tensor indecomposable, (1.10) implies $W_i|\rho_i(L_A')$ is restricted. Then one checks, using (1.12) for SL_2, that for each $s \in \Pi(L_i)$, a nonidentity element from the group U_s occurs in the factorization of $\rho_i(x_\beta(t))$. So, $s|T_A = r_i\beta$, for each such s. (We use (2.5) to see that the p-power is r_i.) Thus $U_{-\gamma}$ affords the unique 1-space of Q_Y/K_γ with $T(L_A')$ weight $q_0\mu_\beta$, and all other $T(L_A')$ weights in Q_Y/K_γ are strictly less than $q_0\mu_\beta$. Also, let $a_i \in \Sigma^+(L_i)$ with $-\Sigma a_i - \gamma \in \Sigma^-(Y)$, and such that $\Sigma a_i + \gamma$ has maximum height with these properties. Set $r_0 = \Sigma a_i + \gamma$. Then, U_{-r_0} affords the unique 1-space of Q_Y/K_γ with $T(L_A')$ weight $-q_0\mu_\beta$. Thus, $x_{-\alpha}(t) = x_{-\gamma}(c_1t^{q_0})u_1$ and $x_{-\alpha-\beta}(t) = x_{-r_0}(c_2t^{q_0})u_2$, for $c_i \in k^*$ and $u_i \in K_\gamma$. Now since Q_Y/K_γ is an irreducible L_Y' module with high weight space $U_{-\gamma}K_\gamma/K_\gamma$, there exists $\delta \in \Pi(L_Y)$ with $[U_\delta,U_{-r_0}] \neq 1$. Clearly, $\delta \in \Pi(L_i)$ for some i with $(\Sigma L_i,\gamma) \neq 0$. The remarks about the factorization of $x_\beta(t)$ imply that a nonidentity element from the group $U_{-r_0+\delta}$ occurs in the expression for $[x_\beta(t),x_{-\alpha-\beta}(t)]$. Since $Q_A' \leq K_\gamma$ and $\dim(Q_A/Q_A') = 2$, a nonidentity element from the group $U_{-r_0+\delta}$ occurs in the factorization of $x_{-\alpha}(t)$. Thus, $-r_0+\delta = -\gamma$ and $\Sigma r_i = \delta$. But then, $Q_Y/K_\gamma \cong U_{-\gamma} \times U_{-\gamma-\delta}$,

contradicting $\dim(Q_Y/K_\gamma) > 2$.□

<u>(2.17)</u> Assume (A,p) is not special and $(A,p) \neq (G_2,2)$. Suppose there exists $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that the following hold:

(i) There exists a unique pair $1 \le i,j \le r$, with $(\Sigma L_i,\gamma) \neq 0 \neq (\Sigma L_j,\gamma)$.

(ii) L_i is of type A_1, L_j is of type A_k, for some k.

(iii) $V_{L_m}(-\gamma) \cong W_m$ or $W_m{}^*$, for m = i,j, where W_m is the natural module for L_m.

(iv) $W_j|L_A{}'$ is tensor indecomposable.

(v) $Q_A \nleq K_\gamma$.

Then there exists q, a power of p, such that q is the field twist on the embedding of $L_A{}'$ in L_i and in L_j and $\gamma|Z_A = q\alpha$.

<u>Proof</u>: Note that p > k. By (2.7), the field twists on the embeddings of $L_A{}'$ in L_i and in L_j are equal. Let q be the associated power of p. Let μ_β be the fundamental dominant weight corresponding to the root β. Then the $L_A{}'$ composition factors of Q_Y/K_γ have high weights $\{q(k+1)\mu_\beta, q(k-1)\mu_\beta\}$. If $k+1 < p$, there exists a unique $L_A{}'$ composition factor isomorphic to a twist of $Q_A{}^\alpha$; moreover, this twist is q and the result follows from (2.4). If $k+1 = p$, the $L_A{}'$ composition factors of Q_Y/K_γ have high weights $\{pq\mu_\beta, q(k-1)\mu_\beta\}$. If A has type B_2 or G_2 and β is short the prime restrictions and the above argument imply the result. If β is long, the result holds unless $\gamma|Z_A = pq\alpha$. But since $-\gamma$ affords $T(L_A{}')$ weight $pq\mu_\beta$, (2.16) implies that this cannot occur. Again the result holds.□

<u>(2.18).</u> Assume (A,p) is not special and $(A,p) \neq (G_2,2)$. Let $\gamma,\delta \in \Pi(Y) - \Pi(L_Y)$ be such that δ is long when p = 2 and the following hold:

(i) $(\gamma,\delta) < 0$, $V_\gamma(Q_Y) \neq 0$ and $Q_A \le K_\delta$.

(ii) There exists a unique i (respectively, j) such that $(\Sigma L_i,\gamma) \neq 0$ $((\Sigma L_j,\delta) \neq 0)$.

(iii) L_i, L_j and W_m are as in (2.17) (ii) and (iii).

(iv) $V_{L_i}(-\gamma) \cong W_i$ or W_i^* and $V_{L_j}(-\delta) \cong W_j$ or W_j^*.

(v) $W_j|L_A'$ is tensor indecomposable.

Then, there exists q, a power of p, such that q is the field twist on the embedding of L_A' in L_i and in L_j and $(\gamma+\delta)|Z_A = q\alpha$.

<u>Proof</u>: Note that Q_Y/K_γ is a 2-dimensional irreducible L_A' module containing a nontrivial image of $Q_A{}^\alpha$, so the prime restrictions imply that β is long. If $-\delta$ is involved in L_A', then (2.10) implies k is even and $\delta|Z_A = 0$. Let $P_Y\hat{} \geq B_Y{}^-$ be the parabolic of Y with Levi factor $L_Y\hat{} = \langle L_Y, U_{\pm\delta}\rangle$. Then $P_A \leq P_Y\hat{}$, $Q_A \leq R_u(P_Y\hat{}) = Q_Y\hat{}$ and $Z_A \leq Z(L_Y\hat{})$. The bound on $\dim V_\gamma(Q_Y\hat{})$ implies that $p > k = 2$. But since $p > 2$, $2\mu_\beta|T(L_A')$ is not linked to the zero weight in the sense of (1.33); so we may assume $-\delta$ is not involved in L_A'. Then (2.11) implies that there is a nontrivial image of $Q_A{}^\alpha$ in $Q_Y(\gamma,\delta)$. The field twists on the embeddings of L_A' in L_i and in L_j are equal, else $Q_Y(\gamma,\delta)$ is a tensor decomposable irreducible L_A' module of dimension greater than 2. Call this twist q. Then, $\gamma|T_A = q\alpha$. As in the previous result, we are done unless $k+1 = p$ and $(\gamma+\delta)|Z_A = pq\alpha$.

So suppose $k+1 = p$ and $(\gamma+\delta)|Z_A = pq\alpha$. Let $L_i = \langle U_{\pm\beta_i}\rangle$ and $r \in \Sigma^+(L_j)$ the root of maximal height. Examining the $T(L_A')$ weight vectors in $Q_Y(\gamma,\delta)$, we have $x_{-\alpha}(t) = x_{-\gamma}(c_1t^q)x_{-\gamma-\delta}(c_2t^{pq})w_1$ and $x_{-\alpha-\beta}(t) = x_{-\gamma-\beta_i}(c_3t^q)x_{-\beta_i-\gamma-\delta-r}(c_4t^{pq})w_2$, where $c_i \in k^*$ and $w_i \in K = \langle U_{-t} \mid t = \Sigma n_\tau\tau, \tau \in \Pi(Y), n_\gamma>1$ or $n_\delta>1$ or $n_\varepsilon>0$ for some $\varepsilon \in \Pi(Y) - \Pi(L_Y), \varepsilon\neq\gamma,\delta\rangle$. We have used here the fact that there is a unique L_A' composition factor of $Q_Y(\gamma,\delta)$ isomorphic to $(Q_A{}^\alpha)^{pq}$.

Now $x_{\beta_i}(t^q)$ occurs in the factorization of $x_\beta(t)$. This observation, together with the earlier assumption about the factorization of $x_\beta(t)$, implies that there is a nontrivial contribution to the root group $U_{-\gamma-\delta-r}$ in the expression for $[x_{-\alpha-\beta}(t),x_\beta(t)]$, which must occur in the factorization of $x_{-\alpha}(t)$ due to the restrictions on the characteristic. But this contradicts the given factorization of $x_{-\alpha}(t)$.□

CHAPTER 3: $Y = F_4$ or G_2

In this chapter, we consider the main problem where $A \leq Y$ are simple algebraic groups, with Y simply connected, having root system of type G_2 or F_4. Let $V = V(\lambda)$ be an irreducible kY module and let T_A, T_Y, λ_i, μ_i be as in (2.0). We first note that the following results were obtained in [12]:

<u>Theorem (7.1)</u> (in [12]): If $\mathrm{rank} A < \mathrm{rank} Y$ in Y of type G_2 and $V|A$ is irreducible then $A = PSL_2$, $\lambda|T_Y = \lambda_1$, $\lambda|T_A = 6\mu_1$ and $p \neq 2,3,5$.

<u>Theorem (4.1)</u> (in [12]): If $\mathrm{rank} A = \mathrm{rank} Y$, then $V|A$ is irreducible if and only if the following conditions hold:

(i) $p=2$ when $Y = F_4$ or $p = 3$ when $Y = G_2$, and

(ii) $\Sigma(A)$ is a subsystem of $\Sigma(Y)$ containing all long roots (respectively, all short roots) of $\Sigma(Y)$, and $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in \Pi(Y)$ with α short (respectively, long).

The remaining cases are handled in the following

<u>Theorem (3.0).</u> (a) If Y has type G_2 and $p \neq 2,3,5$, there exists a subgroup $A \leq Y$, $A \cong PSL_2$, such that $V(\lambda_1)|A$ is a restricted 7-dimensional irreducible.

(b) If Y has type F_4, $\mathrm{rank} A < \mathrm{rank} Y$, and $V|A$ is irreducible, then A has type G_2, $p=7$ and $\lambda|T_A = 2\mu_1$, $\lambda|T_Y = \lambda_4$.

(c) Let Y have type F_4 with $p = 7$. Then there exists a closed, connected subgroup $B < Y$, B of type G_2, with $V(\lambda_4)|B$ irreducible.

<u>Proof of (3.0)(a) and (c):</u> The proof of (c) is contained in [16]. For

(a), let Y be a simple algebraic group of type G_2 with $\Pi(Y) = \{\alpha_1, \alpha_2\}$, labelled as throughout, with irreducible module $V = V(\lambda_1)$. Assume $p \neq 2,3,5$ and consider the subgroup $A = \langle x_\alpha(t), x_{-\alpha}(t) \mid t \in k\rangle$, where $x_\alpha(t) = x_{\alpha_1}(6t)x_{\alpha_2}(10t)x_{\alpha_1+\alpha_2}(30t^2)x_{2\alpha_1+\alpha_2}(120t^3)x_{3\alpha_1+\alpha_2}(-540t^4)\cdot x_{3\alpha_1+2\alpha_2}(-2160t^5)$ and $x_{-\alpha}(t) = x_{-\alpha_1}(t)x_{-\alpha_2}(t)x_{-\alpha_1-\alpha_2}(-\frac{1}{2}t^2)\cdot x_{-2\alpha_1-\alpha_2}((1/3)t^3)x_{-3\alpha_1-\alpha_2}((1/4)t^4)x_{-3\alpha_1-2\alpha_2}((-1/10)t^5)$. Considering first the action of $L(Y)$ on V, we obtain the following description of the root groups of G_2 (in SL_7), where we use E_{ij} to mean the matrix whose $k\ell$ entry is $\delta_{ik}\delta_{j\ell}$ and write I for the identity matrix:

$$x_{\alpha_1}(t) = I + t(E_{12} + 2E_{34} - E_{45} + E_{67}) - t^2E_{35};$$
$$x_{-\alpha_1}(t) = I + t(E_{21} + E_{43} - 2E_{54} + E_{76}) - t^2E_{53};$$
$$x_{\alpha_2}(t) = I - t(E_{23} + E_{56});\ x_{-\alpha_2}(t) = (x_{\alpha_2}(t))^t);$$
$$x_{\alpha_1+\alpha_2}(t) = I + t(E_{13} - 2E_{24} - E_{46} - E_{57}) + t^2E_{26};$$
$$x_{-\alpha_1-\alpha_2}(t) = I + t(E_{31} - E_{42} - 2E_{64} - E_{75}) + t^2E_{62};$$
$$x_{2\alpha_1+\alpha_2}(t) = I + t(2E_{14} + E_{25} + E_{36} - E_{47}) - t^2E_{17};$$
$$x_{-2\alpha_1-\alpha_2}(t) = I + t(E_{41} + E_{52} + E_{63} - 2E_{74}) - t^2E_{71};$$
$$x_{3\alpha_1+\alpha_2}(t) = I + t(E_{15} - E_{37});\ x_{-3\alpha_1-\alpha_2}(t) = (x_{3\alpha_1+\alpha_2}(t))^T;$$
$$x_{3\alpha_1+2\alpha_2}(t) = I + t(E_{16} + E_{27});\ x_{-3\alpha_1-2\alpha_2}(t) = (x_{3\alpha_1+2\alpha_2}(t))^T.$$

Let P be the diagonal matrix diag(360,60,−12,−3,2,−12,−1). Then one checks that $P^{-1}x_\alpha(t)P$ (respectively, $P^{-1}x_{-\alpha}(t)P$) is the 7×7 matrix obtained by considering the action of $\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$ (respectively, $\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$) on the irreducible 7-dimensional module with ordered basis $\{x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6\}$ and action $\begin{pmatrix}a & b\\ c & d\end{pmatrix}(x^iy^j) = (ax+cy)^i(bx+dy)^j$. Hence, $A \cong PSL_2$, $V|A$ is irreducible with the correct high weight and (3.0)(a) holds.□

For the remainder of this chapter, let $Y = F_4$. The proof of (3.0)(b) involves a straightforward reduction to the case where rankA = 2 and then a detailed study of the possible embeddings of a maximal parabolic subgroup of A. We adopt Notation and Hypothesis (2.0), and note that since all components of L_Y' are necessarily of classical type, (1.5) implies

$Z_A \leq Z_Y$.

Suppose p=2. Then there is a surjection (isomorphism of abstract groups) φ: $Y \longrightarrow Y$. (See Section 10 of [14].) We may consider V as a module for $\varphi^{-1}(Y)$.

<u>(3.1).</u> Assume V is p-basic. Then $V|\varphi^{-1}(Y)$ is an algebraic conjugate of a basic module. (See (1.8) for the definition of basic and p-basic.)

<u>Proof</u>: This follows from (2.2) of [12].□

<u>Hypothesis</u>: If p=2, we will assume $\lambda|T_Y$ has short support; i.e., V|Y is basic. For if λ has both long and short support, V|Y is tensor decomposable, by (1.7). If λ has long support, the above remarks and (3.1) give rise to a configuration $(\varphi^{-1}(A),\varphi^{-1}(Y),V)$, where V is a conjugate of a basic module. But then, we reduce to $(\varphi^{-1}(A),\varphi^{-1}(Y),W)$, where W is basic.

<u>(3.2).</u> If rank A = 3 and L_A' is of type B_2 (= C_2), then $\dim V^1(Q_A) = 1$.

<u>Proof</u>: Since P_Y is minimal, $L_Y' = \langle U_{\pm\beta_2}, U_{\pm\beta_3}\rangle$. Also, if p > 2, Q_Y/K_{β_1} and Q_Y/K_{β_4} are nonisomorphic irreducible L_A' modules. Thus, $Q_A \leq K_{\beta_1}$ or $Q_A \leq K_{\beta_4}$ and (2.3) implies $\dim V^1(Q_Y) = \dim V^1(Q_A) = 1$. Thus, if $\dim V^1(Q_A) > 1$, p = 2 and $\lambda|T_Y = \lambda_3$ or $\lambda_3+\lambda_4$. Using induction to determine the possible labellings of V|A, we see that either $\lambda|T_Y = \lambda_4$, with $\lambda|T_A = q\mu_2$, $q\mu_3$ or $q_1\mu_1 + q\mu_3$, or $\lambda|T_Y = \lambda_3+\lambda_4$, with $\lambda|T_A = q\mu_1 + q\mu_2$ or $q\mu_3$. (Here q and q_1 are distinct p-powers.) But Table 1 of [5] implies that $\dim V|A < \dim V|Y$ in each case. Hence, $\dim V^1(Q_A) = 1$.□

<u>(3.3).</u> If rank A = 3 and L_A' is of type A_2, then $\dim V^1(Q_A) = 1$.

<u>Proof</u>: Suppose false; i.e. suppose $\dim V^1(Q_A) > 1$. We first note that size restrictions imply that L_Y' is a simple algebraic group. If p >2, (2.14) implies L_Y' is not of type A_2. Thus, the Main Theorem of [12] and

the above remarks imply $p = 3$ and $L_Y{}'$ is of type B_3. However, the $L_A{}'$ composition factors of Q_Y/K_{β_4} have dimensions 1 and 7, while $\dim(Q_A{}^{\alpha}) =$ 3 or 6. Thus, $Q_A \leq K_{\beta_4}$. But $V_{\beta_4}(Q_Y) \neq 0$, contradicting (2.3). Now if $p = 2$, induction and the labelling of $V|Y$ imply $L_Y{}' = \langle U_{\pm\beta_3}, U_{\pm\beta_4}\rangle$. Using (3.2), (1.23) and Table 1 of [5], we find that $\dim V|A < \dim V|Y$ in every case. Hence, $\dim V^1(Q_A) = 1$.□

The above results imply rank $A = 2$. For the remainder of this chapter we use the following

<u>Notation:</u> Let $\Pi(A) = \{\alpha,\beta\}$ and $\Pi(L_A) = \{\beta\}$. Let μ_α (respectively, μ_β) denote the fundamental dominant weight corresponding to α (respectively, β). We will also use μ_β to mean $\mu_\beta|T(L_A{}')$.

<u>(3.4).</u> There are no examples (A,Y,V) in the main theorem with $p = 2$, A simple, Y of type F_4 and rank $A <$ rank Y.

<u>Proof:</u> Suppose false. Then (3.2) and (3.3) imply rank $A = 2$. By [8], $\dim V|Y = 26$, 246 or 4096 and $\dim V|A = 8^k\cdot 3^\ell$, 4^m, $6^r\cdot 14^s\cdot 64^t$, where A has type A_2, B_2, G_2 respectively and $k,\ell,m,r,s,t \in \mathbb{Z}^+$. So $\lambda|T_Y = \lambda_3+\lambda_4$. If A has type A_2, $\lambda|T_A = (q_1 + q_2 + q_3 + q_4)(\mu_1+\mu_2)$ for distinct p-powers q_1, q_2, q_3 and q_4. Thus, for a fixed maximal parabolic P_A, $\dim V^1(Q_A) = 16$. But there is no parabolic P_Y of Y with $\dim V^1(Q_Y) = 16$, contradicting [13]. Thus, A must have type B_2 or G_2.

Suppose A has type B_2. Let P_A be a maximal parabolic of A with $\dim V^1(Q_A) > 1$. By the main theorem of [12], $\dim V^1(Q_A) = 2$, 4 or 8. In fact since $\dim V|A = 4096$, we must have $\dim V^1(Q_A) = 8$, else $\dim V|A < \dim V|Y$. So P_Y (as in Hypothesis and Notation (2.0)) has type B_3. However, Q_Y/K_{β_4} is then an 8-dimensional irreducible $L_A{}'$ module and hence cannot contain a nontrivial image of $Q_A{}^{\alpha}$, contradicting (2.3).

Finally, we must consider the case where A has type G_2. The above remarks imply $\lambda|T_A = (q_1+q_2)(\mu_1+\mu_2)$ for q_1 and q_2 distinct p-powers.

Now let P_A be the maximal parabolic of A with Levi factor $L_A = \langle U_{\pm\beta}\rangle T_A$ where β is long. By induction, P_Y has Levi factor of type B_2. However, Q_Y/K_{β_4} is then a 4-dimensional irreducible L_A' module which cannot contain a nontrivial image of $Q_A{}^{\alpha}$. But this contradicts (2.3).□

<u>(3.5).</u> (i) If $\langle\lambda,\beta\rangle \neq 0$ and L_Y' is quasisimple, then $(A,p) = (G_2,3)$, L_Y' has type A_1 and $\Pi(L_Y) \neq \{\beta_3\}$ or A has type B_2, $L_Y' = \langle U_{\pm\beta_1}, U_{\pm\beta_2}\rangle$ and β is short.

(ii) Assume β is long, unless $(A,p) = (G_2,3)$, in which case β is arbitrary. If $L_Y' = L_1 \times L_2$, for L_i a simple algebraic group with $V^1(Q_Y) = M_1 \otimes M_2$ where M_i is an irreducible kL_i module, then at most one of M_1 and M_2 is nontrivial.

<u>Proof</u>: Suppose $\langle\lambda,\beta\rangle \neq 0$ and L_Y' is a simple algebraic group. Since $p > 2$, if rank $L_Y' = 1$, there exists $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $V_\gamma(Q_Y) \neq 0$ and $\dim(Q_Y/K_\gamma) = 2$. So if $(A,p) \neq (G_2,3)$, β must be long. Also, (3.4) and (2.14) imply that either $(A,p) = (G_2,3)$ or $L_Y' = \langle U_{\pm\beta_3}\rangle$. If $L_Y' = \langle U_{\pm\beta_3}\rangle$, then $V_{\beta_2}(Q_Y) \neq 0$ so $Q_A \nleq K_{\beta_2}$. But Q_Y/K_{β_2} is a 3-dimensional irreducible L_A' module, while $\dim(Q_AK_{\beta_2}/K_{\beta_2}) = 2$. Thus, if $\mathrm{rank}(L_Y') = 1$, $(A,p) = (G_2,3)$ and $L_Y' \neq \langle U_{\pm\beta_3}\rangle$. If $\mathrm{rank}(L_Y') = 2$, we use the condition that each Q_Y/K_γ have an L_A' composition factor of dimension $\dim(Q_A{}^{\alpha})$, for $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $V_\gamma(Q_Y) \neq 0$. This results in the second configuration of (i).

If L_Y' has type B_3, the L_A' composition factors of Q_Y/K_{β_4} are of dimensions 1 and 7. Hence, L_Y' is not of type B_3. Thus, L_Y' has type C_3. Since $Q_A \nleq K_{\beta_1}$, Q_Y/K_{β_1} must be a reducible L_A' module. Hence, by (7.1) of [12], $V^1(Q_Y)$ is isomorphic to the natural module for L_Y'. Moreover, $\langle\lambda,\beta_1\rangle = 0$ else the bound on $\dim V_{\beta_1}(Q_Y)$ is exceeded. Thus, $\lambda|T_Y = \lambda_4$ and $\dim V|Y \leq 26$. However, $\langle\lambda,\beta\rangle = 5\cdot q_1$ or $q_1 + 2q_2$, for q_1 and q_2 distinct p-powers. Applying (1.23) when A has type A_2, we see that $\dim V|A > \dim V|Y$. Contradiction. This completes the proof of (i).

Let L_i and M_i be as in (ii), and suppose M_1 and M_2 are both nontrivial. Let q_i be the field twist on the embedding of L_A' in L_i for i = 1,2. (This is well-defined as L_i has type A_1 or A_2.) Then $q_1 \neq q_2$ by (2.5) and (2.6). Suppose there exists $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $(\gamma, \Sigma L_i) \neq 0$ for i = 1, 2. Examining the L_A' composition factors of Q_Y/K_γ and recalling that p > 2, we reduce to $\Pi(L_Y) = \{\beta_1, \beta_3, \beta_4\}$. Temporarily assume $(A,p) \neq (G_2,3)$. The L_A' composition factors of Q_Y/K_{β_2} have high weights $(q_1 + 4q_2)\mu_\beta$ and $q_1\mu_\beta$; if p = 3 and $q_1 = 3q_2$, the composition factors have high weights $(2q_1 + q_2)\mu_\beta$, $q_2\mu_\beta$ and $q_1\mu_\beta$. Thus, if $p \neq 3$, $\beta_2|Z_A = q_1\alpha$. If p = 3 and $q_1 = 3q_2$, $\beta_2|Z_A = q_1\alpha$ or $q_2\alpha$. We notice that a nonidentity element from the set $U_{-\beta_2} \cdot U_{-\beta_1-\beta_2}$ must occur in the factorization of some element in $Q_A - Q_A'$, since $\langle \lambda, \beta_1 \rangle \neq 0$. Since $-\beta_2$ (respectively, $-\beta_1-\beta_2$) affords $T(L_A')$ weight $(q_1 + 4q_2)\mu_\beta$ (respectively, $(-q_1 + 4q_2)\mu_\beta$), (2.4) implies p = 3, $q_1 = 3q_2$ and $\beta_2|Z_A = q_2\alpha$. We also note that a nonidentity element from the set $U_{-\beta_2} \cdot U_{-\beta_2-\beta_3} \cdot U_{-\beta_2-\beta_3-\beta_4}$ must occur in the factorization of some element in $Q_A - Q_A'$, since M_2 is nontrivial. However, $-\beta_2-\beta_3$ (respectively, $-\beta_2-\beta_3-\beta_4$) affords $T(L_A')$ weight $(q_1 + 2q_2)\mu_\beta = 5q_2\mu_\beta$ ($q_1\mu_\beta = 3q_2\mu_\beta$), contradicting $\beta_2|Z_A = q_2\alpha$.

Now suppose $(A,p) = (G_2,3)$ with $\Pi(L_Y) = \{\beta_1, \beta_3, \beta_4\}$ and q_1, q_2 as above. Consider the action of L_A' on the 25-dimensional restricted irreducible kY module, $V(\lambda_4)$. There is a 6-dimensional L_A' composition factor with high weight $(\lambda-\beta_2-\beta_3-\beta_4)|T(L_A)$. However, there is no L_A' module of dimension 25 affording such an L_A' composition factor.

Now suppose $\Pi(L_Y) = \{\beta_1, \beta_4\}$ and assume $(A,p) \neq (G_2,3)$. Then $Q_A \nleq K_{\beta_k}$, for k = 1,4, and we find that $\beta_2|Z_A = q_1\alpha$ and $\beta_4|Z_A = q_2\alpha$ (or vice versa). But this contradicts (2.8). If $(A,p) = (G_2,3)$, again consider the action of L_A' on $V(\lambda_4)$. There is a 4-dimensional L_A' composition factor with high weight $(\lambda-\beta_2-2\beta_3-\beta_4)|T(L_A')$. Now argue as before to produce a contradiction.

(3.6). Assume β is long unless $(A,p) = (G_2,3)$, in which case β is arbitrary. If $\langle \lambda,\beta \rangle \neq 0$, one of the following holds:

(a) $(A,p) \neq (G_2,3)$, $\Pi(L_\gamma) = \{\beta_1,\beta_3\}$, $\langle \lambda,\beta_1 \rangle \leq 2$, $\langle \lambda,\beta_2+\beta_3 \rangle = 0$.

(b) $(A,p) \neq (G_2,3)$, $\Pi(L_\gamma) = \{\beta_1,\beta_3\}$, $\langle \lambda,\beta_1 \rangle = 0 = \langle \lambda,\beta_2 \rangle$.

(c) $\Pi(L_\gamma) = \{\beta_1,\beta_4\}$, $\langle \lambda,\beta_1 \rangle \cdot \langle \lambda,\beta_4 \rangle = 0$ and $Q_A \nleq K_{\beta_2}$ and $Q_A \nleq K_{\beta_3}$.

(d) $(A,p) = (G_2,3)$, $\Pi(L_\gamma) = \{\beta_1,\beta_3\}$ and $\langle \lambda,\beta_1 \rangle \cdot \langle \lambda,\beta_3 \rangle = 0$.

(e) $\Pi(L_\gamma) = \{\beta_1,\beta_3,\beta_4\}$, $\lambda|T_\gamma = \lambda_4$, $(A,p) \neq (G_2,3)$.

(f) $\Pi(L_\gamma) = \{\beta_1,\beta_2,\beta_4\}$, $\lambda|T_\gamma = c\lambda_4$, $c \leq 2$, $(A,p) \neq (G_2,3)$.

(g) $\Pi(L_\gamma) = \{\beta_1,\beta_2,\beta_4\}$, $\lambda|T_\gamma = \lambda_2$, $(A,p) \neq (G_2,3)$.

(h) $(A,p) = (G_2,3)$, L_γ' has type A_1 and $\Pi(L_\gamma) \neq \{\beta_3\}$.

Proof: By (3.5), either (h) holds or L_γ' has type $A_1 \times A_1$ or $A_1 \times A_2$ with only one component acting nontrivially on $V^1(Q_\gamma)$.

Case 1: Suppose L_γ' is of type $A_1 \times A_1$.

Applying (1.15), we see that $\Pi(L_\gamma) \neq \{\beta_2,\beta_4\}$. Consider the case $\Pi(L_\gamma) = \{\beta_1,\beta_3\}$ and $(A,p) \neq (G_2,3)$. We claim that $\langle \lambda,\beta_2 \rangle = 0$. Otherwise, a nonidentity element from the root group $U_{-\beta_2}$ occurs in the factorization of some element of $Q_A - Q_A'$. But then $\beta_2|Z_A = q\alpha$, where $-\beta_2$ affords $T(L_A')$ weight $q\mu_\beta$, for some p-power q. This contradicts (2.16). Finally, the condition $\langle \lambda,\beta_1 \rangle \leq 2$ in (a) follows by considering the bound on $\dim V_{\beta_2}(Q_\gamma)$ given in (1.25) and the L_γ' composition factor in $V_{\beta_2}(Q_\gamma)$ afforded by $f_{\beta_1+\beta_2}v^+$. Now, if $\Pi(L_\gamma) = \{\beta_1,\beta_4\}$, $p > 2$, (2.10) and (2.11) imply that $Q_A \nleq K_{\beta_k}$ for $k = 2,3$. This completes the consideration of Case 1.

Case 2: Suppose L_γ' is of type $A_1 \times A_2$.

Assume for now that if $A = G_2$, then $p \neq 3$. If $\Pi(L_\gamma) = \{\beta_1,\beta_3,\beta_4\}$, (1.35), (1.36) and the bound on $\dim V_{\beta_2}(Q_\gamma)$ given in (1.25) imply that $\lambda|T_\gamma = \lambda_4$. If $\Pi(L_\gamma) = \{\beta_1,\beta_2,\beta_4\}$, (1.36) and the bound on $\dim V_{\beta_3}(Q_\gamma)$ imply that $\lambda|T_\gamma = \lambda_1$, λ_2 or $c\lambda_4$, for $c \leq 2$. Recall that $\lambda|T_\gamma \neq \lambda_1$.

It remains to consider the case where $A = G_2$, $p = 3$ and L_γ' has type $A_1 \times A_2$. Note that when $p = 3$, G_2 irreducibles have dimensions $7^k \cdot 27^\ell$ for

$k, \ell \in \mathbb{Z}^+$. Suppose $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_4\}$. If $\langle \lambda,\beta_1\rangle \neq 0$, (1.36) and the bound on $\dim V_{\beta_2}(Q_Y)$ of (1.25) imply $\langle \lambda,\beta_2\rangle = 0$. Since $\lambda|T_Y \neq \lambda_1$, we have $\lambda|T_Y = 2\lambda_1$. However, by [8], $\dim V|A \neq \dim V|Y$. Hence, $\langle \lambda,\beta_1\rangle = 0$. Again the bound on $\dim V_{\beta_2}(Q_Y)$ and (1.35) imply $\langle \lambda,\beta_2\rangle = 0$. So $\lambda|T_Y = \lambda_3$ or λ_4 and by [8] $\dim V|A \neq \dim V|Y$. So $\Pi(L_Y) \neq \{\beta_1,\beta_3,\beta_4\}$.

Finally, we note that if $(A,p) = (G_2,3)$, with $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_4\}$, then in the action of L_A' on the 25-dimensional irreducible kY module, $V(\lambda_4)$, there are no 1-dimensional L_A' composition factors. But every 25-dimensional kA module has 1-dimensional L_A' composition factors.

This completes the proof of (3.6).

(3.7). A is not of type A_2.

Proof: Suppose false; choose β such that $\langle \lambda,\beta\rangle \neq 0$. Then (2.9) rules out the configuration described in (c) of (3.6). Apply (1.23) in the remaining cases to obtain a precise description of $\lambda|T_A$. Then (1.26) and the methods of (1.30) and (1.32) imply $\dim V|A < \dim V|Y$ in the configurations of (a), (f) and (g). If $\lambda|T_Y = \lambda_4$ and $\lambda|T_A = q(2\mu_\alpha + 2\mu_\beta)$ as in (e), we must use (1.33) and Table 1 of [5] to see that the $\dim V|A \neq \dim V|Y$. Thus, we have reduced to $\lambda|T_Y = c\lambda_3 + x\lambda_4$, for $0<c<p$, $0 \leq x < p$ and $\lambda|T_A = q(c\mu_\alpha + c\mu_\beta)$, the configuration implied by (b) of (3.6). However, $\dim V^2(Q_Y) > \dim V^2(Q_A)$. Contradiction.□

(3.8). If A is of type B_2 and β is short, then $\langle \lambda,\beta\rangle = 0$.

Proof: Suppose false; i.e., suppose $\langle \lambda,\beta\rangle \neq 0$. Then (3.4), (3.5) and (7.1) of [12] imply that each component, L_i, of L_Y' has type A_{k_i} for some k_i and if $k_i > 1$, M_i is isomorphic to the natural module (or dual) for L_i. However, we also have $h_\beta(-1) \in Z(A) \leq Z(Y) = 1$; so in fact, L_Y' is quasisimple and by (3.5), $L_Y' = \langle U_{\pm\beta_1},U_{\pm\beta_2}\rangle$. In this case, $Q_A K_{\beta_3}/K_{\beta_3} = Q_Y/K_{\beta_3}$ and (2.3) and (2.13) imply $\langle \lambda,\beta_3+\beta_4\rangle = 0$. Thus, $\lambda|T_Y = \lambda_2$. Also, (3.6) implies $\lambda|T_A = 2q\mu_\beta + cq_0\mu_\alpha$, for some p-powers q and q_0 and c = 0 or 2. By (1.26) and [8], $\dim V|A \le 140 < \dim V|Y$. Contradiction.□

<u>(3.9).</u> A is not of type B_2.

<u>Proof</u>: Suppose false. Then (3.8) implies that if $\langle\lambda,\beta\rangle \ne 0$, then β is long. So (3.6) gives the possible Levi factors, L_Y, of P_Y. Using (1.26) to obtain an upper bound for $\dim V|A$, we see that (3.6)(b) or (c) holds. If (3.6)(b) holds, so $\lambda|T_Y = c\lambda_3 + x\lambda_4$, for $p>c>0$, $p>x\ge 0$ and $\lambda|T_A = cq\mu_\beta$ for q a p-power, then $\dim V^2(Q_Y) > \dim V^2(Q_A)$. Thus, (3.6)(c) holds.

Suppose $\lambda|T_Y = c\lambda_1 + x\lambda_2 + y\lambda_3$, for $c > 0$, $x,y \ge 0$ and $\lambda|T_A = cq\mu_\beta$, for q a p-power. We first claim that x = 0 = y. For otherwise, applying (2.3) and (2.13), we find that $\beta_2|T_A = q_0\alpha$ or $\beta_3|T_A = q_0\alpha$, for q_0 some p-power. Thus, $f_{\beta_2}v^+$ or $f_{\beta_3}v^+$ is a nonzero vector in $V_{T_A}(\lambda - q_0\alpha)$. But $\langle\lambda,\alpha\rangle = 0$. So we now have $\lambda|T_Y = c\lambda_1$. It requires an easy check to see that $\dim V^2(Q_Y) = \dim V^2(Q_A)$ in this case. Thus $[V,Q_A{}^2] = [V,Q_Y{}^2]$. Also, $\dim V^3(Q_A) \le 2c$. But if c > 1, $f_{2\beta_1+2\beta_2}v^+$ and $f_{\beta_1+\beta_2+\beta_3}v^+$ afford L_Y' composition factors in $V^3(Q_Y)$ of dimensions c−1 and 2c, respectively. Thus, c = 1. But then $\dim V|A < \dim V|Y$.

Thus, it remains to consider $L_Y' = \langle U_{\pm\beta_1}\rangle \times \langle U_{\pm\beta_4}\rangle$, $\lambda|T_Y = x\lambda_2 + y\lambda_3 + c\lambda_4$, $\lambda|T_A$ as above. But the same argument as above implies x = 0 = y and c = 1. So $\dim V|A < \dim V|Y$. Contradiction.

<u>(3.10).</u> Let A be of type G_2. Then $p \ne 3$.

<u>Proof:</u> Suppose false; choose $\beta \in \Pi(A)$ such that $\langle\lambda,\beta\rangle \ne 0$. Then (3.6) implies $L_Y' = \langle U_{\pm\beta_k}\rangle$ for k = 1, 2 or 4 or $L_Y' = \langle U_{\pm\beta_i}\rangle \times \langle U_{\pm\beta_j}\rangle$ for $\{i,j\} = \{1,3\}$ or $\{1,4\}$. In the latter case, $\langle\lambda,\beta_i\rangle = 0$ or $\langle\lambda,\beta_j\rangle = 0$. In particular, if $\langle\lambda,\beta_\ell\rangle \le 1$ for all ℓ, then $\dim V|A = 7$ or 49 and $\dim V|A \le 27^2$ in any case. Now [8] implies $\lambda|T_Y \ne \lambda_k$ for any k. Also, $\lambda|T_Y \ne 2\lambda_k$ for any k. For otherwise, [8] and (3.5) imply $\lambda|T_Y = 2\lambda_2$. But then (1.38) implies $\dim V|A < \dim V|Y$. Using (1.32) we see that $\dim V|Y > 49$. So there

exists $1 \le j \le 4$ with $\langle \lambda, \beta_j \rangle = 2$. Let W_0 be the stabilizer in W of λ. If rank $W_0 = 1$, counting only the conjugates of $V_{T_Y}(\lambda)$ and $V_{T_Y}(\lambda - \beta_j)$ (where $\langle \lambda, \beta_j \rangle = 2$) we see that $\dim V|Y > 27^2$. Thus, $\mathrm{rank}(W_0) = 2$. It is now a check to see that $\dim V|Y > 27^2$ in every case, completing the proof of (3.10).□

<u>(3.11).</u> Let A be of type G_2 with short ro β. If $\langle \lambda, \beta \rangle \neq 0$, then $L_Y' = \langle U_{\pm\beta_1} \rangle \times \langle U_{\pm\beta_3}, U_{\pm\beta_4} \rangle$, p = 7 and $\lambda|T_Y = \lambda_4$ or $\lambda|T_Y = 2\lambda_1$.

<u>Proof</u>: By (3.5) and (3.10), L_Y' is not a sir le algebraic group. Consider first the case where L_Y' has type A_1 A_2. If $\gamma \in \Pi(Y) - \Pi(L_Y)$, Q_Y/K_γ contains an L_A' composition factor iso orphic to a twist of $Q_A{}^\alpha$ only if the field twists on the embeddings of L_A' in the two components are equal. Also $p > 2$ implies $V_\gamma(Q_Y) \neq 0$ for $\gamma \in \Pi(Y) - \Pi(L_Y)$. Thus, (2.3), (2.5) and (2.6) imply that only one component acts nontrivially on $V^1(Q_Y)$.

Suppose $\Pi(L_Y) = \{\beta_1, \beta_3, \beta_4\}$. In the action of L_A' on the 26-dimensional kY-module with high weight λ_4, there will be a 5-dimensional composition factor. The only 26-dimensional kA module affording this is a conjugate of the irreducible kA module with high weight $2\mu_\beta$ when p = 7; hence, the prime restriction of the result. Continuing with L_Y' as above, using (1.36) and the bound on $\dim V_{\beta_2}(Q_Y)$ given in (1.25), we reduce to the configurations of the result.

Consider now the case where $\Pi(L_Y) = \{\beta_1, \beta_2, \beta_4\}$. Since $Q_A \nleq K_{\beta_3}$, the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_4} \rangle$ and in $\langle U_{\pm\beta_1}, U_{\pm\beta_2} \rangle$ are equal. Consider the action of L_A' on the 26- dimensional irreducible kY module, $V(\lambda_4)$. One checks that there are two 4-dimensional, three 3-dimensional, four 2-dimensional and one 1- dimensional L_A' composition factors. But there is no 26-dimensional kA module affording such an L_A' composition series. Hence $\Pi(L_Y) \neq \{\beta_1, \beta_2, \beta_4\}$.

It remains to show that L_Y' does not have type $A_1 \times A_1$. Since

$\dim(Q_A{}^{\alpha}) = 4$, $\Pi(L_Y) \neq \{\beta_1,\beta_4\}$. If $\Pi(L_Y) = \{\beta_2,\beta_4\}$ (respectively, $\{\beta_1,\beta_3\}$), $Q_A \leq K_{\beta_1}$ (respectively, K_{β_4}). But then $p > 2$, (2.10) and (2.11) produce a contradiction. This completes the proof of (3.11).

(3.12). Let A be of type G_2 with β long and α short. Then $\langle\lambda,\alpha\rangle \neq 0$.

Proof: Suppose $\langle\lambda,\alpha\rangle = 0$. Then $\langle\lambda,\beta\rangle \neq 0$ for β the long root. Recall, by (3.4) and (3.10), $p > 3$. Thus, (3.6) and (1.10) imply that $V|A$ is basic. Consider the configuration of (3.6)(c); the above remarks imply that $\beta_2|T_A = \alpha = \beta_3|T_A$, while $\beta_1|T_A = \beta = \beta_4|T_A$. So $\langle\lambda,\beta_2+\beta_3\rangle = 0$ since $\langle\lambda,\alpha\rangle = 0$. Now, $V_{T_A}(\lambda-2\beta_1-2\beta_2) \oplus V_{T_A}(\lambda-\beta_1-\beta_2-\beta_3-\beta_4) \oplus V_{T_A}(\lambda-\beta_2-\beta_3-2\beta_4) \oplus V_{T_A}(\lambda-2\beta_3-2\beta_4) \oplus V_{T_A}(\lambda-2\beta_1-\beta_2-\beta_3) \leq V_{T_A}(\lambda-2\alpha-2\beta)$. If $\langle\lambda,\beta\rangle > 1$ (so $\langle\lambda,\beta_1\rangle > 1$ or $\langle\lambda,\beta_4\rangle > 1$), the last weight space has dimension 2 while the sum of weight spaces in $V|Y$ has dimension 3. So $\langle\lambda,\beta\rangle = 1$ and $\dim V|A < \dim V|Y$. Thus, (3.6)(c) does not hold.

Using (1.30), (1.26) and [8], we may argue that $\dim V|A \neq \dim V|Y$ in ther configurations of (3.6) (a), (e), (f) and (g). Thus, the only possible configuration is as described in (3.6) (b); $L_Y' = \langle U_{\pm\beta_1}\rangle \times \langle U_{\pm\beta_3}\rangle$, $\lambda|T_Y = c\lambda_3 + x\lambda_4$, $c > 0$, $x \geq 0$ and $\lambda|T_A = c\mu_\beta$. However, $\dim V_{\beta_2}(Q_Y) + \dim V_{\beta_4}(Q_Y)$ exceeds the bound on $\dim V^2(Q_A)$. Contradiction.

Proof of (3.0)(b): Under the hypotheses of (3.0)(b), results (3.2) – (3.12) imply that $A = G_2$ and $p > 3$. Let $\beta \in \Pi(A)$ be short, so by (3.12), $\langle\lambda,\beta\rangle \neq 0$. As well, by (3.11), $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_4\}$, $p = 7$ and $\lambda|T_Y = \lambda_4$ or $\lambda|T_Y = 2\lambda_1$. In each case, $\langle\lambda,\beta\rangle = 2q$, for q a p-power. If $\lambda|T_Y = \lambda_4$, $\dim V|Y = 26$, so $\langle\lambda,\alpha\rangle = 0$, else the methods of (1.30), (1.32) and (1.35) imply $\dim V|A > \dim V|Y$. Then (1.10) implies the result of (3.0)(b). If $\lambda|T_Y = 2\lambda_1$, then $\langle\lambda,\alpha\rangle \neq 0$, else by (1.26) and [8] $\dim V|A < \dim V|Y$. Thus (3.6) implies that $\langle\lambda,\alpha\rangle = 2q_0$, for a p-power q_0. But then $\dim V|A \neq \dim V|Y$, by [8]. This completes the proof of (3.0)(b).□

CHAPTER 4: THE ONE COMPONENT THEOREM

Let $A < Y$ be simple algebraic groups, with Y simply connected, having root system of type E_n and with rank $Y >$ rank $A > 2$. In this chapter, we prove a result which is a useful tool in finding the configurations of the Main Theorem when rank $A > 2$. Throughout this chapter, we adopt Hypothesis and Notation (2.0), with the following additional restrictions: If A has type B_m, C_m or F_4, then $p>2$. The result we prove is the following:

Theorem 4.0. If, in addition to the above hypotheses, L_Y' is a simple algebraic group of type A_k or D_k, for some k, and if dim $V^1(Q_Y) > 1$, then the pair (A, L_A'), (Y, L_Y') is one of the following:

(i) (C_4, C_3), (E_n, A_5);

(ii) (C_4, A_3), (E_6, A_5);

(iii) (F_4, C_3), (E_6, A_5);

(iv) (F_4, B_3), (E_6, D_4);

(v) (C_5, C_4), (E_8, A_7); or

(vi) (C_3, C_2), (E_8, D_5), $p=5$, $L_Y' = \langle U_{\pm\beta_i} | 1 \le i \le 5 \rangle$, and $\lambda | T_Y = \lambda_1$.

Moreover, in each case, if $\rho: L_A \longrightarrow L_Y$ is the natural homomorphism, the pairs $\rho(L_A') \le L_Y'$ occur as natural embeddings of classical groups.

Remarks. 1. Under the hypotheses of Theorem (4.0), we see that $V_\gamma(Q_Y) \ne 0$ for all $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $(\gamma, \Sigma L_Y) \ne 0$. (See (1.24).)

2. The hypotheses of Theorem (4.0) and (1.5) imply that $Z_A \le Z_Y$.

(4.1). Under the hypotheses of Theorem (4.0), $L_A' \not\cong L_Y'$

Proof: Suppose false. Let q be the field twist on the embedding of L_A' in L_Y'. Then, since distinct L_Y'-irreducibles restrict to distinct L_A' irreducibles, $Q_Y/K_\gamma \cong (Q_A{}^\alpha)^q$ as L_A'-modules, for all $\gamma \in \Pi(Y)-\Pi(L_Y)$ such that $(\gamma,\Sigma L_Y) \neq 0$. In particular, if γ_1, γ_2 are two such roots, $-\gamma_1|(T_Y \cap L_Y') = -\gamma_2|(T_Y \cap L_Y')$. Also, (2.14) implies that there does not exist $\delta \in \Pi(Y)-\Pi(L_Y)$ with $(\delta,\Sigma L_Y) \neq 0$. Thus, L_Y' must be a maximal parabolic of Y. So rank A = rank Y. Contradiction.□

(4.2). Assume L_Y' has type A_k for some k, W is the natural module for L_Y', and $W|L_A' \cong (Q_A{}^\alpha)^q$ or $W^*|L_A' \cong (Q_A{}^\alpha)^q$ for some p-power q. If $\dim V^1(Q_Y) > 1$, then (4.0) (i) or (v) holds.

Proof: By (4.1), A does not have type A_m, for any m and if A is of type B_m, L_A' does not have type A_{m-1}.

(4.2.1) (A,L_A') is not of type (B_3,B_2).

Proof: Suppose false. By assumption, p>2 and $L_Y' = A_4$. Theorem (8.1) of [12] implies that there does not exist $\gamma \in \Pi(Y)-\Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \neq 0$ such that $Q_Y/K_\gamma \cong W\wedge W$ or $W^*\wedge W^*$. Hence $Y = E_7$ or E_8 and $Q_Y/K_\gamma \cong W$ or W^* for all $\gamma \in \Pi(Y)-\Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \neq 0$. So in particular, $Q_AK_\gamma/K_\gamma = Q_Y/K_\gamma$ for such γ. Thus, if q is the field twist on the embedding of L_A' in L_Y', (2.13) implies $\gamma|T_A = q\alpha$ for all $\gamma \in \Pi(Y)-\Pi(L_Y)$ such that $(\gamma,\Sigma L_Y) \neq 0$. In addition, (2.12) implies $\tau|T_A = 0$ for all $\tau \in \Pi(Y)-\Pi(L_Y)$ such that $(\tau,\Sigma L_Y) = 0$. Finally, note that for all $\beta \in \Pi(L_Y)$, $\beta|T_A = q\eta$ for some $\eta \in \Pi(L_A)$.

By Theorem (8.1) of [12], $V^1(Q_Y) \cong W$ (or W^*) or $V^1(Q_Y) \cong W\wedge W$ (or $W^*\wedge W^*$). Thus, $\langle\lambda,\alpha\rangle \neq 0$, else by (1.26) and (1.32), dim V|A < dim V|Y. So there exists $\gamma \in \Pi(Y)-\Pi(L_Y)$ such that $(\gamma,\Sigma L_Y) \neq 0$ and $\langle\lambda,\gamma\rangle \neq 0$, else there is no vector in V|Y with T_A weight $\lambda - q_0\alpha$. (See the preceeding work describing the restriction of β_i to T_A for $1 \le i \le n$.) These remarks, together with (2.13), imply that $\Pi(L_Y) = \{\beta_4,\beta_5,\beta_6,\beta_7\}$ with $\langle\lambda,\beta_2\rangle \neq 0$ if

$Y = E_7$ or $\langle\lambda,\beta_2+\beta_8\rangle \neq 0$ if $Y = E_8$. We now argue that $V^1(Q_Y) \cong W$ or W^* For if $V^1(Q_Y) \cong W\wedge W$ or $W^*\wedge W^*$, we first pass to the parabolic $P_Y\hat{}$ of (2.12), where $P_Y\hat{}$ has Levi factor $L_Y\hat{} = \langle L_Y, U_{\pm\beta_1}\rangle$. If $\langle\lambda,\beta_2\rangle \neq 0$, then $\langle\lambda,\beta_5\rangle = 1$, else $f_{\beta_2}v^+$ and $f_{3456}v^+$ afford $(L_Y\hat{})'$ composition factors of $V^2(Q_A)_{\lambda-q\alpha}$, exceeding the bound of (1.22). With $\langle\lambda,\beta_5\rangle = 1$, $f_{\beta_2}v^+$ affords an $(L_Y\hat{})'$ composition factor of $V^2(Q_A)_{\lambda-q\alpha}$ of dimension 40 unless p=3, in which case the composition factor has dimension 30. (See (1.34).) As well, $f_{345}v^+$ affords an $(L_Y')\hat{}$ composition factor of dimension 20. Hence, p=3. But then (1.34) implies that $\dim V_{T_Y}(\lambda-\beta_2-\beta_4-\beta_5) = 3$, while the multiplicity of this weight in the first composition factor mentioned is only 1. So the bound on $\dim V^2(Q_A)_{\lambda-q\alpha}$ is exceeded. We may argue similarly if $Y = E_8$ with $\langle\lambda,\beta_8\rangle \neq 0$. Hence, $V^1(Q_Y) \cong W$ or W^*.

Using (1.34) carefully, as above, we find that the bound on $\dim V^2(Q_A)_{\lambda-q\alpha}$ is again exceeded unless $Y = E_8$ with $\lambda|T_Y = \lambda_7+y\lambda_8$, for some $y>0$. Now there does not exist a vector in $V|Y$ with weight $\lambda-q_0\alpha$, for $q_0\neq q$, so $V|A$ is a conjugate of a restricted module and hence by (1.10), $\lambda|T_A = a\mu_1+\mu_2$ for some $0<a<p$. Since $((f_{\beta_8})^y)v^+ \in V_{T_A}(\lambda-y\alpha_1)$, $y \leq a$. But if $y<a$, there is no vector in $V|Y$ with T_A weight $\lambda-a\alpha_1$. So $y = a$. Now, let $L_0 = \langle U_{\pm\beta_k} \mid 3\leq k\leq 8\rangle$, a group of type A_6 with natural subgroup, B, of type B_3. We note that v^+ affords an L_0 composition factor of V which restricts to B to produce a composition factor with the same high weight as B_3 module as $V|A$. But L_0 lies in a proper parabolic of Y and hence acts reducibly on V. Thus, $\dim V|A < \dim V|Y$. Contradiction.

<u>(4.2.2).</u> (A,L_A') is not of type (D_4,A_3).

<u>Proof</u>: Suppose false; then $L_Y' = A_5$. Let $\Pi(L_A) = \{\alpha_1,\alpha_2,\alpha_3\}$, so $\Pi(A) - \Pi(L_A) = \{\alpha_4\}$. Then there does not exist $\gamma \in \Pi(Y)-\Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \neq 0$, $Q_Y/K_\gamma \cong W\wedge W$ (or $W^*\wedge W^*$). This follows from Theorem (8.1) of [12] if p > 2. If p=2, though $(W\wedge W)|L_A'$ is reducible, one checks that there is no L_A' composition factor of $W\wedge W$ isomorphic to a twist of $Q_A{}^\alpha$. Let q be the field twist on the embedding of L_A' in L_Y'. Consider

first the case where $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_4, \beta_5,\beta_6\}$. Then, $Q_Y/K_{\beta_2} \cong W\wedge W\wedge W$ as L_Y'-modules. One checks that the L_A' composition factors of $W\wedge W\wedge W$ have high weights $2q\mu_1$ and $2q\mu_3$. Thus, p=2, as $Q_A \nleq K_{\beta_2}$. So by the Main Theorem of [12], $V^1(Q_Y) \cong W$ or W^*. By (1.25), dim $V_{\beta_2}(Q_Y) \leq 36$, so $\langle\lambda,\beta_2\rangle = 0$. Moreover, (2.13) implies that $\langle\lambda,\beta_7\rangle = 0 = \langle\lambda,\beta_8\rangle$ if $Y = E_8$; and recall that $\lambda|T_Y \neq \lambda_1$ if $Y = E_7$. Thus, one of the following holds:

(a) $Y = E_6$, $\lambda|T_Y = \lambda_1$ (or λ_6).

(b) $Y = E_7$, $\lambda|T_Y = \lambda_1 + \lambda_7$ or $\lambda|T_Y = \lambda_6 + x\lambda_7$. (c) $Y = E_8$ and $\lambda|T_Y = \lambda_1$ or λ_6.

Referring to Table 1 of [5], we see that $\dim V|A = 8^k\cdot 26^\ell\cdot 160^m$, for $k,\ell,m \in \mathbb{Z}^+$. So [8] implies that (a) does not hold, if (b) holds $\lambda|T_Y = \lambda_6+\lambda_7$, and if (c) holds $\lambda|T_Y = \lambda_6$. By induction and Theorem (7.1) of [12], $\langle\lambda,\alpha_4\rangle = 0$, q_1, or q_1+q_2, for q_1 and q_2 distinct p-powers. But then $\dim V|A < \dim V|Y$, by (1.32) and (1.38). Thus, $L_Y' \neq \langle U_{\pm\beta_i} \mid i = 1,3,4,5,6\rangle$.

It remains to consider the case where $L_Y' = \langle U_{\pm\beta_i} \mid 4\leq i\leq 8\rangle$. Let q be as above. Then (2.12) and (2.4) imply $\beta_1|T_A = 0$ and $\beta_3|T_A = q\alpha_4 = \beta_2|T_A$. One checks that $\beta_4|T_A = q\alpha_2 = \beta_8|T_A$, $\beta_5|T_A = q\alpha_i = \beta_7|T_A$, for i = 1 or 3, respectively, and $\beta_6|T_A = q(\alpha_3 - \alpha_1)$ or $q(\alpha_1 - \alpha_3)$, respectively. Also, by (2.13), $\langle\lambda,\beta_3\rangle = 0$.

Now, Theorem (8.1) of [12] implies $V^1(Q_Y) \cong W$, W^*, $W\wedge W$, or $W^*\wedge W^*$ (the latter two only if $p\neq 2$). Thus, $\langle\lambda,\alpha_4\rangle \neq 0$, else (1.26) and (1.32) imply that $\dim V|A < \dim V|Y$. So, in particular, $\lambda - q_0\alpha_4$ is a T_A weight in $V|A$, for some p-power q_0. The above remarks imply that $\langle\lambda,\beta_2\rangle \neq 0$, else there is no vector in $V|Y$ with T_A weight $\lambda - q_0\alpha_4$.

We now argue carefully, using (1.34) and the parabolic $P_Y\hat{}$ of (2.12) (as in the proof of (4.3)), to see that the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_4}$ is exceeded in every configuration. This completes the proof of (4.2.2).

<u>(4.2.3).</u> (A, L_A') is not of type (C_3,C_2).

<u>Proof</u>: Suppose false; then $L_Y' = A_3$. Examining L_Y' composition

factors of $W \wedge W$, we see that there does not exist $\gamma \in \Pi(Y)-\Pi(L_Y)$ such that $(\gamma,\Sigma L_Y) \neq 0$ and $Q_Y/K_\gamma \cong W \wedge W$. Thus, $Q_Y/K_\gamma \cong W$ or W^* for all $\gamma \in \Pi(Y)-\Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \neq 0$. So $Q_Y/K_\gamma = Q_A K_\gamma/K_\gamma$ for all such γ. Let $\Pi(A) = \{\alpha_1,\alpha_2,\alpha_3\}$. Thus, if q is the field twist on the embedding of L_A' in L_Y', (2.13) implies $\gamma|T_A = q\alpha_1$ for all $\gamma \in \Pi(Y)-\Pi(L_Y)$ such that $(\gamma,\Sigma L_Y) \neq 0$. In addition, (2.12) implies $\tau|T_A = 0$ for all $\tau \in \Pi(Y)-\Pi(L_Y)$ such that $(\tau,\Sigma L_Y) = 0$. If $\Pi(L_Y) = \{r_1,r_2,r_3\}$ with $(r_i,r_{i+1}) < 0$, for $i = 1,2$, then $r_1|T_A = q\alpha_2 = r_3|T_A$ and $r_2|T_A = q\alpha_3$. Now, there exists at most one $\gamma \in \Pi(Y)-\Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \neq 0$ and $\langle \lambda,\gamma \rangle \neq 0$, else $\dim V_{T_A}(\lambda-q\alpha_1) > 1$, contradicting (1.31). So, in fact, $\langle \lambda,\alpha_1 \rangle = cq$ for some $0 \leq c < p$, as there does not exist a T_Y weight restricting to $\lambda - q_0\alpha_1$ for $q_0 \neq q$. So by (1.10), q=1.

The restriction $Q_Y/K_\gamma \not\cong W \wedge W$ implies $\Pi(L_Y)$ is (a) $\{\beta_1,\beta_3,\beta_4\}$, (b) $\{\beta_4,\beta_5,\beta_6\}$, (c) $\{\beta_5,\beta_6,\beta_7\}$, or (d) $\{\beta_6,\beta_7,\beta_8\}$. In (a), (b), (c), (d), respectively, $\langle \lambda,\beta_4 \rangle = 0$, $\langle \lambda,\beta_4 \rangle = 0$, $\langle \lambda,\beta_5 \rangle = 0$, $\langle \lambda,\beta_6 \rangle = 0$, respectively. Otherwise, $\dim V_{T_A}(\lambda-\alpha_1-\alpha_2) > 2$, contradicting (1.29).

If $\langle \lambda,\alpha_1 \rangle = 0$, then Theorem (8.1) of [12] implies $\lambda|T_A = k\mu_2$, for $0<k<p$ or $\lambda|T_A = a\mu_2 + b\mu_3$, where $a \neq 0 \neq b$ and $a+b = p-1$. In the first case, $\dim V_{T_A}(\lambda-\alpha_1-2\alpha_2-\alpha_3) \leq 3$, and in the second case, $\dim V_{T_A}(\lambda-\alpha_1-\alpha_2-\alpha_3) \leq 2$. (See (1.29).) Thus, the information in the above 2 paragraphs implies that $Y = E_6$, L_Y' is as in (a) or (b) and $\lambda|T_Y = k\lambda_1$ or $k\lambda_6$, respectively. However, this configuration is ruled out by a direct application of (1.23). Thus, $\langle \lambda,\alpha_1 \rangle \neq 0$.

Now, $V_{T_A}(\lambda-\alpha_1) \neq 0$ implies there exists $\gamma \in \Pi(Y)-\Pi(L_Y)$, with $(\gamma,\Sigma L_Y) \neq 0$, and $\langle \lambda,\gamma \rangle \neq 0$. Recall that there exists only one such γ. Applying these remarks, (2.13) and symmetry, we restrict still further to:

(A) $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_4\}$, $\langle \lambda,\beta_2 \rangle \neq 0$, $\langle \lambda,\beta_i \rangle = 0$, $i \geq 4$.

(B) $\Pi(L_Y) = \{\beta_4,\beta_5,\beta_6\}$, $Y = E_7$, $\langle \lambda,\beta_2+\beta_7 \rangle \neq 0$, $\langle \lambda,\beta_1+\beta_3+\beta_4 \rangle = 0$.

(C) $\Pi(L_Y) = \{\beta_5,\beta_6,\beta_7\}$, $Y = E_8$, $\langle \lambda,\beta_8 \rangle \neq 0$, $\langle \lambda,\beta_i \rangle = 0$, $1 \leq i \leq 4$.

(D) $\Pi(L_Y) = \{\beta_4,\beta_5,\beta_6\}$, $Y = E_8$, $\langle\lambda,\beta_2\rangle \neq 0$, $\langle\lambda,\beta_1+\beta_3+\beta_4+\beta_7+\beta_8\rangle = 0$.

Note, (1.23) rules out Y of type E_6, entirely.

By Theorem (8.1) of [12], and the above work, $\lambda|T_A = y\mu_1 + k\mu_2$, for $0<y,k<p$, or $\lambda|T_A = y\mu_1 + a\mu_2 + b\mu_3$, for $0<y,a,b<p$, $a+b = p-1$. We use (1.29) to check that $\dim V_{T_A}(\lambda-\alpha_1-\alpha_2) \leq 2$, $\dim V_{T_A}(\lambda-2\alpha_1-\alpha_2) \leq 2$, and $\dim V_{T_A}(\lambda-\alpha_1-\alpha_2-\alpha_3) \leq 4$. Also, $\dim V_{T_A}(\lambda-\alpha_1-2\alpha_2-\alpha_3) \leq 5$, if $\lambda|T_A = y\mu_1 + k\mu_2$.

Consider the configuration of (B) when $\lambda|T_Y = b\lambda_5 + a\lambda_6 + x\lambda_7$, $0<a,b,x<p$, $a+b = p-1$. Then $f_{245}v^+$, $f_{345}v^+$, $f_{1345}v^+$, $f_{\beta_7}f_{45}v^+$ and $f_{567}v^+$ are five linearly independent vectors in $V_{T_A}(\lambda-\alpha_1-\alpha_2-\alpha_3)$, contradicting the given bound. Using (1.34) to argue in this manner for each case, we reduce to Y of type E_7, $\lambda|T_Y = k\lambda_6 + x\lambda_7$, $\lambda|T_A = y\mu_1 + k\mu_2$, for $0<k,y,x<p$.

We now argue that x=y. Since $0 \neq (f_{\beta_7})^x v^+ \in V_{T_A}(\lambda- x\alpha_1)$, $x\leq y$. Moreover, there does not exist a vector in $V|Y$ with T_A weight $\lambda - z\alpha_1$ where $z>x$. For if $0\neq w$ is a T_Y weight vector with weight $\lambda - \Sigma c_i\beta_i$ and $c_7 > x$, then $c_6 > 0$. So $y\leq x$.

Let $X = \langle U_{\pm\beta_3},U_{\pm\beta_4},U_{\pm\beta_5},U_{\pm\beta_6},U_{\pm\beta_7}\rangle$. Then X contains a natural subgroup, $C \leq X$, of type C_3. The X composition factor of V afforded by v^+ has dimension strictly less than $\dim V|Y$, since X is contained in the Levi factor of a proper parabolic of Y. But the C composition factor of V afforded by v^+ has the same high weight as $V|A$, as C_3 module. Thus, $\dim V|A < \dim V|Y$. Contradiction. This completes the proof of (4.2.3).

<u>(4.2.4).</u> (A, L_A') is not of type (C_3,A_2).

<u>Proof</u>: Suppose false. By assumption, $p>2$ and $L_Y' = A_5$. By the Main Theorem of [12], L_A' acts irreducibly on $W\wedge W$ and on $W^*\wedge W^*$. One checks that $W\wedge W\wedge W$ has no 6-dimensional L_A' composition factor. Thus, $Y = E_8$ and $L_Y' = \langle U_{\pm\beta_i} \mid 4\leq i\leq 8\rangle$. Let q be the field twist on the embedding of L_A' in L_Y'.

Now, one checks that $Q_A K_{\beta_2}/K_{\beta_2} = Q_Y/K_{\beta_2}$ forces an embedding of L_A' in L_Y' which gives the following: $\beta_4|T_A = q\alpha_2 = \beta_5|T_A = \beta_7|T_A$, $\beta_6|T_A = q(\alpha_1 - \alpha_2)$ and $\beta_8|T_A = q\alpha_1$. Also, by (2.13), $\beta_2|T_A = q\alpha_3 = \beta_3|T_A$ and $\beta_1|T_A = 0$. Moreover, (2.13) also implies that $\langle\lambda,\beta_1+\beta_3\rangle = 0$. Thus, either $\langle\lambda,\alpha_3\rangle = 0$ or $\langle\lambda,\beta_2\rangle \neq 0$. If $\langle\lambda,\beta_2\rangle \neq 0$, we argue carefully using (1.34) (as in the proof of (4.2.1)) that the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_3}$ is exceeded in every configuration. Thus, $\langle\lambda,\alpha_3\rangle = 0$. But it is a straightforward check, using induction, (1.26) and (1.32), to see that $\langle\lambda,\alpha_3\rangle = 0$ implies $\dim V|A < \dim V|Y$. This completes the proof of (4.2.4).

It remains to consider the case where $L_Y' = A_7$ and (A,L_A') has type (D_5,D_4) or (F_4,B_3). In the first case, consideration of the quotient Q_Y/K_{β_2}, in view of Theorem (8.1) of [12], implies that $p = 2$ and $V^1(Q_Y) \cong W$ or W^*. But now in each case, the bound on $\dim V_{\beta_2}(Q_Y)$ implies $\lambda|T_Y = \lambda_8$, a contradiction.□

<u>(4.7).</u> Suppose L_Y' is of type A_k, with natural module W, and $(Q_A{}^{\alpha})^q \not\cong W$ or W^*, as L_A' modules, for any p-power q. If $\dim V^1(Q_Y) > 1$, then Theorem (4.0) (ii) or (iii) holds.

<u>Proof</u>: Since L_A' acts irreducibly on $W \not\cong (Q_A{}^{\alpha})^q$, (2.3) implies that there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$, with $(\gamma,\Sigma L_Y) \neq 0$, such that $Q_Y/K_\gamma \cong W$ or W^*. In particular, we have $k \geq 4$. In fact, $k>4$. For otherwise, since $\operatorname{rank} A > 2$, L_A' must be of type B_2 $(= C_2)$ in order to have a 5-dimensional irreducible representation. Moreover, since $W \not\cong (Q_A{}^{\alpha})^q$, $A = C_3$. However, there exists $\gamma \in \Pi(Y)-\Pi(L_Y)$, with $(\gamma,\Sigma L_Y) \neq 0$, such that $Q_Y/K_\gamma \cong W\wedge W$ or $W^*\wedge W^*$, a 10-dimensional irreducible L_A' module. (Recall $p \neq 2$ when $A = C_3$.) Thus $Q_A \leq K_\gamma$, contradicting (2.3).

Consider the case where $L_Y' = A_5$. The existence of a 6-dimensional irreducible L_A'-module not isomorphic to a twist of $Q_A{}^{\alpha}$ implies (A,L_A') is of type (A_3,A_2), (B_3,A_2), (A_4,A_3), (B_4,A_3), (C_4,A_3), or (F_4,C_3). If $L_A' = A_2$, then $p>2$. For the pairs (A_3,A_2), (A_4,A_3), and (B_3,A_2),

$\dim Q_A^\alpha < \dim W$. But the Main Theorem of [12] implies that L_A' acts irreducibly on $W \wedge W$ and on $W^* \wedge W^*$, unless p=2 and $L_A' = A_3$. Even in the latter case, the L_A' composition factors of $W \wedge W$ have dimensions 14 and 1. Hence, there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma, \Sigma L_Y) \neq 0$ and $Q_Y/K_\gamma \cong W \wedge W$ or $W^* \wedge W^*$. Also, by the Main Theorem of [12], $V^1(Q_Y) \cong W$, W^*, $W \wedge W$ or $W^* \wedge W^*$. But now a direct application of (1.23) rules out all possible configurations.

For (A, L_A') of type (B_4, A_3), the preceeding remarks imply $L_Y' = \langle U_{\pm\beta_i} \mid i = 1,3,4,5,6\rangle$. Then, $Q_Y/K_{\beta_2} \cong W \wedge W \wedge W$, as L_Y'-modules. The L_A' composition factors of $W \wedge W \wedge W$ have high weights $2\mu_1$ and $2\mu_3$. But since p>2, this implies $Q_A \leq K_{\beta_2}$, contradicting (2.3). In the (F_4, C_3) and (C_4, A_3) cases, there does not exist an L_A' composition factor of $W \wedge W$ isomorphic to a twist of Q_A^α. Thus, there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma, \Sigma L_Y) \neq 0$ and $Q_Y/K_\gamma \cong W \wedge W$ or $W^* \wedge W^*$. So $Y = E_6$ and we have the result.

Consider now the case where $L_Y' = A_6$. Since W is a 7-dimensional irreducible L_A'-module ($\not\cong (Q_A^\alpha)^q$), either $L_A' = A_2$ and p=3, or (A, L_A') is of type (F_4, B_3). In the latter case, Theorem (8.1) of [12] produces a contradiction to (2.3). So we consider the case where $L_A' = A_2$ and p=3. Let $\Pi(L_A) = \{\alpha_1, \alpha_2\}$. By (1.25), for $\gamma \in \Pi(Y) - \Pi(L_Y)$, $\dim V_\gamma(Q_Y) \leq 21$, if $A = A_3$ or B_3, and $\dim V_\gamma(Q_Y) \leq 42$, if $A = C_3$. This restriction implies that either (a) $Y = E_7$ with $\lambda|T_Y = \lambda_7$ or (b) $Y = E_8$ with $\lambda|T_Y = \lambda_2$ and $A = C_3$. (Note that we used the methods of (1.30), (1.32), and (1.35) to find a lower bound for $\dim V_\gamma(Q_Y)$.) In each case, $\lambda|T_A = q(\mu_1 + \mu_2) + x\mu_3$, for some $x \geq 0$ and some p-power q. If $A = A_3$, (1.23) implies that $x = q \cdot 1$. In the configuration of (a), $\dim V|Y = 56$. However, referring to Table 1 of [5], and using the methods of (1.32) and (1.30), we see that $\dim V|A \neq 56$. In the configuration of (b), $Q_Y/K_{\beta_3} \cong W \wedge W$. But $W \wedge W$ has L_A' composition factors with high weights $3q\mu_1$, $3q\mu_2$, $q(\mu_1+\mu_2)$ and 0. (We have used (2.6) to identify q with the field twist on the embedding of L_A' in L_Y'.)

Thus, there is no L_A' composition factor of Q_Y/K_{β_3} isomorphic to a twist of $Q_A{}^{\alpha_3}$. But this contradicts (2.3) and (2.4). Thus, L_Y' does not have type A_6.

It remains to consider the case where $L_Y' = A_7$. Since W is an 8-dimensional irreducible L_A'-module ($\not\cong (Q_A{}^{\alpha})^q$), $L_A' = A_2$ or D_4, or (A,L_A) has type (B_4,B_3). Note that $V^1(Q_Y) \not\cong W$ or W^*. For otherwise the bound on $\dim V_{\beta_2}(Q_Y)$ of (1.25) implies that $\lambda|T_Y = \lambda_8$. Thus, the Main Theorem of [12] implies that $A = D_5$, $L_A' = D_4$, and $p>2$. However, then L_A' acts irreducibly on the 56-dimensional L_Y' module Q_Y/K_{β_2}. But this implies $Q_A \le K_{\beta_2}$, contradicting (2.3). Thus, L_Y' does not have type A_7.

This completes the proof of (4.7).□

<u>(4.8).</u> If L_Y' is of type D_k, for some $k\ge 4$ and $\dim V^1(Q_Y) > 1$, then Theorem (4.0) (iv) or (vi) holds.

<u>Proof</u>: Let $\mu_1, \ldots, \mu_k$ be the fundamental dominant weights for D_k. Then we will call $W = V(\mu_1)$ the natural module for L_Y'.

Suppose L_A' acts irreducibly on some L_Y' module other than W. Then the Main Theorem of [12] implies that L_A' acts irreducibly on the L_Y'-modules with high weights μ_{k-1} and μ_k. In every case, there exists $\gamma \in \Pi(Y)-\Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \ne 0$ such that the irreducible L_Y'-module Q_Y/K_γ has high weight μ_{k-1} or μ_k. Thus, $Q_A{}^{\alpha}$ is isomorphic to the L_A' irreducible afforded by μ_{k-1} or μ_k. The Main Theorem of [12] then implies that $A = F_4$, $L_A' = B_3$, and $L_Y' = D_4$.

With (A,L_A',L_Y') of type (F_4,B_3,D_4), suppose $Y = E_7$ or E_8. Let q be the field twist on the embedding of L_A' in L_Y'. Then, by (2.13) and (2.6), $\beta_1|T_A = q\alpha_4 = \beta_6|T_A$, $\beta_k|T_A = 0$, and $\langle\lambda,\beta_k\rangle = 0$ for $k>6$. Also, since the embedding of L_A' in L_Y' is the natural embedding of classical groups, $\beta_2|T_A = q\alpha_1$, $\beta_4|T_A = q\alpha_2$ and $\beta_3|T_A = q\alpha_3 = \beta_5|T_A$. We claim that $\langle\lambda,\alpha_4\rangle = \langle\lambda,\beta_1\rangle$. Certainly, $x = \langle\lambda,\beta_1\rangle \le \langle\lambda,\alpha_4\rangle$, as $0 \ne (f_{\beta_1})^x v^+ \in V_{T_Y}(\lambda - xq\alpha_4)$. Also, $\langle\lambda,\beta_1\rangle \ge \langle\lambda,\alpha_4\rangle = y$. For otherwise, there is no vector in $V|Y$ with T_A weight $\lambda - yq\alpha_4$. Now, let $X = \langle U_{\pm\beta_i} | 1 \le i \le 6 \rangle$. Then, X contains a natural subgroup of type F_4, say $A_0 \le X$. Moreover, the X composition factor of V afforded by v^+ is not all of V. But the A_0 composition factor of V afforded by v^+ has the same dimension as $V|A$. Thus, $\dim V|A < \dim V|Y$. Hence, if (A, L_A', L_Y') has type (F_4, B_3, D_4), $Y = E_6$ and the result holds.

Now consider the case where $V^1(Q_Y) \cong W$ and L_A' acts reducibly on every nontrivial, restricted L_Y' module other than W. The Main Theorem of [12] then implies that the triple (L_A', L_Y', p) is one of (A_2, D_4, p), $(B_2, D_5, 5)$, $(B_2, D_7, 3)$, $(C_3, D_7, 3)$ or $(C_3, D_7, 7)$. We first note that L_Y' does not have type D_7. For otherwise, the bound on $\dim V_{\beta_1}(Q_Y)$ implies $\lambda|T_Y = \lambda_8$. As well, (L_A', L_Y') does not have type (A_2, D_4); for one checks that L_A' acts irreducibly on each of the 8-dimensional irreducible L_Y'-modules. So $Q_A \le K_{\beta_1}$, contradicting (2.3). Thus, $L_A' = B_2$ and $L_Y' = D_5$, with $p=5$. For $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that Q_Y/K_γ is one of the restricted irreducible spin modules for L_Y', Q_Y/K_γ has L_A' composition factors of dimensions 12 and 4. Hence, (2.3) implies that $A = C_3$. Also, by (1.25), $\dim V_\gamma(Q_Y) \le 40$, so $\langle \lambda, \gamma \rangle = 0$. An application of (1.23) then rules out Y of type E_6. Moreover, $L_Y' = \langle U_{\pm\beta_i} | 1 \le i \le 5 \rangle$, else Q_Y/K_{β_7} is a 10-dimensional irreducible L_A' module containing a nontrivial image of $Q_A{}^\alpha$. The above remarks and (2.3) imply $\lambda|T_Y = \lambda_1$. Thus, $Y = E_8$ and the result holds.

This completes the proof of (4.8).□

The proof of Theorem (4.0) is now complete.

CHAPTER 5: RANK(A) $\geq$ 3

In this chapter we establish the Main Theorem under the following conditions: Y has type E_n and rank A > 2. We adopt Notation and Hypothesis (2.0) throughout the chapter. Note that Theorem (4.1) of [12] implies rank A < rank Y. Our result is the following.

Theorem (5.0). If $V|A$ is irreducible, then $Y = E_6$ and one of the following holds:

(i) A is the fixed point subgroup of the graph automorphism of Y, so A has type F_4, and $\lambda|T_Y = (p-3)\lambda_1$ or $(p-3)\lambda_6$, for p>3, or $\lambda|T_Y = \lambda_1 + (p-2)\lambda_3$ or $\lambda_6 + (p-2)\lambda_5$, for p>2. Moreover, with A and V as described, $V|A$ is irreducible.

(ii) p>2, A has type C_4, $\lambda|T_A = \mu_2$ and $\lambda|T_Y = \lambda_1$ or λ_6.

Moreover, if p>2, $Y = E_6$ and B is the fixed point subgroup of the automorphism τi_x, where i_x is the inner automorphism associated with $x = h_{\beta_1+2\beta_2+2\beta_3+3\beta_4+2\beta_5+\beta_6}(-1)$, then B has type C_4, and $V(\lambda_1)|B$ and $V(\lambda_6)|B$ are irreducible with high weight μ_2.

Proof of existence statement in (ii): (due to G. Seitz) It is a check to see that the fixed point subgroup of the automorphism τi_x is $B = \langle x_{\pm\beta_1}(t)x_{\pm\beta_6}(t), x_{\pm\beta_3}(t)x_{\pm\beta_5}(t), x_{\pm\beta_4}(t),$ $x_{\pm(\beta_2+\beta_3+\beta_4)}(t)x_{\pm(\beta_2+\beta_4+\beta_5)}(-t) \mid t \in k\rangle$, that this group has type C_4 and that $V(\lambda_1)$ and $V(\lambda_6)$ have a B composition factor with high weight μ_2. But then, p>2 and Table 1 of [5] imply that this composition factor has dimension 27 and so $V(\lambda_1)|B$ and $V(\lambda_6)|B$ are irreducible.□

(5.1) Assume there are no examples (B,Y,W) satisfying the hypotheses of the Main Theorem with Rank(B) = 3. Also, assume (A,p) is not special. If L_A' is of type A_3 and $\dim V^1(Q_A) > 1$, then L_Y' is a simple algebraic group.

Proof. Suppose false; i.e., suppose L_Y' has more than one component. Let $\Pi(L_A) = \{\alpha_1,\alpha_2,\alpha_3\}$, so $\alpha = \alpha_4$. Then by rank restrictions L_Y' has 2 components. In fact L_Y' has type $A_3 \times A_3$, since P_Y is minimal and A_3 has no 5-dimensional irreducible representation. Thus, (1.5) implies $Z_A \leq Z_Y$. By (2.7), we may assume the field twist on the embedding of L_A' in each component of L_Y' is q, for some p-power q.

Consider the case where $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_4,\beta_6,\beta_7,\beta_8\}$. The L_A' composition factors of Q_Y/K_{β_5} have high weights among $\{q(\eta_1+\eta_3),0,2q\eta_1, 2q\eta_3, q\eta_2\}$, where η_1, η_2, η_3 represent the fundamental dominant weights of A_3, labelled as throughout. Since there is a nontrivial image of $Q_A{}^\alpha$ in Q_Y/K_{β_5}, A = D_4 or C_4, or p=2 and A = A_4. Then (1.23) forces $\dim V|A < \dim V|Y$ if p=2 and A has type A_4. Thus, A = D_4 or C_4. Now, $\dim(Q_Y/K_{\beta_2}) < \dim Q_A{}^\alpha$, so $Q_A \leq K_{\beta_2}$. But (1.33) implies that $q\eta_i$ and 0 cannot be high weights of an indecomposable L_A' module, for i = 1 or 3. Hence, we may assume that $-\beta_2$ is not involved in L_A'. So (2.11) implies that there is a nontrivial image of $Q_A{}^\alpha$ in $Q_Y(\beta_5,\beta_2)$. However, the L_A' composition factors of this L_Y' irreducible have high weights among $\{q(\eta_1+\eta_2), q(\eta_2+\eta_3), q\eta_1, q\eta_3\}$. Thus, $\Pi(L_Y) \neq \{\beta_1,\beta_3,\beta_4,\beta_6,\beta_7,\beta_8\}$.

We now have $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_4,\beta_6,\beta_7,\beta_8\}$. The L_A' composition factors of Q_Y/K_{β_5} have high weights among $\{q(\eta_1+\eta_2), \eta_1, q(\eta_2+\eta_3), q\eta_3\}$. Since (2.3) implies that there is a nontrivial image of Q_A in Q_Y/K_{β_5}, A = A_4 or B_4. As above, if $Q_A \leq K_{\beta_1}$ then $-\beta_1$ is not involved in L_A'. But there is no 4-dimensional L_A' composition factor of $Q_Y(\beta_5,\beta_1)$. Hence (2.11) implies $Q_A \nleq K_{\beta_1}$. Since each of the irreducible L_Y' modules Q_Y/K_{β_5} and Q_Y/K_{β_1} must have an L_A' composition factor isomorphic to a twist of $Q_A{}^{\alpha_4}$, the embedding of $T(L_A')$ in $T(L_Y')$ is as follows: $h_{\alpha_1}(c) = h_{\beta_2}(c^q)\cdot h_{\beta_6}(c^q)$, $h_{\alpha_2}(c) = h_{\beta_4}(c^q)\cdot h_{\beta_7}(c^q)$ and $h_{\alpha_3}(c) = h_{\beta_3}(c^q)\cdot h_{\beta_8}(c^q)$. Considering the $T(L_A')$ weight vectors in Q_Y/K_{β_5}, we see that $x_{-\alpha_4}(t) = x_{-245}(c_1t^q)x_{-567}(c_2t^q)x_{-456}(c_3t^q)w$, where $c_i \in k$, some c_i nonzero, and $w \in K_{\beta_5}$. So $\beta_5|T_A = q(\alpha_4-\alpha_1-\alpha_2)$. This implies $\langle\lambda,\beta_k\rangle = 0$ for $3\le k\le 6$, else $(V_{T_Y}(\lambda-q\alpha_4+q\alpha_1+q\alpha_2) \oplus V_{T_Y}(\lambda-q\alpha_4+q\alpha_1) \oplus V_{T_A}(\lambda-q\alpha_4+q\alpha_1) \oplus V_{T_A}(\lambda-q\alpha_4-q\alpha_3+q\alpha_1)) \neq 0$. Also, (2.13) implies $\beta_1|T_A = q\alpha_4$.

Suppose A is of type B_4, so $p>2$. Then $\langle\lambda,\alpha_i\rangle = 0$ for $i=2,3,4$, else rank restrictions imply there is a parabolic subgroup of Y, P_0, containing the B_3 parabolic of A with containment of unipotent radicals and such that the Levi factor of P_0 is a simple algebraic group. However, Theorem (4.0) and the induction hypothesis imply that no such configuration occurs. Thus $\lambda|T_A = q(a\mu_1)$ for some $0<a<p$. Comparing this with the information in the preceeding paragraph, we have $\lambda|T_Y = x\lambda_1+a\lambda_2$, for some $x\ge0$. However, $0 \neq f_{\beta_2+\beta_4+\beta_5}v^+ \in V_{T_A}(\lambda-q\alpha_4)$. Contradiction.

So we have reduced to the case where $A = A_4$. Applying (1.23) and (1.10) and the above remarks we find that either $\lambda|T_A = a\mu_1+a\mu_4$ and $\lambda|T_Y = x\lambda_1+a\lambda_2$, for some $a>0$, $x\ge0$ or $\lambda|T_A = b\mu_2+b\mu_3$ and $\lambda|T_Y = y\lambda_1+b\lambda_7+b\lambda_8$, for some $b>0$, $y\ge0$. In the first case, $a > 1$, else $\dim V|A < \dim V|Y$. But then $f_{245}f_2v^+$ and $f_{2456}v^+$ are linearly independent vectors in $V_{T_A}(\lambda-\alpha_1-\alpha_4)$, which is a 1-dimensional weight space. In the second case, $0 \neq f_{567}v^+ \in V_{T_A}(\lambda-\alpha_4)$, contradicting $\langle\lambda,\alpha_4\rangle = 0$.

This completes the proof of (5.1).□

<u>(5.2).</u> Let (A,Y,V) be as in the Main Theorem with Y of type E_n and rank$A > 3$. Assume there are no examples (B,Y,W) satisfying the hypotheses of the Main Theorem with rank(B) = 3. Also assume (A,p) is not special. Then Y has type E_6 and one of the following holds:

(i) A is of type C_4, so $p>2$, $\lambda|T_A = \mu_2$, and $\lambda|T_Y = \lambda_1$ (or λ_6).

(ii) A is of type F_4.

Proof: Choose P_A such that $\dim V^1(Q_A) > 1$. Consider first the case where rankA = 4. By (5.1) and size restrictions, L_Y' has one component. Then, the induction hypothesis implies that Theorem (4.0) applies. In particular, (1.5) implies $Z_A \le Z_Y$.

Consider the case where $A = C_4$ and $L_A' = A_3$, as in (4.0) (ii). Then $Y = E_6$ and $L_Y' = A_5$. The embedding of L_A' in L_Y' is the natural embedding of A_3 in A_5. The Main Theorem of [12] implies that one of the following holds:

(a) $\lambda|T_Y = \lambda_1+x\lambda_2$ or $\lambda_6+x\lambda_2$, for $0\le x<p$ and $\lambda|T_A = q\mu_2+y\mu_4$, where q is some p-power and $y\ge 0$.

(b) $\lambda|T_Y = x\lambda_2+\lambda_3$ or $x\lambda_2+\lambda_5$ and $\lambda|T_A = q\mu_1+q\mu_3+y\mu_4$, where x, q and y are as in (a).

In either case, the C_3 parabolic of A acts nontrivially on $\langle v^+\rangle$. Thus, Theorem (4.0) and the induction hypothesis imply that the C_3 parabolic of A is contained in a conjugate of P_Y, with containment of unipotent radicals. Theorem (8.1) in [12] rules out the configuration of (b) and forces $y = 0$ in the configuration of (a). Thus $V|A$ is a conjugate of a basic module (recall that $p>2$ when $A = C_4$), and so by (1.10), $V|A$ has high weight μ_2. Moreover $x=0$, else (1.26) and (1.32) imply that $\dim V|A < \dim V|Y$. Thus $\lambda|T_Y = \lambda_1$ and we have the configuration of (i) in the statement of the result.

Consider now the case where $A = C_4$, $L_A' = C_3$, $Y = E_n$ and $L_Y' = A_5$, as in (4.0) (i). The embedding of L_A' in L_Y' is the natural embedding of C_3 in A_5. Theorem (8.1) of [12] implies that the A_3 parabolic of A acts nontrivially on $\langle v^+\rangle$, so we may reduce to the first case. Thus, if rankA=4, (i) or (ii) holds.

Suppose rankA = 5 and consider the case where L_Y' is a simple algebraic group of classical type; so by (4.0), $A = C_5$, $L_A' = C_4$, and

$L_Y' = A_7$, as in (4.0) (v). The embedding of L_A' in L_Y' is the natural embedding of classical groups. By (8.1) of [12], the A_4 parabolic of A acts nontrivially on $\langle v^+\rangle$. Thus, size restrictions and the preceeding two paragraphs imply that Theorem (4.0) applies. But there are no examples (C_4,A_4), (Y,L_Y') in Theorem (4.0). Thus, if rankA = 5, $L_Y' = E_6$ or E_7. The only possible configuration, given inductively by the preceeding two paragraphs would be: $A = C_5$, $L_A' = C_4$, and $L_Y' = E_6$. However, Q_Y/K_{β_7} is a 27-dimensional irreducible L_A' module on which Z_A induces scalars. But then, $Q_AK_{\beta_7}/K_{\beta_7}$ is an 8-dimensional L_A' submodule of Q_Y/K_{β_7}, by (2.4). Thus rankA $\neq 5$. Also, by induction and Theorem (4.0), rank A $\neq 6,7$.

This completes the proof of (5.2).□

<u>(5.3).</u> Suppose (A,Y,V) are as in the Main Theorem and the pair (A,Y) has type (F_4,E_6), with $p>2$. Then V is a basic module for A. Moreover, A is the fixed point subgroup under the graph automorphism of Y.

<u>Proof</u>: Let P_1 (respectively, P_2) be the maximal parabolic of A, containing B_A^-, corresponding to the simple root α_4 (respectively, α_1). If $P_i = L_iQ_i$ is the Levi decomposition for P_i, then $L_1' = B_3$ and $L_2' = C_3$. Rank restrictions imply that Theorem (4.0) applies whenever $\dim V^1(Q_i) > 1$. In particular, Theorem (8.1) in [12] implies that $V^1(Q_i)$ is a tensor indecomposable L_i' module. Thus, if V|A is tensor decomposable, the above remarks and (1.7) imply that $\lambda|T_A = q_1a\mu_1+q_2b\mu_4$, where $p>a,b>0$ and q_1 and q_2 are distinct p-powers. However, Theorem (8.1) of [12] implies that $V^1(Q_1)$ cannot have high weight $(q_1a\mu_1)|T(L_1')$. Thus, V|A is a tensor indecomposable module. Then $p>2$ and (1.10) imply that V|A is basic. Hence Proposition (2.8) of [12] holds.

Let P_Y^1 and P_Y^2 be as in Proposition (2.8) of [12]. That is, if $P_Y^i = L_Y^iQ_Y^i$ is the Levi decomposition of P_Y^i, then $P_i \leq P_Y^i$, $Q_i \leq Q_Y^i$, $L_i \leq L_Y^i$ and $Z_i = Z(L_i)° \leq Z_Y^i = Z(L_Y^i)°$ for i=1,2. Moreover, the fixed maximal torus T_Y is contained in L_Y^i. Now V|Y nontrivial implies that

$\dim V^1(Q_i) > 1$ for $i = 1$ or 2. Induction and the Main Theorem of [12] imply $\dim V^1(Q_2) > 1$ in every possible configuration, and that $(L_Y{}^2)'$ has type A_5.

Choose a base $\Pi(Y) = \{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\}$ of $\Sigma(Y)$, labelled as throughout, such that $\Pi((L_Y{}^2)) = \{\beta_1,\beta_3,\beta_4,\beta_5,\beta_6\}$ and $Q_Y{}^2 = \langle U_r \mid r \in \Sigma^-(Y) - \Sigma((L_Y{}^2)')\rangle$. Then, $x_{\alpha_2}(t) = x_{\beta_4}(t)$, $x_{\alpha_3}(t) = x_{\beta_3}(t)x_{\beta_5}(t)$ and $x_{\alpha_4}(t) = x_{\beta_1}(t)x_{\beta_6}(t)$. Also, examining the $T(L_2{}')$ weights in $Q_Y{}^2/K_{\beta_2}$, we see that $x_{-\alpha_1}(t) = x_{-\beta_2}(at)u$, where $a \in k^*$, $u \in K_{\beta_2}$.

We claim that $(L_Y{}^1)'$ has type D_4. This follows from the Main Theorem and induction, if $\dim V^1(Q_1) > 1$. So suppose $\dim V^1(Q_1) = 1$. Then, $\lambda|T_Y = a\lambda_1 + x\lambda_2$ and $\lambda|T_A = a\mu_4$. In fact, $x=0$ as there is no nontrivial embedding of B_3 in A_4. If $(L_Y{}^1)'$ is not of type D_4, $(L_Y{}^1)'$ is a conjugate of $\langle U_{\pm\beta_2}, U_{\pm\beta_3}, U_{\pm\beta_4}, U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$, and $\dim V^2(Q_Y{}^1) \geq 16$. However, (1.25) implies $\dim V^2(Q_1) \leq 8$. Thus, $(L_Y{}^1)'$ has type D_4, as claimed. Also, $L_1{}' \leq (L_Y{}^1)'$ must be the natural embedding of classical groups.

So there exists $\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\} \subseteq \Sigma(Y)$ such that $(L_Y{}^1)' = \langle U_{\pm\gamma_1}, U_{\pm\gamma_2}, U_{\pm\gamma_3}, U_{\pm\gamma_4}\rangle$ is of type D_4 and such that $x_{-\alpha_1}(t) = x_{-\gamma_1}(b_1t)$, $x_{\alpha_2}(b_2t) = x_{\gamma_2}(t)$ and $x_{\alpha_3}(t) = x_{\gamma_3}(b_4t)x_{\gamma_4}(b_5t)$, for some $b_i \in k^*$. Comparing this with the known information about the factorization of these elements, we see that $\gamma_1 = \beta_2$, $\gamma_2 = \beta_4$, $\{\gamma_3,\gamma_4\} = \{\beta_3,\beta_5\}$ and $b_4 = b_5$. Then, A is in fact the fixed point subgroup under the graph automorphism of Y.□

In the following few results, we will determine on which modules V, the fixed point subgroup of the graph automorphism of $Y = E_6$ acts irreducibly. We are forced to do tedious calculations within the universal enveloping algebra of L(Y). Thus, it is necessary to know the structure constants of L(Y); i.e., we need a set of consistent signs for the commutator relations among elements of a Chevalley basis of L(Y). (Since all root strings in $\Sigma(Y)$ have length 1, the constants involved are either 1 or −1.) In Section 4.2 of [6], a method for constructing a set of structure constants is described. It involves choosing a set of "extraspecial" pairs (r,s) of roots. The structure constants, $N_{r,s}$, where $[e_r,e_s] = N_{r,s}e_{r+s}$, may be chosen arbitrarly for these pairs. Then, using Theorem 4.1.2 of [6], one generates the remaining structure constants.

The set of extraspecial pairs is chosen by first fixing a total ordering on the space $\mathcal{V}$ containing the roots. We do this as follows. Let $v_1 = \beta_1$, $v_2 = \beta_3$, $v_3 = \beta_4$, $v_4 = \beta_2$, $v_5 = \beta_5$, $v_6 = \beta_6$. Then, we say $0 \prec \Sigma c_i v_i$ if and only if the first nonzero coefficient c_i is positive. An ordered pair (r,s) is said to be special if $r+s \in \Sigma(Y)$ and $0 \prec r \prec s$. An ordered pair (r,s) is said to be extraspecial if (r,s) is special and if for all special pairs (r_1,s_1) with $r+s = r_1+s_1$, we have $r \preceq r_1$. Then, every root in $\Sigma^+(Y)$ which is the sum of two roots in $\Sigma^+(Y)$ can be uniquely expressed as a sum of an extraspecial pair. We choose $N_{r,s} = 1$ for all extraspecial pairs.

(5.4). Let (A,Y,V) be as in (5.3), so p>2. Then one of the following holds: (i) $\lambda|T_Y = (p-3)\lambda_1$ or $(p-3)\lambda_6$, for p>3.

(ii) $\lambda|T_Y = \lambda_1 + (p-2)\lambda_3$ or $\lambda_6 + (p-2)\lambda_5$, for p>2.

Proof: In view of (5.3) and (1.1), it will suffice to work with the Lie algebras $L(A) \le L(Y)$. Actually, we do all computations inside of the universal enveloping algebra of L(Y), where we view L(Y) as a subalgebra of its universal enveloping algebra. (See Section 17 of [9].) Let $\Pi(A) = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ and $\Pi(Y) = \{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\}$ be labelled as throughout. Let $\{e_{\beta_i},f_{\beta_i} | 1 \le i \le 6\}$ be the corresponding elements of L(Y). Let $\langle v^+\rangle$ be the unique 1-space of V such that $e_{\beta_i}v^+ = 0$ for $1 \le i \le 6$. Then we have the elements e_{α_i} of L(A) as follows:

(5.4.1). $e_{\alpha_1} = e_{\beta_2}$, $e_{\alpha_2} = e_{\beta_4}$,

$e_{\alpha_3} = e_{\beta_3} + e_{\beta_5}$, $e_{\alpha_4} = e_{\beta_1} + e_{\beta_6}$

In the following, we will consider the possible modules V|Y given inductively by repeated applications of Theorem (8.1) of [12]. In all cases except the two in the statement of the result, we will produce a vector $w \in V - \langle v^+ \rangle$, which is a maximal vector for $L^+(A)$, the Lie subalgebra of $L(A)$ generated by e_{α_i}, for $1 \le i \le 4$. Thus, $L(A)$ acts reducibly on $V|Y$ except in these two configurations. (Note that (1.34) is frequently used in the check that w is a maximal vector.)

By Theorem (8.1) of [12], the C_3 parabolic of A will act nontrivially on $\langle v^+ \rangle$ in every possible configuration. (For convenience, we will refer to (8.1) of [12] simply as (8.1) for the next few results.) Suppose $\langle \lambda, \beta_k \rangle = 0$ for $k=3,4,5,6$, as in the first configuration of (8.1) (c). Then, (8.1)(d) implies that $\lambda|T_Y = a\lambda_1$, for some $a>0$. Consider the vector $w = (-f_{(1,1,2,2,1,0)}f_1 - f_{12345}f_{134} + f_{(1,1,1,2,1,0)}f_{13} + f_{1234}f_{1345}$ $-(a+3)f_{(1,1,2,2,1,1)} + (a+3)f_{(1,1,1,2,2,1)})v^+$. Then w is a maximal vector for $L^+(A)$. Moreover, if $a \neq p-3$, then $w \neq 0$, as $e_{(1,1,1,2,2,1)}w =$ $(a+3)h_{(1,1,1,2,2,1)}v^+ = a(a+3)v^+$. Thus, if $\lambda|T_Y = a\lambda_1$, (i) holds.

In the configuration where $\lambda|T_Y = b\lambda_3 + x\lambda_2$, for $1 \neq b = p-1$ and $p>x\ge 0$, (8.1)(d) implies that $x=0$ or $x\neq 0$ and $x + b + 2 \equiv 0 \pmod p$. Suppose $x = 0$. Consider the vector $w = (f_{(1,1,2,2,1,0)} + f_{(1,1,1,2,1,0)}f_3 +$ $f_{12345}f_{34} + f_{1345}f_{234} - 2f_{2345}f_{134} + f_{(0,1,1,2,1,1)}f_3 - f_{23456}f_{34} +$ $f_{3456}f_{234} + f_{(0,1,1,2,2,1)})v^+$. (Recall, $p>2$.) Using the fact that $b=p-1$, we can show that $f_{13}f_{34}v^+ = f_{134}f_3v^+$. Then, applying other elements of $L(Y)$ to this equation and using commutator relations, we obtain other such dependence relations. These are necessary to show that, in fact, $L^+(A)w = 0$. Moreover, $w\neq 0$ since $e_{(0,1,1,2,2,1)}w = h_{(0,1,1,2,2,1)}v^+ = bv^+$. Thus, $L(A)$ does not act irreducibly on V. Suppose now that $\lambda|T_Y = b\lambda_3 +$ $x\lambda_2$ and $x\neq 0$. Then, since $x + b + 2 \equiv 0 \pmod p$, and $b = p-1$, we have $x = b = p-1$. Let $w = (f_{1234}f_3 - f_{234}f_{13} - f_{12345} + f_{1345}f_2 - f_{23456} +$ $f_{3456}f_2)v^+$. Then $L^+(A)w = 0$ and $w \neq 0$ as $e_{12345}w = -bv^+$. Thus, $L(A)$ does not act irreducibly on V.

Consider next the configuration where $\lambda|T_Y = c\lambda_1 + b\lambda_3 + x\lambda_2$, for $c \neq 0 \neq b$, $c+b = p-1$ and $p > x \geq 0$. If $x=0$, let $w = ((b+2)f_{(0,1,1,2,2,1)} + f_{(0,1,1,2,1,1)}f_3 + f_{3456}f_{234} - f_{23456}f_{34} + f_{(0,1,1,2,1,0)}f_{13} + (b+2)f_{(1,1,2,2,1,0)} + f_{1234}f_{345} - f_{2345}f_{134})v^+$. Then $L^+(A)w = 0$ and $e_{(0,1,1,2,2,1)}w = (b+2)h_{(0,1,1,2,2,1)}v^+ = (b+2)bv^+$; so if $b \neq p-2$, then $w \neq 0$ and (ii) holds. Suppose λ is as above with $x>0$. Then (8.1)(d) implies that $b+x+2 \equiv 0 \pmod p$. Let $w = (f_{234}f_1 + (b+1)f_{1234} + f_{245}f_1 + cf_{2456})v^+$. Then $L^+(A)w = 0$ and $w \neq 0$, as $e_{34}w = f_2f_1v^+ \neq 0$.

Finally, we must consider the case where $\lambda|T_Y = b\lambda_3 + a\lambda_4 + x\lambda_2$, for $a \neq 0 \neq b$, $a+b = p-1$ and $p > x \geq 0$. Theorem (8.1) implies that $x = 0$. Let $w = (f_{2456}f_{45} - f_{456}f_{245} + af_{(1,1,1,2,1,0)} - (a+1)f_{1234}f_{45} + af_{1345}f_{24} + f_{134}f_{245} + (a+1)f_{23456}f_4 - f_{3456}f_{24} - af_{2456}f_{34} + f_{1234}f_{34} - f_{134}f_{234})v^+$. Then $L^+(A)w = 0$ and $w \neq 0$, as $e_{(0,1,1,2,1,1)}w = a(a+1)v^+$.

This completes the proof of (5.4).□

Definition: If $\mu = \lambda - \Sigma c_i\beta_i$ is a T_Y weight in V, then the <u>level</u> of μ is Σc_i. We define the level of a T_A weight $\nu = \lambda - \Sigma d_i\alpha_i$ similarly. For each $1 \leq i \leq 6$, $\beta_i|T_A = \alpha_j$ for some $1 \leq j \leq 4$, so level is preserved under restriction.

<u>(5.5).</u> Let Y have type E_6 and let $A<Y$ be the fixed point subgroup of the graph automorphism of Y and assume $p>2$. Suppose V is a restricted irreducible, rational kY-module with high weight λ. Let $\langle v^+ \rangle$ be the unique 1-space of V|Y invariant under B_Y. If V|A is reducible, there exists a maximal vector for B_A, $w \in V - \langle v^+ \rangle$, such that one of the following holds:

(i) $\lambda|T_Y = (p-3)\lambda_1$, for $p>3$, and $e_{(1,1,2,2,1,1)}w \in \langle v^+ \rangle$.

(ii) $\lambda|T_Y = \lambda_1 + (p-2)\lambda_3$, for $p>2$, and $e_{(1,1,2,2,1,0)}w \in \langle v^+ \rangle$ or $e_{(1,1,2,2,1,1)}w \in \langle v^+ \rangle$.

<u>Proof</u>: We first prove the following

Claim: There exists a maximal vector for B_A, $w \in V - \langle v^+ \rangle$.

Note that the long word for the Weyl group of E_6, w_0, lies in A. That is, there exists a coset representative, n_0, such that $w_0 = n_0 N_Y(T_Y)$ and $n_0 \in A$. Indeed, one checks that $w_0 = (s_{\beta_1+2\beta_2+2\beta_3+3\beta_4+2\beta_5+\beta_6} \cdot s_{\beta_4} \cdot s_{\beta_3+\beta_4+\beta_5} \cdot s_{\beta_1+\beta_3+\beta_4+\beta_5+\beta_6}) N_Y(T_Y)$, where s_r is the reflection corresponding to the root r. Now, since $V|A$ is reducible, there exists $0<W<V$, an irreducible A-submodule of V. If $v^+ \notin W$, the result follows. So suppose $v^+ \in W$. Since $n_0 \in A$, $n_0 v^+ \in W$. Also, $n_0 v^+$ is a maximal vector for $B_Y{}^- = \langle U_{-r} | r \in \Sigma^-(Y)\rangle T_Y$.

Consider now the kY-module V^*, with high weight $-w_0\lambda$, by (1.11). Write $V = \langle n_0 v^+\rangle \oplus V_0$ and define a vector $f^+ \in V^*$ as follows: $f^+(n_0 v^+) = 1$ and $f^+(v_0) = 0$ for $v_0 \in V_0$. One checks that f^+ is a maximal vector for $B_Y{}^+$ and has weight $-w_0\lambda$. Thus, $\langle f^+\rangle$ is the unique 1-space of V^* with these properties. Moreover, $f^+ \notin \mathrm{Ann}(W)$ as $n_0 v^+ \in W$. Hence, in $V^*|A$, Ann(W) is an invariant submodule, not containing the maximal vector f^+. So there exists a vector $g^+ \in V^* - \langle f^+\rangle$, such that g^+ is a maximal vector for B_A. So the claim holds for V^*. But $V^*|A \cong V|A$, and so the claim holds in general.

We now pass to the level of Lie algebras. Thus, there exists a vector $w \in V - \langle v^+\rangle$ such that $L^+(A)w = 0$, where $L^+(A)$ is the Lie algebra span of the elements e_{α_i}, for $1 \le i \le 4$. And we may assume, $w \in V_{T_A}(\nu)$, for some dominant weight ν. However, $w \notin V_{T_Y}(\mu)$, for any weight $\mu = \lambda - \Sigma d_i \beta_i$. For otherwise, (5.4.1) and the linear independence of weight vectors with distinct weights would imply $w \in \langle v^+\rangle$. Choose w to have minimal level. Since $w \notin \langle v^+\rangle$, w is not a maximal vector for $L^+(Y)$, the Lie algebra span of the elements e_{β_i}, for $1 \le i \le 6$. Thus, (5.4.1) implies that $e_{\beta_1} w \neq 0$ or $e_{\beta_3} w \neq 0$.

Case I: $e_{\beta_1} w \neq 0$.

Note that $e_{\beta_1} w \notin \langle v^+\rangle$, else $w \in V_{T_A}(\lambda - \alpha_4) = V_{T_Y}(\lambda - \beta_1)$, contradicting the opening remarks of the proof. Thus, by minimality, $e_{\beta_1} w$ is not a maximal vector for $L^+(A)$. Since $e_{\alpha_i} e_1 w = 0$ for i=1, 2 and 4,

$e_{\alpha_3}e_1w = (e_3+e_5)e_1w = e_{13}w \neq 0$. Here also $e_{13}w \notin \langle v^+ \rangle$, else $w \in V_{T_Y}(\lambda-\beta_1-\beta_3)$. So by minimality, $e_{13}w$ is not a maximal vector for $L^+(A)$. Now, $e_{\alpha_i}e_{13}w = 0$ for i=1, 3 and 4, so $e_{\alpha_2}e_{13}w = e_{134}w \neq 0$. As above, $e_{134}w \notin \langle v^+ \rangle$, and so is not a maximal vector for $L^+(A)$. But, $e_{\alpha_i}e_{134}w = 0$ for i = 2 and 4. Thus, $e_{\alpha_1}e_{134}w \neq 0$ or $e_{\alpha_3}e_{134}w \neq 0$.

Suppose $e_{\alpha_1}e_{134}w = e_{1234}w \neq 0$. Since the only T_Y weight restricting to $\lambda-\alpha_1-\alpha_2-\alpha_3-\alpha_4$ is $\lambda-\beta_1-\beta_2-\beta_3-\beta_4$, $e_{1234}w \notin \langle v^+ \rangle$ and so is not a maximal vector for $L^+(A)$. Since $e_{\alpha_i}e_{1234}w = 0$ for i=1, 2 and 4, $e_{\alpha_3}e_{1234}w = e_{12345}w \neq 0$. Suppose now that $e_{\alpha_3}e_{134}w = e_{1345}w \neq 0$. Note that $e_{1345}w \notin \langle v^+ \rangle$, else the opening remarks of the proof imply that $\lambda|T_Y = \lambda_1+(p-2)\lambda_3$ and $w \in V_{T_Y}(\lambda-\beta_1-\beta_3-\beta_4-\beta_5) \oplus V_{T_Y}(\lambda-\beta_3-\beta_4-\beta_5-\beta_6) \oplus V_{T_Y}(\lambda-\beta_1-2\beta_3-\beta_4)$. So $w = af_{1345}v^+ + bf_{3456}v^+ + cf_{134}f_3v^+$, for some $a,b,c \in k$. But $e_{\alpha_4}w = 0$ implies that $b = 0$, $e_{\alpha_2}w = 0$ implies that $c = 0$ and $e_{\alpha_3}w = 0$ implies that $a = 0$, contradicting the original choice of w. So by minimality $e_{1345}w$ is not a maximal vector for $L^+(A)$. Now $e_{\alpha_i}e_{1345}w = 0$ for i=2, 3. In fact $e_{\alpha_4}e_{1345}w = e_{13456}w = 0$, as $e_{13456} \in L^+(A)$. (We use here the fact that p>2.) Thus, $e_{\alpha_1}e_{1345}w = e_{12345}w \neq 0$, in this case also.

Now, if $e_{12345}w \in \langle v^+ \rangle$, $w \in V_{T_A}(\lambda-\alpha_1-\alpha_2-2\alpha_3-\alpha_4)$. But, $\lambda-\alpha_1-\alpha_2-2\alpha_3-\alpha_4$ is not a dominant weight, with λ as given. Thus, $e_{12345}w \notin \langle v^+ \rangle$, and so is not a maximal vector for $L^+(A)$. Now, $e_{\alpha_i}e_{12345}w = 0$ for i = 1 and 3. In fact, $e_{\alpha_4}e_{12345}w = e_{123456}w = 0$, as $e_{123456} \in L^+(A)$. Hence $e_{\alpha_2}e_{12345}w = e_{(1,1,1,2,1,0)}w \neq 0$. Note that $e_{(1,1,1,2,1,0)}w \notin \langle v^+ \rangle$, else $w \in V_{T_A}(\lambda-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4)$. But $\lambda-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$ is not a dominant weight. Thus, by minimality, $e_{(1,1,1,2,1,0)}w$ is not a maximal vector for $L^+(A)$. Now, $e_{\alpha_i}e_{(1,1,1,2,1,0)}w = 0$ for i=1 and 2. In fact $e_{\alpha_4}e_{(1,1,1,2,1,0)}w = 0$, as $e_{(1,1,1,2,1,1)} \in L^+(A)$. Hence $e_{\alpha_3}e_{(1,1,1,2,1,0)}w = e_{(1,1,2,2,1,0)}w \neq 0$. Suppose $e_{(1,1,2,2,1,0)}w \in \langle v^+ \rangle$. Then $w \in V_{T_A}(\lambda-\alpha_1-2\alpha_2-3\alpha_3-\alpha_4)$. Since $\lambda-\alpha_1-2\alpha_2-3\alpha_3-\alpha_4$ is dominant only if $\lambda|T_Y = \lambda_1 + (p-2)\lambda_3$, we have one of the configurations of the result.

Suppose now that $e_{(1,1,2,2,1,0)}w \notin \langle v^+\rangle$ and so is not a maximal vector for $L^+(A)$. Since $e_{\alpha_i}e_{(1,1,2,2,1,0)}w = 0$ for i=1, 2 and 3, we have $e_{\alpha_4}e_{(1,1,2,2,1,0)}w = e_{(1,1,2,2,1,1)}w \neq 0$. Note that $e_{\alpha_i}e_{(1,1,2,2,1,1)}w = 0$ for $1 \le i \le 4$, as $e_{(1,1,2,2,2,1)} \in L^+(A)$. So by minimality, $e_{(1,1,2,2,1,1)}w \in \langle v^+\rangle$.

This completes the consideration of Case I.

Case II: $e_{\beta_3}w \neq 0$.

Note that $e_{\beta_3} w \notin \langle v^+\rangle$, else $w \in V_{T_A}(\lambda-\alpha_3) = V_{T_Y}(\lambda-\beta_3)$, contradicting the opening remarks of the proof. Thus, by minimality, $e_{\beta_3}w$ is not a maximal vector for $L^+(A)$. Since, $e_{\alpha_i}e_{\beta_3}w = 0$ for i = 1,3, we have $e_{\alpha_2}e_{\beta_3}w = e_{34}w \neq 0$ or $e_{\alpha_4}e_{\beta_3}w = -e_{13}w \neq 0$. If $e_{13}w \neq 0$, we may refer to Case I. So suppose $e_{34}w \neq 0$. Note that $e_{34}w \notin \langle v^+\rangle$, else $w \in V_{T_Y}(\lambda-\beta_3-\beta_4)$, contradicting the opening remarks of the proof. Hence, by minimality, $e_{34}w$ is not a maximal vector for $L^+(A)$.

Now $e_{\alpha_2}e_{34}w = 0$, and in fact, $e_{\alpha_3}e_{34}w = 0$, as $e_{345} \in L^+(A)$. Thus, $e_{\alpha_1}e_{34}w = e_{234}w \neq 0$ or $e_{\alpha_4}e_{34}w = -e_{134}w \neq 0$. If $e_{134}w \neq 0$ we may refer to Case I. So suppose $e_{234}w \neq 0$. Then $e_{234}w \notin \langle v^+\rangle$, as above, and so is not a maximal vector for $L^+(A)$. But $e_{\alpha_i}e_{234}w = 0$ for i = 1,2. In fact, $e_{\alpha_3}e_{234}w = 0$, as $e_{2345} \in L^+(A)$. Hence, $e_{\alpha_4}e_{234}w = -e_{1234}w \neq 0$. But now we may refer to Case I.

This completes the proof of (5.5).□

(5.6). Let (A,Y) be as in (5.5), so p>2. If $\lambda|T_Y = (p-3)\lambda_1$, for p>3, V|A is irreducible.

Proof: Suppose false; i.e., suppose V|A is reducible. Then (5.5) implies that there exists $w \in V - \langle v^+\rangle$, a maximal vector for B_A, such that $e_{(1,1,2,2,1,1)}w \in \langle v^+\rangle$. Hence, $e_{\alpha_i}w = 0$ for $1 \le i \le 4$, where we now view V as a module for L(A). Now, $w \in V_{T_A}(\lambda-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4)$. The nontrivial T_Y weights restricting to $\lambda-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4$ are

$\lambda-2\beta_1-\beta_2-2\beta_3-2\beta_4-\beta_5$, $\lambda-\beta_1-\beta_2-2\beta_3-2\beta_4-\beta_5-\beta_6$, and $\lambda-\beta_1-\beta_2-\beta_3-2\beta_4-2\beta_5-\beta_6$. The last two weights are conjugate to $\lambda-\beta_1$ and so have multiplicity 1. A spanning set for the weight space $V_{T_Y}(\lambda-2\beta_1-\beta_2-2\beta_3-2\beta_4-\beta_5)$ is $w_1 = f_{(1,1,2,2,1,0)}f_{\beta_1}v^+$, $w_2 = f_{12345}f_{134}v^+$, $w_3 = f_{(1,1,1,2,1,0)}f_{13}v^+$ and $w_4 = f_{1234}f_{1345}v^+$. Hence $w = \Sigma c_i w_i$, $1\le i\le 6$, for some $c_i \in k$ and $w_5 = f_{(1,1,2,2,1,1)}v^+$ and $w_6 = f_{(1,1,1,2,2,1)}v^+$.

Applying e_{α_i}, $1\le i\le 4$, and e_{2345} (an element of $L^+(A)$) to w, we find that $L^+(A)w = 0$ only if $c_5 = 0 = c_6$. So $w \in V_{T_Y}(\lambda-2\beta_1-\beta_2-2\beta_3-2\beta_4-\beta_5)$. But, the linear independence of weight vectors with distinct weights and (5.4.1) imply that $e_{\beta_i}w = 0$, for $1\le i\le 6$. Since $w \notin \langle v^+\rangle$, this is a contradiction.

This completes the proof of (5.6).□

<u>(5.7).</u> Let (A,Y) be as in (5.5), so $p>2$. If $\lambda|T_Y = \lambda_1 + (p-2)\lambda_3$, for $p>2$, then $V|A$ is irreducible.

<u>Proof</u>: Suppose false. Then (5.5) implies that there exists $w \in V - \langle v^+\rangle$ a maximal vector for B_A such that $e_{(1,1,2,2,1,0)}w \in \langle v^+\rangle$ or $e_{(1,1,2,2,1,1)}w \in \langle v^+\rangle$. In particular, $L^+(A)w = 0$.

Case I: $e_{(1,1,2,2,1,0)}w \in \langle v^+\rangle$.

Then $w \in V_{T_A}(\lambda-\alpha_1-2\alpha_2-3\alpha_3-\alpha_4)$. The nontrivial T_Y weights restricting to $\lambda-\alpha_1-2\alpha_2-3\alpha_3-\alpha_4$ are $\mu_1 = \lambda-\beta_2-\beta_3-2\beta_4-2\beta_5-\beta_6$, $\mu_2 = \lambda-\beta_2-2\beta_3-2\beta_4-\beta_5-\beta_6$, $\mu_3 = \lambda-\beta_1-\beta_2-3\beta_3-2\beta_4$, and $\mu_4 = \lambda-\beta_1-\beta_2-2\beta_3-2\beta_4-\beta_5$. Let $w_1 = f_{(0,1,1,2,2,1)}v^+$, $w_2 = f_{(0,1,1,2,1,1)}f_3v^+$, $w_3 = f_{3456}f_{234}v^+$, $w_4 = f_{23456}f_{34}v^+$, $w_5 = f_{1234}f_{34}f_3v^+$, $w_6 = f_{(1,1,1,2,1,0)}f_3v^+$, $w_7 = f_{(1,1,2,2,1,0)}v^+$, $w_8 = f_{1234}f_{345}v^+$, and $w_9 = f_{2345}f_{134}v^+$. Then $V_{T_Y}(\mu_1) = \langle w_1\rangle$, $V_{T_Y}(\mu_2) = \langle w_2,w_3,w_4\rangle$, $V_{T_Y}(\mu_3) = \langle w_5\rangle$ and $V_{T_Y}(\mu_4) = \langle w_i \mid 6 \le i \le 9\rangle$. Thus, $w = \Sigma c_i w_i$, for some $c_i \in k$, $1\le i\le 9$.

Applying e_{2345}, e_{345}, e_{β_2}, $e_{34}- e_{45}$ and $e_{134}+ e_{456}$ (all elements of

$L^+(A)$) to w, we find that $L^+(A)w = 0$ only if $c_6 = -c_7 = c_8 = -c_9$, $c_1 = 0 = c_5$ and $c_2 = c_3 = -c_4$.

We now claim that $w \in \langle v^+ \rangle$, which will contradict the choice of w. It suffices to show that $e_{\beta_i}w = 0$ for $1 \le i \le 6$. By hypothesis, $e_{\beta_2}w = 0 = e_{\beta_4}w$. Consider now $e_{\beta_1}w = c_6(f_{(0,1,1,2,1,0)}f_{\beta_3} + f_{234}f_{345} - f_{2345}f_{34})v^+$. It is a straightforward check that $e_{\beta_i}(e_{\beta_1}w) = 0$ for $1 \le i \le 6$. Thus $e_{\beta_1}w = 0$; so $e_{\beta_6}w = 0$.

Now $e_{\alpha_3}w = e_{\beta_3}w + e_{\beta_5}w = 0$ and since $e_{\beta_3}w \in V_{T_Y}(\lambda-\beta_2-\beta_3-2\beta_4-\beta_5-\beta_6) \oplus V_{T_Y}(\lambda-\beta_1-\beta_2-\beta_3-2\beta_4-\beta_5)$ and $e_{\beta_5}w \in V_{T_Y}(\lambda-\beta_1-\beta_2-2\beta_3-2\beta_4)$, we must have $e_{\beta_3}w = 0 = e_{\beta_5}w$. Hence, $e_{\beta_i}w = 0$ for $1 \le i \le 6$, and $w \in \langle v^+ \rangle$ as claimed. This completes the consideration of Case I.

Case II: $e_{(1,1,2,2,1,1)}w \in \langle v^+ \rangle$.

Then, $w \in V_{T_A}(\lambda-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4)$. The nontrivial T_Y weights restricting to $\lambda-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4$ are $\mu_1 = \lambda-2\beta_1-\beta_2-3\beta_3-2\beta_4$, $\mu_2 = \lambda-\beta_1-\beta_2-\beta_3-2\beta_4-2\beta_5-\beta_6$, $\mu_3 = \lambda-\beta_1-\beta_2-2\beta_3-2\beta_4-\beta_5-\beta_6$ and $\mu_4 = \lambda-2\beta_1-\beta_2-2\beta_3-2\beta_4-\beta_5$. Also, $V_{T_Y}(\mu_1) = \langle w_1 \rangle$, $V_{T_Y}(\mu_2) = \langle w_2 \rangle$, $V_{T_Y}(\mu_3) = \langle w_i | 3 \le i \le 7 \rangle$ and $V_{T_Y}(\mu_4) = \langle w_k | 8 \le k \le 10 \rangle$, where $w_1 = f_{1234}f_{13}f_{34}v^+$, $w_2 = f_{(1,1,1,2,2,1)}v^+$, $w_3 = f_{(1,1,2,2,1,1)}v^+$, $w_4 = f_{13456}f_{234}v^+$, $w_5 = f_{23456}f_{134}v^+$, $w_6 = f_{123456}f_{34}v^+$, $w_7 = f_{(0,1,1,2,1,1)}f_{13}v^+$, $w_8 = f_{(1,1,2,2,1,0)}f_1v^+$, $w_9 = f_{12345}f_{134}v^+$, $w_{10} = f_{(1,1,1,2,1,0)}f_{13}v^+$.

Applying e_{α_4}, e_{24}, e_{2345}, $e_{1234} + e_{2456}$ and $e_{134} + e_{456}$ (all elements of $L^+(A)$), we see that $L^+(A)w = 0$ only if $c_i = 0$ for $i = 1,2,8,9,10$. So $w \in V_{T_Y}(\mu_3)$. But (5.4.1) and the linear independence of weight vectors with distinct weights implies $e_{\beta_i}w = 0$ for $1 \le i \le 6$. But this implies $w \in \langle v^+ \rangle$. Contradiction.

This completes the proof of (5.7).□

The remainder of this chapter is devoted to proving that there are no examples (A,Y,V) in the main theorem with A simple, of rank 3 and Y of type E_n, nor with (A,p) special and rankA > 2. This will complete the proof of Theorem (5.0).

<u>(5.8).</u> Suppose rank A = 3 with (A,p) not special, L_A' is of type A_2 with $\dim V^1(Q_A) > 1$. Then $L_Y' = L_1 \times L_2$, with L_i of type A_2 or D_4. Moreover, if L_i has type D_4, then $\langle \lambda, \beta_j \rangle = 0$ for $1 \le j \le 5$.

<u>Proof</u>: If L_Y' is quasisimple, Theorem (4.0) implies that L_Y' is of exceptional type. Then (6.0) implies that $L_Y' = E_6$ and Q_Y/K_{β_7} is a 27-dimensional irreducible L_A' module. Thus, Z_A induces scalars on Q_Y/K_{β_7}, forcing $Q_AK_{\beta_7}/K_{\beta_7}$ to be an L_A' submodule of Q_Y/K_{β_7}. Since $\dim Q_A < 27$, $Q_A \le K_{\beta_7}$, contradicting (2.3). Thus, L_Y' is not a quasisimple.

By size retrictions, $L_Y' = L_1 \times L_2$, where L_i is a simple algebraic group of classical type. Hence, (1.5) implies $Z_A \le Z_Y$. Notice, if A_2 is embedded in D_5, acting reducibly on the natural module for D_5, A_2 lies in a proper parabolic of D_5. Thus, the minimality of P_Y and the Main Theorem of [12] imply that L_i has type A_2 or D_4. Suppose $L_i = D_4$. Since P_Y is minimal, $\rho_i(L_A')$ is either irreducible on Q_Y/K_{β_1}, or p=3 and L_A' acts on Q_Y/K_{β_1} with composition factors of dimensions 1 and 7. Thus, $Q_A \le K_{\beta_1}$ and by (2.3), $\langle \lambda, \beta_j \rangle = 0$ for $1 \le j \le 5$.□

<u>(5.9).</u> If (A, L_A') is of type (B_3, B_2) and p>2, then $\dim V^1(Q_A) = 1$.

<u>Proof</u>: Suppose $\dim V^1(Q_A) > 1$. Then L_Y' is not a simple algebraic group. For otherwise, Theorem (4.0) implies L_Y' is of exceptional type. But the result of Chapter 7 indicates that there is no such embedding. Thus $Y = E_8$ and L_Y' is of type $A_3 \times A_3$ or $A_3 \times A_4$. Also, the projection of L_A' into each component of L_Y' must act irreducibly on the natural module for that component, as P_Y is minimal. Note that $h_{\alpha_3}(-1) \in Z(A) \le Z(Y) = \{1\}$. Since Y is simply connected, $h_{\alpha_3}(-1)$ must be in the kernel of the 4-dimensional representation of L_A'. But it is easy to check that this is not the case. Contradiction.□

<u>(5.10).</u> If A is of type A_3 or B_3, with (A,p) not special, and L_A' is of type A_2, then $\dim V^1(Q_A) = 1$.

<u>Proof</u>: Suppose false. Let $\Pi(L_A) = \{\alpha_1,\alpha_2\}$, so $\Pi(A) - \Pi(L_A) = \{\alpha_3\}$. By (5.8), $L_Y' = L_1 \times L_2$, with L_i of type A_2 or D_4. Let q_i be the field twist on the embedding of L_A' in L_i for i = 1,2. Thus, (1.5) implies $Z_A \le Z_Y$. Suppose $L_1 = D_4$. Then $Y = E_8$, $\Pi(L_1) = \{\beta_2,\beta_3,\beta_4,\beta_5\}$, $\Pi(L_2) = \{\beta_7,\beta_8\}$, and $\langle\lambda,\beta_j\rangle = 0$ for $1 \le j \le 5$. By (1.23) and (5.9), $\lambda|T_A = q(c\mu_1+d\mu_2+c\mu_3)$ if $A = A_3$ or $\lambda|T_A = qc\mu_1$ if $A = B_3$, with $\lambda|T_Y = x\lambda_6+c\lambda_j+d\lambda_k$ where $\{j,k\} = 7,8$ and q is some p–power. If p=3, $0 \le c,d \le 2$ and (1.26) and (1.32) imply $\dim V|A < \dim V|Y$. Hence, $p \ne 3$.

Since $p \ne 3$, $\rho_1(L_A')$ acts irreducibly on $V_{L_1}(-\beta_6)$ and (2.7) implies that $q_1=q_2$. One checks that if $\rho_2(h_{\alpha_1}(c)) = h_{\beta_8}(c^{q_2})$ and $\rho_2(h_{\alpha_2}(c)) = h_{\beta_7}(c^{q_2})$, the L_A' composition factors of Q_Y/K_{β_6} have high weights $q_2(\mu_1+2\mu_2)$, $q_2(2\mu_1)$ and $q_2\mu_2$. By symmetry, if $\rho_2(h_{\alpha_1}(c)) = h_{\beta_7}(c^{q_2})$ and $\rho_2(h_{\alpha_2}(c)) = h_{\beta_8}(c^{q_2})$, there is no L_A' composition factor of Q_Y/K_{β_6} isomorphic to a twist of $Q_A^{\alpha_3}$. Thus, L_A' must project into L_2 in the first way described.

We claim $\langle\lambda,\beta_6+\beta_7\rangle = 0$. For otherwise, since $V_{T_Y}(\lambda-\beta_6) \oplus V_{T_Y}(\lambda-\beta_6-\beta_7) \ne 0$, some nonidentity element from the set $U_{-\beta_6} \cdot U_{-\beta_6-\beta_7}$ must occur in the factorization of an element in $Q_A - Q_A'$. However, $-\beta_6$ (respectively, $-\beta_6-\beta_7$) affords $T(L_A')$ weight $q_2(\mu_1+2\mu_2)$ (respectively, $2q_2\mu_1$). But no such weight vectors occur in $Q_AK_{\beta_6}/K_{\beta_6}$. Thus, if L_Y' has type $D_4 \times A_2$, $\lambda|T_Y = c\lambda_8$, for some $1<c<p$. Then (1.23) implies $\lambda|T_A = q_2(c\mu_1+c\mu_3)$ if $A = A_3$, and (5.9) implies $\lambda|T_A = q_2c\mu_1$, if $A = B_3$. In any case, $\dim V_{\beta_6}(Q_Y) \le 3/2(c+1)(c+2)$, by (1.25) and (1.12). However, $f_{\beta_6+\beta_7+\beta_8}v^+$ affords an L_Y' composition factor in $V_{\beta_6}(Q_Y)$ of dimension $4c(c+1)$. Contradiction.

Now consider the case where L_i has type A_2 for i = 1,2, and L_1 and L_2 are separated by exactly two nodes of the Dynkin diagram. For convenience, temporarily label as follows:

$L_Y' = \langle U_{\pm\gamma_1}, U_{\pm\gamma_2}\rangle \times \langle U_{\pm\gamma_5}, U_{\pm\gamma_6}\rangle = L_1 \times L_2$, $\gamma_3, \gamma_4 \in \Pi(Y) - \Pi(L_Y)$, with $(\gamma_i, \gamma_{i+1}) < 0$ for $1 \le i \le 5$. Then (2.8), (2.5) and (2.6) imply that only one of L_1 and L_2 act nontrivially on $V^1(Q_Y)$. Say $L_1v^+ \ne v^+$, $L_2v^+ = v^+$. Since $Q_A \not\le K_{\gamma_3}$, we compare high weights of the L_A' modules $Q_AK_{\gamma_3}/K_{\gamma_3}$ and Q_Y/K_{γ_3} to see that $\rho_1(h_{\alpha_1}(c)) = h_{\gamma_1}(c^{q_1})$ and $\rho_1(h_{\alpha_2}(c)) = h_{\gamma_2}(c^{q_1})$. It is a check to see that the L_Y' composition factor of Q_Y afforded by $U_{-\gamma_3-\gamma_4}$ has no 3-dimensional L_A' submodule isomorphic to a twist of $Q_A^{\alpha_3}$. But by (1.33), we may assume that if $Q_A \le K_{\gamma_4}$, $-\gamma_4$ is not involved in L_A'. Hence, (2.11) implies that $Q_A \not\le K_{\gamma_4}$.

Now, $Q_A \not\le K_{\gamma_3}$ and $Q_A \not\le K_{\gamma_4}$ and (2.9) imply that $A = B_3$. Moreover, (2.13) and (2.8) imply $\gamma_3|T_A = q_1\alpha_3 = \gamma_4|T_A$. By (5.9), $\langle\lambda,\gamma_i\rangle = 0$ for $2 \le i \le 6$. Also, by (1.10), $q_1=1$ and so, if $\langle\lambda,\gamma_1\rangle = c$, for $0<c<p$, then $\langle\lambda,\alpha_1\rangle = c$. Now, the subgroup $X = \langle U_{\pm\gamma_1}, U_{\pm\gamma_2}, U_{\pm\gamma_3}, U_{\pm\gamma_4}, U_{\pm\gamma_5}, U_{\pm\gamma_6}\rangle$ of type A_6 has a natural subgroup, B, of type B_3. Moreover, the X composition factor of $V|Y$ afforded by $\langle v^+\rangle$ is not all of $V|Y$ as X is contained in the Levi factor of a proper parabolic of Y. But the B composition factor of $V|Y$ afforded by $\langle v^+\rangle$ has the same high weight, as B_3 module, as does $V|A$. Thus, $\dim V|A < \dim V|Y$. Hence, this configuration does not occur.

Consider now the case where L_1 and L_2 are separated by more than two nodes of the Dynkin diagram. Thus $Y = E_8$, $\Pi(L_1) = \{\beta_1,\beta_3\}$ and $\Pi(L_2) = \{\beta_7,\beta_8\}$. Then (2.13) and (2.3) imply that $\langle\lambda,\beta_k\rangle = 0$ for $k =$ 2,4,5,6. We first note that only one of L_1 and L_2 acts nontrivially on $V^1(Q_Y)$. For otherwise (2.13) implies $\beta_4|T_A = q_1\alpha_3$, $\beta_6|T_A = q_2\alpha_3$, and $\beta_2|T_A = 0 = \beta_5|T_A$. As well, we have $\beta_1|T_A = q_1\alpha_1$, $\beta_3|T_A = q_1\alpha_2$, $\beta_8|T_A = q_2\alpha_1$, $\beta_7|T_A = q_2\alpha_2$. Thus, there is no vector in $V|Y$ with T_A weight $\lambda - q\alpha_3$, for any p-power q, contradicting (2.14).

Note that A does not have type B_3. For otherwise, $\lambda|T_A = cq_i\mu_1$, for $i = 1$ or 2, and $\lambda|T_Y = c\lambda_1$ or $c\lambda_8$. And as above, considering the usual embedding of B_3 in A_6 we have $\dim V|A < \dim V|Y$. Also, (1.23) and (1.10) imply $V|A$ is restricted; say $\lambda|T_A = c\mu_1+d\mu_2+c\mu_3$ Let $P_Y\hat{} > B_Y^-$ be the parabolic subgroup of Y with Levi factor $L_Y\hat{} = \langle L_Y, U_{\pm\beta_5}\rangle$. Then $Q_A \le Q_Y\hat{} = R_u(P_Y\hat{})$ and $Z_A \le Z(L_Y\hat{})^\circ$ by (2.12). For the argument which follows, we may assume $\langle\lambda,\beta_1+\beta_3\rangle = 0$ and $\langle\lambda,\beta_7+\beta_8\rangle \ne 0$. So $\lambda|T_Y = d\lambda_7+c\lambda_8$, $\beta_5|T_A = 0$, $\beta_6|T_A = \alpha_3$, $\beta_7|T_A = \alpha_2$ and $\beta_8|T_A = \alpha_1$. If $Q_A \le K_{\beta_4}$, by (1.33) we may assume that $-\beta_4$ is not involved in L_A'. Hence, by (2.11), there is a nontrivial image of $Q_A{}^{\alpha_3}$ in $Q_Y\hat{}(\beta_6,\beta_4)$. But $Q_Y\hat{}(\beta_6,\beta_4)$ has no L_A' composition factor isomorphic to a twist of $Q_A{}^{\alpha_3}$. Thus, $Q_A \nleq K_{\beta_4}$ and $\beta_4|T_A = q_1\alpha_3$. Now, if $V_{T_A}(\lambda-\alpha_2-x\alpha_3) \ne 0$, then conjugating by s_{α_3}, we see that $V_{T_A}(\lambda-\alpha_2-(c+1-x)\alpha_3) \ne 0$. So $x \le c+1$. In particular, $q_1 = 1$, else $0 \ne f_{4567}v^+ \in V_{T_A}(\lambda-\alpha_2-(q+1)\alpha_3)$. So $\beta_1|T_A = \alpha_1$, $\beta_3|T_A = \alpha_2$, and $\beta_4|T_A = \alpha_3$. Now, if $\langle\lambda,\beta_7\rangle \ge 2$, $(f_{67})^2v^+$, $(f_{567})^2v^+$, $f_{4567}f_7v^+$ and $f_{34567}v^+$ are 4 linearly independent vectors in $V_{T_A}(\lambda-2\alpha_2-2\alpha_3)$. But (1.28) implies $\dim V_{T_A}(\lambda-2\alpha_2-2\alpha_3) \le 3$. Thus, $\langle\lambda,\beta_7\rangle = 1$. Also, $\langle\lambda,\beta_8\rangle > 1$, else (1.23) implies $\dim V|A < \dim V|Y$. If $\langle\lambda,\beta_8\rangle = c = \langle\lambda,\alpha_1\rangle = \langle\lambda,\alpha_3\rangle$, we may assume $c \ne p-2$, else by (1.35), $\dim V_{T_A}(\lambda-\alpha_2-\alpha_3) = 1$, but $f_{67}v^+$ and $f_{567}v^+$ are 2 linearly independent vectors in this weight space. Now by Theorem 3 of [3] and (1.28), if $c+1 < p-1$,

$\dim V_{T_Y}(\lambda-\beta_7-\beta_8) = \dim V_{T_Y}(\lambda-2\beta_7-2\beta_8) = \dim V_{T_Y}(\lambda-\beta_7-2\beta_8) = 2.$

Thus, $\dim\big(V_{T_Y}(\lambda-2\beta_6-2\beta_7-2\beta_8) + V_{T_Y}(\lambda-2\beta_5-2\beta_6-2\beta_7-2\beta_8) + V_{T_Y}(\lambda-\beta_5-2\beta_6-2\beta_7-2\beta_8) + V_{T_Y}(\lambda-\beta_4-\beta_5-\beta_6-2\beta_7-2\beta_8) + V_{T_Y}(\lambda-\beta_3-\beta_4-\beta_5-\beta_6-\beta_7-2\beta_8) + V_{T_Y}(\lambda-\beta_1-\beta_3-\beta_4-\beta_5-\beta_6-\beta_7-\beta_8)\big) > 9$.

But each of these weight spaces lies in $V_{T_A}(\lambda-2\alpha_1-2\alpha_2-2\alpha_3)$, which has dimension at most 9, by (1.28). Hence, we may assume $c+1 > p-1$, so $c = p-1$. Now, (1.33) implies $\dim V_{T_Y}(\lambda-\beta_7-2\beta_8) = 2$ and since $\lambda-2\beta_7-2\beta_8$ is conjugate to $\lambda-\beta_7-2\beta_8$, $\dim V_{T_Y}(\lambda-2\beta_7-2\beta_8) = 2$. And again the dimension of the given sum of T_Y weight spaces exceeds 9. Thus, $\Pi(L_Y) \ne \{\beta_1,\beta_3,\beta_7,\beta_8\}$.

It remains to consider the case where L_i is of type A_2 for $i=1,2$ and L_1 and L_2 are separated by exactly one node of the Dynkin diagram. For convenience, temporarily label as follows: $\Pi(L_1) = \{\gamma_1,\gamma_2\}$, $\Pi(L_2) = \{\gamma_4,\gamma_5\}$, $\gamma_3 \in \Pi(Y) - \Pi(L_Y)$ and $(\gamma_i,\gamma_{i+1}) < 0$ for $1 \le i \le 4$. Then (2.7) implies that the field twists on the embeddings of L_A' in L_1 and in L_2 are equal. Call this twist q. Thus, by (2.5) and (2.6), only one of L_1 and L_2 acts nontrivially on $V^1(Q_Y)$. Say, $L_1v^+ \neq v^+$ and $L_2v^+ = v^+$.

Now we may assume p>2. For otherwise, $A = A_3$ and (1.23), (1.26) and (1.32) imply $\dim V|A < \dim V|Y$. Then, we find that Q_Y/K_{γ_3} has an L_A' composition isomorphic to a twist of $Q_A{}^{\alpha_3}$ if and only if $h_{\alpha_1}(c) = h_{\gamma_2}(c^q)\cdot h_{\gamma_4}(c^q)$ and $h_{\alpha_2}(c) = h_{\gamma_1}(c^q)h_{\gamma_5}(c^q)$. Then, considering the $T(L_A')$ weight vectors in the module Q_Y/K_{γ_3}, we see that $x_{-\alpha_3}(t) = x_{-\gamma_2-\gamma_3}(c_1t^q)\cdot x_{-\gamma_3-\gamma_4}(c_2t^q)w$, where $c_i \in k$, with c_1 and c_2 not both zero, and $w \in K_{\gamma_3}$. Thus, $\gamma_3|T_A = q(\alpha_3 - \alpha_1)$. In particular, $\langle\lambda,\gamma_3\rangle = 0$. So if $\langle\lambda,\gamma_1\rangle = d$, $\langle\lambda,\gamma_2\rangle = c$, for some $0 \le c,d < p$, then $\langle\lambda,\alpha_1\rangle = cq$ and $\langle\lambda,\alpha_2\rangle = dq$. If $A = B_3$, d=0 by (5.9). In fact, c=0 also, else $f_{\gamma_2+\gamma_3}v^+$ is a nonzero vector in $V_{T_A}(\lambda - q\alpha_3)$. Thus, $A = A_3$ and (1.23) implies that $\langle\lambda,\alpha_3\rangle = cq$. Let $P \ge B_Y{}^-$ be the proper parabolic subgroup of Y with Levi factor $\langle U_{\pm\gamma_i} | 1 \le i \le 5\rangle T_Y$. Then, considering the usual embedding of A_3 in A_5, we see that $\dim V|A \le \dim V^1(R_u(P)) < \dim V|Y$. This completes the proof of (5.10).□

<u>(5.11).</u> If A has type C_3, with p>2, and L_A' has type A_2 with $\dim V^1(Q_A) > 1$, then Y has type E_8, L_Y' has type $A_2 \times D_4$ and $\lambda|T_Y = c\lambda_7 + d\lambda_8$, $d \neq 0$, $\langle\lambda,\alpha_1\rangle = cq$, $\langle\lambda,\alpha_2\rangle = dq$, for some p-power q.

<u>Proof</u>: By (5.8), $L_Y' = L_1 \times L_2$, where L_i has type A_2 or D_4. Thus, (1.5) implies $Z_A \le Z_Y$. Consider first the case where $L_1 = D_4$. Then $Y = E_8$, L_Y' has type $D_4 \times A_2$ and $\langle\lambda,\beta_i\rangle = 0$ for $1 \le i \le 5$.

Let q_i be the field twist on the embedding of L_A' in L_i, for i=1,2. If $q_1 \neq q_2$, L_A' acts irreducibly on Q_Y/K_{β_6} or p=3 and Q_Y/K_{β_6} has L_A' composition factors of dimensions 21 and 3. Thus, since $Q_A \not\le K_{\beta_6}$, $q_1=q_2$. If $\rho_2(h_{\alpha_1}(c)) = h_{\beta_7}(c^{q_1})$ and $\rho_2(h_{\alpha_2}(c)) = h_{\beta_8}(c^{q_1})$, the L_A' composition factors of Q_Y/K_{β_6} have high weights $\{q_1(2\mu_1+\mu_2), q_1\mu_1, q_12\mu_2\}$. By symmetry, if $\rho_2(h_{\alpha_1}(c)) = h_{\beta_8}(c^{q_1})$ and $\rho_2(h_{\alpha_2}(c)) = h_{\beta_7}(c^{q_1})$, there is no L_A' composition factor of Q_Y/K_{β_6} isomorphic to a twist of $Q_A{}^{\alpha_3}$. Thus, we see that L_A' must project onto L_2 in the first way described and $\beta_6|Z_A = q_1\alpha_3$. Note that $\langle\lambda,\beta_6\rangle = 0$ else a nonidentity element from $U_{-\beta_6}$ must occur in the factorization of some element of $Q_A{}^{\alpha_3}$; but $-\beta_6$ does not afford a $T(L_A')$ weight in $(Q_A{}^{\alpha_3})^{q_1}$. (See (2.4).) Also, if $\lambda|T_Y = c\lambda_7+d\lambda_8$, then $\langle\lambda,\alpha_1\rangle = cq_1$ and $\langle\lambda,\alpha_2\rangle = dq_1$. Suppose $d = 0 \neq c$. Then (1.26) and (1.25) imply $\dim V_{\beta_6}(Q_Y) \leq 3(c+1)(c+2)$. Now, $f_{\beta_6+\beta_7}v^+$ affords an L_Y' composition factor of $V_{\beta_6}(Q_Y)$ of dimension $8\cdot\dim V_0$, where V_0 is the A_2 module with high weight $(c-1)\eta_1+\eta_2$ and η_1, η_2 are the fundamental dominant weights for A_2. The corollary to Theorems 3 and 4 in [3] implies $\dim V_0 = c(c+2)$ if $c < p-1$ or $\frac{1}{2}(c+1)(c+2) + (c-1)$ if $c= p-1$. But $8\cdot\dim V_0 > 3(c+1)(c+2)$ in each case, contradicting the bound on $\dim V_{\beta_6}(Q_Y)$. Thus $d \neq 0$.

Now consider the case where L_i is of type A_2 for $i=1,2$. Say $L_1v^+ \neq v^+$. Since $p>2$, $Q_A{}^{\alpha_3}$ is a 6-dimensional irreducible L_A' module. So there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\Sigma L_1) \neq 0$ and such that Q_Y/K_γ is isomorphic to a 3-dimensional irreducible L_1 module. Thus, there exists $\gamma_0 \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma_0,\Sigma L_i) \neq 0$ for $i=1,2$. For convenience, temporarily label as follows: $\Pi(L_1) = \{\gamma_1,\gamma_2\}$, $(\gamma_2,\gamma_0) \neq 0$, $\Pi(L_2) = \{\gamma_3,\gamma_4\}$, $(\gamma_3,\gamma_0) \neq 0$. Then (2.7) implies that the field twists on the embeddings of L_A' in L_1 and in L_2 are equal. Call this twist q. Considering the L_A' composition factors of Q_Y/K_{γ_0}, we have $h_{\alpha_1}(c) = h_{\gamma_1}(c^q)h_{\gamma_4}(c^q)$ and $h_{\alpha_2}(c)= h_{\gamma_2}(c^q)\, h_{\gamma_3}(c^q)$.

We first claim that there does not exist $\tau \in \Pi(Y) - \Pi(L_Y)$ with $(\tau,\gamma_4) \neq 0$. For if there exists such a τ, Q_Y/K_τ is a 3-dimensional irreducible L_A' module and so $Q_A \leq K_\tau$. By (2.10), $-\tau$ is not involved in L_A', so (2.11) implies that there is a nontrivial image of $Q_A{}^{\alpha_3}$ in $Q_Y(\gamma_0,\tau)$, which is also a 3-dimensional irreducible L_A' module. Contradiction. As well, there does not exist $\tau \in \Pi(Y) - \Pi(L_Y)$ with $(\tau,\gamma_3) \neq 0$. For if there exists such a τ, as above, $Q_A \le K_\tau$, $-\tau$ is not involved in L_A' and (2.11) applies. But $Q_Y(\gamma_0,\tau)$ has L_A' composition factors of dimensions 8 and 1 (7 and 1, if p=3).

The above work implies $Y = E_6$ and $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_5,\beta_6\}$. We may assume, by symmetry, that $\langle\lambda,\beta_5+\beta_6\rangle = 0$, and by (2.3), $\langle\lambda,\beta_2\rangle = 0$. Also, one checks that $x_{-\alpha_3}(t) = x_{-\beta_4}(c_1t^q)w$, where $c_1 \in k^*$ and $w \in K_{\beta_4}$. Thus, by (2.4), $\beta_4|T_A = q\alpha_3$; and (2.12) implies $\beta_2|T_A = 0$. Thus, $\langle\lambda,\beta_4\rangle = 0$, else f_4v^+ and $f_{24}v^+$ are two linearly independent vectors in $V_{T_A}(\lambda-q\alpha_3)$.

We claim that $\langle\lambda,\alpha_3\rangle = 0$. For since $\beta_1|T_A = q\alpha_1 = \beta_6|T_A$, $\beta_3|T_A = q\alpha_2 = \beta_5|T_A$, $\beta_4|T_A = q\alpha_3$, and $\langle\lambda,\beta_4\rangle = 0$, there does not exist a vector in $V|Y$ with T_A weight $\lambda - q_0\alpha_3$ for any p-power q_0. So we have $\lambda|T_A = q(c\mu_1 + d\mu_2)$ and $\lambda|T_Y = c\lambda_1 + d\lambda_3$ for some $0 \le c,d < p$.

Now let $X = \langle U_{\pm\beta_1},U_{\pm\beta_3},U_{\pm\beta_4},U_{\pm\beta_5},U_{\pm\beta_6}\rangle$. Then X has a natural subgroup, C, of type C_3. Moreover, v^+ affords an X composition factor of $V|Y$ with dimension strictly less than $\dim V|Y$, as X is contained in the Levi factor of a proper parabolic of Y. But v^+ affords a C composition factor of $V|Y$ with the same high weight as $V|A$, as C_3 module. Thus, $\dim V|A < \dim V|Y$. Contradiction.□

<u>(5.12).</u> If (A,L_A') is of type (C_3,C_2) with p>2, then $\dim V^1(Q_A) = 1$.

<u>Proof</u>: Suppose false; i.e., suppose $\dim V^1(Q_A) > 1$. We first claim that L_Y' is not simple. The work of Chapter 7 indicates that L_Y' is not a simple algebraic group of exceptional type. Hence, if L_Y' is simple, Theorem (4.0) implies $Y = E_8$, $\lambda|T_Y = \lambda_1$ and $\langle\lambda,\alpha_2\rangle = 2q$, for some p-power q. However, this contradicts (5.11). Thus, $Y = E_8$ and L_Y' is of type $A_3 \times A_3$ or $A_3 \times A_4$.

We first claim that $\langle\lambda,\alpha_2\rangle \ne 0$. For otherwise, (5.11) and the Main Theorem of [12] imply $\lambda|T_A = q\mu_3$ and $\dim V|A < \dim V|Y$. Hence, (5.11) implies $\lambda|T_Y = c\lambda_7+d\lambda_8$ with $d\ne0$. Now $\langle U_{\pm\beta_j} \mid j = 6, 7, 8\rangle$ is a component of L_Y' with the embedding of L_A' in L_Y' the natural embedding of classical groups, up to some twist. Thus, $\langle\lambda,\alpha_2\rangle = dq_0$ and

$\langle\lambda,\alpha_3\rangle = cq_0$ for some p-power q_0. In fact, since $d \neq 0$, (5.11) implies $\langle\lambda,\alpha_1\rangle = cq_0$, so by (1.10), $q_0 = 1$. Now, let $X = \langle U_{\pm\beta_j} \mid 2 \leq j \leq 8\rangle$. Then X has a natural subgroup of type C_3, call it C. (See IV_8 on Table 1 of [12].) Moreover, v^+ affords an X composition factor of V|Y of dimension strictly less than dimV|Y. But v^+ affords a C composition factor of V|Y with the same high weight as V|A, as C_3 modules. Thus, dimV|A < dimV|Y. Contradiction.□

<u>(5.13).</u> If (A,Y,V) is an example in the main theorem, with Y of type E_n and rankA > 2, then (A,p) is not special.

<u>Proof:</u> Suppose false

<u>Case I:</u> Suppose p = 2 and A has type B_k or C_k.

We first claim that $\langle\lambda,\alpha_j\rangle > 0$ for some j > 1. For otherwise, applying induction to a maximal parabolic of A corresponding to α_k, rank restrictions, (5.1) – (5.11), (9.4) and the Main Theorem of [12] imply that $\lambda|T_A = q\mu_1$ or $(q+q_0)\mu_1$ and dimV|A = 2k or $(2k)^2$, respectively. Also, k < rankY, by Theorem (4.1) of [12]. But then (1.32) and [8] imply dimV|Y $\neq$ dimV|A. So $\langle\lambda,\alpha_j\rangle > 0$ for some j > 1, as claimed.

For the remainder of Case I considerations, let $\alpha = \alpha_1$; so L_A' has type B_{k-1} or C_{k-1}, and $\dim V^1(Q_A) > 1$. Suppose L_Y' has type D_ℓ, for some $\ell \geq 4$. Then by (1.5), $Z_A \leq Z_Y$ and by (4.1) of [12], rank $L_A' = k-1 <$ rankL_Y'. If L_A' acts irreducibly on some module other than W, the natural module for L_Y', the Main Theorem of [12] implies that either L_A' acts irreducibly on the 2 fundamental spin modules for L_Y', or $L_A' = B_3$, $L_Y' = D_4$ and L_A' acts irreducibly on 2 of the 3 restricted 8-dimensional irreducible L_Y' modules. But there exists $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $V_\gamma(Q_Y) \neq 0$ and Q_Y/K_γ isomorphic to one of these two spin modules; while I_α is never isomorphic to one of these modules. Thus, W is the only L_Y' module on which L_A' acts irreducibly. So $V^1(Q_Y) \cong W$. If $L_Y' = D_7$, then $\dim V^1(Q_Y) = 14$, so k = 4. However, the bound on $\dim V_{\beta_1}(Q_Y)$ implies $\lambda|T_Y = \lambda_8$. By (4.1) of [12], the only remaining possibility is that

$L_Y' = D_4$, $k = 4$ and $\lambda|T_A = x\mu_1+q\mu_4$, for some p-power q. Induction, applied to the maximal parabolic of A corresponding to α_k, implies that x = 0, q_1 or q_1+q_2, for q_1 and q_2 distinct p-powers. So dimV|A = 16, 128, or 1024. As well, $\langle\lambda,\beta_2+\beta_3+\beta_5\rangle = 1$ and $\langle\lambda,\beta_4\rangle = 0$. Finally, using (2.3) to obtain more information about $\lambda|T_Y$, and (1.32) and [8], we see that dimV|A $\neq$ dimV|Y. Thus, no component of L_Y' has type D_ℓ.

Suppose A = B_3 or C_3. The above work and (9.4) imply that L_i has type A_3 for all i. Note that $\lambda|T_Y \neq \lambda_\ell$ for any ℓ. For otherwise, $\lambda|T_A = q_2x\mu_1+q_2\mu_2$ or $q_1x\mu_1+q_2\mu_3$ for x = 0 or 1 and for some p-powers q_1 and q_2. Then Table 1 of [5] implies dimV|A = 8, 14, 48 or 64; so dimV|A $\neq$ dimV|Y by [8]. Now, in general, induction and rank restrictions imply dimV|A $\leq 6^2\cdot 8$ if Y = E_6 or E_7 and dimV|A $\leq 6\cdot 8^3$ if Y = E_8. But then the above remarks, (1.32) and [8] imply dimV|A < dimV|Y. Hence, A $\neq$ B_3 or C_3.

Now, suppose A = C_4 or B_4. A straightforward argument shows that $L_Y' = A_5$ and $\lambda|T_A = x\mu_1+q\mu_2$ for some p-power q. If Y = E_6 or E_7, induction (applied to the A_3 maximal parabolic of A) implies that x = 0 or q. So dimV|A = 26 or 112. But then (1.32) and [8] imply dimV|A $\neq$ dimV|Y. So Y = E_8. By induction, x = 0, q_1, q, or q_1+q, for some p-power $q_1 \neq q$. Then, dimV|A = 26, 112, 208, or 896. But [8] and (1.32) imply dimV|A < dimV|Y. So Y $\neq$ B_4 or C_4.

Suppose A = B_5 or C_5. The previous work of this result and the Main Theorem of [12] imply that $L_Y' = A_7$. The bound on $\dim V_{\beta_2}(Q_Y)$ implies that $\lambda|T_Y = \lambda_1$. Also, $\lambda|T_A = x\mu_1+q\mu_2$. However, induction (applied to the A_4 maximal parabolic of A) provides a contradiction. Thus, A does not have type B_5 or C_5. But this fact, together with rank restrictions implies that A does not have type B_6 or C_6, and consequently neither can A have type B_7 or C_7.

<u>Case II:</u> A has type F_4 and p = 2.

First note that $\langle\lambda,\alpha_j\rangle > 0$ for some $1 \leq j \leq 3$, else by induction, the

Main Theorem of [12] and the previous work of this result, $\lambda|T_A = q\mu_4$, for some p-power q, and $\dim V|A = 26 < \dim V|Y$. Let $\alpha = \alpha_4$, so $L_A' = B_3$ and $\dim V^1(Q_A) > 1$. If $L_Y' = D_7$, $V^1(Q_Y) \cong W$, the natural module for L_Y'. For otherwise, by the Main Theorem of [12], Q_Y/K_{β_1} is a tensor decomposable, irreducible L_A' module containing a nontrivial image of $Q_A{}^\alpha$. But now, the bound on $\dim V_{\beta_1}(Q_Y)$ implies $\lambda|T_Y = \lambda_8$. If $L_Y' = A_7$, the bound on $\dim V_{\beta_2}(Q_Y)$ implies $\lambda|T_Y = \lambda_1$, while $\lambda|T_A = q\mu_3 + x\mu_4$. Hence, $\dim V|A = 246\cdot 26^k$ or $4096\cdot 26^k$ for some $k \geq 0$. (See [8].) But then $\dim V|A \neq \dim V|Y$ by [8].

Suppose $L_Y' = A_5$. Since L_A' acts irreducibly on W, the natural module for L_Y', (2.3) implies that there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $Q_Y/K_\gamma \cong W$ or W^*. Also, by induction $V^1(Q_Y) \cong W$ or W^*. Hence, if $Y = E_6$, the bound on $\dim V_{\beta_2}(Q_Y)$ implies $\lambda|T_Y = \lambda_1$ or λ_6. But then [8] implies $\dim V|A \neq \dim V|Y$. So $Y = E_7$, $\Pi(L_Y) = \{\beta_j \mid j = 2,4,5,6,7\}$, and $\lambda|T_Y = x\lambda_3 + \lambda_\ell$ where $\ell = 2$ or 7. Also, $\lambda|T_A = q\mu_1 + y\mu_4$ and by [8], $\dim V|A = 26^k$; so $x \neq 0$. By induction, $\langle\lambda,\alpha_4\rangle = 0$ or q_1; so $\dim V|A \leq 26^2 < \dim V|Y$, by (1.32). So $L_Y' \neq A_5$.

It remains to consider the case where L_Y' has type D_4 and by induction $\lambda|T_A = xq\mu_1 + q\mu_3 + y\mu_4$ for some p-power q and x = 0 or 1. Also, since Q_Y/K_{β_i} must contain a nontrivial image of $Q_A{}^\alpha$, for i = 1 and 5, $\langle\lambda,\beta_3+\beta_5\rangle = 1$, $\langle\lambda,\beta_4\rangle = 0$ and $\langle\lambda,\beta_2\rangle = x$. As well, (2.3) implies $\langle\lambda,\beta_\ell\rangle = 0$ for $\ell \geq 7$. Applying induction to the C_3 maximal parabolic of A, we find that $Y = E_8$, $\lambda|T_Y = a\lambda_1 + \lambda_3$, and $\lambda|T_A = q\mu_3 + q\mu_4$. However, [8] and (1.32) imply $\dim V|A < \dim V|Y$. This completes the proof of (5.13).□

CHAPTER 6: INITIAL RANK TWO RESULTS

In this chapter, we will prove the Main Theorem in case $A = A_2$, $p>2$ and $Y = E_n$. The method of proof depends almost entirely on restricting our attention to the embedding of one of the two maximal parabolics of A. In fact, we will actually be studying the embedding of the maximal parabolic of any rank 2 group whose Levi factor is $\langle U_{\pm\beta} \rangle$, for β long. (Though we must assume $p > 3$, if $A = G_2$.) We establish a reasonably short list of possible such embeddings. (See (6.9).) For $A = A_2$, repeated applications of (1.23) usually enable us to determine the structure of $V|A$, by knowing this one embedding. The A_2 result is the following.

<u>Theorem 6.0.</u> (a) Let A be a simple algebraic group of type A_2, Y a simply connected, simple algebraic group of type E_n. Suppose $p>2$, $A < Y$ and $V|A$ is irreducible, for $V = V(\lambda)$ a nontrivial, restricted irreducible kY-module. Then, $p = 5$, $Y = E_6$ and $\lambda|T_Y = \lambda_1$ (or λ_6). Moreover, $\lambda|T_A = 2\mu_1 + 2\mu_2$, where μ_1 and μ_2 are the fundamental dominant weights corresponding to a fixed set of simple roots for $\Sigma(A)$.

(b) If $p \neq 2,5$ and $Y = E_6$, there exists a closed subgroup $B < Y$, $B \cong PSL_3(k)$ such that $V(\lambda_1)|B$ is irreducible.

<u>Remark:</u> The proof of (6.0)(b) is given in [16].

<u>Hypothesis:</u> For the remainder of this chapter we adopt Notation and Hypothesis (2.0) with rank $A = 2$, $\Sigma(Y)$ of type E_n, $p > 2$ and $p \neq 3$ when $A = G_2$. So (A,p) is not special. However, we will write $\Pi(A) = \{\alpha,\beta\}$, so $L_A = \langle U_{\pm\beta} \rangle T_A$ and write μ_α and μ_β for the corresponding fundamental

dominant weights. We take β to be the long root. Finally, assume $V|A$ is irreducible and $\langle \lambda, \beta \rangle \neq 0$.

(6.1). $L_Y{}'$ is not a simple algebraic group.

Proof: Suppose false. First note that Theorem (7.1) of [12] implies $L_Y{}'$ is not of type E_k. Thus, by (1.5), $Z_A \leq Z_Y$. Consider first the case $L_Y{}' = A_k$ for some k. By (2.14), k>1. Let W be the natural module for $L_Y{}'$. Then by (7.1) of [12], $V^1(Q_Y) \cong W$ or W^*. Thus, there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $(\gamma, \Sigma L_Y) \neq 0$ and $Q_Y/K_\gamma \cong W$ or W^*. Moreover, since $W \wedge W$ (or $W^* \wedge W^*$) has all even weights as an $L_A{}'$ module and $p > 2$, there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $(\gamma, \Sigma L_Y) \neq 0$ and $Q_Y/K_\gamma \cong W \wedge W$ or $W^* \wedge W^*$. These remarks imply that $L_Y{}' = A_{n-1}$, where $Y = E_n$, $n = 6, 7, 8$. Moreover, by (1.25), $\dim V_\gamma(Q_Y) \leq 2n$ in each case. However, it is an easy check to see that $\dim V_{\beta_2}(Q_Y) \geq \frac{1}{2}n(n-1)$ in each case. Contradiction.

Suppose $L_Y{}' = D_5$ and $V^1(Q_A)$ has high weight $(3q_1 + 3q_2)\mu_\beta$, as described in Theorem (7.1)(c) of [12]. Then there exists $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $(\gamma, \Sigma L_Y) \neq 0$ and Q_Y/K_γ is a 16-dimensional irreducible $L_A{}'$-module. Thus $Q_A \leq K_\gamma$, contradicting $\langle \lambda, \beta \rangle \neq 0$.

Since p>2, it remains to consider the case where $L_Y{}' = D_k$ for some k and $V^1(Q_Y) \cong W$, the natural module for $L_Y{}'$. Note that by (1.14), W is a tensor decomposable $L_A{}'$ module and $L_Y{}' = D_4$ or D_6. If $L_Y{}' = D_6$, $f_{134567}v^+$ affords an $L_Y{}'$ composition factor of $V_{\beta_1}(Q_Y)$ of dimension 32. However this exceeds the bound of (1.25). Hence, $L_Y{}' = D_4$ and $V^1(Q_A)$ has high weight $(q_1 + 3q_2)\mu_\beta$, for q_1 and q_2 distinct p-powers. One checks that $L_A{}'$ acts irreducibly on two of the three fundamental 8-dimensional irreducible $L_Y{}'$-modules. Hence, $L_A{}'$ acts irreducibly on Q_Y/K_{β_1} or on Q_Y/K_{β_6}, forcing $Q_A \leq K_{\beta_1}$ or K_{β_6}. But this contradicts (2.3).

This completes the proof of (6.1).□

Remark (6.2). (1) If there exists $1 \le i \le r$ with L_i of exceptional type, then $\dim V^1(Q_Y) > 1$, induction and Theorem (7.1) of [12] imply that $Y = E_8$, $L_Y' = L_1 \times L_2$, where $L_1 = E_6$ and $L_2 = \langle U_{\pm\beta_8}\rangle$ and $\lambda|T_Y = x\lambda_7 + c\lambda_8$, for some $p>x\ge 0$ and $p>c>0$. Now, $\dim V^2(Q_Y) \ge 27c$. Thus, the bound on $\dim V^2(Q_A)_{\lambda - q\alpha}$ and the description of $V^2(Q_A)$ in (1.22) imply that $k \ge 7$, where $V|A = V_1{}^{q_1} \otimes \cdots \otimes V_k{}^{q_k}$, as in (2.0). In particular, if $V|A$ is tensor indecomposable and $V^1(Q_A)$ is nontrivial, all components of L_Y' have classical type.

(2) We will often use without reference the fact that $U_{-2\alpha-\beta}\cdot U_{-3\alpha-\beta}\cdot U_{-3\alpha-2\beta} \le Q_A'$. This follows from the stated prime restrictions.

(6.3). If L_i has type D_{k_i} for some i and k_i, then Y has type E_8, $\lambda|T_Y = x\lambda_7 + c\lambda_8$, for $c>0$ and $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_4,\beta_5,\beta_8\}$ or $\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_8\}$.

Proof: Since all components of L_Y' are necessarily of classical type, (1.5) implies $Z_A \le Z_Y$. Moreover, by (6.1), L_Y' is not simple, so size restrictions imply $L_i = D_4$ or D_5 and Y has type E_7 or E_8.

Case I: Suppose M_i is nontrivial.

Consider first the case where $L_i = D_5$. Then, (7.1) of [12] and (1.14) imply $\rho_i(L_A')$ acts irreducibly on the two fundamental spin representations of L_i. Hence, $\langle U_{\pm\beta_1}\rangle \le L_i$, else Q_Y/K_{β_1} is a 16-dimensional irreducible L_A' module containing a nontrivial image of Q_A. The same argument applied to Q_Y/K_{β_6} implies $\langle U_{\pm\beta_7}\rangle \le L_Y'$. Thus, $L_Y' = L_i \times L_j$ where $L_j = \langle U_{\pm\beta_7}\rangle$ or $\langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$. Also, (2.7) implies M_j is trivial. Now, using (1.30) and (1.32), and recalling that $p>3$ in this configuration, we see that the bound on $\dim V_{\beta_6}(Q_Y)$ of (1.25) is exceeded. Thus, $L_i = D_4$.

Since $p\ne 2$, (7.1) of [12] and (1.14) imply $M_i|L_A'$ has high weight $(q_1+3q_2)\mu_\beta$, for q_1 and q_2 distinct p-powers. By considering $\rho_i(h_\beta(c))$, one checks that $\rho_i(L_A')$ acts irreducibly on two of the three restricted

8-dimensional irreducible L_i modules. On the third $\rho_i(L_A')$ acts with composition factors of dimensions 3 and 5. Thus, $Q_A \leq K_{\beta_1}$, contradicting (2.3). This completes the consideration of Case I.

Case II: Suppose M_i is trivial.

Note that there does not exist $\gamma \in \Pi(Y)-\Pi(L_Y)$ and $1 \leq j \leq r$, $j \neq i$ such that $(\Sigma L_i, \gamma) \neq 0 \neq (\Sigma L_j, \gamma)$. For otherwise, by size restrictions, M_j is nontrivial and (1.32) and (1.36) imply that the bound on $\dim V_\gamma(Q_Y)$ of (1.25) is exceeded in every possible such configuration. Thus, $Y = E_8$ and $L_Y' = L_i \times \langle U_{\pm\beta_8}\rangle$, where $L_i = \langle U_{\pm\beta_j} \mid 2 \leq j \leq 5\rangle$ or $\langle U_{\pm\beta_j} \mid 1 \leq j \leq 5\rangle$. Also, $\langle \lambda, \gamma\rangle = 0$ for all $\gamma \in \Pi(Y)-\Pi(L_Y)$ such that $(\gamma, \Sigma L_i) \neq 0$, else the bound on $\dim V_\gamma(Q_Y)$ is exceeded. Thus, (6.3) holds.□

(6.4). Let $\gamma \in \Pi(Y)-\Pi(L_Y)$ and $1 \leq i \neq j \leq r$ such that $(\Sigma L_i, \gamma) \neq 0 \neq (\Sigma L_j, \gamma)$. Then M_i or M_j is trivial.

Proof: Suppose false; i.e., suppose M_i and M_j are both nontrivial. By Theorem (7.1) of [12] and (6.3), L_i and L_j have type A_{k_i}, A_{k_j}, respectively, for some $k_i, k_j \geq 1$. Thus, by (1.5), $Z_A \leq Z_Y$. Let W_k denote the natural module for L_k, $k = i$, j. By (7.1) of [12], if rank $L_k > 1$, $M_k \cong W_k$ or W_k^* for $k = i$, j.

Case I: $V_{L_\ell}(-\gamma) \cong W_\ell$ or W_ℓ^* for $\ell = i$ and j.

By (2.5) and (2.7) and the preceeding remarks, $\gamma = \beta_4$ and there exists $k \neq i$, j, such that M_k is trivial and $(\Sigma L_k, \beta_4) \neq 0$. Then, (1.15) implies that $L_i \times L_j \times L_k$ has type $A_1 \times A_1 \times A_m$, $m = 1$ or 3, or $A_1 \times A_2 \times A_m$, $m = 2$, 3 or 4 (with a possible reordering of the triples). Then (1.36) implies that the bound on $\dim V_{\beta_4}(Q_Y)$ of (1.25) is exceeded unless the triples are $\{A_1, A_1, A_m\}$, $m = 1$ or 3, or $\{A_1, A_2, A_2\}$ with the A_1 component acting trivially on $V^1(Q_Y)$.

Suppose rank $L_m = 1$ for $m = i$, j, k. Let $\gamma_m \in \Pi(Y)$ such that $L_m = \langle U_{\pm\gamma_m}\rangle$ for $m = i$, j, k. Let q_i (respectively, q_j) be the field twist on the embedding of L_A' in $\langle U_{\pm\gamma_i}\rangle$ (respectively, $\langle U_{\pm\gamma_j}\rangle$), where $q_i \neq q_j$.

Then, by (2.7), we may assume that the field twist on the embedding of L_A' in $\langle U_{\pm\gamma_k}\rangle$ is also q_j. Then the L_A' composition factors of Q_Y/K_{β_4} have high weights $(2q_j + q_i)\mu_\beta$ and $q_i\mu_\beta$. Since $V_{T_Y}(\lambda-\gamma_i-\beta_4) \neq 0$, some element from the group $U_{-\beta_4}\cdot U_{-\gamma_i-\beta_4}$ must appear in the factorization of an element in Q_A-Q_A'. But $-\beta_4$ (respectively, $-\gamma_i-\beta_4$) affords $T(L_A')$ weight $(2q_j + q_i)\mu_\beta$ (respectively, $(2q_j - q_i)\mu_\beta$). Neither of these weights occurs in $(Q_A{}^\alpha)^{q_i}$. Thus, rank $L_m > 1$ for $m = i$, j, or k.

We consider now the case where $L_i\times L_j\times L_k$, in some reordering, has type $A_1\times A_1\times A_3$, so $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_5,\beta_6,\beta_7\}$. Let $\langle U_{\pm\beta_5},U_{\pm\beta_6},U_{\pm\beta_7}\rangle = L_0$. Let q_1 (respectively, q_2) be the field twist on the embedding of L_A' in $\langle U_{\pm\beta_2}\rangle$ (respectively, $\langle U_{\pm\beta_3}\rangle$). Since $p>2$, (1.15) implies $V_{L_0}(-\beta_4)|L_A'$ is tensor indecomposable; in particular, $p>3$. Let the field twist on the embedding of L_A' in L_0 be q_3. By (2.5), (2.6) and (2.7), $\{q_1,q_2,q_3\}$ consists of two distinct powers of p. It is then an easy check to see that there is no 2-dimensional L_A' composition factor of Q_Y/K_{β_4}. Thus $Q_A \leq K_{\beta_4}$, contradicting (2.3).

We have, therefore, $\Pi(L_i\times L_j\times L_k) = \{\beta_1,\beta_3,\beta_2,\beta_5,\beta_6\}$ with $\langle\lambda,\beta_2\rangle = 0$. The bound on $\dim V_{\beta_4}(Q_Y)$ of (1.25) implies that $\langle\lambda,\beta_1\rangle = 1 = \langle\lambda,\beta_6\rangle$ and $\langle\lambda,\beta_m\rangle = 0$ for $2\leq m\leq 5$. Now, $f_{134}v^+$ affords an L_Y' composition factor in $V_{\beta_4}(Q_Y)$ of dimension 14, if p=3, or 16 otherwise. Also, $\dim V_{T_Y}(\lambda-\beta_1-\beta_3-\beta_4-\beta_5-\beta_6) \geq 4$, by (1.34) and if p=3, a 1-space from this weight space occurs in the above mentioned composition factor, and otherwise, a 2-space from this weight space occurs. Hence, 3 (respectively, 2) distinct composition factors of $V_{\beta_4}(Q_Y)$ of dimension 2 are afforded by vectors in $V_{T_Y}(\lambda-\beta_1-\beta_3-\beta_4-\beta_5-\beta_6)$, if $p\neq 3$ (respectively, p=3). In either case, the bound on $\dim V_{\beta_4}(Q_Y)$ is exceeded. Thus, $L_i\times L_j\times L_k$ does not have type $A_2\times A_2\times A_1$ and this completes the consideration of Case I.

Case II: For $\ell = i$ or j, $V_{L_\ell}(-\gamma) \not\cong W_\ell$ or $W_\ell{}^*$.

Since W_ℓ is an irreducible L_A' module for $\ell = i$, j, if rank $L_\ell > 1$, there does not exist $\delta \in \Pi(Y)-\Pi(L_Y)$ such that $Q_Y/K_\delta \cong W_\ell$ or $W_\ell{}^*$, else $Q_A \leq K_\delta$. Applying (1.36) and the various techniques for obtaining lower bounds on dimensions of modules (e.g., (1.30), (1.32) and (1.35)), the bound on $\dim V_\gamma(Q_Y)$ of (1.25) restricts the situation still further. We find that $\Pi(L_i \times L_j) = \{\beta_1,\beta_2,\beta_4,\beta_5\}$ and p=3. In particular, $W_j|L_A{}'$ is tensor decomposable. Let q_1 be the field twist on the embedding of $L_A{}'$ in $\langle U_{\pm\beta_1}\rangle$. Suppose $W_j|L_A{}'$ has high weight $(q_2+q_3)\mu_\beta$, for q_2 and q_3 distinct p-powers, with $q_2 \neq q_1 \neq q_3$. Then, Q_Y/K_{β_3} has $L_A{}'$ composition factors with high weights $(q_1+2q_2)\mu_\beta$ and $(q_1+2q_3)\mu_\beta$. In particular, there is no 2-dimensional $L_A{}'$ composition factor of Q_Y/K_{β_3}, contradicting (2.3).

This completes the consideration of Case II and the proof of (6.4).□

<u>(6.5).</u> Suppose there exist distinct $1\leq i,j,k\leq r$ such that $(\Sigma L_m,\beta_4) \neq 0$ for m = i, j, k and suppose M_m is nontrivial for m = i, j or k. Then one of the following holds:

(i) $A \neq G_2$, $Y = E_6$, $\lambda|T_Y = \lambda_1$ or λ_6 and $\Pi(L_Y) = \{\beta_i \mid i \neq 4\}$.

(ii) $A \neq A_2$, $Y = E_8$, $\lambda|T_Y = \lambda_1$ and $\Pi(L_i \times L_j \times L_k) = \{\beta_1,\beta_3,\beta_2,\beta_5,\beta_6\}$.

(iii) $Y = E_7$, $p\neq 3$, $L_i \times L_j \times L_k$ has type $A_1 \times A_1 \times A_3$ and $\lambda|T_Y = x\lambda_1 + \lambda_7$, $p>x\geq 0$.

(iv) $A \neq A_2$, rank $L_m = 1$ for m = i, j, k, and $\dim(M_i \otimes M_j \otimes M_k) = 2 = \dim V^1(Q_Y)$ and $\langle\lambda,\beta_4\rangle = 0$.

<u>Proof</u>: Since all components of $L_Y{}'$ are necessarily of classical type, (1.5) implies $Z_A \leq Z_Y$. Let W_m denote the natural module for L_m, m = i, j, k. Then Theorem (7.1) of [12] implies that if rank$L_m > 1$ and M_m is nontrivial, $M_m \cong W_m$ or $W_m{}^*$, and (6.4) implies that only one of M_i, M_j and M_k is nontrivial. As well, (1.15) restricts the situation somewhat. We then use the bound on $\dim V_{\beta_4}(Q_Y)$ of (1.25) in conjunction with (1.36) to see that $\langle\lambda,\beta_4\rangle = 0$ and one of the following holds:

(a) rank $L_m = 1$ for m = i, j, k and $\dim(M_i \otimes M_j \otimes M_k) = 2$.

(b) $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_5,\beta_6,\beta_7\}$, $\langle\lambda,\beta_m\rangle = 0$ for $2\le m\le 6$ and $\langle\lambda,\beta_7\rangle = 1$.

(c) $L_i \times L_j \times L_k$ has type $A_2\times A_1\times A_2$, the A_1 component acts trivially on $V^1(Q_Y)$ and if M_ℓ is nontrivial for $\ell \in \{i,j,k\}$, $M_\ell \cong (V_{L_\ell}(-\gamma))^*$.

(d) $L_i \times L_j \times L_k$ has type $A_2\times A_1\times A_3$, $\langle\lambda,\beta_m\rangle = 0$ for $1\le m\le 6$, $\langle\lambda,\beta_7\rangle = 1$ and $V^1(Q_A)$ is a tensor decomposable L_A' module.

(Recall, $\lambda|T_Y \ne \lambda_8$ if $Y = E_8$.) In the configuration of (d), $Y = E_7$, else Q_Y/K_{β_8} is a 4-dimensional irreducible L_A'-module containing a nontrivial image of $Q_A{}^\alpha$. But now, using (1.23), (1.32) and (1.35), we find that $\dim V|A \ne \dim V|Y$. Thus the configuration of (d) does not occur.

Consider now the configuration of (c). If $Y = E_6$, $\dim V|Y = 27$. Using the methods of (1.30), (1.32), (1.33) and (1.35) we see that if A has type G_2, $\dim V|A > 27$. Thus, (i) holds. If $Y = E_7$, $Q_A \le K_{\beta_7}$ since Q_Y/K_{β_7} is a 3-dimensional irreducible L_A'-module. But then, $\lambda|T_Y = \lambda_1$. Consider the case where $Y = E_8$. We must study the image of Q_A in Q_Y/K_{β_4}. For the purposes of this argument we may assume $\langle\lambda,\beta_1\rangle = 1$, $\langle\lambda,\beta_m\rangle = 0$ for $2\le m\le 6$. Let q_1, q_2 and q_3 be the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle$, $\langle U_{\pm\beta_2}\rangle$ and $\langle U_{\pm\beta_5},U_{\pm\beta_6}\rangle$, respectively. By (2.7), q_1, q_2 and q_3 are not all distinct. We will show that, in fact, $q_1 = q_3$.

Suppose $q_1 = q_2 \ne q_3$. Then the L_A' composition factors of Q_Y/K_{β_4} have high weights $(3q_2+2q_3)\mu_\beta$ and $(q_1+2q_3)\mu_\beta$. (If p=3 and $3q_1 = q_3$, the weights are $3q_3\mu_\beta$, $q_3\mu_\beta$ and $(q_1+2q_3)\mu_\beta$.) Thus, (2.4) implies that p=3, $3q_1 = q_3$ and $\beta_4|Z_A = 9q_1\alpha$ or $3q_1\alpha$. Since $V_{T_Y}(\lambda-\beta_1-\beta_2-\beta_3-\beta_4) \ne 0$, a nonidentity element from the set $U_{-24}\cdot U_{-234}\cdot U_{-1234}$ must appear in the factorization of some element in $Q_A - Q_A'$. Also, $-\beta_2-\beta_4$ (respectively, $-\beta_2-\beta_3-\beta_4$, $-\beta_1-\beta_2-\beta_3-\beta_4$) affords $T(L_A')$ weight $7q_1\mu_\beta$ (respectively, $5q_1$, $3q_1$). So (2.4) implies $\beta_4|Z_A = 3q_1\alpha = q_3\alpha$. Then, examining the $T(L_A')$ weights of root group elements in Q_Y/K_{β_4}, we see that $x_{-\alpha}(t) = x_{-1234}(c_1t^{q_3})u_1$ and $x_{-\alpha-\beta}(t) = x_{-12345}(c_2t^{q_3})u_2$, where $c_i \in k^*$, $u_i \in K_{\beta_4}$. However, there is then a nontrivial contribution to the root group U_{-1345} in the expression for $[x_\beta(t), x_{-\alpha-\beta}(t)]$, contradicting (6.2)(2) and the given factorization of $x_\alpha(t)$. Thus, we do not have $q_1 = q_2 \neq q_3$. A similar argument shows that the configuration $q_1 \neq q_2 = q_3$ cannot occur. In fact, $q_1 = q_3$, regardless of the labelling of $\lambda|T_Y$.

If $Q_A \leq K_{\beta_7}$, $\lambda|T_Y = \lambda_1$. Suppose $Q_A \nleq K_{\beta_7}$. Then $\langle U_{\pm\beta_8}\rangle \leq L_Y'$, else Q_Y/K_{β_7} is a 3-dimensional irreducible L_A' module containing a nontrivial image of $Q_A{}^\alpha$. Then (2.7) implies that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_8}\rangle$ is $q_1 = q_3$. Thus, (2.5) implies $\langle\lambda,\beta_8\rangle = 0$. Moreover, $\langle\lambda,\beta_7\rangle = 0$ else the bound on $\dim V_{\beta_7}(Q_Y)$ of (1.25) is exceeded. Thus, $\lambda|T_Y = \lambda_1$ or λ_6. In either case, $A \neq A_2$, as (1.23) implies $\dim V|A < \dim V|Y$. If $\lambda|T_Y = \lambda_6$, (7.1) of [12] implies $\langle\lambda,\alpha\rangle = 0, q, 2q, 3q, 4q, 5q, q+q_0$ or $q+2q_0$, for q and q_0 distinct p-powers. Then (1.27) implies $\dim V|A \leq 2\cdot3^3\cdot7^2\cdot11$. But by (1.38) $\dim V|Y > \dim V|A$. Hence, $\lambda|T_Y \neq \lambda_6$ and (ii) holds.

For L_Y as in (b), $p\neq3$, else $V^1(Q_Y)$ is tensor decomposable and (1.15) is contradicted. Also $Y = E_7$, else Q_A/K_{β_7} is a 4-dimensional irreducible L_A'-module containing a nontrivial image of Q_A. Thus, (iii) holds.

Finally, we must consider the case where $\mathrm{rank}(L_m) = 1$ for $m = i, j, k$. Suppose q_0 is the field twist on the embedding of L_A' in two of the components L_i, L_j, L_k and q_1 is the twist on the embedding in the third. If $q_0 \neq q_1$, the L_A' composition factors of Q_A/K_{β_4} have high weights $(2q_0 + q_1)\mu_\beta$ and $q_1\mu_\beta$. Since $p>2$, (2.4) implies $\beta_4|Z_A = q_1\alpha$.

Temporarily label as follows: $\Pi(L_m) = \{\gamma_m\}$ for $m = i, j, k$, and let $\langle\lambda,\gamma_i\rangle \neq 0$. So $\langle\lambda,\gamma_j+\gamma_k\rangle = 0$. Let q_m be the field twist on the embedding of L_A' in L_m for $m = i, j, k$. Suppose $q_j = q_k \neq q_i$. From above, $\beta_4|Z_A = q_i\alpha$. However, $V_{T_Y}(\lambda-\gamma_i-\beta_4) \neq 0$ implies that a nonidentity element from the set $U_{-\beta_4}\cdot U_{-\gamma_i-\beta_4}$ must appear in the factorization of some element in Q_A-Q_A'. But $-\beta_4$ (respectively, $-\gamma_i-\beta_4$) affords $T(L_A')$ weight $(2q_j + q_i)\mu_\beta$ (respectively, $(2q_j - q_i)\mu_\beta$) which is not a weight in $(Q_A{}^\alpha)^{q_i}$ if $q_i \neq q_j$. Thus, $q_i = q_j$ or $q_i = q_k$.

We are now able to show that there does not exist $1 \le \ell \le r$, $\ell \notin \{i,j,k\}$ such that M_ℓ is nontrivial. For, suppose there exists such an ℓ. Then, $L_\ell = \langle U_{\pm\beta_7}\rangle$, $\langle U_{\pm\beta_8}\rangle$ or $\langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$. If $L_\ell = \langle U_{\pm\beta_7}\rangle$, (2.17) and (2.4) imply $Q_A \le K_{\beta_6}$, contradicting (2.3). If $L_\ell = \langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$, (2.17) implies that the field twists on the embeddings of $L_A{}'$ in $\langle U_{\pm\beta_5}\rangle$ and in L_ℓ are equal. Also, if this twist is q, $\beta_6|Z_A = q\alpha$. The above work on field twists and (2.5) imply $\beta_4|Z_A = q\alpha$. However, $\dim V_{\beta_4}(Q_Y) = 4\dim M_\ell = \dim V^2(Q_A)_{\lambda - q\alpha}$, by (1.22). But there is a nontrivial contribution to $V^2(Q_A)_{\lambda - q\alpha}$ from $V_{\beta_6}(Q_Y)$. Contradiction. Thus, $L_\ell = \langle U_{\pm\beta_8}\rangle$. If $Q_A \nleq K_{\beta_6}$, (2.8) implies that the field twists on the embedding of $L_A{}'$ in $\langle U_{\pm\beta_5}\rangle$ and in L_ℓ are equal. If $Q_A \le K_{\beta_6}$, we consider the $L_A{}'$ composition factors of $Q_Y(\beta_7,\beta_6)$. Since $p>2$, (2.10) and (2.11) give the same result Now proceed as before to produce a contradiction. We have, therefore, $\dim V^1(Q_Y) = 2$. Moreover, (1.23) implies $\dim V|A < \dim V|Y$ if $A = A_2$. Thus, (iv) holds and the proof of (6.5) is complete.□

<u>(6.6).</u> Let $\gamma \in \Pi(Y) - \Pi(L_Y)$. Suppose there exists a unique pair $1 \le i,j \le r$ such that $(\Sigma L_i,\gamma) \ne 0 \ne (\Sigma L_j,\gamma)$, L_i, L_j are of type A_{k_i}, A_{k_j} for some $k_i, k_j \ge 1$ and $\dim(M_i \otimes M_j) > 1$. Then, one of the following holds:

(i) $A = G_2$ (so $p>3$), $Y = E_8$, $\Pi(L_i \times L_j) = \{\beta_1,\beta_2,\beta_3,\beta_4,\beta_6\}$, $\lambda|T_Y = \lambda_1$, $\langle\lambda,\beta\rangle = 4q_1$ and $\langle\lambda,\alpha\rangle = q+q_1$, $2q_1+q$ or $3q_1+q$, for q and q_1 distinct p-powers

(ii) $A \ne A_2$, $Y = E_7$ or E_8, $\langle U_{\pm\beta_1}\rangle$ and $\langle U_{\pm\beta_2}, U_{\pm\beta_4}, U_{\pm\beta_5}\rangle$ are components of $L_Y{}'$, and $\langle U_{\pm\beta_7}\rangle \le L_Y{}'$. Also, $\langle\lambda,\beta_2\rangle = 1$, $\langle\lambda, \beta_\ell\rangle = 0$ for $\ell = 1, 3, 4, 5, 6, 7$ and $\dim V^1(Q_Y) = 4$.

(iii) $L_i \times L_j$ has type $A_1 \times A_2$ and $\langle\lambda,\gamma\rangle = 0$.

(iv) $A = A_2$ or B_2, $p>3$, $Y = E_7$, $\lambda|T_Y = \lambda_7$, $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_5,\beta_6,\beta_7\}$ and $V^1(Q_A)$ is a tensor indecomposable $L_A{}'$ module.

(v) $A = B_2$ or G_2, $Y = E_8$, $\lambda|T_Y = \lambda_2$ and $\Pi(L_Y) =$ $\{\beta_1,\beta_2,\beta_4,\beta_6,\beta_7,\beta_8\}$. Moreover, $\beta_3|Z_A = q\alpha = \beta_5|Z_A$, where q is the field twist on the embedding of L_A' in each component of L_Y'.

(vi) $A = B_2$ or G_2, $Y = E_7$ or E_8, L_i (respectively, L_j) has type A_1 (respectively, A_3), and $\langle\lambda,\gamma\rangle = 0$. $V_{L_m}(-\gamma) \cong W_m$ or W_m^*, where W_m is the natural module for L_m, for $m = i,j$. If $L_iv^+ \neq v^+$, then $L_jv^+ = v^+$ and $M_i \cong W_i$. If $L_jv^+ \neq v^+$, then $L_iv^+ = v^+$ and $M_j \cong (V_{L_j}(-\gamma))^*$. Moreover, in each case, $V_{L_j}(-\gamma)|L_A'$ is tensor decomposable and $\dim V^1(Q_Y) = \dim(M_i \otimes M_j)$.

In addition, if there exists $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\Sigma L_i) \neq 0$, $(\delta,\Sigma L_k) = 0$ for $k \neq i$ and $Q_Y/K_\delta \cong M_i$ or M_i^*, then either $k_i = 1$ or $V_\delta(Q_Y) = 0$.

<u>Proof</u>: Since all components of L_Y' are necessarily of classical type, (1.5) implies $Z_A \leq Z_Y$. Let W_k denote the natural module for L_k, $k = i, j$. Theorem (7.1) of [12] implies that if rank $L_k > 1$ and M_k is nontrivial, then $M_k \cong W_k$ or W_k^* for $k = i, j$. Also (6.4) implies that only one of M_i and M_j is nontrivial.

<u>Case I</u>: $V_{L_k}(-\gamma) \cong W_k \wedge W_k$ (or $W_k^* \wedge W_k^*$) for $k = i$ or j.

The bound on $\dim V_\gamma(Q_Y)$ of (1.25) and (1.15) restrict the situation considerably. We are left with the following possibilities for the type of $L_i \times L_j$: $A_1 \times A_k$, $k = 3, 4, 5$ with the A_1 component acting trivially on $V^1(Q_Y)$, or $A_3 \times A_3$. The cases $L_i \times L_j$ of type $A_3 \times A_3$ or $A_1 \times A_5$ are easily ruled out by standard arguments.

Consider now the case $L_i \times L_j$ of type $A_1 \times A_4$. The bound on $\dim V_\gamma(Q_Y)$ of (1.25) implies $\langle\lambda,\beta_2\rangle = 0 = \langle\lambda,\gamma\rangle$. Thus, if $Y = E_6$, $\lambda|T_Y = \lambda_1$ or λ_6 and $\dim V|Y = 27$. However, if we recall that $p \geq 5$ and apply (1.29), (1.32) and (1.23), it is not difficult to see that $\dim V|A > 27$ in every case. Thus, $Y = E_7$ or E_8. Moreover, standard arguments imply $\Pi(L_i \times L_j) = \{\beta_1,\beta_2,\beta_3,\beta_4,\beta_6\}$ and $\langle\lambda,\beta_1\rangle = 1$, $\langle\lambda,\beta_i\rangle = 0$ for $2 \leq i \leq 6$. We consider the image of Q_A in Q_Y/K_{β_5}. Let q_1 be the field twist on the embedding of L_A' in the A_4 component and let q_2 be the field twist on the embedding of L_A' in $\langle U_{\pm\beta_6}\rangle$. Examining the L_A' composition factors of

Q_Y/K_{β_5} we see that if $q_1 \neq q_2$ we must have $p = 5$ and $5q_1 = q_2$. Moreover, the only composition factor isomorphic to a twist of $Q_A{}^{\alpha}$ has twist q_1, so $\beta_5|Z_A = q_1\alpha$. Since $V_{T_Y}(\lambda-\beta_1-\beta_3-\beta_4-\beta_5) \neq 0$, a nonidentity element from the set $U_{-5}\cdot U_{-45}\cdot U_{-345}\cdot U_{-1345}$ must appear in the factorization of some element in $Q_A - Q_A{}'$. However, $-\beta_5$ (respectively, $-\beta_4-\beta_5$, $-\beta_3-\beta_4-\beta_5$, $-\beta_1-\beta_3-\beta_4-\beta_5$) affords $T(L_A{}')$ weight $11q_1\mu_\beta$ (respectively, $9q_1\mu_\beta$, $7q_1\mu_\beta$, $5q_1\mu_\beta$), none of which occur in $(Q_A{}^{\alpha})^{q_1}$. Thus, (2.4) implies $q_1 = q_2$.

We now claim that $\beta_5|Z_A = q_1\alpha$. Examining the $L_A{}'$ composition factors of Q_Y/K_{β_5} we see that this is clear if $p>7$. If $p=5$, possibly $\beta_5|Z_A = 5q_1\alpha$. But, in this case, examining the $T(L_A{}')$ weight vectors in Q_Y/K_{β_5}, we have $x_{-\alpha}(t) = x_{-45}(c_1t^{5q_1})x_{-56}(c_2t^{5q_1})w_1$ and $x_{-\alpha-\beta}(t) = x_{-(1,1,1,2,1,1)}(d_1t^{5q_1})\cdot x_{-(1,1,2,2,1,0)}(d_2t^{5q_1})w_2$, where c_i, $d_i \in k$, c_1 or c_2 nonzero, d_1 or d_2 nonzero, and $w_i \in K_{\beta_5}$. However, since a nonidentity element from the group U_{β_4} appears in the factorization of $x_\beta(t)$, there is a nontrivial contribution to the root group $U_{-123456}$ in the expression for $[x_{-\alpha-\beta}(t), x_\beta(t)]$. This contradicts the given factorization of $x_{-\alpha}(t)$. Thus, $\beta_5|Z_A \neq 5q_1\alpha$. If $p=7$, the $L_A{}'$ composition factors of Q_Y/K_{β_5} have high weights $7q_1\mu_\beta$, $5q_1\mu_\beta$, $3q_1\mu_\beta$, and $q_1\mu_\beta$ and $-\beta_5$ affords $T(L_A{}')$ weight $7q_1\mu_\beta$. By (2.16) $\beta_5|Z_A \neq 7q_1\mu_\beta$. So $\beta_5|Z_A = q_1\alpha$, as claimed.

Suppose $Y = E_7$; so $\lambda|T_Y = \lambda_1 + x\lambda_7$ where $p>x>0$. Thus, $Q_A \nleq K_{\beta_7}$ and since Q_Y/K_{β_7} is isomorphic to $(Q_A{}^{\alpha})^{q_1}$, $\beta_7|T_A = q_1\alpha$. However, $f_{\beta_7}v^+$ and $f_{1345}v^+$ afford 2 distinct $L_Y{}'$ composition factors in the Z_A weight space $V^2(Q_A)_{\lambda-q_1\alpha}$, exceeding the dimension bound of (1.22). So $Y \neq E_7$.

We now have $Y = E_8$. If $\langle U_{\pm\beta_8}\rangle$ is a component of $L_Y{}'$, (2.17) implies $Q_A \leq K_{\beta_7}$, so $\lambda|T_Y = \lambda_1$. If $\langle U_{\pm\beta_8}\rangle$ is not a component of $L_Y{}'$, $\langle\lambda,\beta_8\rangle = 0$ by (2.3) and $\langle\lambda,\beta_7\rangle = 0$ by (2.13). Again $\lambda|T_Y = \lambda_1$. Also, $f_{1345}v^+$ affords an $L_Y{}'$ composition factor in $V^2(Q_A)_{\lambda-q_1\alpha}$ of dimension 10, which is the upper bound on this dimension, by (1.22). So $\dim V^2(Q_A)_{\lambda-q_1\alpha} = 10$. This can occur only if q_1 has nonzero coefficient in the p-adic expansion of

$\langle\lambda,\alpha\rangle$. (See (1.22) for a description of $V^2(Q_A)_{\lambda-q_1\alpha}$.) Suppose now that $V|A$ is tensor indecomposable. We first claim that $Q_A \nleq K_{\beta_7}$. Otherwise, we have $P_A \leq P_Y\hat{} \geq B_Y^-$, a parabolic subgroup with Levi factor $L_Y\hat{} = \langle L_Y, U_{\pm\beta_7}\rangle$ and $Q_A \leq R_u(P_Y\hat{}) = Q_Y\hat{}$. But $\dim V^2(Q_Y\hat{})$ exceeds the bound on $\dim V^2(Q_A)$. So $Q_A \nleq K_{\beta_7}$, as claimed. This implies that $L_Y' = L_i \times L_j$. Moreover, $\beta_7|Z_A = q_1\alpha$ as in E_7. Also, $Q_A \leq K_{\beta_8}$, so if $P_2 \geq B_Y^-$ is the parabolic subgroup of Y with Levi factor $L_2 = \langle L_Y, U_{\pm\beta_8}\rangle$, then $P_A \leq P_2$ and $Q_A \leq R_u(P_2) = Q_2$. Now $[V,Q_A{}^2] = [V,Q_2{}^2]$, since $V_{\beta_5}(Q_2) = V_{\beta_5}(Q_Y) = V^2(Q_A)_{\lambda-q_1\alpha} = V^2(Q_A)$. By (1.20) $\dim V^3(Q_A) \leq 20$. But $f_{(1,1,1,2,2,1,0,0)}v^+$ and $f_{134567}v^+$ afford distinct L_2 composition factors in $V^3(Q_2)$ of dimensions 15 and 10, respectively. Contradiction. Hence, $V|A$ is tensor decomposable.

By induction and the above remarks, $\langle\lambda,\alpha\rangle = q_1+q$, q_1+2q, $2q_1+q$, $q+q_1+q_0$, q_1+3q or $3q_1+q$, for q and q_0 distinct p-powers, each distinct from q_1. So (1.23) implies that A does not have type A_2. By [8], $\dim V|Y = 3875$. Now, (1.27) implies $\dim V|A < \dim V|Y$ if $A = B_2$. If $A = G_2$, [8] and the methods of (1.30) and (1.32) imply $\dim V|A > 3875$ if $\langle\lambda,\alpha\rangle = q+q_0+q_1$, q_1+3q or q_1+2q. Thus, (6.6)(i) holds.

Consider now $L_i \times L_j$ of type $A_1 \times A_3$. Since the A_3 component acts nontrivially on $V^1(Q_Y)$, $\Pi(L_i \times L_j) = \{\beta_1,\beta_2,\beta_4,\beta_5\}$. Otherwise, Q_Y/K_{β_1} is a 4-dimensional irreducible L_A'-module with proper submodule $Q_AK_{\beta_1}/K_{\beta_1}$. A similar argument and (6.4) imply $Y = E_7$ or E_8 and $\langle U_{\pm\beta_7}\rangle$ is contained in a component of L_Y' which acts trivially on $V^1(Q_Y)$. The bound of (1.25) on $\dim V_{\beta_3}(Q_Y)$ and $\dim V_{\beta_6}(Q_Y)$ implies $\langle\lambda,\beta_2\rangle = 1$ and $\langle\lambda,\beta_k\rangle = 0$ for $k = 1$, 3, 4, 5, 6. Moreover, A does not have type A_2, else (1.23), (1.26) and (1.32) imply $\dim V|A < \dim V|Y$. Thus, if $L_i \times L_j$ has type $A_1 \times A_3$, (6.6)(ii) holds.

This completes the consideration of Case I.

<u>Case II</u>: $V_{L_k}(-\gamma) \cong W_k$ or $W_k{}^*$ for $k = i$, j.

We first note that since $p>2$, (1.15) allows us to reduce to the following pairs $L_i \times L_j$: $A_1 \times A_i$, $i = 2, 3, 4$, $A_2 \times A_3$ and $A_3 \times A_3$. Moreover, the bound on $\dim V_{\gamma}(Q_Y)$ and (1.36) imply $\langle\lambda,\gamma\rangle = 0$ in every case. Again, standard arguments show that $L_i \times L_j$ does not have type $A_3 \times A_3$.

Consider the configuration where (L_i,L_j) has type (A_2,A_3). Then $p>2$ and (1.15) imply that $W_j|L_A'$ is tensor indecomposable and so $p>3$. Let q be the field twist on the embedding of L_A' in L_i and in L_j. (There is only one twist by (2.7).) Temporarily label as follows: $\Pi(L_i) = \{\gamma_1,\gamma_2\}$, $\Pi(L_j) = \{\gamma_3,\gamma_4,\gamma_5\}$ with $(\gamma_2,\gamma) \neq 0 \neq (\gamma_3,\gamma)$ and $(\gamma_3,\gamma_4) \neq 0 \neq (\gamma_4,\gamma_5)$. Then there does not exist $\delta \in \Pi(Y)-\Pi(L_Y)$ such that $(\delta,\gamma_5) \neq 0$. For otherwise, $(\delta,\Sigma L_k) = 0$ for all $k\neq i$ so $Q_Y/K_\delta \cong W_i$ or W_i^*. Thus, $Q_A \leq K_\delta$ and M_i is trivial. But since $p>2$, (2.10) and (2.11) imply that there is a nontrivial image of $Q_A{}^\alpha$ in $Q_Y(\gamma,\delta)$, a 3-dimensional irreducible L_A' module. Contradiction. Similarly, there does not exist $\tau \in \Pi(Y)-\Pi(L_Y)$, $\tau\neq\gamma$ such that $(\tau,\gamma_3) \neq 0$. Otherwise, as above, $Q_A \leq K_\tau$ and (2.11) applies. However, $Q_Y(\gamma,\tau)$ has all even $T(L_A')$ weights and there can be no nontrivial image of $Q_A{}^\alpha$ in this module. (See (1.15).)

The considerations of the preceeding paragraph imply that either $Y = E_7$ with $\Pi(L_i \times L_j) = \{\beta_1,\beta_3,\beta_5,\beta_6,\beta_7\}$ or $Y = E_8$ with $\Pi(L_i \times L_j) = \{\beta_2,\beta_4,\beta_6,\beta_7,\beta_8\}$, $\{\beta_2,\beta_4,\beta_5,\beta_7,\beta_8\}$ or $\{\beta_3,\beta_4,\beta_6,\beta_7,\beta_8\}$. If $\Pi(L_i \times L_j) = \{\beta_3,\beta_4,\beta_6,\beta_7,\beta_8\}$, Q_Y/K_{β_k} is a 3-dimensional irreducible L_A' module, for $k = 1$ and 2, so $Q_A \leq K_{\beta_k}$. Hence $\langle\lambda,\beta_\ell\rangle = 0$ for $1\leq\ell\leq 4$. But then the bound on $\dim V_{\beta_5}(Q_Y)$ implies $\lambda|T_Y = \lambda_8$. If $\Pi(L_i \times L_j) = \{\beta_2,\beta_4,\beta_5,\beta_7,\beta_8\}$, standard arguments allow us to reduce to a special case of (6.6)(ii). If $\Pi(L_i \times L_j) = \{\beta_1,\beta_3,\beta_5, \beta_6,\beta_7\}$, the bound on $\dim V_{\beta_4}(Q_Y)$ and (2.3) imply $\lambda|T_Y = \lambda_1$ or λ_7. Thus, $\lambda|T_Y = \lambda_7$ and $\dim V|Y = 56$. Using (1.30) and (1.32), it is not difficult to see that $\dim V|A > 56$ if $A = G_2$. Thus, (6.6)(iv) holds.

Finally, suppose $\Pi(L_i \times L_j) = \{\beta_2,\beta_4,\beta_6,\beta_7,\beta_8\}$. Consider $Q_A K_{\beta_5}/K_{\beta_5} \leq Q_Y/K_{\beta_5}$. If we examine the L_A' composition factors of Q_Y/K_{β_5} and recall that $p>3$, we see that $\beta_5|Z_A = q\alpha$, if $p>5$. If $p=5$, $-\beta_5$ affords $T(L_A')$ weight $5q\mu_\beta$ so $\beta_5|Z_A \neq 5q\alpha$, by (2.16). So again

$\beta_5|Z_A = q\alpha$. Note that $\langle U_{\pm\beta_1}\rangle$ is a component of $L_Y{}'$. Otherwise, $\langle\lambda,\beta_1\rangle = 0$ by (2.3) and $\langle\lambda,\beta_i\rangle = 0$ for $i = 2, 3, 4$ as $Q_A \le K_{\beta_3}$; but then the bound on $\dim V_{\beta_5}(Q_Y)$ implies $\lambda|T_Y = \lambda_8$. Notice that if $Q_A \nleq K_{\beta_3}$, (2.17) implies that the field twist on the embedding of $L_A{}'$ in $\langle U_{\pm\beta_1}\rangle$ is also q, and that $\beta_3|Z_A = q\alpha$. Thus, $\langle\lambda,\beta_1\rangle = 0$, by (2.5). As well, the bound on $\dim V^2(Q_A)_{\lambda - q\alpha}$ of (1.22) implies $\lambda|T_Y = \lambda_2$. Also, if we apply (1.23) when $A = A_2$, we see that $\dim V|A < \dim V|Y$. Thus (6.6)(v) holds.

Now, consider $L_i \times L_j$ of type $A_1 \times A_4$. Temporarily label as follows: $\Pi(L_i \times L_j) = \{\gamma_0,\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$ with $(\gamma_0,\gamma) \ne 0 \ne (\gamma,\gamma_1)$ and $(\gamma_i,\gamma_{i+1}) \ne 0$ for $i = 1, 2, 3$. We have already $\langle\lambda,\gamma\rangle = 0$. Actually, the bound on $\dim V_\gamma(Q_Y)$ implies $\langle\lambda,\gamma_k\rangle = 0$ for $k = 0, 1, 2, 3$ and $\langle\lambda,\gamma_4\rangle = 1$. Let W be the natural module for the A_4 component. Then $L_A{}'$ acts irreducibly on W and W^*. Moreover, $L_A{}'$ acting on $W \wedge W$ (or $W^* \wedge W^*$) has all even weights. Thus, since $p>2$, there does not exist $\tau \in \Pi(Y) - \Pi(L_Y)$ such that $(\tau,\gamma_i) \ne 0$ for some $1 \le i \le 4$ and $(\tau,\Sigma L_k) = 0$ for all $k \ne i, j$. These considerations imply that $Y = E_8$ and either (a) $\Pi(L_Y) = \{\beta_i \mid i = 3, 5 \le i \le 8\}$, with $\langle\lambda,\beta_8\rangle = 1$ or (b) $\Pi(L_i \times L_j) = \{\beta_i \mid i = 2, 5 \le i \le 8\}$, with $\langle\lambda,\beta_8\rangle = 1$ or (c) $\Pi(L_i \times L_j) = \{\beta_2,\beta_4,\beta_5, \beta_6,\beta_8\}$, with $\langle\lambda,\beta_2\rangle = 1$. As well, (2.17) implies that there exists a p-power, q, such that q is the field twist on the embedding of $L_A{}'$ in L_i and in L_j and such that $\gamma|Z_A = q\alpha$.

In the configuration of (a), suppose that $Q_A \nleq K_{\beta_1}$. Then Q_Y/K_{β_1} is isomorphic to $(Q_A{}^\alpha)^q$ as $L_A{}'$-module,so (2.4) implies $\beta_1|Z_A = q\alpha$. But $\beta_4|Z_A = q\alpha$; so the bound on $\dim V^2(Q_A)_{\lambda - q\alpha}$ of (1.22) implies $\langle\lambda,\beta_1\rangle = 0$. Also, if $Q_A \le K_{\beta_1}$, $\langle\lambda,\beta_1\rangle = 0$. But then in either case $\lambda|T_Y = \lambda_8$.

If $L_i \times L_j$ is as in (b), $\langle U_{\pm\beta_1}\rangle \le L_Y{}'$, else (2.3) and the preceeding remarks imply $\lambda|T_Y = \lambda_8$. If $Q_A \nleq K_{\beta_3}$, (2.8) implies $\beta_3|Z_A = q\alpha$. Since Q_Y/K_{β_3} is a 2-dimensional $L_A{}'$ irreducible, the field twist on the embedding of $L_A{}'$ in $\langle U_{\pm\beta_1}\rangle$ is q and therefore $\langle\lambda,\beta_1\rangle = 0$, by (2.5). Moreover, $\langle\lambda,\beta_3\rangle = 0$, else the bound on $\dim V^2(Q_A)_{\lambda - q\alpha}$ is exceeded. Also, if $Q_A \le K_{\beta_3}$, $\langle\lambda,\beta_1 + \beta_3\rangle = 0$. But in either case $\lambda|T_Y = \lambda_8$.

If $L_i \times L_j$ is as in (c), previous remarks imply $\langle U_{\pm\beta_1}\rangle \le L_Y{}'$. However, now we have a configuration of Case I which was ruled out. This completes the consideration of $L_i \times L_j$ of type $A_1 \times A_4$.

We now consider (L_i, L_j) of type (A_1, A_3). We have $\langle \lambda, \gamma \rangle = 0$. Moreover, the bound on $\dim V_\gamma(Q_Y)$ of (1.25) implies that if M_i is nontrivial, $M_i \cong W_i$ and if M_j is nontrivial $M_j \cong (V_{L_j}(-\gamma))^*$ ($\cong W_j$ or $W_j{}^*$ by the first paragraph of the proof). As well, (1.15) implies $W_j|L_A{}'$ is tensor decomposable. Temporarily label as follows: $\Pi(L_i) = \{\gamma_0\}$, $\Pi(L_j) = \{\gamma_1, \gamma_2, \gamma_3\}$, $(\gamma_1, \gamma) \ne 0$, $(\gamma_i, \gamma_{i+1}) \ne 0$, $i = 1, 2$. Let q be the field twist on the embedding of $L_A{}'$ in L_i. Say $V_{L_j}(-\gamma)|L_A{}'$ has high weight $(q + q_0)\mu_\beta$. (The power q appears by (2.7).) Then the $L_A{}'$ composition factors of Q_Y/K_γ have high weights $(2q + q_0)\mu_\beta$ and $q_0\mu_\beta$. Since $p>2$, $\gamma|Z_A = q_0\alpha$, by (2.4).

<u>Claim:</u> If $\dim M_k > 1$ for some k, then k = i or j.

Reason: Suppose false; i.e., suppose there exists $k \ne i, j$ such that $\dim M_k > 1$. If L_k is separated from L_i or L_j by exactly one node of the Dynkin diagram, size restrictions, (6.4) and the work of this result thus far imply $Y = E_8$ and $\Pi(L_Y) = \{\beta_1, \beta_4, \beta_5, \beta_6, \beta_8\}$. Also, $\langle \lambda, \beta_1 \rangle = 1 = \langle \lambda, \beta_8 \rangle$ and $\langle \lambda, \beta_i \rangle = 0$, $3 \le i \le 7$. Let q_1, q_2 be the distinct field twists on the embeddings of $L_A{}'$ in $\langle U_{\pm\beta_1}\rangle$, $\langle U_{\pm\beta_8}\rangle$, respectively. (They must be distinct by (2.5).) Then, by the preceeding paragraph, Q_Y/K_{β_2} is an irreducible $L_A{}'$-module with high weight $(q_1 + q_2)\mu_\beta$ and $\beta_3|Z_A = q_2\alpha$ and $\beta_7|Z_A = q_1\alpha$. For the purposes of this argument, we may assume $h_\beta(c) = h_{\beta_1}(c^{q_1})h_{\beta_4}(c^{q_1+q_2})h_{\beta_5}(c^{2q_2})h_{\beta_6}(c^{q_1+q_2})h_{\beta_8}(c^{q_2})$. Examining the $T(L_A{}')$ weight vectors in Q_Y/K_{β_3} and Q_Y/K_{β_7}, we find that $x_{-\alpha}(t) = x_{-13}(c_1t^{q_2})\cdot x_{-34}(c_2t^{q_2})x_{-78}(d_1t^{q_1})x_{-567}(d_2t^{q_1})w$, where $c_i, d_i \in k$, c_1 or c_2 nonzero, d_1 or d_2 nonzero, and $w \in K_{\beta_3} \cap K_{\beta_7}$. Suppose $c_2 = 0$. Since $V_{T_Y}(\lambda - \beta_1 - \beta_3 - \beta_4) \ne 0$, a nonidentity element from the group U_{-134} must occur in the factorization of some element from $Q_A - Q_A{}'$. However, $-\beta_1 - \beta_3 - \beta_4$ affords $T(L_A{}')$ weight $(q_2 - 2q_1)\mu_\beta$, which does not occur in

$(Q_A{}^{\alpha})^{q_2}$, contradicting (2.4). Thus, $c_2 \neq 0$. A similar argument shows that $d_2 \neq 0$. But $c_2 d_2 \neq 0$ contradicts (2.8).

Thus, L_k is separated from L_i and L_j by more than one node of the Dynkin diagram. Size restrictions imply $Y = E_7$ or E_8 and $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_5,\beta_6,\beta_7\}$. Since $Q_A \nleq K_{\beta_3}$ and Q_Y/K_{β_3} is a 2-dimensional irreducible $L_A{}'$-module, $\beta_3|T_A = q_0\alpha$ where q_0 is the field twist on the embedding of $L_A{}'$ in $\langle U_{\pm\beta_1}\rangle$. Then (2.8) implies $\beta_4|Z_A = q_0\alpha$. Thus, if $L_j = \langle U_{\pm\beta_5}, U_{\pm\beta_6}, U_{\pm\beta_7}\rangle$, $V_{L_j}(-\beta_4)|L_A{}'$ has high weight $(q+q_0)\mu_\beta$ and the field twist on the embedding of $L_A{}'$ in $\langle U_{\pm\beta_2}\rangle$ is q. (See the previous general work on $Q_A K_\gamma/K_\gamma \leq Q_Y/K_\gamma$, in this configuration.) Thus, (2.5) implies that M_j is trivial and so $\langle\lambda,\beta_2\rangle = 1$, $\langle\lambda,\beta_i\rangle = 0$ for $4 \leq i \leq 7$. Now, using (1.36) if $\langle\lambda,\beta_3\rangle \neq 0$, we see that the bound on $\dim V^2(Q_A)_{\lambda - q_0\alpha}$ of (1.22) is exceeded. This completes the proof of the Claim.

Now apply (1.23) when $A = A_2$ to find that $\dim V|A < \dim V|Y$, unless $Y = E_7$ and $\lambda|T_Y = \lambda_7$. But then $\dim V|Y = 56$ and $\dim V|A = 8$ or 64 (7 or 49 if p=3). (See (1.36).) Thus, A does not have type A_2. Moreover, (1.23) also implies Y does not have type E_6. Thus, (6.6)(vi) holds. This completes the consideration of Case II.

The final statement of (6.6) follows from (2.3).□

<u>(6.7).</u> Suppose there exists $1 \leq i \leq r$ such that L_i is separated from all other components of $L_Y{}'$ by more than one node of the Dynkin diagram and such that M_i is nontrivial. Then rank $L_i = 1$ and $\dim V^1(Q_Y) = \dim M_i$.

<u>Proof</u>: Since all components of $L_Y{}'$ are necessarily of classical type, (1.5) implies $Z_A \leq Z_Y$. Moreover, by (6.3), L_i has type A_k for some k. Let W denote the natural module for L_i. Theorem (7.1) of [12] implies $M_i \cong W$ or W^* if rank$L_i > 1$. Thus, if rank $L_i > 1$, there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $(\gamma, \Sigma L_i) \neq 0$ and $V_{L_i}(-\gamma) \cong W$ or W^*, as Q_Y/K_γ would be a (rank(L_i)+1)-dimensional irreducible $L_A{}'$-module containing a nontrivial image of Q_A. Thus, if rank $L_i > 1$, $\Pi(L_i) = \{\beta_1,\beta_2,\beta_3,\beta_4\}$.

However, (1.15) implies that $Q_A \leq K_{\beta_5}$, contradicting (2.3). So rank $L_i = 1$. Let q be the field twist on the embedding of L_A' in L_i. We will suppose $\dim V^1(Q_Y) > \dim M_i$. In particular, there exists $1 \leq m \leq r$, $m \neq i$, such that M_m is nontrivial. Thus, by (6.3) each component of L_Y' has type A_{m_k}, for some $m_k \geq 1$.

Now, (6.5), (6.6) and the above remarks imply that either

(a) there exists $k \neq i,m$ such that $L_m \times L_k$ has type $A_1 \times A_2$ (or$A_2 \times A_1$), ΠL_m and ΠL_k are separated by exactly one node of the Dynkin diagram, corresponding to a simple root γ, and $(\Sigma L_j, \gamma) = 0$ for $j \neq k,m$, or

(b) rank$L_m = 1$ and $\Pi(L_m)$ is separated by more than one node of the Dynkin diagram from all other components of L_Y'.

<u>Claim.</u> There does not exist $\delta, \tau \in \Pi(Y) - \Pi(L_Y)$ such that $(\delta, \tau) < 0$, $(\delta, \Sigma L_i) \neq 0$ and $(\tau, \Sigma L_Y) = 0$.

Reason: Suppose false. Then by (2.13) and (2.3), $\langle \lambda, \delta \rangle = 0 = \langle \lambda, \tau \rangle$, $\delta|T_A = q\alpha$ and $\tau|T_A = 0$. Also, by (2.15), ΠL_i corresponds to an end node of the Dynkin diagram and $\Pi L_i \neq \{\beta_1\}$. If $\Pi L_i = \{\beta_2\}$, (2.15) implies $\tau = \beta_5$ and $\Pi(L_Y) = \{\beta_2, \beta_1, \beta_7\}$. If $\Pi L_i = \{\beta_7\}$, we argue similarly that $\Pi(L_Y) = \{\beta_7, \beta_1, \beta_2\}$ or $\{\beta_7, \beta_1, \beta_3, \beta_2\}$. Finally, if $\Pi L_i = \{\beta_8\}$, $\Pi(L_Y) = \{\beta_8, \beta_1, \beta_2, \beta_4\}$.

Consider the case where $\Pi(L_Y) = \{\beta_1, \beta_2, \beta_7\}$. Then $Q_A \nleq K_{\beta_3}$ and $Q_A \nleq K_{\beta_4}$; otherwise, $p > 2$, (2.10) and (2.11) imply that there is a nontrivial image of $Q_A{}^{\alpha}$ in $Q_Y(\beta_3, \beta_4)$, contradicting (1.15). Now (2.8) implies that the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_1} \rangle$ and in $\langle U_{\pm\beta_2} \rangle$ are equal; call this twist q. (In particular, only one of $\langle \lambda, \beta_1 \rangle$ and $\langle \lambda, \beta_2 \rangle$ is nonzero.) Let $q_0 \neq q$ be the field twist on the embedding of L_A' in $\langle U_{\pm\beta_7} \rangle$. Then $\beta_3|T_A = q\alpha = \beta_4|T_A$, $\beta_5|T_A = 0$ and $\beta_6|T_A = q_0\alpha$. Moreover, $\langle \lambda, \beta_4 + \beta_5 + \beta_6 \rangle = 0$. If $\langle \lambda, \beta_2 \rangle \neq 0$, $\langle \lambda, \beta_3 \rangle = 0$, else $(V_{\beta_3}(Q_Y) \oplus V_{\beta_4}(Q_Y)) \leq V^2(Q_A)_{\lambda - q\alpha}$ and the bound on $\dim V^2(Q_A)_{\lambda - q\alpha}$ is exceeded. But now we see that there is no vector in $V|Y$ of weight $\lambda - q_0\alpha$, contradicting (2.14). Hence, $\langle \lambda, \beta_1 \rangle \neq 0$ and $\langle \lambda, \beta_2 \rangle = 0$. If $\langle \lambda, \beta_3 \rangle \neq 0$, $f_{34}v^+$ and $f_{345}v^+$ are 2 linearly independent vectors in $V_{T_A}(\lambda - 2q\alpha)$, contradicting (1.31). So $\langle\lambda,\beta_3\rangle = 0$ and again (2.14) is contradicted. Hence, $\Pi(L_Y) \neq \{\beta_1,\beta_2,\beta_7\}$.

If $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_2,\beta_7\}$, let $P_Y\hat{}$ be the parabolic of (2.12), so $P_Y\hat{}$ has Levi factor $L_Y\hat{} = \langle L_Y, U_{\pm\beta_5}\rangle$. Then the bound on $\dim V_{\beta_4}(R_u(P_Y\hat{}))$ implies that $\langle\lambda,\beta_2+\beta_3+\beta_4\rangle = 0$ and $\langle\lambda,\beta_1\rangle = 1$. By (2.17), there is a p-power q_0 which is the field twist on the embedding of L_A' in $\langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle$ and $\langle U_{\pm\beta_2}\rangle$ and such that $\beta_4|Z_A = q_0\alpha$. Examining the $T(L_A)$ weight vectors in Q_Y/K_{β_4}, we have $x_{-\alpha}(t) = x_{-\beta_3-\beta_4}(c_1t^{q_0})\cdot x_{-\beta_2-\beta_4}(c_2t^{q_0})u_0$, for $c_i\in k$, c_1 or c_2 nonzero and $u_0\in K_{\beta_4}$. Hence $(\beta_3+\beta_4)|T_A = q_0\alpha$ and $\beta_4|T_A = q_0(\alpha-\beta)$. But now one can check that there is no vector in $V|Y$ with weight $\lambda-q\alpha$, where q is the field twist on the embedding of L_A' in L_i, contradicting (2.14).

Finally, consider the case where $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_4,\beta_8\}$. Let q_0 be the field twist on the embedding of L_A' in $\langle U_{\pm\beta_1}\rangle$ and $\langle U_{\pm\beta_2},U_{\pm\beta_4}\rangle$. Then $Q_A \leq K_{\beta_5}$, so $\langle\lambda,\beta_2+\beta_4+\beta_5+\beta_6+\beta_7\rangle = 0$. Also, $-\beta_5$ is not involved in L_A', else $\beta_5|Z_A = 0$ and the bound on $\dim V^2(Q_A)_{\lambda-q_0\alpha}$ is exceeded. Let $P_Y\hat{}$ be the parabolic of (2.12); so $P_Y\hat{}$ has Levi factor $L_Y\hat{} = \langle L_Y, U_{\pm\beta_6}\rangle$. Then, $Q_A \leq K_{\beta_5} \leq R_u(P_Y\hat{}) = Q_Y\hat{}$ and (2.11) implies that there is a nontrivial image of $Q_A{}^{\alpha}$ in $Q_Y\hat{}(\beta_7,\beta_5)$. But this implies that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_8}\rangle$ is also q_0, contradicting (2.5).

This completes the proof of the claim.

Now consider the configuration of (a). Note that by size retrictions and the above claim $\Pi(L_i)$ is separated by exactly two nodes of the Dynkin diagram from $\Pi(L_k\times L_m)$. In fact, $\Pi(L_k\times L_m) = \{\beta_5,\beta_7,\beta_8\}$ or $\{\beta_5,\beta_6,\beta_8\}$, $\Pi(L_i) = \{\beta_1\}$ and in each case $\langle U_{\pm\beta_2}\rangle$ is a component of L_Y'. For in the other possible configurations (2.8), (2.17), and (2.18) would force the field twist on the embedding of L_A' in L_m to be q, contradicting (2.5) and (2.6). In fact, $\Pi(L_k\times L_m) \neq \{\beta_5,\beta_6,\beta_8\}$. For otherwise, $Q_A \nleq K_{\beta_4}$ and we argue that $L_m = \langle U_{\pm\beta_8}\rangle$ and the field twist on the embedding of L_A' in L_m is q. This again produces a contradiction.

Consider now $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_5,\beta_7,\beta_8\}$, with $\langle\lambda,\beta_1\rangle \neq 0$. Since $p>2$, (1.15) implies $Q_A \leq K_{\beta_4}$. However, $-\beta_4$ is not involved in L_A'; for otherwise, $\beta_4|Z_A = 0$ and using the parabolic $P_Y\hat{}$ of (2.11), we see that the bound on $\dim V_{\beta_3}(Q_Y\hat{})$ is exceeded. If q is the field twist on the embedding of L_A' in $\langle U_{\pm\beta_1}\rangle$, then $q \neq q_0$, where q_0 is the field twist on the embedding of L_A' in $\langle U_{\pm\beta_5}\rangle$ and $\langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$. By (2.11), there is a nontrivial image of $Q_A{}^\alpha$ in $Q_Y(\beta_6,\beta_4)$; so the field twist on the embedding of L_A' in $\langle U_{\pm\beta_2}\rangle$ is also q_0. Now there is a nontrivial image of $Q_A{}^\alpha$ in the L_Y' module $Q_Y(\beta_3,\beta_4)$, which has L_A' composition factors of high weights $(2q_0+q)\mu_\beta$ and $q\mu_\beta$. Hence, $(\beta_3+\beta_4)|Z_A = q\alpha$ and $\beta_4|Z_A = 0$. But as above, the bound on $\dim V_{\beta_3}(Q_Y\hat{})$ of (1.25) is exceeded. Thus, the configuration of (a) cannot occur.

It remains to consider the case where $\operatorname{rank}(L_m) = 1$ and L_m is separated from all other components of the Dynkin diagram by more than one node of the Dynkin diagram. Note that the above claim implies that if ΠL_i or $\Pi L_m = \{\beta_\ell\}$, then $\ell \notin \{3,4,5\}$ and if $Y = E_8$, $\ell \neq 6$. Also, (2.8) implies that $\Pi(L_i)$ and $\Pi(L_m)$ are not separated by exactly two nodes of the Dynkin diagram. Finally, there does not exist $\tau \in \Pi(L_Y)$ such that $L_Y' = L_i \times L_m \times \langle U_{\pm\tau}\rangle$ and $\Pi(L_i)$ and $\Pi(L_m)$ are separated from τ by exactly two nodes of the Dynkin diagram. For otherwise, $p>2$, (2.18) and (2.8) imply that the field twists on the embeddings of L_A' in L_i, $\langle U_{\pm\tau}\rangle$ and L_m are all equal. These remarks and the above claim allow us to reduce to $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_5,\beta_8\}$. Then since $p>2$, $Q_A \leq K_{\beta_4}$, so $\Pi(L_i \times L_m) = \{\beta_1,\beta_8\}$. Let q (respectively, q_0) be the field twist on the embedding of L_A' in $\langle U_{\pm\beta_1}\rangle$ (respectively, $\langle U_{\pm\beta_8}\rangle$). Then (2.18) and (2.8) imply that q_0 is also the field twist on the embedding of L_A' in $\langle U_{\pm\beta_5}\rangle$. As in the previous case, (2.11) implies that there is a nontrivial image of $Q_A{}^\alpha$ in $Q_Y(\beta_3,\beta_4)$. Thus, the field twist on the embedding of L_A' in $\langle U_{\pm\beta_2}\rangle$ is either q or q_0.

Since $p>2$, examining the L_A' composition factors of $Q_Y(\beta_3,\beta_4)$ gives that $(\beta_3+\beta_4)|Z_A = q\alpha$ or $q_0\alpha$ (depending on the field twist on the embedding of L_A' in $\langle U_{\pm\beta_2}\rangle$). But if $(\beta_3+\beta_4)|Z_A = q\alpha$, then $\beta_4|Z_A = 0$, as $\beta_3|Z_A = q\alpha$. Using the parabolic $P_Y\hat{}$ of (2.11), we see that the bound on $\dim V_{\beta_3}(Q_Y\hat{})$ is exceeded. Thus, $(\beta_3+\beta_4)|Z_A = q_0\alpha$. However, using (1.36) if $\langle\lambda,\beta_1\rangle$ or $\langle\lambda,\beta_8\rangle = p-1$, we see that the contribution of $V_{\beta_7}(Q_Y)$ and the L_Y' composition factor(s) afforded by $f_{\beta_3+\beta_4}v^+$ and/or $f_{\beta_1+\beta_3+\beta_4}v^+$ exceed the bound on $\dim V^2(Q_A)_{\lambda-q_0\alpha}$.

This completes the proof of (6.7).□

<u>(6.8).</u> Let $\gamma \in \Pi(Y)-\Pi(L_Y)$ such that there exists a unique pair $1\le i,j\le r$ with $(\Sigma L_i,\gamma) \neq 0 \neq (\Sigma L_j,\gamma)$ and L_i has type A_1, L_j has type A_2 and $\dim(M_i\otimes M_j) > 1$. Then $\dim V^1(Q_Y) \le 3$ and if $A = A_2$, then $Y = E_6$.

<u>Proof</u>: Suppose there exists $k \neq i$, j such that M_k is nontrivial. By size restrictions, L_m is necessarily of classical type for all $1\le m\le r$, so by (1.5), $Z_A \le Z_Y$. Also, by size restrictions, L_k has type A_{n_k} for some n_k. So (6.5), (6.6) and (6.7) imply L_k has type A_1 or A_2 and is separated by exactly one node of the Dynkin diagram, corresponding to a root $\delta \in \Pi(Y)-\Pi(L_Y)$, from a component of L_Y' of type A_2 or A_1, respectively. Moreover, there are exactly two components of L_Y' whose root systems are not orthogonal to δ. By size restrictions, $(\delta,\Sigma(L_i\times L_j)) \neq 0$. However, (2.17) implies that the field twists on the embeddings of L_A' in L_i, L_j and L_k are equal, contradicting (2.5) and (2.6). Hence, M_ℓ is trivial for all $\ell \neq i$, j.

Now (6.6) implies $\langle\lambda,\gamma\rangle = 0$. Moreover, if $L_i = \langle U_{\pm\gamma_i}\rangle$ for $\gamma_i \in \Pi(L_Y)$, and if $\langle\lambda,\gamma_i\rangle \neq 0$, the bound on $\dim V_\gamma(Q_Y)$ implies $\langle\lambda,\gamma_i\rangle \le 2$. Thus, $\dim V^1(Q_Y) = \max\{ \dim M_i, \dim M_j \} \le 3$. Then, if $Y = E_7$ or E_8, A does not have type A_2, as (1.23) shows $\dim V|A \le 27 < \dim V|Y$. This completes the proof of (6.8).□

<u>Proof of (6.0)</u>: Let α_1, α_2, μ_1 and μ_2 be as in the statement of (6.0). Without loss of generality, we may assume $\langle\lambda,\alpha_1\rangle \neq 0$. Let

$L_A = \langle U_{\pm\alpha_1}\rangle T_A$ and fix notation as before. By (6.1), L_Y' is not a simple algebraic group.

Suppose there exists $1 \le i \le r$ such that L_i has type D_k for some k. Then by (6.3), (1.23) and (1.10), $Y = E_8$, $\lambda|T_Y = x\lambda_7 + c\lambda_8$, $\lambda|T_A = c\mu_1 + c\mu_2$ and $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_4,\beta_5,\beta_8\}$ or $\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_8\}$. Also, (2.9) implies that $Q_A \le K_{\beta_6}$. Then using the parabolic $P_Y\hat{}$ of (2.11) and applying (1.36), we see that the bound on $\dim V^2(Q_A)$ implied by (1.22) is exceeded. Thus, there does not exist i such that L_i has type D_k.

Suppose there exist distinct $1 \le i,j,k \le r$ such that $(\Sigma L_m,\beta_4) \ne 0$ for m = i, j, k and such that M_m is nontrivial for m = i, j or k. Then (6.5)(i) or (iii) holds. If (6.5)(i) holds, it is established in the proof that the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$ and in $\langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$ are equal. Call this twist q. Then by (1.23) and (1.10), $\lambda|T_A = 2\mu_1 + 2\mu_2$. Now, $\dim V|Y = 27$; but (1.36) implies $\dim V|A < 27$ if p=5. Hence, (6.0)(a) holds.

Now, consider the configuration described in (6.5)(iii). Then, by (1.23), $\lambda|T_A = (q_1+q_2)(\mu_1 + \mu_2)$ for q_1 and q_2 distinct p-powers, or $\lambda|T_A = q(3\mu_1 + 3\mu_2)$, for some p-power q, p>3. If $x \ne 0$, (1.32) and (1.26) imply $\dim V|A < \dim V|Y$. Thus, x=0 and $\dim V|Y = 56$. If $\lambda|T_A = (q_1+q_2)(\mu_1 + \mu_2)$, $\dim V|A = 64$ or p=3 and $\dim V|A = 49$. (Use (1.35).) Hence, $\lambda|T_A = q(3\mu_1 + 3\mu_2)$ and p>3. So, by (1.10), q=1. The Weyl module for A with high weight $3\mu_1 + 3\mu_2$ has dimension 64 by (1.27). Using (1.33) and the fact that p>3, we see that $\dim V|A < 64$ if and only if p=7 and $\dim V_{T_A}(\lambda-\alpha_1-\alpha_2) = 1$ or p=5 and $\dim V_{T_A}(\lambda-3\alpha_1-3\alpha_2) < 4$. However, if p=7 and $\dim V_{T_A}(\lambda-\alpha_1-\alpha_2) = 1$, (1.33) implies $\dim V|A \le 64 - \dim V(2\mu_1 + 2\mu_2) < 56$. And if p=5, $\dim V|A \ge 64 - 4 = 60$. Hence $\dim V|A \ne 56$. Thus, the configuration of (6.5)(iii) does not occur.

Suppose there exists $\gamma \in \Pi(Y)-\Pi(L_Y)$ such that there exists a unique pair $1 \le i,j \le r$ with $(\Sigma L_i,\gamma) \ne 0 \ne (\Sigma L_j,\gamma)$ and M_i or M_j is nontrivial. Then L_i and L_j have type A_{k_i}, respectively A_{k_j} for some $k_i, k_j \ge 1$ and

(6.6)(iii) or (iv) holds. If (6.6)(iv) holds, $p>3$, (1.23) and (1.10) imply $\lambda|T_A = 3\mu_1 + 3\mu_2$. Then the argument of the preceeding paragraph shows $\dim V|A \neq \dim V|Y$.

Suppose (6.6)(iii) holds. Then by (6.8), $Y = E_6$ and $\dim V^1(Q_Y) \leq 3$. If $\Pi(L_i \times L_j) = \{\beta_1,\beta_4,\beta_5\}$, (6.6) and (6.8) imply $\langle\lambda,\beta_k\rangle = 0$ for $2\leq k\leq 6$ and $\langle\lambda,\beta_1\rangle = c \leq 2$. By (1.23), $\lambda|T_A = q(c\mu_1)$. But then $\dim V|A < \dim V|Y$. Suppose $\Pi(L_i \times L_j) = \{\beta_3,\beta_5,\beta_6\}$. Recall that the field twists on the embeddings of L_A' in L_i and in L_j are equal. Call this twist q. Then, (2.17) implies $\beta_4|Z_A = q\alpha$. Moreover, $Q_A \nleq K_{\beta_1}$, else $p > 2$, (2.10) and (2.11) imply that there is a nontrivial image of $Q_A{}^\alpha$ in $Q_Y(\beta_4,\beta_1)$, a 3-dimensional irreducible L_A'-module. Hence, by (2.4), $\beta_1|Z_A = q\alpha$, also. This implies $\langle\lambda,\beta_1\rangle = 0$ and $\langle\lambda,\beta_3\rangle \leq 1$, else the bound on $\dim V^2(Q_A)_{\lambda-q\alpha}$ of (1.22) is exceeded. Hence by (1.23) and (2.3), $\lambda|T_Y = \lambda_3$ or λ_5 and $\lambda|T_A = q(2\mu_1 + \mu_2)$ or $\lambda|T_Y = \lambda_6$ and $\lambda|T_A = q(2\mu_1)$. However, in each case, $\dim V|A < \dim V|Y$. Consider next the configuration $\Pi(L_i \times L_j) = \{\beta_1,\beta_2,\beta_4\}$. If $L_Y' = L_i \times L_j$, $Q_A \leq K_{\beta_5}$, $\lambda|T_Y = c\lambda_1$ and by (1.23) $\lambda|T_A = q(c\mu_1)$, for $c = 1$ or 2. However, $\dim V|A < \dim V|Y$. Thus, $\langle U_{\pm\beta_6}\rangle \leq L_Y'$. If $\langle\lambda,\beta_2+\beta_4\rangle \neq 0$, $Q_A \nleq K_{\beta_5}$, so (2.17) implies that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_6}\rangle$ is also q and that $\beta_5|Z_A = q\alpha$. Thus, we see that $\lambda|T_Y = \lambda_2$, else the bound on $\dim V^2(Q_A)_{\lambda-q\alpha}$, of (1.22), is exceeded. But this is a contradiction. Hence, $\langle\lambda,\beta_1\rangle \neq 0$. Applying (6.8) and (1.23), we see that $\dim V|A < \dim V|Y$.

So if (6.6)(iii) holds, we may assume by symmetry that $\Pi(L_i \times L_j) = \{\beta_1,\beta_3,\beta_2\}$. If $L_Y' = L_i \times L_j$, $\langle\lambda,\beta_5+\beta_6\rangle = 0$. But (1.23) then implies $\dim V|A < \dim V|Y$. Thus, $L_Y' = L_i \times L_j \times \langle U_{\pm\beta_6}\rangle$ and by (6.8), $\langle\lambda,\beta_6\rangle = 0$. In fact, $\langle\lambda,\beta_5\rangle = 0$, for otherwise, $Q_A \nleq K_{\beta_5}$, and (2.8) implies $\beta_5|Z_A = q\alpha$ which means the bound on $\dim V^2(Q_A)_{\lambda - q\alpha}$, of (1.22), is exceeded. But now (1.23) implies $\dim V|A < \dim V|Y$. Hence, the hypothesis of (6.6) cannot be satisfied.

Consider now the possibility that there exists $1\leq i\leq r$ such that L_i is separated from all other components of L_Y' by more than one node of the Dynkin diagram. Then all components of L_Y' are necessarily of classical type, so by (1.5), $Z_A \le Z_Y$. By (6.7), rank $L_i = 1$ and $\dim V^1(Q_Y) = \dim M_i = c+1$, for some $p>c>0$. Let q be the field twist on the embedding of L_A' in L_i. Consider first the case where $Y = E_7$ or E_8. Then (1.23) and (1.10) imply $\lambda|T_A = c\mu_1 + c\mu_2$. Let $\gamma, \delta \in \Pi(Y)-\Pi(L_Y)$ such that $(\Sigma L_i,\gamma) \neq 0$, $(\gamma,\delta) < 0$. Then, by (2.4), $\gamma|Z_A = \alpha$ and by (2.3) and (2.9), $Q_A \le K_\delta$. If $-\delta$ is not involved in L_A', (2.11) implies $(\gamma+\delta)|Z_A = \alpha$. Thus, $\delta|Z_A = 0$. If $-\delta$ is involved in L_A', (2.10) implies $\delta|Z_A = 0$. Hence, we may use the parabolic $P_Y\hat{}$ of (2.11) to see that the bound on $\dim V_\gamma(Q_Y)$ and (1.36) imply $\langle\lambda,\gamma\rangle = 0$. Moreover, $(\Sigma L_Y,\delta) = 0$, else $c\le 2$ and $\dim V|A \le 27 < \dim V|Y$. Now (2.15) implies $L_i = \langle U_{\pm\beta_2}\rangle$, $\gamma = \beta_4$ and $\delta = \beta_5$ or $L_i = \langle U_{\pm\beta_n}\rangle$ if $Y = E_n$. Now, if $\langle\lambda,\beta_j\rangle \neq 0$ for some $\beta_j \in \Pi(Y) - \Pi(L_i)$, then $\beta_j \in \Pi(Y) - \Pi(L_Y)$ and the bound on $\dim V^2(Q_A)$ is exceeded. So, $\langle\lambda,\beta_j\rangle = 0$ for all $\beta_j \notin \Pi(L_i)$. In each case, there is a parabolic of Y with Levi factor, L, of type A_5 such that v^+ affords an L composition factor with dimension $(c+1)(c+2)(c+3)\cdot(c+4)(c+5)/5!$ by (1.12). Since $\dim V|A \le (c+1)^3$, by (1.27), $c\le 3$. Using the methods of (1.30) and (1.32), it is easy to check that $\dim V|A < \dim V|Y$. Hence, $Y = E_6$.

Since L_Y' is not simple, we may take $L_i = \langle U_{\pm\beta_j}\rangle$ for j = 1, 2, 3. In fact, by (2.15), we may exclude j=3. If $L_i = \langle U_{\pm\beta_2}\rangle$, (2.15) implies $\langle U_{\pm\beta_1}\rangle \le L_Y'$. By (2.9), $Q_A \le K_{\beta_3}$. However, since $p > 2$, this contradicts (2.18). Hence, $L_i = \langle U_{\pm\beta_1}\rangle$. Then (2.15) implies $(\Sigma L_Y,\beta_4) \neq 0$, so by (2.9), $Q_A \le K_{\beta_4}$ and $V_{\beta_4}(Q_Y) = 0$. The considerations of the case where $L_i = \langle U_{\pm\beta_2}\rangle$ and previous general remarks imply that $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_5\}$ or $\{\beta_1,\beta_5,\beta_6\}$ or $\{\beta_1,\beta_2,\beta_5,\beta_6\}$. In the second and third cases, (1.23) and Theorem (7.1) of [12] imply $\lambda|T_Y = \lambda_1$ and $\lambda|T_A = q\mu_1 + 2q_0\mu_2$ for some p-power q_0. However, $\dim V|A \le 18 < \dim V|Y$. In the first case, by considering the action of A on V^*, we see that $\langle\lambda,\beta_3\rangle = 0$. Then (1.23) and (1.10) imply $\lambda|T_A = c\mu_1 + y\mu_2$ and $\lambda|T_Y = c\lambda_1 + y\lambda_6$. Hence, if $y\neq 0$, (2.4) implies $\beta_6|Z_A = \alpha$ and the bound on $\dim V^2(Q_A)$ is exceeded. So $y=0$. But then clearly $\dim V|A < \dim V|Y$. Therefore, the conditions of (6.7) cannot be satisfied.

It remains to consider the possibility that there exists $1 \le i \le r$ such that L_i has exceptional type. Then $Y = E_8$, $\Pi(L_Y) = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_8\}$ and by (7.1) of [12], $\lambda|T_Y = x\lambda_7 + c\lambda_8$ for $p>c>0$. Then (1.23) and (1.10) imply $\lambda|T_A = c\mu_1 + c\mu_2$. However, this contradicts Remark (6.2).

This completes the proof of (6.0).□

(6.9): Summary of Results in Chapter 6

Description of V|Y	$\langle\lambda,\beta\rangle$	Reference
0 0 0 0 0 0 1	$3q, q+q_1$	(6.6)(ii)
0 0 0 0 0 0 a $a \geq 0$ 1	$3q, q+q_1$	(6.6)(ii)
0 0 0 0 0 b c $c>0, b\geq 0$ 0	cq	(6.2) or (6.3)
1 0 0 0 0 0	2q	(6.5)(i)
1 0 0 0 0 0 0 0	2q,4q	(6.5)(ii) or (6.6)(i)
a 0 0 0 0 1 $a\geq 0$ 0	$3q, q+q_1$	(6.5)(iii)
0 0 0 0 0 1 0	3q	(6.6)(iv)
0 0 0 0 0 0 0 1	2q	(6.6)(v)
Partial description of V|Y		
0 0 0 or 1 0 0 or 0 0 1 1 0 0	q	(6.5)(iv)
···1 0 0 0··· or ···0 1 0 0···	2q	(6.8)
···c 0 0 0··· $c\leq 2$	cq	(6.8)
···1 0 0 0 0···	$q, q+q_1$	(6.6)(vi)
···c··· $c\geq 0$	cq	(6.7)

CHAPTER 7: $A = B_2$

In this chapter, we will prove that there are no examples in the solution of the main problem with A of type B_2, Y of type E_n and $p \neq 2$. We adopt Notation and Hypothesis (2.0), with the following additions and/or modifications. With $\Pi(A) = \{\alpha_1, \alpha_2\}$, with α_1 long, we will take $L_A = \langle U_{\pm\alpha_2} \rangle T_A$. Note that since $p \neq 2$, $Q_A{}^{\alpha_1}$ is a 3-dimensional irreducible L_A' module.

<u>Remark (7.1).</u> Note that $h_{\alpha_2}(-1) \in Z(A) \leq Z(Y)$. Since $Z(Y) \cong \mathbb{Z}_3$ (respectively, $\mathbb{Z}_2$, 1) if $Y = E_6$ (respectively, E_7, E_8), $h_{\alpha_2}(-1) \neq 1$ will imply Y has type E_7. In particular, if L_i has type A_1 for some $1 \leq i \leq r$, then $Y = E_7$.

<u>(7.2).</u> Let q be a p-power.

(1) If $p > 3$ and $\langle \lambda, \alpha_2 \rangle = 3q$, then $\dim V|A \neq 56$.

(2) If $p \geq 3$ and $\langle \lambda, \alpha_2 \rangle = q$, then $\dim V|A = 56$ if and only if $\langle \lambda, \alpha_1 \rangle = 2q_0$ for some p-power $q_0 \neq q$.

(3) If $\lambda | T_Y = \lambda_j$ for some j, then $\langle \lambda, \alpha_2 \rangle \neq 2q$.

<u>Proof</u>: Suppose $p > 3$, $\langle \lambda, \alpha_2 \rangle = 3q$ and $\dim V|A = 56$. Then (1.27) implies that $\langle \lambda, \alpha_1 \rangle \neq 0$. The methods of (1.30) and (1.32) imply that $V|A$ is tensor indecomposable, else $\dim V|A > 56$. So $\lambda | T_A = xq\mu_1 + 3q\mu_2$, for some $x < p$. If $x > 1$, the methods of (1.30), (1.32) and (1.35) imply that $\dim V|A > 56$. So we may assume $\lambda | T_A = \mu_1 + 3\mu_2$. Now, $p \neq 7$, else (1.27) and (1.35) imply that $\dim V|A \leq 64 - \dim V(3\mu_2) < 56$. Also, $p \neq 5$, else the last proposition of [4] implies $\dim V|A \leq 64 - \dim V(\mu_1 + \mu_2) < 56$. But now, $p > 7$ and (1.33) imply $\dim V|A = 64$. This completes the proof of (1).

Now suppose $p>2$, $\langle\lambda,\alpha_2\rangle = q$ and $\dim V|A = 56$. We first claim that $V|A$ is tensor decomposable. For if $\lambda|T_A = xq\mu_1+q\mu_2$ for some $0\leq x<p$, (1.27) implies $x\geq 3$. In fact $x=3$, else the methods of (1.30) and (1.32) imply $\dim V|A > 56$. By (1.33), $\dim V(3\mu_1+\mu_2) < 80$ only if $p=13$ or $p=7$. If $p=7$, $\dim V|A \geq 80 - 4(\dim V_{T_A}(\lambda-3\alpha_1-3\alpha_2)) \geq 80 - 16$. (Use (1.29) to find a spanning set for the indicated weight space.) If $p=13$, $\dim V|A < 80$ only if $\dim V_{T_A}(\lambda-2\alpha_1-\alpha_2) < 2$. However, $\dim V_{T_A}(\lambda-2\alpha_1-\alpha_2) \geq 1$, so $\dim V|A \geq 80 - \dim V(3\mu_2) \geq 60$. ($\lambda-2\alpha_1-2\alpha_2 = 3\mu_2$.) Thus, $V|A$ is tensor decomposable as claimed. It is now easy to see that $\langle\lambda,\alpha_1\rangle = 2q_0$ for some p-power $q_0 \neq q$. This completes the proof of (2).

Suppose $\langle\lambda,\alpha_2\rangle = 2q$ and $\lambda|T_Y = \lambda_j$ for some $1 \leq j \leq \operatorname{rank}Y$. Then $\langle\lambda,\alpha_1\rangle \neq 0$ else $\dim V|A < \dim V|Y$. If $\langle\lambda,\alpha_1\rangle = 1\cdot q_0$ for some p-power q_0, $\dim V|A \leq 50$; so (1.32) implies that $Y = E_6$ and $\dim V|A = 27$. So $q=q_0$. But now applying the last proposition of [4], we see that $\dim V(\mu+2\mu_2) \neq 27$. Consider now the possibility that $\langle\lambda,\alpha_1\rangle = 2q_0$, for some p-power q_0. If $q \neq q_0$, using (1.27), (1.30) and (1.32), we find that $117 \leq \dim V|A \leq 140$. However, using (1.32) and [8], we see that $\dim V|Y > 140$ or $\dim V|A < 117$. Thus, $q=q_0$, and by (1.27), (1.30) and (1.32), $36 \leq \dim V|A \leq 81$. Thus, [8] implies that $Y = E_7$, $\lambda|T_Y = \lambda_7$ and $\dim V|A = 56$. Now, $\dim V|A < 81$ and (1.33) imply $p=5$ or $p=7$ (recall $p \neq 2$). If $p=7$, (1.33) and the last proposition of [4] imply $\dim V|A = 81 - t$, where t is the dimension of the irreducible kA-module with high weight $2\mu_2$. Using (1.33) again for this module when $p=7$, we see that $t = 10$ and $\dim V|A = 71 \neq 56$. So $p=5$. Then (1.33) and [4] imply that $\dim V|A \geq (81-14) - x$, where x is the multiplicity of the weight $\lambda-3\alpha_1-4\alpha_2$ in the Weyl module $W(\lambda)$. But by (1.29), a spanning set for this weight space has size 8. Thus $x \leq 8$ and $\dim V|A > 56$. So, $\dim V|A \neq 56$, as claimed. Hence, $\langle\lambda,\alpha_1\rangle \neq 2q_0$, for q_0 a p-power.

For the remaining possibilities, we refer to (6.9) and the configurations in which $\lambda|T_Y$ is more explicitly described. In each case, the methods of (1.30), (1.32) and (1.35), (1.27), [8] and the work of the preceeding paragraphs show that $\dim V|A \neq \dim V|Y$. This completes the proof of (3) and of (7.2).□

(7.3). If $\dim V^1(Q_Y) > 1$, L_Y' is not a simple algebraic group.

Proof: Suppose false. Then Theorem (7.1) of [12] implies L_Y' is of classical type, so by (1.5), $Z_A \leq Z_Y$. Consider first the case where L_Y' has type D_k for some $k \geq 4$. Arguing as in the proof of (6.1), we reduce to L_Y' of type D_6, with $V^1(Q_Y)$ isomorphic to the natural module for L_Y. Moreover, $Y = E_7$, else Q_Y/K_{β_8} is a 12-dimensional irreducible L_A' module containing a nontrivial image of $Q_A{}^{\alpha_1}$. The bound on $\dim V_{\beta_1}(Q_Y)$, (1.32) and (1.34) imply $\langle \lambda,\beta_1 \rangle = 0$. Therefore, $\dim V|Y = 56$ and $\langle \lambda,\alpha_2 \rangle = q_1+5q_2$ or $q_1+q_2+2q_3$, for q_1, q_2 and q_3 distinct p-powers. Using the methods of (1.30) and (1.32), we see that $\dim V|A > 56$ in either case. Hence, L_Y' has type A_k, for some k.

Assume for now that rank $L_Y' > 2$. Then (7.1) of [12] implies $V^1(Q_Y) \cong W$ or W^*, where W is the natural module for L_Y'. Since L_Y' acts irreducibly on W (and W^*), there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $V_{L_Y'}(-\gamma) \cong W$ or W^*. If L_Y' has type A_{n-1} when Y has type E_n, the bound on $\dim V_{\beta_2}(Q_Y)$ of (1.25) implies $\lambda|T_Y = \lambda_n$. In particular, $Y \neq E_8$. But now induction, (1.30) and (1.32) imply that $\dim V|A > \dim V|Y$ in each case.

Now, consider $L_Y' = \langle U_{\pm\beta_j} \mid 1 \leq j \leq 4 \rangle$. The bound on $\dim V_{\beta_5}(Q_Y)$ and (2.3) imply $\langle \lambda,\beta_j \rangle = 0$ for $j \geq 5$. Also, $\langle \lambda,\alpha_2 \rangle = 4q$, for some p-power q. If $Y = E_6$ (respectively, E_7), $\lambda|T_Y = \lambda_1$ (respectively, λ_2). Thus, $Y \neq E_6$ as $\dim V|A > 27 = \dim V|Y$. So $Y = E_7$ and $\lambda|T_Y = \lambda_2$, or $Y = E_8$ and $\lambda|T_Y = \lambda_1$ or λ_2. The L_A' composition factors of Q_Y/K_{β_5} have high weights $6q\mu_2$ and $2q\mu_2$; if $p = 5$, the high weights are $(q+5q)\mu_2$ and $2q\mu_2$. Thus, (2.12) applies to give $\beta_j|Z_A = 0$ for $j \geq 6$. Then, using the parabolic $P_Y\hat{}$ of (2.12), we see that the bound on $\dim V_{\beta_5}(Q_Y\hat{})$, of (1.25), is exceeded. So, $L_Y' \neq \langle U_{\pm\beta_j} \mid 1 \leq j \leq 4 \rangle$. But if $L_Y' = \langle U_{\pm\beta_j} \mid j \neq 1,3 \rangle$ in Y of type E_8, the bound on $\dim V_{\beta_3}(Q_Y)$, of (1.25), implies $\lambda|T_Y = \lambda_8$. Also,

$L_Y' \neq \langle U_{\pm\beta_j} \mid j = 2,4,5,6,7\rangle$ as there is no 3-dimensional composition factor of Q_Y/K_{β_3}, contradicting (2.3). Thus, we have shown that $\mathrm{rank}(L_Y') \leq 2$.

Note that rank $L_Y' \neq 1$, else $\dim(Q_Y/K_\gamma) \leq 2$ for all $\gamma \in \Pi(Y) - \Pi(L_Y)$ and $Q_A \leq K_\gamma$ for all such γ, contradicting (2.3). Hence, L_Y' has type A_2. Let q be the field twist on the embedding of L_A' in L_Y'. Then $\beta|T_A = q\alpha_2$ for $\beta \in \Pi(L_Y)$. Since $Q_AK_\gamma/K_\gamma = Q_Y/K_\gamma$ for all $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $(\gamma,\Sigma L_Y) \neq 0$, $\gamma|T_A = q\alpha_1$, for all such γ. By (2.12), $\delta|T_A = 0$ for all $\delta \in \Pi(Y) - \Pi(L_Y)$ such that $(\delta,\Sigma L_Y) = 0$. Also, $\langle\lambda,\alpha_1\rangle \neq 0$, else $\dim V|A < \dim V|Y$. So there exists $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \neq 0$ and $\langle\lambda,\gamma\rangle \neq 0$, else $V_{T_A}(\lambda-q\alpha_1) = 0$. Moreover, there exists a unique such γ, else $\dim V_{T_A}(\lambda-q\alpha_1) \geq 2$, contradicting (1.31). Also, (2.13) implies γ corresponds to an end node of the Dynkin diagram. Finally, we need to note that $V|A$ is a conjugate of a restricted module as there are no nontrivial T_Y weights in $V|Y$ restricting to $\lambda - q_0\alpha_1$, for $q_0 \neq q$. So by (1.10), $\lambda|T_A = x\mu_1 + 2\mu_2$ for $p > x > 0$. Then by (1.29), $\dim V_{T_A}(\lambda-\alpha_1-2\alpha_2) \leq 3$. However, it is easy to check that if the above conditions are satisfied, there exist 4 linearly independent vectors in $V|Y$ which lie in $V_{T_A}(\lambda-\alpha_1-2\alpha_2)$. This completes the proof of (7.3).

(7.4). If $\dim V^1(Q_Y) > 1$, each L_i has type A_{k_i}, for some $k_i \geq 1$.

Proof: Suppose false. Then L_i has type D_k for some i and k. Otherwise, by (7.1) of [12], $Y = E_8$ and L_Y' has type $E_6 \times A_1$. But this contradicts Remark (7.1). Since $p > 2$, (7.1) of [12], (7.3) and size restrictions imply L_i has type D_4 or D_5 and $Y = E_7$ or E_8. Since all components of L_Y' are necessarily of classical type, (1.5) implies $Z_A \leq Z_Y$.

Arguing as in the proof of (6.3) and applying (7.1) and (7.3), we reduce to L_Y' of type D_4. Note that M_i is nontrivial. For otherwise, (7.1), (7.3) and the bound on $\dim V_{\beta_6}(Q_Y)$ and on $\dim V_{\beta_1}(Q_Y)$ imply $Y = E_8$ and

$\lambda|T_Y = \lambda_8$. Then, M_i nontrivial, $p>2$, (7.1) of [12], and (1.14) imply L_A' acts on M_i ($\cong$ the "natural" module for L_i) with high weight $(q_1 + 3q_2)\mu_2$, for q_1 and q_2 distinct p-powers. Since Q_Y/K_{β_1} and Q_Y/K_{β_6} must each have a 3-dimensional L_A' composition factor, the remarks in the proof of (6.3) imply that $\langle\lambda,\beta_3\rangle = 0$, $V_{L_i}(-\beta_6)|L_A' \cong M_i|L_A'$ and $\langle U_{\pm\beta_7}\rangle \le L_Y'$. Moreover, (1.15) implies $Y = E_8$ and $\langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle \le L_Y'$. Then by (2.7), $\langle\lambda,\beta_7\rangle = 0 = \langle\lambda,\beta_8\rangle$ and the bound on $\dim V_{\beta_1}(Q_Y)$ and on $\dim V_{\beta_6}(Q_Y)$ implies $\lambda|T_Y = \lambda_2$. By (6.9), $\langle\lambda,\alpha_1\rangle = 0$, $3q$, $q_0 + q$, $2q$, or q, for q and q_0 distinct p-powers. But (1.27) and (1.38) imply that, in every case, $\dim V|A < \dim V|Y$. Contradiction.

This completes the proof of (7.4).□

<u>(7.5).</u> If $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_3,\beta_5,\beta_7\}$, then $\dim V^1(Q_Y) = 1$.

<u>Proof</u>: Suppose false. By (1.5), $Z_A \le Z_Y$. Let q_1, q_2, q_3 and q_4 be the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$, $\langle U_{\pm\beta_2}\rangle$, $\langle U_{\pm\beta_5}\rangle$, $\langle U_{\pm\beta_7}\rangle$, respectively. Note that (7.1) implies that $Y = E_7$.

<u>Claim 1.</u> If $\langle\lambda,\beta_1+\beta_2+\beta_3+\beta_5\rangle > 0$, exactly one of $\langle\lambda,\beta_1+\beta_3\rangle$, $\langle\lambda,\beta_2\rangle$ and $\langle\lambda,\beta_5\rangle$ is nonzero.

Reason: Suppose false. Then (2.7), (2.5) and (2.6) imply that $\{q_1,q_2,q_3\}$ consists of exactly two distinct p-powers. If $q_2 = q_3$, the L_A' composition factors of Q_Y/K_{β_4} have high weights $(2q_1 + 2q_2)\mu_2$ and $2q_1\mu_2$. Hence, $\beta_4|Z_A = q_1\alpha_1$. Since $\langle\lambda,\beta_1\rangle$ or $\langle\lambda,\beta_3\rangle$ is nonzero, a nonidentity element from the set $U_{-4}\cdot U_{-34}\cdot U_{-134}$ must occur in the factorization of some element in $Q_A - Q_A'$. However, $-\beta_4$ (respectively, $-\beta_3-\beta_4$, $-\beta_1-\beta_3-\beta_4$) affords $T(L_A')$ weight $(2q_1 + 2q_2)\mu_2$ (respectively, $2q_2\mu_2$, $(-2q_1 + 2q_2)\mu_2$). And since $p > 2$ and $q_1 \ne q_2$, none of these weights occurs in $(Q_A{}^{\alpha_1})^{q_1}$. Hence, we may assume $q_1 = q_2$. Then, the L_A' composition factors of Q_Y/K_{β_4} have high weights $(3q_1 + q_3)\mu_2$ and $(q_1 + q_3)\mu_2$. If $p = 3$ and $3q_1 = q_3$, the weights are $2q_3\mu_2$, $(q_1 + q_3)\mu_2$ and 0. Thus, $p = 3$, $3q_1 = q_3$ and $\beta_4|Z_A = q_3\alpha_1$. Then, we find that

$x_{-\alpha_1}(t) = x_{-\beta_4}(c_1t^{q_3})u_1$, $x_{-\alpha_1-\alpha_2}(t) = x_{-1234}(c_2t^{q_3})x_{-45}(c_3t^{q_3})u_2$, and $x_{-\alpha_1-2\alpha_2}(t) = x_{-12345}(c_4t^{q_3})\, u_3$, where $c_1, c_4 \in k^*$, $c_2, c_3 \in k$, c_2 or c_3 nonzero, and $u_i \in K_{\beta_4}$. Then, there is a nontrivial contribution to the root group U_{-1345} in the expression for $[x_{\alpha_2}(t), x_{-\alpha_1-2\alpha_2}(t)]$, contradicting the given factorizations of $x_{-\alpha_1}(t)$ and $x_{-\alpha_1-\alpha_2}(t)$. This completes the proof of Claim 1.

Claim 2. $\langle\lambda,\beta_1+\beta_2+\beta_3+\beta_5\rangle = 0$.

Reason: Suppose false; then Claim 1 implies that exactly one of $\langle\lambda,\beta_1+\beta_3\rangle$, $\langle\lambda,\beta_2\rangle$, $\langle\lambda,\beta_5\rangle$ is nonzero. By (2.7), $\{q_1,q_2,q_3\}$ consists of at most two distinct p-powers. Arguing as in the proof of Claim 1, we find that $q_2=q_3$ and $\beta_4|Z_A = q_1\alpha_1$.

Suppose $\langle\lambda,\beta_1+\beta_3\rangle \neq 0$. Then, the proof of Claim 1 shows that $q_1=q_2$. Also, the bound on $\dim V_{\beta_4}(Q_Y)$ implies $\langle\lambda,\beta_4+\beta_3\rangle = 0$. Now, if $Q_A \nleq K_{\beta_6}$, (2.17) implies that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_7}\rangle$ is also q_1. Thus, by (2.5) and (2.6), $\langle\lambda,\beta_7\rangle = 0$. Also, $\langle\lambda,\beta_6\rangle = 0$, else the bound on $\dim V_{\beta_6}(Q_Y)$ is exceeded. But now we have $\lambda|T_Y = \lambda_1$. So $\langle\lambda,\beta_1+\beta_3\rangle = 0 \neq \langle\lambda,\beta_2+\beta_5\rangle$.

Suppose $\langle\lambda,\beta_2\rangle \neq 0$, so $\langle\lambda,\beta_\ell\rangle = 0$ for $\ell = 1, 3, 5$. If $Q_A \nleq K_{\beta_6}$, (2.17) implies that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_7}\rangle$ is $q_3 = q_2$. Then (2.5) and (2.6) imply $\langle\lambda,\beta_7\rangle = 0$. Moreover, $\langle\lambda,\beta_6\rangle = 0$, else the bound on $\dim V_{\beta_6}(Q_Y)$ is exceeded. Finally, using (1.36) and the bound on $\dim V_{\beta_4}(Q_Y)$, we see that $\lambda|T_Y = \lambda_2$, and $\langle\lambda,\alpha_2\rangle = 1\cdot q_2$. By (6.9), $\langle\lambda,\alpha_1\rangle = 0$, q, $q + q_0$, $3q$ or $2q$, for q and q_0 distinct p-powers. In every case, (1.27) and [8] imply $\dim V|A < \dim V|Y$. Thus, $\langle\lambda,\beta_1+\beta_2+\beta_3\rangle = 0$ and $\langle\lambda,\beta_5\rangle \neq 0$.

The arguments of the preceeding paragraph imply $\lambda|T_Y = \lambda_5 + x\lambda_6$, for $x \geq 0$ and by (2.17), $\beta_6|T_A = q_2\alpha_1$. Moreover, $q_1 \neq q_2$, else the bound on $\dim V^2(Q_A)_{\lambda-q_2\alpha_1}$, of (1.22), is exceeded. So $V^2(Q_A)_{\lambda-q_1\alpha_1} \neq 0 \neq V^2(Q_A)_{\lambda-q_2\alpha_1}$ and $V|A$ is tensor decomposable. In particular, $\langle\lambda,\alpha_1\rangle \neq 0$. By (6.9), $\langle\lambda,\alpha_1\rangle = q$, $2q$ or xq, for some p-power $q \neq q_2$. Then by (1.27) and (1.32), $\dim V|A < \dim V|Y$ unless $\langle\lambda,\alpha_1\rangle = xq$ with $x \neq 0$. However, then $\dim V^2(Q_A)_{\lambda-q_2\alpha_1} = 2$ while $f_{\beta_6}v^+$ affords an L_Y' composition factor in $V^2(Q_A)_{\lambda-q_2\alpha_1}$ of dimension 6. Contradiction. This completes the proof of Claim 2.

Claim 2 implies that $\langle\lambda,\beta_7\rangle \neq 0$; so by (2.7), $q_3 = q_4$.

Claim 3. $Q_A \leq K_{\beta_4}$.

Reason: Suppose $Q_A \nleq K_{\beta_4}$. We examine the image of $Q_A{}^{\alpha_1}$ in Q_Y/K_{β_4}. Arguing as in the proof of Claim 1, we see that $q_2 = q_3$ and $\beta_4|Z_A = q_1\alpha_1$. Suppose $q_1 \neq q_3$. Examining the $T(L_A')$ weights of Q_Y/K_{β_4}, we see that $x_{-\alpha_1}(t) = x_{-45}(c_1t^{q_1})x_{-24}(c_2t^{q_1})x_{-\beta_6}(c_3t^{q_3})u_0$, where $c_i \in k$, c_1 or c_2 nonzero, $c_3 \neq 0$, and $u_0 \in K_{\beta_4} \cap K_{\beta_6}$. In fact, $c_1c_2 \neq 0$, else there is a nontrivial contribution to the root group $U_{-\beta_4}$ in the expression for $[x_{\alpha_2}(t),x_{-\alpha_1}(t)]$. We also find that $x_{-\alpha_1-2\alpha_2}(t) = x_{-1345}(d_1t^{q_1})\cdot x_{-1234}(d_2t^{q_1})x_{-567}(d_3t^{q_3})u_1$, where $d_i \in k$, $d_3 \neq 0$, d_1 or d_2 nonzero and $u_1 \in K_{\beta_4} \cap K_{\beta_6}$. Thus, in the expression for $[x_{-\alpha_1}(t),x_{-\alpha_1-2\alpha_2}(t)]$, there is a nontrivial contribution to the root group U_{-24567}. Contradiction.

So $q_1 = q_3$ and again examining the $T(L_A')$ weight vectors in Q_Y/K_{β_4}, we find that $x_{-\alpha_1}(t) = x_{-34}(a_1t^{q_3})x_{-24}(a_2t^{q_3})x_{-45}(a_3t^{q_3})x_{-\beta_6}(a_4t^{q_3})w_0$ and $x_{-\alpha_1-2\alpha_2}(t) = x_{-1345}(b_1t^{q_3})x_{-1234}(b_2t^{q_3})x_{-2345}(b_3t^{q_3})x_{-567}(b_4t^{q_3})w_1$, where $a_i, b_i \in k$, $a_4b_4 \neq 0$, $w_0, w_1 \in K_{\beta_4} \cap K_{\beta_6}$ and $a_ib_j \neq 0$ for some $1 \leq i, j \leq 3$. In fact, at least two of a_1, a_2, a_3 are nonzero, else there is a nontrivial contribution to the root group $U_{-\beta_4}$ in the expression for $[x_{-\alpha_1}(t),x_{\alpha_2}(t)]$. So a_1 or a_2 is nonzero. But then there is a nontrivial contribution to the group $U_{-24567}\cdot U_{-34567}$ in the expression for $[x_{-\alpha_1}(t),x_{-\alpha_1-2\alpha_2}(t)]$. Contradiction. This completes the proof of Claim 3.

Now, $Q_A \leq K_{\beta_4}$ implies that $\lambda|T_Y = x\lambda_6 + c\lambda_7$, for $x \geq 0$, $c>0$ and $\langle\lambda,\alpha_2\rangle = cq_4$. Referring to (6.9), we see that if $x \neq 0$ or if $c>1$, $\langle\lambda,\alpha_1\rangle = 0$, q, $2q$, xq or cq. So $\dim V^2(Q_A) \leq 5c+3$, by (1.22). Let $w = f_{456}v^+$ if $\langle\lambda,\beta_6\rangle \neq 0$, or $w = f_{4567}v^+$ if $\langle\lambda,\beta_6\rangle = 0$. Then $w \notin [V, K_{\beta_4}{}^2]$, so $w \notin [V,Q_A{}^2]$; hence, w affords an L_Y' composition factor in $V^2(Q_A)$.

Adding the dimension of the L_Y' composition factor afforded by $f_{\beta_6}v^+$ (or $f_{67}v^+$), we find that if $x \neq 0$ or $c>1$, $\dim V^2(Q_A) > 5c+3$. (Use (1.36).) Hence, $\lambda|T_Y = \lambda_7$ and (7.2) implies that $\langle\lambda,\alpha_1\rangle = 2q$, for some p-power $q \neq q_4$.

One checks that in the action of L_A' on the kA module $V(2q\mu_1+q_4\mu_2)$, if $q_4 \neq 3q$, there is an 8- or 10-dimensional L_A' composition factor. However, the given embedding of L_A' in P_Y affords no such L_A' composition factor on $V(\lambda_7)$. So p=3 and $q_4 = 3q$. Then, there are exactly six 6-dimensional L_A' composition factors of $V(2q\mu_1 + q_4\mu_2)$, two of which have distinct high weights. However, one checks that the given embedding of L_A' in P_Y does not afford such an L_A' composition series of $V(\lambda_7)$.

This completes the proof of (7.5).□

<u>(7.6)</u>. Let $\gamma \in \Pi(Y)-\Pi(L_Y)$ and $1 \leq i,j \leq r$ such that $(\Sigma L_i,\gamma) \neq 0 \neq (\Sigma L_j,\gamma)$.

(i) Then M_i or M_j is trivial.

(ii) If in addition there exists $k \neq i,j$ such that $(\Sigma L_k,\gamma) \neq 0$, then $\dim(M_\ell) = 1$ for $\ell = i,j,k$.

<u>Proof</u>: By (7.4), all components of L_Y' have classical type, so by (1.5), $Z_A \leq Z_Y$. Let W_m denoted the natural module for L_m. By (7.1) of [12], if $\dim M_m > 1$ and $\operatorname{rank} L_m > 1$, $M_m \cong W_m$ or $W_m{}^*$. Consider first the case where there exists k as in (ii), so $\gamma = \beta_4$. Then (2.5) and (2.7) imply that at most two of M_i, M_j, and M_k are nontrivial. Since $\operatorname{rank}(L_m) = 1$ for m = i, j or k, (7.1) implies that $Y = E_7$ and $\langle U_{\pm\beta_3}\rangle$ is not a component of L_Y'. This observation, together with (1.15), implies that $L_i \times L_j \times L_k$ has type $A_1\times A_2\times A_3$ or $\Pi(L_i\times L_j\times L_k) = \{\beta_1,\beta_3,\beta_2,\beta_5\}$. In the second case, $L_Y' = L_i \times L_j\times L_k\times \langle U_{\pm\beta_7}\rangle$, else $h_{\alpha_2}(-1)$ does not centralize U_{β_6}. But this configuration does not occur, by (7.5). If $L_i\times L_j\times L_k$ has type $A_1\times A_2\times A_3$, p>2, the bound on $\dim V_{\beta_4}(Q_Y)$ and (1.36), imply $\lambda|T_Y = \lambda_1$ or λ_7. But

$\lambda|T_Y \neq \lambda_1$, so $\lambda|T_Y = \lambda_7$. Also, considering Q_Y/K_{β_4} in view of (1.15), we see that $V^1(Q_A)$ is tensor indecomposable; so $\langle\lambda,\alpha_2\rangle = 3q$, for some p-power, q, and $p > 3$. But this contradicts (7.2). Hence, (ii) holds.

Now suppose M_i and M_j are both nontrivial. We may assume $V_{L_i}(-\gamma) \not\cong W_i$ or W_i^*. For otherwise, (2.5), (2.6) and (2.7) imply that there exists $k \neq i,j$ with $(\Sigma L_k,\gamma) \neq 0$. But then (ii) implies $\dim(M_i) = 1 = \dim(M_j)$. Hence, $\mathrm{rank}(L_i) > 2$ and $V_{L_i}(-\gamma) \cong W_i \wedge W_i$ or $W_i^* \wedge W_i^*$. Moreover, since L_A' acts irreducibly on W_i, there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ such that $Q_Y/K_\delta \cong W_i$ or W_i^*, else $Q_A \leq K_\delta$, contradicting (2.3). Since $(W_i \wedge W_i)|L_A'$ has all even $T(L_A')$ weights, (1.15) implies that L_j has type A_2 or L_j has type A_3 and $M_j|L_A'$ is tensor decomposable. Thus L_i has type A_4 and $p > 3$. Let q_1 be the field twist on the embedding of L_A' in L_i. Then $(W_i \wedge W_i)|L_A'$ has composition factors with high weights $6q_1\mu_2$ and $2q_1\mu_2$ (or $(5q_1+q_1)\mu_2$ and $2q_1\mu_2$, if p=5). Using (2.5) and (2.6), and the above remarks, it is a check to see that there is no composition factor of Q_Y/K_γ isomorphic to a twist of $Q_A^{\alpha_1}$. Thus, $Q_A \leq K_\gamma$, contradicting (2.3). Hence, (i) holds.

This completes the proof of (7.6).

<u>(7.7)</u>. Let $\gamma \in \Pi(Y) - \Pi(L_Y)$ and $1 \leq i, j \leq r$, such that $(\Sigma L_i,\gamma) \neq 0 \neq (\Sigma L_j,\gamma)$. Then $\dim(M_i \otimes M_j) = 1$.

<u>Proof</u>. Suppose $\dim(M_i \otimes M_j) > 1$. By (7.4), each component of L_Y' has type A_{k_i} for some $k_i \geq 1$; so (1.5) implies $Z_A \leq Z_Y$. Let W_m denote the natural module for L_m, m = i, j. By (7.1) of [12], if M_m is nontrivial and $\mathrm{rank}(L_m) > 1$, $M_m \cong W_m$ or W_m^*. Also, (7.6) implies that only one of M_i and M_j is nontrivial.

<u>Case I</u>: Suppose $V_{L_i}(-\gamma) \not\cong W_i$ or W_i^*.

Then $\mathrm{rank}(L_i) > 2$ and $V_{L_i}(-\gamma) \cong W_i \wedge W_i$ or $W_i^* \wedge W_i^*$. Since $(W_i \wedge W_i)|L_A'$ has all even weights, p>2 and (1.15) implies L_j has type A_2 or A_3, so $Y = E_7$ or E_8. Note that if M_m is nontrivial and $\mathrm{rank}(L_m) > 2$, there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ such that $Q_A \nleq K_\delta$, $(\delta,\Sigma L_m) \neq 0$ and $Q_Y/K_\delta \cong W_m$ or W_m^*. These remarks and the bound on $\dim V_\gamma(Q_Y)$ imply that $\gamma = \beta_5$ and either $Y = E_8$, with $\Pi(L_Y) = \{\beta_i \mid i \neq 5,8\}$ and $\lambda|T_Y = \lambda_1 + x\lambda_8$, or $Y = E_7$ (respectively, E_8), with $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_4,\beta_6,\beta_7\}$ and $\lambda|T_Y = \lambda_7$ (respectively, $\lambda_7+x\lambda_8$, for $x \geq 0$).

In the first case, the argument in the second paragraph of the proof of (7.6) implies that the field twists on the embeddings of L_A' in the two components of L_Y' are equal. Call this twist q. Then, as p>3, the only L_A' composition factors of Q_Y/K_{β_5} isomorphic to a twist of $Q_A{}^{\alpha_1}$ have high weight $2q\mu_2$. Thus, $\beta_5|Z_A = q\alpha_1$. If $x \neq 0$, $Q_A \nleq K_{\beta_8}$ and Q_Y/K_{β_8} is the irreducible L_A' module with high weight $2q\mu_2$, so $\beta_8|Z_A = q\alpha_1$ also. However, the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_1}$, of (1.22), is exceeded. Thus, $x = 0$. By (6.9), $\langle\lambda,\alpha_1\rangle = 0$, $2q_0$ or q_0, for some p-power q_0. But (1.27) and [8] imply $\dim V|A < \dim V|Y$. Thus, the first configuration does not occur.

In the second case, (7.2) implies that $Y \neq E_7$. If $x \neq 0$, one checks that $V_{T_Y}(\lambda-\beta_8) \leq V_{T_A}(\lambda-q\alpha_1)$, where $\langle\lambda,\alpha_2\rangle = 2q$, for some p-power q. By (6.9), $\langle\lambda,\alpha_1\rangle = 0$, q, 2q or xq. But then $9 \geq \dim V^2(Q_A) \geq V^2(Q_Y) \geq \dim(V_{\beta_8}(Q_Y) + V_{\beta_5}(Q_Y)) \geq 12$. Hence $x = 0$, contradicting (7.2).

This completes the consideration of Case I.

<u>Case II</u>: Suppose $V_{L_m}(-\gamma) \cong W_m$ or W_m^* for m = i, j.

By (7.6), we have $(\Sigma L_k,\gamma) = 0$ for $k \neq i, j$. Also, (1.15) and (7.1) imply $L_i \times L_j$ has type $A_1 \times A_\ell$, $\ell = 1$ or 3 and $Y = E_7$, or $L_i \times L_j$ has type $A_2 \times A_\ell$, $\ell = 2$, 3 or 4, or $A_3 \times A_3$. Actually, $L_i \times L_j$ cannot have type $A_3 \times A_3$, else $Y = E_8$ and $Q_A \leq K_{\beta_2}$; so $\langle\lambda,\beta_\ell\rangle = 0$ for $1 \leq \ell \leq 4$. But then the bound on $\dim V_\gamma(Q_Y)$ of (1.25) implies $\lambda|T_Y = \lambda_8$.

Now consider $L_i \times L_j$ of type $A_2 \times A_4$. Using (2.3) and the bound on $\dim V_\gamma(Q_Y)$ of (1.25), we restrict the possibilities for λ. We are left with $Y = E_8$, $\lambda|T_Y = \lambda_1$ and $\langle\lambda,\alpha_2\rangle = 2q$ or 4q, for some p-power q. But (7.2) and a dimension argument from Case I imply that $\dim V|A \neq \dim V|Y$.

Suppose $L_i \times L_j$ has type $A_2 \times A_3$. Temporarily label as follows:

$L_i = \langle U_{\pm\gamma_1}, U_{\pm\gamma_2}\rangle$, $L_j = \langle U_{\pm\gamma_k} \mid k = 3,4,5\rangle$, $(\gamma_2,\gamma) \neq 0 \neq (\gamma,\gamma_3)$, $(\gamma_k,\gamma_{k+1}) < 0$, $k = 3, 4$. Note that (1.15) implies $V_{L_j}(-\gamma)\,|L_A'$ is tensor decomposable. Let $V_{L_j}(-\gamma)$ have high weight $(q_1 + q_2)\mu_2$ as L_A' module, where q_1 and q_2 are distinct p-powers. Then, by (2.7) and (2.6) we may assume that the field twist on the embedding of L_A' in L_i is q_1. Then Q_Y/K_γ has L_A' composition factors with high weights $(3q_1 + q_2)\mu_2$ and $(q_1+ q_2)\mu_2$. Thus, $p = 3$ and $3q_1 = q_2$, else there is no 3-dimensional composition factor. In this case, $\gamma|Z_A = q_2\alpha_1$, and we find that

$$x_{-\alpha_1}(t) \in U_{-\gamma}K_\gamma,$$

$$x_{-\alpha_1-\alpha_2}(t) \in (U_{-\gamma-\gamma_3}\cdot U_{-\gamma-\gamma_3-\gamma_4}\cdot U_{-\gamma_1-\gamma_2-\gamma-\gamma_3}\cdot U_{-\gamma_1-\gamma_2-\gamma-\gamma_3-\gamma_4})K_\gamma, \text{ and}$$

$$x_{-\alpha_1-2\alpha_2}(t) = x_{-\gamma_1-\gamma_2-\gamma-\gamma_3-\gamma_4-\gamma_5}(ct^{q_2})u_0,$$

where $c \in k^*$, $u_0 \in K_\gamma$. Then, there is a nontrivial contribution to the root group $U_{-\gamma_2-\gamma-\gamma_3-\gamma_4-\gamma_5}$ in the expression for $[x_{\alpha_2}(t), x_{-\alpha_1-2\alpha_2}(t)]$, contradicting the given information about $x_{-\alpha_1}(t)$ and $x_{-\alpha_1-\alpha_2}(t)$. Thus, $L_i{\times}L_j$ does not have type $A_2{\times}A_3$.

Consider now the pair $A_1{\times}A_3$ in Y of type E_7. Temporarily label as follows: $L_i = \langle U_{\pm\gamma_0}\rangle$, $L_j = \langle U_{\pm\gamma_k} \mid 1 \le k \le 3\rangle$, $(\gamma,\gamma_1) \neq 0$ and $(\gamma_k,\gamma_{k+1}) < 0$ for $k = 1, 2$. By (1.15), $W_j\,|L_A'$ is tensor indecomposable, so $p > 3$. Note that there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$, with $(\delta,\gamma_0) \neq 0$ and $(\delta,\Sigma L_k) = 0$ for $k \neq i$. For otherwise, since Q_Y/K_δ is a 2-dimensional irreducible L_A' module and $Q_Y(\gamma,\delta)$ is a 4-dimensional irreducible L_A' module, (2.11) implies $-\delta$ is involved in L_A'. But this cannot occur as $p\neq 2$. (See (2.10).) Arguing similarly, one shows that there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\gamma_3) \neq 0$ and $(\delta,\Sigma L_k) = 0$ for all $k \neq j$. These remarks, and the bound on $\dim V_\gamma(Q_Y)$, together with (1.36), imply that $\langle\lambda,\gamma\rangle = 0$ and either (a) $\Pi(L_i{\times}L_j) = \{\beta_2,\beta_5,\beta_6,\beta_7\}$ or (b) $\Pi(L_i{\times}L_j) = \{\beta_2,\beta_4,\beta_5,\beta_7\}$. By (2.17), there exists a p-power q, which is the field twist on the embeddings of L_A' in L_i and in L_j and such that $\gamma|Z_A = q\alpha_1$.

In each case, $L_Y' = L_i{\times}L_j$, else $h_{\alpha_2}(-1) \notin Z(Y)$. In case (a), the bound on $\dim V_{\beta_5}(Q_Y)$ and (2.3) imply $\lambda|T_Y = c\lambda_2$ for $1 \le c \le 3$ or $\lambda|T_Y = \lambda_5$ or λ_7. If $\lambda|T_Y = c\lambda_2$ for c = 2 or 3, then $\langle \lambda, \alpha_1 \rangle = 0$ or cq_1 for some p-power q_1. But (1.27) and (1.32) imply $\dim V|A < \dim V|Y$. In the remaining cases, (6.9) implies $\langle \lambda, \alpha_1 \rangle = 0$, q_1, $2q_1$, $3q_1$ or q_1+q_2, for q_1 and q_2 distinct p-powers. Now (1.27) and [8] imply $\dim V|A \neq \dim V|Y$ unless $\lambda|T_Y = \lambda_7$. In this case $\dim V|Y \neq \dim V|A$ by (7.2). Thus the configuration of (a) does not occur. If $L_Y' = L_i \times L_j$ as in (b), $Q_A \le K_{\beta_3}$ as Q_Y/K_{β_3} has L_A' composition factors of dimensions 5 and 1. Hence $\lambda|T_Y = c\lambda_7$, for $1 \le c \le 3$. If $\lambda|T_Y \neq \lambda_7$, we may argue as above to see that $\dim V|A \neq \dim V|Y$. If $\lambda|T_Y = \lambda_7$, (7.2) implies $\lambda|T_A = 2q_0\mu_1 + q\mu_2$, for q and q_0 distinct p-powers. However, the Z_A weight space $V^2(Q_A)_{\lambda - q\alpha_1}$ has dimension 2, while $0 \neq w \in V_{T_Y}(\lambda - \beta_7 - \beta_6)$ affords an L_Y' composition factor of $V^2(Q_A)_{\lambda - q\alpha_1}$ of dimension 4. Thus, $L_i \times L_j$ does not have type $A_1 \times A_3$.

Consider now the case where $L_i \times L_j$ has type $A_1 \times A_1$ in Y of type E_7. Temporarily label as follows: let $\gamma_i, \gamma_j \in \Pi(L_Y)$, with $L_k = \langle U_{\pm\gamma_k} \rangle$, k = i, j and let q be the field twist on the embeddings of L_A' in L_i and in L_j. (See (2.17).) Then, it is easy to check that $\gamma|T_A = q\alpha_1$. As in the previous case, there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ such that $(\delta, \gamma_i) \neq 0$, $(\delta, \Sigma L_k) = 0$ for all $k \neq i$. Similarly, there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ such that $(\delta, \gamma_j) \neq 0$, $(\delta, \Sigma L_k) = 0$ for all $k \neq j$. These remarks imply that $\{\gamma_i, \gamma_j\} = \{\beta_2, \beta_5\}$ or $\{\beta_5, \beta_7\}$.

Consider the case where $\{\gamma_i, \gamma_j\} = \{\beta_2, \beta_5\}$. Then $\langle U_{\pm\beta_1} \rangle$ is not a component of L_A', else $h_{\alpha_2}(-1) \notin Z(Y)$, contradicting (7.1). Thus, (2.12) implies $\beta_1|T_A = 0 = \beta_3|T_A$. This forces $\langle \lambda, \beta_4 \rangle = 0$, else $f_{34}v^+$ and $f_{\beta_4}v^+$ are linearly independent vectors in $V_{T_Y}(\lambda - q\alpha_1)$, contradicting (1.31). Suppose $\langle \lambda, \beta_2 \rangle \neq 0$. Then $f_{24}v^+$, $f_{234}v^+$ and $f_{1234}v^+$ are 3 linearly independent vectors in $V_{T_Y}(\lambda - q\alpha_1 - q\alpha_2)$, contradicting (1.37). Thus, $\langle \lambda, \beta_2 \rangle = 0$. A similar argument shows that $\langle \lambda, \beta_5 \rangle = 0$. So $\{\gamma_i, \gamma_j\} \neq \{\beta_2, \beta_5\}$.

Consider now the case where $\{\gamma_i, \gamma_j\} = \{\beta_5, \beta_7\}$. The previous remarks imply that there exists another component of L_Y', say L_k, with $(\Sigma L_k,\beta_4) \neq 0$. Note that $\langle U_{\pm\beta_3}\rangle$ is not a component of L_Y', else $h_{\alpha_2}(-1)$ does not centralize U_{β_1}. So if $U_{\pm\beta_3} \leq L_Y'$, $\langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle$ is a component of L_Y'. In fact, if $\langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle$ is a component of L_Y' then $\langle U_{\pm\beta_2}\rangle$ is also, else $h_{\alpha_2}(-1)$ does not centralize U_{β_4}. Thus, (7.5) implies $L_Y' = L_i \times L_j \times \langle U_{\pm\beta_2}\rangle$.

By the previous case, $\langle\lambda,\beta_2+\beta_5\rangle = 0$ and $\langle\lambda,\beta_7\rangle \neq 0$. We claim that $Q_A \leq K_{\beta_4}$. Otherwise, (2.17) implies that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_2}\rangle$ is also q and we find that $x_{-\alpha_1}(t) = x_{-\beta_4}(c_1t^q)\cdot x_{-\beta_6}(c_2t^q)u_1$, $x_{-\alpha_1-2\alpha_2}(t) = x_{-245}(c_3t^q)x_{-567}(c_4t^q)u_2$, where $c_i \in k^*$, $u_i \in K_{\beta_4} \cap K_{\beta_6}$. But then there is a nontrivial contribution to the group $U_{-2456}\cdot U_{-4567}$ in the expression for $[x_{-\alpha_1}(t),x_{-\alpha_1-2\alpha_2}(t)]$. So $Q_A \leq K_{\beta_4}$ and (2.3) implies $\langle\lambda,\beta_k\rangle = 0$ for $1\leq k\leq 5$.

If $-\beta_4$ is involved in L_A', $\beta_4|Z_A = 0$ by (2.10). Otherwise, (2.11) implies that there is a nontrivial image of $Q_A{}^{\alpha_1}$ in $Q_Y(\beta_6,\beta_4)$. So the field twist on the embedding of L_A' in $\langle U_{\pm\beta_2}\rangle$ is q, $(\beta_4+\beta_6)|Z_A = q\alpha_1$ and again $\beta_4|Z_A = 0$. Using the parabolic $P_Y{}^\wedge$ of (2.11), we see that the bound on $\dim V_{\beta_6}(Q_Y{}^\wedge)$ is exceeded unless $\langle\lambda,\beta_6\rangle = 0$ and $\langle\lambda,\beta_7\rangle \leq 3$. (Refer to (1.36) in case $\langle\lambda,\beta_7\rangle = p-1$.) So $\lambda|T_Y = c\lambda_7$, $c \leq 3$ and $\langle\lambda,\alpha_2\rangle = cq$. By (6.9), $\langle\lambda,\alpha_1\rangle = 0$ or cq_0 if $c > 1$, or $\langle\lambda,\alpha_1\rangle = 0$, $3q_0$, $q_1 + q_0$, q_0 or $2q_0$, for q_0 and q_1 distinct p-powers, if $c = 1$. But (1.27) and (1.38) imply $\dim V|A < \dim V|Y$ if $c>1$. Thus, $c = 1$, $\dim V|Y = 56$, and by (7.2), $\lambda|T_A = 2q_0\mu_1 + q\mu_2$ where $q_0 \neq q$. Note that the Z_A weight space $V^2(Q_A)_{\lambda-q\alpha_1}$ has dimension 2. But $V_{T_Y}(\lambda-\beta_7-\beta_6)$, $V_{T_Y}(\lambda-\beta_7-\beta_6-\beta_5)$ and $V_{T_Y}(\lambda-\beta_7-\beta_6-\beta_5-\beta_4)$ are 3 nonzero weight spaces lying in $V^2(Q_A)_{\lambda-q\alpha_1}$. Contradiction.

It remains to consider the case where $L_i \times L_k$ has type $A_2\times A_2$. We first claim that there does not exist a third component of L_Y'. For if L_Y' has 3 components, size restrictions and the fact that $(\gamma,\Sigma L_k) = 0$ for $k \neq i, j$, imply that the third component has type A_1. Then by (7.1), $Y = E_7$ and $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_4,\beta_6,\beta_7\}$. Now, (1.15) implies $Q_A \leq K_{\beta_3}$, so

$\langle\lambda,\beta_\ell\rangle = 0$ for $1 \le \ell \le 4$ and $\langle\lambda,\beta_6\rangle$ or $\langle\lambda,\beta_7\rangle$ is nonzero. Since $p>2$ and all $T(L_A')$ weights in Q_Y/K_{β_3} are odd, (2.10) implies that $-\beta_3$ is not involved in L_A'. Thus (2.11) implies that there is a nontrivial image of $Q_A{}^{\alpha_1}$ in $Q_Y(\beta_5,\beta_3)$, an L_A' module with no 3-dimensional composition factor. Thus, $L_Y' = L_i \times L_j$, as claimed.

Now, the bound on $\dim V_\gamma(Q_Y)$, of (1.25), implies that $\langle\lambda,\gamma\rangle = 0$. Hence, there exists $\delta \in \Pi(Y) - \Pi(L_Y)$ with $\langle\lambda,\delta\rangle \neq 0$. For otherwise, $\lambda|T_Y = \lambda_\ell$ for some ℓ and $\langle\lambda,\alpha_2\rangle = 2q$, contradicting (7.2). Then $\delta \neq \gamma$ and by (2.3), $(\delta,\Sigma L_Y) \neq 0$. Say $(\delta,\Sigma L_i) \neq 0$. Let q be the field twist on the embedding of L_A' in L_i and in L_j. (Use (2.7) to get equal twists.) Then the L_A' composition factors of Q_Y/K_γ have high weights $4q\mu_2$, $2q\mu_2$ and 0. Thus, $\gamma|Z_A = q\alpha_1$. Moreover, $Q_AK_\delta/K_\delta = Q_Y/K_\delta$ and by (2.13), $\delta|T_A = q\alpha_1$. Then, the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_1}$, of (1.22), implies that M_i is nontrivial. Also, by (2.13), there does not exist $\delta_1 \in \Pi(Y) - \Pi(L_Y)$ with $(\delta_1,\delta) < 0$. So δ corresponds to an end node of the Dynkin diagram. We now claim that there does not exist $\gamma_1 \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\gamma_1) < 0$. For if there exists such a γ_1, $Q_A \le K_{\gamma_1}$ and (2.12) and the above remarks imply that $\gamma_1|Z_A = 0$. Then using the parabolic $P_Y\hat{}$ of (2.11), we find that $\dim V_\gamma(Q_Y\hat{}) + \dim V_\delta(Q_Y\hat{})$ exceeds the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_1}$, of (1.22). Finally, we note that there does not exist $\delta_1 \in \Pi(Y) - \Pi(L_Y)$ with $\gamma \neq \delta_1 \neq \delta$ and $(\delta_1,\Sigma L_i) \neq 0$. For, as with δ, $\delta_1|T_A = q\alpha_1$ and the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_1}$ is exceeded. These remarks imply that $Y = E_8$ and $\delta = \beta_8$. The bound on $\dim V^2(Q_A)_{\lambda-q\alpha_1}$ implies, even more explicitly, that $\lambda|T_Y = \lambda_7 + x\lambda_8$ for $p > x > 0$.

By (6.9), $\langle\lambda,\alpha_1\rangle = 0$, xq_0, q_0 or $2q_0$, for some p-power q_0. Then (1.27) and (1.32) imply $\langle\lambda,\alpha_1\rangle = xq_0$, else $\dim V|A < \dim V|Y$. In fact, since $\beta_8|T_A = q\alpha_1$, $f_{\beta_8}v^+$ is a nonzero vector in $V_{T_A}(\lambda-q\alpha_1)$; so $q_0 = q$ and by (1.10), $\lambda|T_A = x\mu_1 + 2\mu_2$. Now, let $P_0 \ge B_Y{}^-$ be the parabolic subgroup of Y with Levi factor $L_0 = \langle U_{\pm\beta_\ell} \mid 5 \le \ell \le 8\rangle T_Y$. Then, L_0 has a natural subgroup, B, of type B_2. Moreover, $V^1(R_u(P_0))|B$ has a composition factor with the same high weight, as a B_2-module, as that of $V|A$. Thus, $\dim V|A < \dim V|Y$. Contradiction.

This completes the proof of (7.7).

(7.8). Suppose there exists $1 \le i \le r$ such that L_i is separated from all other components of L_Y' by more than one node of the Dynkin diagram. Then M_i is trivial.

Proof: Suppose false; i.e., suppose L_i is as described and M_i is nontrivial. By (7.4), each component of L_Y' is of classical type, so (1.5) implies $Z_A \le Z_Y$. Let W be the natural module for L_i (of type A_k). Then by (7.1) of [12], if $\mathrm{rank}(L_i) > 1$, $M_i \cong W$ or W^*.

Case I: Suppose $\mathrm{rank}(L_i)>2$.

Arguing as in the proof of (6.7) and applying (7.1) and (7.3), we see that $L_i = \langle U_{\pm\beta_j} | 1 \le j \le 4\rangle$ and $L_Y' = L_i \times \langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$. Let q_1 (respectively, q_2) be the field twist on the embedding of L_A' in L_i (respectively, $\langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$). Then the L_A' composition factors of Q_Y/K_{β_5} have high weights $6q_1\mu_2$ and $2q_1\mu_2$. So $\beta_5|Z_A = q_1\alpha_1$. Since $Q_Y/K_{\beta_6} \cong (Q_A{}^{\alpha_1})^{q_2}$ as L_A' modules, if $Q_A \not\le K_{\beta_6}$, $\beta_6|Z_A = q_2\alpha_1$. Then (2.8) implies that $q_1 = q_2$. Thus, either $Q_A \le K_{\beta_6}$ and $\langle \lambda, \beta_k\rangle = 0$ for k = 6, 7, 8, or $q_1 = q_2$ and (2.5) and (2.6) imply $\langle \lambda, \beta_7 + \beta_8\rangle = 0$. In fact, even in the second case, $\langle \lambda, \beta_6\rangle = 0$, else the bound on $\dim V^2(Q_A)_{\lambda - q_1\alpha_1}$ of (1.22) is exceeded. Also, $\langle \lambda, \beta_5\rangle = 0$, else the bound on $\dim V_{\beta_5}(Q_Y)$ of (1.25) is exceeded. So $\lambda|T_Y = \lambda_1$ or λ_2 and $\langle \lambda, \alpha_2\rangle = 4q_1$. Referring to (6.9), we find that if $\lambda|T_Y = \lambda_1$, then $\langle \lambda, \alpha_1\rangle = 0$, 2q, q, or $q+q_0$, for q and q_0 distinct p-powers. However in each case, by (1.27) and [8], $\dim V|A < \dim V|Y$. Thus, $\lambda|T_Y = \lambda_2$. Now, by (6.9), $\langle \lambda, \alpha_1\rangle = 0$, 3q, 2q, q, or $q+q_0$. However, (1.27) and (1.38) imply $\dim V|A < \dim V|Y$. This completes the consideration of Case I.

Case II: Suppose $\mathrm{rank}(L_i) \le 2$.

Then, in fact, $\mathrm{rank}(L_i) = 2$, else there exists a 2-dimensional L_A'

irreducible, Q_Y/K_δ, containing a nontrivial image of $Q_A^{\alpha_1}$. Suppose there exists $1 \le k \le r$, $k \ne i$ with M_k nontrivial. Then (7.7) implies that L_k is separated from all other components of L_Y' by more than one node of the Dynkin diagram. Then previous remarks of this result imply that L_k is of type A_2. Let q_i (respectively, q_k) be the field twist on the embedding of L_A' in L_i (L_k). So $q_i \ne q_k$. If $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\Sigma L_j) \ne 0$, for $j = i$ or $j = k$, then $Q_Y/K_\delta \cong (Q_A^{\alpha_1})^{q_j}$ and $\delta|T_A = q_j\alpha_1$. (See (2.4).) By (2.8), L_i and L_k are separated by more than two nodes of the Dynkin diagram. So $Y = E_8$ and $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_7,\beta_8\}$. Moreover, by (2.13) and (2.3), $\langle\lambda,\beta_j\rangle = 0$ for $j = 2,4,5,6$. If we take $L_i = \langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle$ and $L_k = \langle U_{\pm\beta_7},U_{\pm\beta_8}\rangle$, the above remarks imply $\beta_4|T_A = q_i\alpha_1$ and $\beta_6|T_A = q_k\alpha_1$. Also, (2.13) implies $\beta_2|T_A = 0 = \beta_5|T_A$. We also have $\beta_1|T_A = q_i\alpha_2 = \beta_3|T_A$ and $\beta_7|T_A = q_k\alpha_2 = \beta_8|T_A$. In particular, $V_{T_A}(\lambda-q\alpha_1) = 0$ for all p-powers q. Thus, $\langle\lambda,\alpha_1\rangle = 0$ and $\lambda|T_A = (2q_i + 2q_k)\mu_2$; so $\dim V|A \le 100 < \dim V|Y$ by (1.27) and (1.32). Hence, there does not exist $1 \le k \le r$, $k \ne i$ with M_k nontrivial.

Now (7.2) implies that there exists $\delta \in \Pi(Y) - \Pi(L_Y)$ with $\langle\lambda,\delta\rangle \ne 0$. By (2.3), $(\delta,\Sigma L_Y) \ne 0$. We claim that $(\delta,\Sigma L_i) \ne 0$. Otherwise, the bound on $\dim V_\delta(Q_Y)$ implies that there exists a unique $1 \le k \le r$, $k \ne i$ with $\mathrm{rank}(L_k) \le 2$ and with $(\Sigma L_k,\delta) \ne 0$. Actually, L_k has type A_2, else Q_Y/K_δ is a 2-dimensional irreducible L_A' module containing a nontrivial image of $Q_A^{\alpha_1}$. If L_i and L_k are separated by more than two nodes of the Dynkin diagram, $L_i \times L_k$ is as in the above paragraph. Then $\langle\lambda,\delta\rangle \ne 0$ contradicts (2.13). Thus, L_i and L_k are separated by exactly two nodes of the Dynkin diagram. Let $\gamma_i,\gamma_k \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma_i,\gamma_k) < 0$, $(\gamma_j,\Sigma L_j) \ne 0$ for $j = i,k$. Let q be the field twist on the embedding of L_A' in L_i. Note that if $\gamma_k|Z_A = 0$, (so $\gamma_k \ne \delta$) then $0 \ne w \in V_{T_Y}(\lambda-\delta)$ affords an $L_Y\hat{}$ composition factor of $V_\delta(R_u(P_Y\hat{}))$ which exceeds the bound of (1.25), where $P_Y\hat{} \ge B_Y^-$ is the parabolic of Y with Levi factor $L_Y\hat{} = \langle L_Y,U_{\pm\gamma_k}\rangle$. Hence, if $Q_A \le K_{\gamma_k}$, so $\delta \ne \gamma_k$, (2.10) implies that $-\gamma_k$ is not involved in L_A' and by (2.11), there is a nontrivial image of $Q_A^{\alpha_1}$ in $Q_Y(\gamma_i,\gamma_k)$. Hence, the field twist on the embedding of L_A' in L_k is q. If $Q_A \nleq K_{\gamma_k}$, (2.8) implies that $\gamma_k|Z_A = q\alpha_1$, which in turn implies, by (2.6) that the field twist on the embedding of L_A' in L_k is again q. Thus, $Q_Y/K_\delta \cong (Q_A{}^{\alpha_1})^q$ and $\delta|T_A = q\alpha_1$. (See (2.4).) But then the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_1}$ of (1.22) is exceeded. Thus, if $\delta \in \Pi(Y) - \Pi(L_Y)$ with $\langle\lambda,\delta\rangle \neq 0$, then $(\delta,\Sigma L_i) \neq 0$ as claimed.

Now, there exists a unique such δ. For otherwise, $\dim V_{T_A}(\lambda-q\alpha_1) > 1$, contradicting (1.31). Hence, $\lambda|T_Y = \lambda_\ell + x\lambda_m$ for some ℓ,m. Moreover, the nodes of the Dynkin diagram corresponding to β_ℓ and β_m are separated by at most one node. By (6.9), $\langle\lambda,\alpha_1\rangle = 0, q_0, 2q_0, 3q_0, q_0+q_1$ or xq_0 for distinct p-powers q_0 and q_1. Using (1.32) and (1.27), we see that $\dim V|A < \dim V|Y$ unless $\langle\lambda,\alpha_1\rangle = xq_0$. Moreover, since $f_\delta v^+ \in V_{T_A}(\lambda-q\alpha_1)$, $q_0 = q$ and by (1.10), $V|A$ is restricted.

Temporarily label as follows: $\Pi L_i = \{\gamma_1,\gamma_2\}$ and $(\delta,\gamma_1) < 0$. We claim that γ_2 must correspond to an end node of the Dynkin diagram. For otherwise, if $\delta_0 \in \Pi(Y) - \Pi(L_Y)$ with $(\delta_0,\gamma_2) < 0$, arguing as above, $\delta_0|T_A = \alpha_1$. The bound on $\dim V^2(Q_A)$ implies $\langle\lambda,\gamma_1\rangle = 1$ and $\langle\lambda,\gamma_2\rangle = 0$. Consider the subgroup $L_0 = \langle U_{\pm\delta}, U_{\pm\gamma_1}, U_{\pm\gamma_2}, U_{\pm\delta_0}\rangle$. Then L_0 has a natural subgroup, B, of type B_2. Moreover, v^+ affords an L_0 composition factor of V which restricted to B produces a B composition factor with the same high weight as $V|A$, as B_2 module. But L_0 lies in a proper parabolic of Y and so acts reducibly on V. Hence, $\dim V|A < \dim V|Y$. Thus, γ_2 corresponds to an end node, as claimed. Also, (2.13) implies that $L_i \neq \langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$ and if $L_i = \langle U_{\pm\beta_2}, U_{\pm\beta_4}\rangle$, $\delta = \beta_5$ and $Y = E_7$ or E_8. In fact, $L_i \neq \langle U_{\pm\beta_2}, U_{\pm\beta_4}\rangle$. For otherwise, (2.12) implies that $\beta_1|T_A = 0$ and using the parabolic $P_Y\hat{}$ of (2.12), we see that the bound on $\dim V^2(Q_A)$ is exceeded. Hence, either $Y = E_7$ with $\lambda|T_Y = x\lambda_5+\lambda_6$ or $x\lambda_5+\lambda_7$ or $Y = E_8$, with $\lambda|T_Y = x\lambda_6+\lambda_7$ or $x\lambda_6+\lambda_8$.

In Y of type E_7, let $L_1 = \langle U_{\pm\beta_1}, U_{\pm\beta_3}, U_{\pm\beta_4}, U_{\pm\beta_5}\rangle$, a group of type A_4, which has a natural subgroup, B, of type B_2. Then $f_{245}f_6v^+$ affords an L_1 composition factor of V which restricts to B to produce a composition factor having the same high weight, as B_2 module, as $V|A$. Similarly, in Y of type E_8, let $L_2 = \langle U_{\pm\beta_3}, U_{\pm\beta_4}, U_{\pm\beta_5}, U_{\pm\beta_6} \rangle$. Here, the vector $f_{123456}f_7v^+$ serves the same purpose as above. In each case, L_i lies in a proper parabolic of Y, so acts reducibly on V. Hence, $\dim V|A < \dim V|Y$ and the result of (7.8) holds.□

(7.9). There are no examples in the Main Theorem with Y of type E_n and A of type B_2, when $p>2$

Proof: Suppose false; i.e., suppose $V|Y$ is a nontrivial kY-module. Then, (7.3), (7.7), and (7.8) imply $\langle \lambda, \alpha_2 \rangle = 0$. So $\langle \lambda, \alpha_1 \rangle \neq 0$. Let $P \geq B_A^-$ be the parabolic subgroup of A with Levi factor $L = \langle U_{\pm\alpha_1} \rangle T_A$. Let R be a parabolic subgroup of Y with $P \leq R$ and $Q = R_u(P) \leq Q_0 = R_u(R)$. Let R be minimal with these properties. Let L_0 be a Levi complement of Q_0 in R such that $T_0 \leq L_0$, for some maximal torus of R, with $T_A \leq T_0$. Fix a base $\Pi_0(Y)$ of the root system, $\Sigma_0^+(Y)$, of Y such that $L \cap U_A \leq Q_0(L_0 \cap U_0)$, where U_0 is the product of T_0 root subgroups corresponding to roots in $\Sigma_0^+(Y)$ and Q_0 is the product of T_0 root subgroups corresponding to roots in $\Sigma_0^-(Y) - \Sigma(L_0)$. Let $\Pi_0(Y) = \{\gamma_1, \ldots, \gamma_n\}$, with Dynkin diagrams labelled as throughout. Let $\langle w^+ \rangle$ be the unique 1-space fixed by U_0; let λ be the T_0 weight of w^+. Then by (6.9), $\dim V|A < \dim V|Y$ unless $L_0' = L_1 \times L_2$, with L_1 a simple algebraic group of type A_1 and L_2 a semisimple algebraic group acting trivially on $V^1(Q_0)$. So if $L_1 = \langle U_{\pm\beta} \rangle$ for some $\beta \in \Pi_0(Y)$, then $\langle \lambda, \beta \rangle = c = \langle \lambda, \alpha_1 \rangle$, for some $p>c>o$. (Use (1.10).) It is easy to check that $\dim V^2(Q) = c$, in this case. Thus, if $\gamma \in \Pi_0(Y) - \Pi(L_Y)$, $\langle \lambda, \gamma \rangle = 0$. (Use (1.36) if $(\gamma, \beta) \neq 0$.) Also, $\beta \neq \gamma_4$, else $f_{\gamma_3+\gamma_4}w^+$, $f_{\gamma_2+\gamma_4}w^+$ and $f_{\gamma_5+\gamma_4}w^+$ afford distinct L_0' composition factors of $V^2(Q_0)$, exceeding $\dim V^2(Q)$. Hence we may choose $\gamma_j, \gamma_k, \gamma_\ell \in \Pi_0(Y)$ with $(\beta, \gamma_j) < 0$, $(\gamma_j, \gamma_k) < 0$ and $(\gamma_k, \gamma_\ell) < 0$. The subgroup $N = \langle U_{\pm\gamma_j}, U_{\pm\gamma_k}, U_{\pm\gamma_\ell}, U_{\pm\beta} \rangle \leq Y$ has type A_4, and therefore has a natural subgroup of type B_2, say A_0. Also, the N-composition factor of V afforded by v^+ is not all of $V|Y$, as N is contained in the Levi factor of a proper parabolic of Y. But the A_0 composition factor of $V|Y$ has the same high weight, as B_2 module, as does $V|A$. Thus, $\dim V|A < \dim V|Y$. Contradiction.□

CHAPTER 8: $A = G_2$

Let $A < Y$ be simple algebraic groups, with Y simply connected, having a root system of type E_n. Let $V = V(\lambda)$ be a restricted irreducible kY-module. In this chapter, we consider the main problem in case A has type G_2. Let T_A (respectively, T) be a fixed maximal torus of A (respectively, Y) with $T_A \leq T$. Let $\Pi(A) = \{\alpha_1,\alpha_2\}$ be a base of the root system $\Sigma(A)$ and $\Pi(Y)$ a base of $\Sigma(Y)$. Label the Dynkin diagrams of $\Sigma(A)$ and $\Sigma(Y)$ as throughout. Let $\{\mu_1,\mu_2\}$ (respectively, $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$) denote the fundamental dominant weights corresponding to the given ordered bases. The result is the following:

<u>Theorem (8.0).</u> (a) If $V|A$ is irreducible, then $p \neq 2,7$, $Y = E_6$, $\lambda|T = \lambda_1$ (or λ_6), $\lambda|T_A = 2\mu_1$.

(b) If $p \neq 2,7$ and $Y = E_6$, then there exists a closed subgroup $B < Y$, B of type G_2, such that $V(\lambda_1)|B$ is irreducible.

<u>Remark:</u> The proof of (8.0)(b) is given in [16]. We prove (8.0)(a) in this chapter in case $p > 3$. The case where $p = 2$ or 3 is handled in Chapter 9.

We adopt Notation and Hypothesis (2.0) throughout this chapter, with the additional conditions: Assume <u>$p > 3$</u> and $L_A = \langle U_{\pm\alpha_1}\rangle T_A$, so $Q_A{}^{\alpha} = Q_A{}^{\alpha_2}$ is a 4-dimensional, tensor indecomposable $L_A{}'$ module.

The following technical lemma which will be used in many of the successive results.

(8.1). (i) For $p \geq 5$, $\dim V(4\mu_2) > 156$.

(ii) $\dim V = 27$ if and only if $\lambda|T_A = 2q\mu_1$, for some p-power q, and $p \neq 7$.

(iii) There does not exist an irreducible kA-module of dimension 56; so if Y has type E_7, $\lambda|T_Y \neq \lambda_7$.

Proof: By applying the methods of (1.30) and (1.33), repeatedly, and recalling that the Weyl group of A has order 12, we obtain (i).

Now, suppose $\dim V = 27$. Then (1.32) implies V is tensor indecomposable, so we may assume V is restricted. Suppose $\langle\lambda,\alpha_1\rangle = a \neq 0 \neq b = \langle\lambda,\alpha_2\rangle$. Then, by [8], a>1 or b>1. Breaking the argument up into separate cases for a=2 or a>2, and b=2 or b>2, the methods of (1.30), (1.32) and (1.35) show that $\dim V > 27$. Thus, a=0 or b=0. Moreover, by (1.27), if $a \neq 0$, a>1 and if $b \neq 0$, b>1. Since p>3, [8] implies $b \neq 2$. Then, the methods of (1.30) and (1.32) imply that $\dim V > 27$ if $b \neq 0$. So $a \neq 0$ and b=0. If a=2, [8] implies the result. By [8], $a \neq 3$. But if a>3, the methods of (1.30) and (1.32) imply $\dim V > 27$. Thus, (ii) holds. Arguing similarly, we obtain (iii).□

(8.2). If $\dim V^1(Q_Y) > 1$, L_Y' is not a simple algebraic group.

Proof: Suppose false. Then Theorem (7.1) of [12] implies that L_Y' is of classical type, so by (1.5), $Z_A \leq Z_Y$. Consider first the case where L_Y' has type D_k for some $k \geq 4$. We may argue as in the proof of (6.1), to obtain: $L_Y' = D_6$, $Y = E_7$ and $V^1(Q_Y) \cong W$, the natural module for L_Y'. Also, $\langle\lambda,\beta_1\rangle = 0$, else the bound on $\dim V_{\beta_1}(Q_Y)$, of (1.25), is exceeded. But now $\lambda|T_Y = \lambda_7$, contradicting (8.1). Thus, L_Y' does not have type D_k for $k \geq 4$.

If L_Y' has type A_k for k>3, we may argue as in the proof of (6.1) to reduce to L_Y' of type A_{n-1} in Y of type E_n, with $\lambda|T_Y = \lambda_n$, contradicting (8.1) and previous general remarks.

We have, therefore, L_Y' of type A_k for $k \leq 3$. Actually, k=3, else there exists $\gamma \in \Pi(Y)-\Pi(L_Y)$, with Q_Y/K_γ an irreducible L_A'-module of dimension 2 or 3, containing a nontrivial image of $Q_A{}^{\alpha_2}$. Note also that there does not exist $\delta \in \Pi(Y)-\Pi(L_Y)$ with $V_{L_Y'}(-\gamma) \cong W\wedge W$. For $(W\wedge W)|L_A'$ has composition factors of dimensions 5 and 1 or two factors of dimension 3. Thus, for each $\gamma \in \Pi(Y)-\Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \neq 0$, $Q_Y/K_\gamma \cong W$ or W^*. Since $p > 2$, $Q_A{}^{\alpha_2}$ is tensor indecomposable, so $W|L_A'$ is tensor indecomposable. Suppose $W|L_A'$ has high weight $3q\mu_2$ for some p-power q. Then comparing high weight vectors in Q_Y/K_γ and $Q_A{}^{\alpha_2}$, we see that $\gamma|T_A = q\alpha_2$ for all $\gamma \in \Pi(Y)-\Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \neq 0$. Also, (2.12) implies $\tau|T_A = 0$ for all $\tau \in \Pi(Y)-\Pi(L_Y)$ with $(\tau,\Sigma L_Y) = 0$. As in the proof of (2.16), $\beta|T_A = q\alpha_1$, for each $\beta \in \Pi(L_Y)$.

Now $\langle\lambda,\alpha_2\rangle \neq 0$, else dim $V|A \neq$ dim $V|Y$. (Use [8], (1.30) and (1.32).) Hence, there exists $\gamma \in \Pi(Y)-\Pi(L_Y)$ with $(\gamma,\Sigma L_Y) \neq 0$ and $\langle\lambda,\gamma\rangle \neq 0$. For otherwise, there is no vector in $V|Y$ with T_A weight $\lambda - q_0\alpha_2$, for any p-power q_0. By (2.13), γ must correspond to an end node of the Dynkin diagram. Applying this restriction and the bound on $\dim(V^2(Q_A)_{\lambda-q\alpha_2})$, we reduce to the following:

(a) $Y = E_6$, $L_Y' = \langle U_{\pm\beta_i} \mid i = 1,3,4\rangle$, $\lambda|T_Y = \lambda_1 + x\lambda_2$ $x>0$.

(b) $Y = E_6$, $L_Y' = \langle U_{\pm\beta_i} \mid 4\leq i\leq 6\rangle$, $\lambda|T_Y = \lambda_6 + x\lambda_2$, $x>0$.

(c) Y E_7, $L_Y' = \langle U_{\pm\beta_i} \mid 4\leq i\leq 6\rangle$, $\lambda|T_Y = \lambda_6 + x\lambda_7$, $x>0$.

(d) $Y = E_8$, $L_Y' = \langle U_{\pm\beta_i} \mid 5\leq i\leq 7\rangle$, $\lambda|T_Y = \lambda_7 + x\lambda_8$, $x>0$.

Actually, the configurations of (a) and (b) can be ruled out by (1.23).

Now, $V|A$ is a conjugate of a basic module since there is no vector in $V|Y$ with T_A weight $\lambda-q_0\alpha_2$ for $q_0 \neq q$. So by (1.10), q=1 and $\langle\lambda,\alpha_1\rangle = 3$. By (1.29), dim $V_{T_A}(\lambda-3\alpha_1-\alpha_2) \leq 4$. Thus, $Y = E_7$; for otherwise, $f_{4567}v^+$, $f_{34567}v^+$, $f_{134567}v^+$, $f_{24567}v^+$ and $f_{5678}v^+$ are five linearly independent vectors in $V_{T_A}(\lambda-3\alpha_1-\alpha_2)$. Now, one checks that in the action of L_A' on the 56-dimensional irreducible kY-module $V(\lambda_7)$, there are no 2- or 3-dimensional composition factors, and all composition factors are tensor indecomposable. But there is no 56-dimensional kA-module which affords such an L_A' composition series. Hence, L_Y' does not have type A_3.

This completes the proof of (8.2).□

(8.3). If $\dim V^1(Q_Y) > 1$, each L_i has type A_{k_i} for some $k_i \geq 1$.

Proof: We first claim that each L_i has classical type. For otherwise, $Y = E_8$ and L_Y' has type $E_6 \times A_1$. By Theorem (7.1) of [12], $\lambda|T_Y = x\lambda_7 + c\lambda_8$ and $\langle\lambda,\alpha_1\rangle = c \cdot q$, for $c>0$ and some p-power q. By (6.9) and (1.22), $\dim V^2(Q_A) \leq 11c + 8$. But $f_{\beta_7}v^+$ and/or $f_{\beta_7+\beta_8}v^+$ afford(s) L_Y' composition factors in $V^2(Q_A)$, forcing $\dim V^2(Q_A) \geq 27c$. (Use (1.36) if $x \neq 0$ and $c = p-1$.) Thus, each component of L_Y' has classical type, so (1.5) implies $Z_A \leq Z_Y$.

Suppose L_i has type D_k for some k. Arguing as in the proof of (6.3), we see that M_i is trivial. Now $\langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$ is not a component of L_Y', else the bounds on $\dim V_{\beta_6}(Q_Y)$ and $\dim V_{\beta_1}(Q_Y)$ imply $\lambda|T_Y = \lambda_8$. Hence, $L_Y' = L_i \times L_j$ with L_j of type A_1 and M_j nontrivial. Moreover, $(\Pi L_i, \Pi L_j) \neq 0$, else there exists $\delta \in \Pi(Y)-\Pi(L_Y)$ with $(\delta,\Sigma L_j) \neq 0$ and Q_Y/K_δ a 2-dimensional irreducible L_A'-module containing a nontrivial image of $Q_A{}^{\alpha_2}$. Let $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\Sigma L_i) \neq 0 \neq (\gamma,\Sigma L_j)$. Now (1.36) and the bound on $\dim V_\gamma(Q_Y)$ of (1.25) imply $L_i = D_4$. So $\Pi(L_Y) = \{ \beta_7, \beta_m \mid 2 \leq m \leq 5 \}$, with $\langle\lambda,\beta_m\rangle = 0$ for $2 \leq m \leq 5$ and $\langle\lambda,\beta_7\rangle > 0$. The previous remarks imply that $Y = E_7$. But the bound on $\dim V_{\beta_1}(Q_Y)$ and on $\dim V_{\beta_6}(Q_Y)$ (in conjunction with (1.36)) implies that $\lambda|T_Y = \lambda_7$, contradicting (8.1). This completes the proof of (8.3).□

(8.4). Suppose there exists $\gamma \in \Pi(Y)-\Pi(L_Y)$ and $1 \leq i,j \leq r$ such that $(\gamma,\Sigma L_i) \neq 0 \neq (\gamma,\Sigma L_j)$. Then M_i or M_j is trivial.

Proof: Suppose false; i.e., suppose M_i and M_j are both nontrivial. By (8.3), each component, L_k, of L_Y' has type A_{m_k} for some $m_k \geq 1$; so (1.5) implies $Z_A \leq Z_Y$. Let W_m denote the natural module for L_m, m = i, j. By (7.1) of [12], if $\operatorname{rank}(L_m) > 1$, $M_m \cong W_m$ or $W_m{}^*$.

Case I: Suppose $V_{L_i}(-\gamma) \not\cong W_i$ or W_i^*.

Then $\mathrm{rank}(L_i) \geq 3$ and $V_{L_i}(-\gamma) \cong W_i \wedge W_i$ or $W_i^* \wedge W_i^*$. Now (1.15) implies that L_j cannot have type A_2. Also, since L_A' acts irreducibly on W_m, for $m = i, j$, if $\mathrm{rank}L_m \neq 3$, there does not exist $\delta \in \Pi(Y)-\Pi(L_Y)$ with $(\delta, \Sigma L_m) \neq 0$ and $Q_Y/K_\delta \cong W_m$ or W_m^* for $m = i$ or j. Finally, the bound on $\dim V_\gamma(Q_Y)$, restricts the situation still further. (Use (1.34), (1.36) and $p>3$.) These remarks imply that $L_i \times L_j$ has type $A_3 \times A_1$, $A_3 \times A_3$ or $A_4 \times A_1$.

If $L_i \times L_j$ has type $A_4 \times A_1$ with $q_1 \neq q_2$ the field twists on the embeddings of L_A' in L_i and L_j respectively, then one checks that there is no 4-dimensional L_A' composition factor of Q_Y/K_γ. But this contradicts (2.3); so $L_i \times L_j$ does not have type $A_4 \times A_1$. If $\Pi(L_i \times L_j) = \{\beta_k \mid k \neq 1,5\}$ in E_8, the bound on $\dim V_{\beta_5}(Q_Y)$ implies $\lambda | T_Y = x\lambda_1 + \lambda_\ell + \lambda_8$ where $\ell = 2$ or 3. However, $f_{\beta_\ell + \beta_4 + \beta_5} v^+$ and $f_{\beta_5 + \beta_6 + \beta_7 + \beta_8} v^+$ afford distinct L_Y' composition factors of $V_{\beta_5}(Q_Y)$ of dimensions 60 and 20, respectively, exceeding the bound of (1.25). Hence, $L_i \times L_j$ does not have type $A_3 \times A_3$.

Finally, consider the case where $L_i \times L_j$ has type $A_3 \times A_1$. We first note that $W_i | L_A'$ is tensor indecomposable. For otherwise, if $W_i | L_A'$ has high weight $(q_1 + q_2)\mu_1$ and if q_3 is the field twist on the embedding of L_A' in L_j, for q_1, q_2, q_3 distinct powers of p, the L_A' composition factors of Q_Y/K_γ are 6 dimensional. But this implies $Q_A \leq K_\gamma$, contradicting (2.3). So $W_i | L_A'$ has high weight $3q\mu_1$ for some $q \neq q_3$. However, now the L_A' composition factors of Q_Y/K_γ have dimensions 10 and 2 so again $Q_A \leq K_\gamma$, contradicting (2.3). This completes the consideration of Case I.

Case II: $V_{L_m}(-\gamma) \cong W_m$ or W_m^* for $m = i, j$.

Then (2.7) implies that there exists $k \neq i, j$ with $(\Sigma L_k, \gamma) \neq 0$ and $\dim M_k = 1$. Thus, $\gamma = \beta_4$. We first claim that $\langle U_{\pm\beta_3} \rangle$ is not a component of L_Y'. For otherwise, $Q_A \leq K_{\beta_1}$ since Q_Y/K_{β_1} is a 2-dimensional irreducible L_A'-module. Also $p > 2$ and (2.10) imply that $-\beta_1$ is not involved in L_A. Hence, by (2.11), there is a nontrivial image of $Q_A^{\alpha_2}$ in $Q_Y(\beta_4, \beta_1)$. But $Q_Y(\beta_4, \beta_1)$ is an irreducible, tensor decomposable

L_A'-module. Thus, $\langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$ and $\langle U_{\pm\beta_2}\rangle$ are components of L_Y'. Then (1.15) and p>3 imply that the third component adjacent to β_4, say L_0, has type A_{k_0} for $k_0 = 2$, 3 or 4. If L_0 has type A_3, again by (1.15), $V_{L_0}(-\beta_4)|L_A'$ is tensor decomposable. The bound on $\dim V_{\beta_4}(Q_Y)$ of (1.25), together with (1.36), implies $\langle\lambda,\beta_2\rangle = 0$ in case L_0 has type A_3 or A_4. Previous remarks then imply that if L_0 has type A_2 or A_3, Q_Y/K_γ has no L_A' composition factor isomorphic to a twist of $Q_A{}^{\alpha_2}$. Hence, $\Pi(L_Y) = \{\beta_m \mid m \neq 4\}$ in E_8 and $\lambda|T_Y = \lambda_1 + \lambda_8$. (The labelling of λ is given by the bound on $\dim V_{\beta_4}(Q_Y)$.) But now, $f_{134}v^+$ and $f_{45678}v^+$ afford distinct L_Y' composition factors of $V_{\beta_4}(Q_Y)$ of dimensions at least 46 and 16, respectively, exceeding the bound of (1.25).

This completes the proof of (8.4).□

<u>(8.5).</u> Suppose there exist distinct $1 \leq i, j, k \leq r$ such that $(\Sigma L_\ell, \beta_4) \neq 0$ for $\ell = i, j, k$ and $\dim(M_i \otimes M_j \otimes M_k) > 1$. Then $Y = E_6$, $\lambda|T_Y = \lambda_1$ (or λ_6) and $\lambda|T_A = 2\mu_1$. Moreover, $p \neq 7$.

<u>Proof</u>: Since each component of L_Y' is necessarily of classical type, (1.5) implies $Z_A \leq Z_Y$. Let W_m denote the natural module for L_m, $m = i,j,k$. By (7.1) of [12], if $\mathrm{rank}(L_m) > 1$ and M_m is nontrivial, $M_m \cong W_m$ or $W_m{}^*$, for $m = i,j,k$. By (8.4), only one of M_i, M_j and M_k is nontrivial.

Since $p > 2$, (1.15) implies $L_i \times L_j \times L_k$ has type $A_1 \times A_1 \times A_\ell$, $\ell = 1$ or 3, or $A_2 \times A_1 \times A_\ell$, $\ell = 2$, 3, or 4. If $L_i \times L_j \times L_k$ has type $A_1 \times A_1 \times A_\ell$, $Q_A \leq K_{\beta_1}$, as Q_Y/K_{β_1} is a 2-dimensional irreducible L_A'-module. Moreover, (2.10) implies that $-\beta_1$ is not involved in L_A'. Hence, by (2.11) and (1.15), applied to $Q_Y(\beta_4,\beta_1)$, $\ell \neq 1$. And in the case where $L_i \times L_j \times L_k$ has type $A_1 \times A_1 \times A_3$, (1.15) applied to Q_Y/K_{β_4} implies $V_{L_k}(-\beta_4)|L_A'$ is tensor indecomposable. But (2.11) and (1.15) (applied to $Q_Y(\beta_4,\beta_1)$) produce a contradiction. Thus, $L_i \times L_j \times L_k$ has type $A_2 \times A_1 \times A_\ell$, $\ell = 2$, 3, 4.

Consider the case where $L_i \times L_j \times L_k$ has type $A_2 \times A_1 \times A_4$. The bound on $\dim V_{\beta_4}(Q_Y)$ implies $\lambda|T_Y = \lambda_1$, and $\langle\lambda,\alpha_1\rangle = 2q$ for some p-power q.

By (6.9), $\langle\lambda,\alpha_2\rangle = 0$, $2q_0$, or q_0, for some p-power q_0. But then [8] and (1.27) imply $\dim V|A < \dim V|Y$. Thus, L_Y' does not have type $A_2 \times A_1 \times A_4$. Consider now the case where $L_i \times L_j \times L_k$ has type $A_2 \times A_1 \times A_3$. Then (1.15) implies that if L_k has type A_3, $V_{L_k}(-\beta_4)|L_A'$ is tensor decomposable. So if $Y = E_8$, Q_Y/K_{β_8} has no L_A' composition factor isomorphic to a twist of $Q_A{}^{\alpha_2}$; so $\langle\lambda,\beta_5+\beta_6+\beta_7+\beta_8\rangle = 0$. Moreover, the bound on $\dim V_{\beta_4}(Q_Y)$ implies that either $\langle\lambda,\beta_1\rangle = 1$ and $\langle\lambda,\beta_\ell\rangle = 0$ for $7 \ge \ell > 1$ or $\langle\lambda,\beta_7\rangle = 1$ and $\langle\lambda,\beta_\ell\rangle = 0$ for $\ell < 7$. In the first case, $Y = E_8$. But we may argue as in the previous case to see that $\dim V|A < \dim V|Y$. In the second case, previous remarks imply that $Y = E_7$, contradicting (8.1).

Finally, we must consider $L_i \times L_j \times L_k$ of type $A_2 \times A_1 \times A_2$. The bound on $\dim V_{\beta_4}(Q_Y)$ and (1.36) imply that $\langle\lambda,\beta_1\rangle = 1$ and $\langle\lambda,\beta_m\rangle = 0$ for $2 \le m \le 6$ or $\langle\lambda,\beta_6\rangle = 1$ and $\langle\lambda,\beta_m\rangle = 0$ for $1 \le m \le 5$. So if $Y = E_6$, $\dim V|Y = 27$ and the result follows from (8.1) and (1.10). If $Y = E_7$, then $Q_A \le K_{\beta_7}$, as Q_Y/K_{β_7} is a 3-dimensional irreducible L_A'-module. But then $\langle\lambda,\beta_k\rangle = 0$ for $k = 5, 6, 7$ and $\lambda|T_Y = \lambda_1$.

Suppose $Y = E_8$. Let q_1 (respectively, q_2, q_3) be the field twist on the embedding of L_A' in $\langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle$ (respectively, $\langle U_{\pm\beta_2}\rangle$, $\langle U_{\pm\beta_5},U_{\pm\beta_6}\rangle$). Then, (2.7) implies that q_1, q_2, q_3 are not all distinct. If $q_1 = q_2 \ne q_3$ or if $q_1 \ne q_2 = q_3$, the L_A' composition factors of Q_Y/K_{β_4} have dimensions 12 and 6. If $q_1 = q_3 \ne q_2$, the L_A' compositions factors of Q_Y/K_{β_4} have dimensions 10, 6, and 2. Thus, $q_1 = q_2 = q_3$. The L_A' composition factors of Q_Y/K_{β_4} have high weights $5q_1\mu_1$, $3q_1\mu_1$ and $q_1\mu_1$. Thus, $\beta_4|Z_A = q_1\alpha_2$. Examining the $T(L_A')$ weights in Q_Y/K_{β_4} we see that $x_{-\alpha_2}(t) = x_{-\beta_2-\beta_4}(c_1t^{q_1})x_{-\beta_4-\beta_5}(c_2t^{q_1})x_{-\beta_3-\beta_4}(c_3t^{q_1})u_0$, for $c_i \in k$, c_1, c_2, c_3 not all zero, and $u_0 \in K_{\beta_4}$. Since $\beta_\ell|T_A = q_1\alpha_1$ for $\ell = 1, 2, 3, 5, 6$, $\beta_4|T_A = q_1(\alpha_2 - \alpha_1)$.

Let $L_i = \langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle$ and $L_j = \langle U_{\pm\beta_5},U_{\pm\beta_6}\rangle$. Note that if $Q_A \not\le K_{\beta_7}$, $\dim(Q_Y/K_{\beta_7}) \ge 4$; so $\langle U_{\pm\beta_8}\rangle \le L_Y'$ We first claim that M_j is trivial. For suppose M_j is nontrivial; in particular, $Q_A \not\le K_{\beta_7}$. Then (2.17) implies that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_8}\rangle$ is also q_1 and that $\beta_7|Z_A = q_1\alpha_2$. Thus, (2.5) and (2.6) imply $\langle\lambda,\beta_8\rangle = 0$. Now, the bound on $\dim V^2(Q_A)_{\lambda-q_1\alpha_2}$, of (1.22), implies $\lambda|T_Y = \lambda_6$. By (6.9), $\langle\lambda,\alpha_2\rangle = 0,2q$, q, or $q + q_0$, for q and q_0 distinct p-powers. However, (1.27) and (1.32) imply $\dim V|A < \dim V|Y$. Thus M_j is trivial; so M_i is nontrivial and $\langle\lambda,\beta_1\rangle = 1$, $\langle\lambda,\beta_\ell\rangle = 0$ for $2\le\ell\le6$. If $\lambda|T_Y = \lambda_1$, we may argue as in the $A_1 \times A_2 \times A_4$ case to produce a contradiction. Thus, $\langle\lambda,\beta_7+\beta_8\rangle \ne 0$. Argue as in the previous paragraph to get $\langle\lambda,\beta_8\rangle = 0$. But then the bound on $\dim V_{\beta_7}(Q_Y)$ is exceeded.

This completes the proof of (8.5).□

(8.6). Let $\gamma \in \Pi(Y)-\Pi(L_Y)$. Suppose there exists a unique pair $1\le i,j\le r$ with $(\Sigma L_i,\gamma) \ne 0 \ne (\Sigma L_j,\gamma)$ and $\dim(M_i\otimes M_j) > 1$. Then $L_i\times L_j$ has type $A_1\times A_2$ and only one of M_i and M_j is nontrivial. Moreover, if $\Pi(L_i) = \{\gamma_0\}$, $\Pi(L_j) = \{\gamma_1,\gamma_2\}$, with $(\gamma_1,\gamma) < 0$, then there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\gamma_0) \ne 0$ (respectively, $(\delta,\gamma_2) \ne 0$) and $(\delta,\Sigma L_m) = 0$ for all $m \ne i$ (respectively, $m\ne j$).

Proof: By (8.3), each component, L_k, of L_Y' has type A_{m_k} for some $m_k\ge1$, so (1.5) implies $Z_A \le Z_Y$. Let W_m denote the natural module for L_m, for m = i, j. By (7.1) of [12], if M_m is nontrivial and $\mathrm{rank}(L_m) > 1$, $M_m \cong W_m$ or W_m^*, for m = i or j. By (8.4), only one of M_i and M_j is nontrivial.

Case I: Suppose $V_{L_i}(-\gamma) \not\cong W_i$ or W_i^*.

Then (1.15) and size restrictions imply that $L_i\times L_j$ has type $A_3\times A_\ell$, for ℓ = 1 or 3, $A_4\times A_\ell$, for ℓ = 1 or 3, $A_5\times A_1$ or $A_6\times A_1$. If $L_i\times L_j$ has type $A_m\times A_1$ for m = 4, 5 or 6, the bound on $\dim V_\gamma(Q_Y)$, together with (1.36), implies that the A_1 component acts trivially on $V^1(Q_Y)$. But if $L_i\times L_j$ has type $A_6\times A_1$, the bound implies $\lambda|T_Y = \lambda_8$. Also, if $L_i\times L_j$ has type $A_5\times A_1$, then $Y = E_7$, else Q_Y/K_{β_8} is a 6-dimensional irreducible L_A'-module containing a nontrivial image of $Q_A^{\alpha_2}$. The bound on

$\dim V_{\beta_3}(Q_Y)$ implies that $\lambda|T_Y = \lambda_7$, contradicting (8.1). Thus, $L_i \times L_j$ does not have type $A_6 \times A_1$ or $A_5 \times A_1$.

Consider now the case where $L_i \times L_j$ has type $A_4 \times A_1$ (so the A_1 component acts trivially on $V^1(Q_Y)$). The bound on $\dim V_\gamma(Q_Y)$ and (1.23) imply that if $Y = E_6$, $\lambda|T_Y = \lambda_2$; thus, $Y = E_7$ or E_8. If $Y = E_7$, Q_Y/K_{β_7} is an irreducible L_A'-module of dimension 2 or 5; so $Q_A \leq K_{\beta_7}$. Thus, $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_3,\beta_4,\beta_6\}$. By (2.10), $-\beta_7$ is not involved in L_A'. However, then (2.11) implies that there is a nontrivial image of $Q_A^{\alpha_2}$ in $Q_Y(\beta_5,\beta_7)$, an L_A'-module with no 4-dimensional, tensor indecomposable compositon factor. Hence, $Y = E_8$. Let q_1 (respectively, q_2) be the field twist on the embedding of L_A' in L_i (respectively, L_j). The L_A' composition factors of Q_Y/K_γ have high weights $(6q_1 + q_2)\mu_1$ anad $(2q_1 + q_2)\mu_1$; if p=5 and $5q_1 = q_2$, the high weights are $(2q_2 + q_1)\mu_1$, $q_1\mu_1$, and $(2q_1 + q_2)\mu_1$; if $q_1 = q_2$ the high weights are $7q_1\mu_1$, $5q_1\mu_1$, $3q_1\mu_1$ and $q_1\mu_1$. Thus, $q_1 = q_2$ and $\gamma|Z_A = q_1\alpha_2$.

Now standard arguments (using (2.5), (2.6), (2.17) and (1.22)) imply that $\Pi(L_i \times L_j) \neq \{\beta_k \mid k = 1,2,4,5,6\}$. We have, therefore, $\Pi(L_i \times L_j) = \{\beta_1,\beta_2, \beta_3,\beta_4,\beta_6\}$. As well, $\langle U_{\pm\beta_8}\rangle$ is a component of L_Y', else we can argue as in E_7 to produce a contradiction. Also, (1.15) implies $Q_A \leq K_{\beta_7}$. If $-\beta_7$ is involved in L_A', $\beta_7|Z_A = 0$. Otherwise, (2.11) implies $Q_Y(\beta_5,\beta_7)$ contains a nontrivial image of $Q_A^{\alpha_2}$. Arguing as with Q_Y/K_{β_5} (in the previous paragraph), we see that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_8}\rangle$ is q_1 and $(\beta_5 + \beta_7)|Z_A = q_1\alpha_2$. So again $\beta_7|Z_A = 0$. Using the parabolic $P_Y\hat{}$ of (2.11), we see that $\lambda|T_Y = \lambda_1$, else the bound on $\dim V_{\beta_5}(Q_Y\hat{})$ is exceeded. By (6.9), $\langle\lambda,\alpha_2\rangle = 0$, 2q or q, for some p-power q. But [8] and (1.27) imply that $\dim V|A \neq \dim V|Y$. Thus, $L_i \times L_j$ does not have type $A_4 \times A_1$.

If $\Pi(L_i \times L_j) = \{\beta_k | k \neq 5\}$ in E_8, the bound on $\dim V_{\beta_5}(Q_Y)$ implies that $\lambda|T_Y = \lambda_1$ and $\langle\lambda,\alpha_1\rangle = 4q$ for some p-power q. Arguing as in the previous paragraph, we have $\dim V|A \neq \dim V|Y$. Thus, $L_i \times L_j$ does not have type $A_4 \times A_3$.

Consider the case where $\Pi(L_i) = \{\beta_2,\beta_3,\beta_4\}$ and $\Pi(L_j) = \{\beta_6,\beta_7,\beta_8\}$ in E_8. Note that $Q_A \nleq K_{\beta_1}$. For otherwise, $\langle\lambda,\beta_k\rangle = 0$ for $1 \le k \le 4$ and the bound on $\dim V_{\beta_5}(Q_Y)$ implies that $\lambda | T_Y = \lambda_8$. Since $Q_A \nleq K_{\beta_1}$, $W_i | L_A'$ is tensor indecomposable. Then (1.15) (applied to Q_Y/K_{β_5}) implies that $W_j | L_A'$ is also tensor indecomposable. Let q_1 (respectively, q_2) be the field twist on the embedding of L_A' in L_i (respectively, L_j). Then the L_A' composition factors of Q_Y/K_{β_5} have high weights $(4q_1 + 3q_2)\mu_1$ and $3q_2\mu_1$, if $q_1 \neq q_2$; if $q_1 = q_2$, the high weights are $7q_1\mu_1$, $5q_1\mu_1$, $3q_1\mu_1$ and $q_1\mu_1$. Thus, $\beta_5 | Z_A = q_2\alpha_2$. Also, since $Q_Y/K_{\beta_1} \cong (Q_A{}^{\alpha_2})^{q_1}$, $\beta_1 | T_A = q_1\alpha_2$. If $q_1 = q_2$, the bound on $\dim V^2(Q_A)_{\lambda - q_1\alpha_2}$, of (1.22), implies that $\lambda | T_Y = \lambda_8$. Thus $q_1 \neq q_2$. Examining $T(L_A')$ weight vectors in Q_Y/K_{β_1} and in Q_Y/K_{β_5}, we have $x_{-\alpha_2}(t) = x_{-\beta_1}(at^{q_1})x_{-245}(b_1t^{q_2})\cdot x_{-345}(b_2t^{q_2})u$, where $a \in k^*$, $b_i \in k$, $u \in K_{\beta_1} \cap K_{\beta_5}$. In fact $b_2 \neq 0$, else there is a nontrivial contribution to the root group $U_{-\beta_4-\beta_5}$ in the expression for $[x_{\alpha_1}(t), x_{-\alpha_2}(t)]$. However, $b_2 \neq 0$ and $q_1 \neq q_2$ contradicts (2.8). Thus, $L_i \times L_j$ does not have type $A_3 \times A_3$.

We must now consider $L_i \times L_j$ of type $A_3 \times A_1$. Suppose $\Pi(L_i \times L_j) = \{\beta_2,\beta_3,\beta_4,\beta_6\}$. If $Q_A \le K_{\beta_1}$, $-\beta_1$ is not involved in L_A'; else $\beta_1 | Z_A = 0$, and using the parabolic $P_Y\hat{}$ of (2.11), we see that the bound on $\dim V_{\beta_5}(Q_Y\hat{})$ is exceeded. Thus, (2.11) implies $Q_A \nleq K_{\beta_1}$, as $Q_Y(\beta_5,\beta_1)$ has no 4-dimensional L_A' composition factor. So if $\Pi(L_i) = \{\beta_2,\beta_3,\beta_4\}$, $V_{L_i}(-\beta_1)$ is tensor indecomposable. We also note that the field twists on the embeddings of L_A' in L_i and L_j are equal, else there is no 4-dimensional L_A' composition factor of Q_Y/K_{β_5}. Call this twist q. Then the L_A' composition factors of Q_Y/K_{β_5} have high weights $5q\mu_1$, $3q\mu_1$ and $q\mu_1$. So $\beta_5 | Z_A = q\alpha$. Then, examining the $T(L_A)$ weight vectors in Q_Y/K_{β_1} and in Q_Y/K_{β_5}, we have

(1) $x_{-\alpha_2}(t) = x_{-\beta_1}(at^q)x_{-45}(a_1t^q)x_{-56}(a_2t^q)w$,

(2) $x_{-\alpha_1-\alpha_2}(t) = x_{-13}(bt^q)x_{-345}(b_1t^q)x_{-245}(b_2t^q)x_{-456}(b_3t^q)v$,

(3) $x_{-3\alpha_1-\alpha_2}(t) = x_{-1234}(ct^q)x_{-23456}(c_1t^q)x_{-\beta_2-\beta_3-2\beta_4-\beta_5}(c_2t^q)u$, where a, b, $c \in k^*$, a_i, b_i, $c_i \in k$, a_1 or a_2, c_1 or c_2 and some b_i nonzero, u, v, $w \in K_{\beta_1} \cap K_{\beta_5}$,

Note that $a_2 \neq 0$ else there is a nontrivial contribution to the root group $U_{-\beta_5}$ in the expression for $[x_{-\alpha_2}(t),x_{\alpha_1}(u)]$. Then, this implies $b_3 \neq 0$, else no nonidentity element from $U_{-\beta_5-\beta_6}$ occurs in the factorization of $[x_{-\alpha_1-\alpha_2}(t),x_{\alpha_1}(u)]$. But if $b_3 \neq 0$, there is a nontrivial contribution to the root group $U_{-\beta_1-\beta_2-\beta_3-2\beta_4-\beta_5-\beta_6}$ in the factorization of $[x_{-\alpha_1-\alpha_2}(t), x_{-3\alpha_1-\alpha_2}(u)]$. Contradiction. Hence, $\Pi(L_i \times L_j) \neq \{\beta_2,\beta_3,\beta_4,\beta_6\}$.

It remains to consider the case where $\Pi(L_i \times L_j) = \{\beta_2,\beta_4,\beta_5,\beta_1\}$. The above argument implies that $Y = E_7$ or E_8 and $\langle U_{\pm\beta_7}\rangle \leq L_Y'$. Suppose $\langle U_{\pm\beta_7}\rangle$ is a component of L_Y'. Then Q_Y/K_{β_6} has no 4-dimensional L_A' composition factor, so $Q_A \leq K_{\beta_5}$. Also, if $Y = E_8$, $\dim(Q_Y/K_{\beta_8}) = 2$ implies $Q_A \leq K_{\beta_8}$. So $\langle \lambda,\beta_j\rangle = 0$ for j = 2, j ≥ 4. In fact, the bound on $\dim V_{\beta_3}(Q_Y)$ implies $\lambda|T_Y = c\lambda_1$, where c = 1 or 2. If c = 1, $Y = E_8$ and by [8], $\dim V|Y = 3875$. However, referring to (6.9), we have $\langle \lambda,\alpha_2\rangle = 0, q_0$ or $2q_0$, for some p-power q_0. In each case, $\dim V|A < \dim V|Y$. Hence, c = 2. Then by (6.9), $\langle \lambda,\alpha_2\rangle = 0$ or $2q_0$. But (1.38) and (1.27) imply $\dim V|A < \dim V|Y$. Hence $\langle U_{\pm\beta_7}\rangle$ is not a component of L_Y'.

So $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_4,\beta_5,\beta_7,\beta_8\}$. If $Q_A \leq K_{\beta_6}$, argue as in the preceding paragraph to produce a contradiction. Hence, $Q_A \nleq K_{\beta_6}$. Then (1.15) implies that $V_{L_i}(-\beta_6)|L_A'$ is tensor indecomposable, where $\Pi(L_i) = \{\beta_2,\beta_4,\beta_5\}$. Then previous remarks and (2.7) imply that the field twists on the embeddings of L_A' in L_i, L_j and $\langle U_{\pm\beta_7},U_{\pm\beta_8}\rangle$ are equal, say q. So, $\langle \lambda,\beta_7+\beta_8\rangle = 0$. The L_A' composition factors of Q_Y/K_{β_6} have high weights $5q\mu_1$, $3q\mu_1$ and $q\mu_1$. Thus, $\beta_6|Z_A = q\alpha_2$. The bound on $\dim V^2(Q_A)_{\lambda-q\alpha_2}$ implies that $\lambda|T_Y = \lambda_2$ or $c\lambda_1$, for c = 1 or 2. If $\lambda|T_Y = c\lambda_1$, argue as above to produce a contradiction. If $\lambda|T_Y = \lambda_2$, refer to (6.9) to see that $\langle \lambda,\alpha_2\rangle = 0$, $q_1 + q_2$, $3q_1$, $2q_1$ or q_1, for q_1 and q_2 distinct p-powers. But by (1.27) and (1.38), dim $V|A <$ dim $V|Y$. Thus, this configuration cannot occur.

This completes the consideration of Case I.

Case II: Suppose $V_{L_k}(-\gamma) \cong W_k$ or W_k^* for $k = i$ and j.

Then Q_Y/K_γ has a 4-dimensional, tensor indecomposable L_A' composition factor only if $L_i \times L_j$ has type $A_1 \times A_k$, for $k = 2$ or 4 or $A_2 \times A_3$. Suppose that $L_i \times L_j$ has type $A_1 \times A_2$. Let γ_0, γ_1, and γ_2 be as in the statement of the result. If there exists $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\gamma_0) < 0$ and $(\delta,\Sigma L_m) = 0$ for all $m \neq i$, then Q_Y/K_δ is a 2-dimensional irreducible L_A' module, so $Q_A \leq K_\delta$. By (2.10), $-\delta$ is not involved in L_A', so (2.11) implies that there is a nontrivial image of $Q_A{}^{\alpha_2}$ in $Q_Y(\gamma,\delta)$. But $Q_Y(\gamma,\delta)$ is a 3-dimensional irreducible L_A' module. Thus, no such δ exists. Arguing similarly, we show that there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\gamma_2) \neq 0$ and $(\delta,\Sigma L_m) = 0$ for all $m \neq j$. In this case, we use $p > 2$ and (1.33) to see that if such a δ exists, we may assume $-\delta$ is not involved in L_A'. Thus, if $L_i \times L_j$ has type $A_1 \times A_2$, the result holds.

We now consider the case where L_i (respectively, L_j) has type A_1 (respectively, A_4). Then (2.17) implies that there exists a p-power, q, which is the field twist on the embeddings of L_A' in both L_i and L_j and such that $\gamma|Z_A = q\alpha_2$. Temporarily label as follows: $L_i = \langle U_{\pm\gamma_0}\rangle$, $L_j = \langle U_{\pm\gamma_k} \mid 1 \leq k \leq 4\rangle$, with $(\gamma,\gamma_1) < 0$, $(\gamma_k,\gamma_{k+1}) < 0$ for $k = 1, 2, 3$. We first note that there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\gamma_0) \neq 0$ and $(\delta,\Sigma L_k) = 0$ for all $k \neq i$. For if there exists such a δ, Q_Y/K_δ is a 2-dimensional irreducible L_A'-module, so $Q_A \leq K_\delta$. By (2.10), $-\delta$ is not involved in L_A', so (2.10) implies that there is a nontrivial image of $Q_A{}^{\alpha_2}$ in $Q_Y(\gamma,\delta)$, which is a 5-dimensional irreducible L_A'-module. Arguing similarly, we show that there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\gamma_4) \neq 0$ and $(\delta,\Sigma L_k) = 0$ for all $k \neq i$. In this case, we must use $p > 3$ and (1.33) to see that we may assume $-\delta$ is not involved in L_A'. Also, if there exists $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\Sigma L_j) \neq 0$ and $(\delta,\Sigma L_\ell) = 0$ for all $\ell \neq j$, then

$Q_A \leq K_\delta$. For neither $W_j|L_A'$ nor $(W_j \wedge W_j)|L_A'$ has a 4-dimensional tensor indecomposable composition factor. In particular, if such a δ exists, M_j is trivial.

If $Y = E_7$, consider the action of L_A' on the 56-dimensional irreducible kY-module, $V(\lambda_7)$. One checks that there is an L_A' composition factor with high weight $6q\mu_1$ and one with high weight $4q\mu_1$. But there is no 56-dimensional kA-module which affords such an L_A' composition series. Hence $Y = E_8$. Moreover, previous remarks imply that either $\Pi(L_i \times L_j) = \{\beta_2, \beta_k \mid 5 \leq k \leq 8\}$ or $\Pi(L_i \times L_j) = \{\beta_k \mid k = 2,4,5,6,8\}$. In the first configuration, we note that $L_Y' = L_i \times L_j$. For if $\langle U_{\pm\beta_1} \rangle$ is a component of L_Y', Q_Y/K_{β_3} is a 2-dimensional irreducible L_A'-module; so $Q_A \leq K_{\beta_3}$. By (2.10), $-\beta_3$ is not involved in L_A', so (2.11) implies that there is a nontrivial image of $Q_A{}^{\alpha_2}$ in $Q_Y(\beta_4,\beta_3)$, contradicting (1.15). Thus, $L_Y' = L_i \times L_j$ and (2.11) implies that $\beta_3|Z_A = 0$. Using the parabolic $P_Y{}^\wedge$ of (2.11), the bound on $\dim V_{\beta_4}(Q_Y{}^\wedge)$, together with (1.36), implies that $\lambda|T_Y = \lambda_8$. Thus, the first configuration cannot occur.

Consider now the second configuration. If $Q_A \nleq K_{\beta_3}$, $\langle U_{\pm\beta_1} \rangle$ is a component of L_Y'. The work of Case I then implies that $\langle \lambda, \beta_j \rangle = 0$ for $j = 1,2,4,5,6$ and the bound on $\dim V_{\beta_3}(Q_Y)$ implies that $\langle \lambda, \beta_3 \rangle = 0$. If $Q_A \leq K_{\beta_3}$, (2.3) implies $\langle \lambda, \beta_j \rangle = 0$ for $j \leq 6$. As well, (1.36) and the bound on $\dim V_{\beta_7}(Q_Y)$ imply that $\lambda|T_Y = c\lambda_8$, for $1 < c \leq 4$, and $\langle \lambda, \alpha_1 \rangle = c \cdot q$. By (6.9), $\langle \lambda, \alpha_2 \rangle = 0$ or $c \cdot q_0$, for some p-power q_0. However, (1.38) and (1.27) imply $\dim V|A < \dim V|Y$ in each case.

We must now consider $L_i \times L_j$ of type $A_2 \times A_3$. Temporarily label as follows: $L_i = \langle U_{\pm\gamma_1}, U_{\pm\gamma_2} \rangle$, $L_j = \langle U_{\pm\gamma_3}, U_{\pm\gamma_4}, U_{\pm\gamma_5} \rangle$, with $(\gamma_2, \gamma) \neq 0 \neq (\gamma, \gamma_3)$ and $(\gamma_k, \gamma_{k+1}) < 0$ for $k = 3, 4$. By (1.15), $V_{L_j}(-\gamma)|L_A'$ is tensor indecomposable. Let q be the field twist on the embeddings of L_A' in L_i and in L_j. (The twists are equal by (2.7).) Then the L_A' composition factors of Q_Y/K_γ have high weights $5q\mu_1$, $3q\mu_1$ and $q\mu_1$. Thus, $\gamma|Z_A = q\alpha_2$. Moreover, examining the $T(L_A')$ weight vectors in

Q_Y/K_γ, we see that $x_{-\alpha}(t) = x_{-\gamma_2-\gamma}(c_1t^q)x_{-\gamma-\gamma_3}(c_2t^q)u_1$, for $c_i \in k$, c_1 or c_2 nonzero, and $u_1 \in K_\gamma$. Since $\gamma_k|T_A = q\alpha_1$, for $1 \le k \le 5$, $\gamma|T_A = q(\alpha_2-\alpha_1)$. In particular, $\langle\lambda,\gamma\rangle = 0$.

We now point out various restrictions on the possible configurations which may arise with $L_i \times L_j$ as above. First, note that if there exists $\delta \in \Pi(Y)-\Pi(L_Y)$ with $(\delta,\Sigma L_i) \ne 0$ and $(\delta,\Sigma L_k) = 0$ for all $k \ne i$, then $Q_A \le K_\delta$. For Q_Y/K_δ is a 3-dimensional irreducible L_A'-module. In particular, if such a δ exists, M_i is trivial and $\langle\lambda,\delta\rangle = 0$. Also, if there exists $\delta \in \Pi(Y)-\Pi(L_Y)$ with $(\delta,\gamma_4) \ne 0$ and $(\delta, \Sigma L_k) = 0$ for all $k \ne j$, then $Q_A \le K_\delta$. For Q_Y/K_γ has L_A' composition factors of dimensions 5 and 1. In particular, if such a δ exists, M_j is trivial and $\langle\lambda,\delta\rangle = 0$. Now, the bound on $\dim V_\gamma(Q_Y)$ implies that $\langle\lambda,\gamma\rangle = 0$ and if M_j is nontrivial, $\langle\lambda,\gamma_3+\gamma_4\rangle = 0$ and $\langle\lambda,\gamma_5\rangle = 1$. Also, note that if there exists $\delta \in \Pi(Y)-\Pi(L_Y)$ with $(\delta,\gamma_3) \ne 0$ or $(\delta,\gamma_5) \ne 0$ and $(\delta,\Sigma L_k) = 0$ for all $k \ne j$, then $Q_A \not\le K_\delta$. For otherwise, (2.10) implies that $-\delta$ is not involved in L_A', and so by (2.11), there is a nontrivial image of $Q_A{}^{\alpha_2}$ in $Q_Y(\gamma,\delta)$, contradicting (1.15). Then since $Q_Y/K_\delta \cong (Q_A{}^{\alpha_2})^q$ as L_A'-modules and $Q_A \not\le K_\delta$, comparing high weight vectors we have $\delta|T_A = q\alpha_2$. Thus, if $\langle\lambda,\delta\rangle \ne 0$, the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_2}$ implies M_j is nontrivial and $(\delta,\gamma_5) \ne 0$.

Suppose $Y = E_7$. The above remarks and (8.1) imply $\lambda|T_Y = \lambda_3$ or λ_6 and $\langle\lambda,\alpha_1\rangle = 2q$. By (6.9), $\langle\lambda,\alpha_2\rangle = 0$, q_0 or $2q_0$ for some p-power q_0. But in each case, [8] implies $\dim V|A \ne \dim V|Y$. Thus, $Y = E_8$.

Suppose $\Pi(L_i) = \{\beta_1,\beta_3\}$ and $\Pi(L_j) = \{\beta_5,\beta_6,\beta_7\}$. We have $\beta_2|Z_A = 0$, by (2.11). Using the parabolic $P_Y\hat{}$ of (2.11), we see that if M_i is nontrivial $\langle\lambda,\beta_1\rangle = 1$ and $\langle\lambda,\beta_3\rangle = 0$, else the bound on $\dim V_{\beta_4}(Q_Y\hat{})$ is exceeded. Previous remarks imply $\lambda|T_Y = \lambda_1$ and by (6.9), $\langle\lambda,\alpha_2\rangle = 0$, q_0 or $2q_0$ for some p-power q_0. But [8] and (1.27) imply $\dim V|A < \dim V|Y$ in each case. Thus, M_j is nontrivial and $\lambda|T_Y = \lambda_7 + x\lambda_8$, for $x \ge 0$, and $\langle\lambda,\alpha_1\rangle = 3q$. Moreover, if $x \ne 0$, $0 \ne f_{\beta_8}v^+ \in V_{T_A}(\lambda - q\alpha_2)$, so in the p-adic expansion for $\langle\lambda,\alpha_2\rangle$, q has nonzero coefficient. By (6.9), $\langle\lambda,\alpha_2\rangle = 0$, $x\cdot q$, q_0 or $2q_0$, for q_0 some power of p. Then (1.27) and (1.32) imply $\dim V|A < \dim V|Y$, unless $x\neq 0$ and $\langle\lambda,\alpha_2\rangle = x\cdot q$. By the above remarks and (1.10), $\lambda|T_A = 3\mu_1 + x\mu_2$. Now, consider the subgroup $D \leq Y$, $D = \langle U_{\pm\beta_k} \mid 2\leq k\leq 8\rangle$, of type D_7. The D composition factor of V afforded by v^+ has dimension strictly less than dim V|Y. Also, D has a natural subgroup of type G_2, say A_0 (found by letting G_2 act on its Lie algebra). Moreover, v^+ affords an A_0 composition factor of V with the same high weight as V|A, as G_2-module. Thus dim V|A < dim V|Y. Hence, $\Pi(L_i\times L_j) \neq \{\beta_1,\beta_3,\beta_5,\beta_6,\beta_7\}$.

Suppose $\Pi(L_i) = \{\beta_7,\beta_8\}$, so $\Pi(L_j) = \{\beta_2,\beta_4,\beta_5\}$ or $\{\beta_3,\beta_4,\beta_5\}$. The work of Case I, the general remarks of the $A_2\times A_3$ work and the bound on $\dim V_\delta(Q_Y)$ for $\delta\in\Pi(Y)-\Pi(L_Y)$ imply $\lambda|T_Y = \lambda_7$. So $\langle\lambda,\alpha_1\rangle = 2q$ and (6.9) implies $\langle\lambda,\alpha_2\rangle = 0$, q_0 or $2q_0$ for some p-power q_0. But [8] and (1.32) imply $\dim V|A < \dim V|Y$. Similar arguments rule out $\Pi(L_i\times L_j) = \{\beta_6,\beta_7,\beta_1,\beta_3,\beta_4\}$.

The general remarks about $L_i\times L_j$ of type $A_2\times A_3$ imply that it remains to consider $\Pi(L_i) = \{\beta_2,\beta_4\}$ and $\Pi(L_j) = \{\beta_6,\beta_7,\beta_8\}$. Now $Q_A \not\leq K_{\beta_3}$, else $\lambda|T_Y = \lambda_8$. Thus, $\langle U_{\pm\beta_1}\rangle$ is a component of L_Y' and (2.17) implies that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_1}\rangle$ is also q. Thus, by (2.5) and (2.6), $\langle\lambda,\beta_1\rangle = 0$. Also, by (2.17), $\beta_3|Z_A = q\alpha_2$. Thus, the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_2}$ implies $\lambda|T_Y = \lambda_2$, and $\langle\lambda,\alpha_1\rangle = 2q$. By (6.9), $\langle\lambda,\alpha_2\rangle = 0$, $3q_0$, q_0+q_1, $2q_0$ or q_0, for q_0 and q_1 distinct p-powers. But in each case, (1.27) and (1.38) imply dim V|A < dim V|Y. Thus, $L_i\times L_j$ does not have type $A_2\times A_3$.

This completes the proof of (8.6).□

<u>(8.7).</u> Suppose there exists $1\leq i\leq r$ such that L_i is separated from all other components of L_Y' by more than one node of the Dynkin diagram. Then M_i is trivial.

<u>Proof</u>: Suppose false; i.e., with i as given, suppose $\dim(M_i) > 1$. By

(8.3) and (1.5), $Z_A \leq Z_Y$ and L_i has type A_{k_i} for some k_i. Let W_i denote the natural module for L_i. Arguing as in the proof of (6.7), we may reduce to L_i of rank 3. Also, (8.2) and the working hypotheses imply $Y = E_7$ or E_8. Moreover, there does not exist $\delta \in \Pi(Y)-\Pi(L_Y)$ with $Q_Y/K_\delta \cong W_i \wedge W_i$. For $(W_i \wedge W_i)|L_A'$ has no 4-dimensional composition factor. Finally, note that M_i is tensor indecomposable, as there exists $\delta \in \Pi(Y) - \Pi(L_Y)$ with $Q_Y/K_\delta \cong M_i$ or M_i^* and $Q_A \nleq K_\delta$.

Consider now the possibility that $L_Y' = L_i \times L_j$, where L_j has type A_1 and is separated by exactly 2 nodes of the Dynkin diagram from L_i. Temporarily label as follows: $L_i = \langle U_{\pm\gamma_1}, U_{\pm\gamma_2}, U_{\pm\gamma_3}\rangle$, $L_j = \langle U_{\pm\gamma_0}\rangle$, where $(\gamma_k, \gamma_{k+1}) < 0$ for $k = 1, 2$. Let $\delta_0, \delta_1 \in \Pi(Y)-\Pi(L_Y)$ with $(\delta_0, \delta_1) < 0$, $(\delta_0, \gamma_0) \neq 0 \neq (\delta_1, \gamma_1)$. Then $Q_A \leq K_{\delta_0}$, as Q_Y/K_{δ_0} is a 2-dimensional irreducible L_A'-module. Moreover, by (2.10), $-\delta_0$ is not involved in L_A'. Thus, (2.11) implies that there is a nontrivial image of $Q_A{}^{\alpha_2}$ in $Q_Y(\delta_1, \delta_0)$. However, this contradicts (1.15). Hence, $L_Y' \neq L_i \times L_j$ as described. In particular, $Y = E_8$.

Suppose $\Pi(L_i) = \{\beta_1, \beta_3, \beta_4\}$. Then (8.50, (8.6) and previous remarks imply $V^1(Q_Y) \cong M_i$. As well, since $\dim(Q_Y/K_{\beta_k}) < 4$ for $k = 6,7,8$, $\langle \lambda, \beta_6+\beta_7+\beta_8\rangle = 0$. So $\lambda|T_Y = \lambda_j + x\lambda_2 + y\lambda_5$, where $j = 1$ or 4. In fact, either $x = 0$ or $y = 0$, else $f_{\beta_2}v^+$ and $f_{\beta_5}v^+$ are 2 linearly independent vectors in $V_{T_A}(\lambda - q\alpha_2)$, contradicting (1.31). Let $z = \langle \lambda, \beta_2+\beta_5\rangle$. If $z \neq 0$, q has a nonzero coefficient in the p-adic expansion of $\langle \lambda, \alpha_2\rangle$. By (6.9), $\langle \lambda, \alpha_2\rangle = 0$, zq, q_0 or $2q_0$ for some p-power q_0. Now (1.32) and [8] imply $\dim V|A \neq \dim V|Y$ unless $z \neq 0$ and $\langle \lambda, \alpha_2\rangle = zq$; so by (1.10), $\lambda|T_A = 3\mu_1 + z\mu_2$. If $\lambda|T_Y = \lambda_4 + z\lambda_k$, $k = 2$ or 5, then $V_{T_Y}(\lambda-\beta_4-\beta_2) \oplus V_{T_Y}(\lambda-\beta_4-\beta_5) \leq V_{T_A}(\lambda-\alpha_1-\alpha_2)$. So (1.35) implies $z = p-2$. But then $\dim V_{T_A}(\lambda-\alpha_1-\alpha_2) = 1$, also by (1.35), contradicting the above containment. So $\lambda|T_A = \lambda_1 + z\lambda_k$ for $k = 2$ or 5. In this case, $z = 1$ else $V_{T_Y}(\lambda-2\beta_k-\beta_1) \oplus V_{T_Y}(\lambda-\beta_2-\beta_4-\beta_5) \oplus V_{T_Y}(\lambda-2\beta_k-\beta_4)$ is a 3-dimensional subspace of $V_{T_A}(\lambda-\alpha_1-2\alpha_2)$, contradicting (1.29). But now $\dim V|A < \dim V|Y$.

It remains to consider the case where $\Pi(L_i) = \{\beta_6,\beta_7,\beta_8\}$. We first claim that $\lambda|T_Y = x\lambda_5 + \lambda_k$, where $k = 6$ or 8. For if $\langle\lambda,\beta_j\rangle \neq 0$ for some $1 \leq j \leq 4$, either $\beta_j \in \Pi(L_Y)$ or Q_Y/K_{β_j} has an L_A' composition factor isomorphic to a twist of $Q_A{}^{\alpha_2}$. Then the work of this proof, (8.7) and (1.15) imply $\Pi(L_Y) = \{\beta_i \mid i \neq 4,5\}$ and $Q_A \nleq K_{\beta_4}$ So (2.17) implies that there is a power of p, say q_0, such that q_0 is the field twist on the embedding of L_A in $\langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle$ and in $\langle U_{\pm\beta_2}\rangle$, and such that $\beta_4|Z_A = q_0\alpha_2$. Then (2.8) implies that $q_0 = q$. Thus, by (2.5) and (2.6), $\langle\lambda,\beta_k\rangle = 0$ for $k = 1, 2, 3$, so $\langle\lambda,\beta_4\rangle \neq 0$. But then the bound on $\dim V_{\beta_4}(Q_Y)$ is exceeded. So $\lambda|T_Y = x\lambda_5 + \lambda_k$, for $k = 6$ or 8, as claimed. As well, if $x \neq 0$, $f_{\beta_5}v^+ \in V_{T_A}(\lambda - q\alpha_2)$, so q has a nonzero coefficient in the p-adic expansion of $\langle\lambda,\alpha_2\rangle$. By (6.9), $\langle\lambda,\alpha_2\rangle = 0$, q_0, $2q_0$, q_0+q_1 or xq, for distinct p-powers q_0 and q_1. Then by (1.27) and (1.32), $\dim V|A < \dim V|Y$ unless $x>1$ and $\langle\lambda,\alpha_2\rangle = xq$; so by (1.10) $q = 1$. Now (2.13) implies $\langle U_{\pm\beta_2},U_{\pm\beta_3}\rangle \cap L_Y \neq \{1\}$ and previous work of this proof implies $\dim(Q_Y/K_{\beta_4}) > 2$. If $\lambda|T_Y = x\lambda_5 + \lambda_6$, $\dim([V,Q_Y]/[V,Q_Y{}^3]) \geq 84$ if $x \neq p-2$, or ≥ 60 if $x = p-2$. If $\lambda|T_Y = x\lambda_5+\lambda_8$, $\dim([V,Q_Y]/[V,Q_Y{}^3]) \geq 92$. (See Table 1 of [5].) But (1.20) and (1.35) imply $\dim([V,Q_A]/[V,Q_A{}^3]) \leq 80$ if $x\neq p-2$, and ≤ 55 if $x = p-2$. Contradiction.

This completes the proof of (8.7).□

<u>(8.8).</u> Let $\gamma \in \Pi(Y)-\Pi(L_Y)$. Suppose there exists a unique pair $1\leq i,j\leq r$ with $(\Sigma L_i,\gamma) \neq 0 \neq (\Sigma L_j,\gamma)$. Assume L_i is of type A_2 and L_j is of type A_1. Then M_i and M_j are trivial.

<u>Proof</u>: Note that (1.5) implies $Z_A \leq Z_Y$. Also, if $Q_A \nleq K_\gamma$, (2.17) implies that there is a p-power q_0 which is the field twist on the embedding of L_A' in L_i and in L_j and $\gamma|Z_A = q_0\alpha_2$.

<u>Claim 1:</u> If L_Y' has components $L_k = \langle U_{\pm\beta_1}\rangle$ and $L_m = \langle U_{\pm\beta_2}, U_{\pm\beta_4}\rangle$, then $\dim V^1(Q_Y) = 1$.

Proof: Suppose $\dim V^1(Q_Y) > 1$. Let q be the field twist on the embedding of L_A' in L_k and L_m. If $\langle\lambda,\beta_k\rangle = 0$ for k = 1, 2, 4, then (8.6) and (8.7) imply $\langle U_{\pm\beta_6}\rangle$ is a component of L_Y' and $\langle\lambda,\beta_6\rangle \neq 0$. Then $Y = E_6$, else Q_Y/K_{β_7} has no L_A' composition factor isomorphic to a twist $Q_A{}^{\alpha_2}$. Thus, by symmetry and (8.6), we may assume $\langle\lambda,\beta_k\rangle \neq 0$ for a unique $k \in \{1,2,4\}$.

Suppose $\langle U_{\pm\beta_6}\rangle \nleq L_Y'$. Then Q_Y/K_{β_5} is a 3-dimensional irreducible L_A' module, so $Q_A \leq K_{\beta_5}$ and $\langle\lambda,\beta_k\rangle = 0$ for k = 2,4,5. Hence, (1.23) implies $Y \neq E_6$. In fact, $\langle\lambda,\beta_k\rangle = 0$ for k > 5 also, as (8.6) and (8.7) imply $V^1(Q_Y) \cong M_k \otimes M_m$ and $\dim(Q_Y/K_{\beta_j}) < 4$ for j > 5. If $-\beta_5$ is involved in L_A' then $\beta_5|Z_A = 0$. Otherwise, by (2.11) there is a nontrivial image of $Q_A{}^{\alpha_2}$ in $Q_Y(\beta_3,\beta_5)$. Since $Q_Y(\beta_3,\beta_5)$ has L_A' composition factors with high weights $3q\mu_1$ and $q\mu_1$, $(\beta_3+\beta_5)|Z_A = q\alpha_2$ and again $\beta_5|Z_A = 0$. So we may use the parabolic $P_Y\hat{}$ of (2.11). The bound on $\dim V^2(Q_A)_{\lambda - q\alpha_2}$ implies $\langle\lambda,\beta_3\rangle = 0$ and $\langle\lambda,\beta_1\rangle \leq 2$. If $\langle\lambda,\beta_1\rangle = 1$, (6.9) implies that $\langle\lambda,\alpha_2\rangle = 0$, q_0, or $2q_0$ for some p-power q_0. In each case, $\dim V|A \neq \dim V|Y$, by (1.27) and [8]. So $\langle\lambda,\beta_1\rangle = 2$. Then $\langle\lambda,\alpha_2\rangle = 0$ or $2q_0$ and $\dim V|A < \dim V|Y$. Thus the assumption that $\langle U_{\pm\beta_6}\rangle \nleq L_Y'$ was incorrect.

Suppose $\langle U_{\pm\beta_6}\rangle$ is a component of L_Y'. Then $Q_A \nleq K_{\beta_5}$. For otherwise, (2.10) implies that $-\beta_5$ is not involved in L_A', so by (2.11), there is a nontrivial image of $Q_A{}^{\alpha_2}$ in $Q_Y(\beta_3,\beta_5)$, contradicting (1.15). Now (2.17) implies that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_6}\rangle$ is also q. Moreover, $\beta_3|Z_A = q\alpha_2 = \beta_5|Z_A$. Examining the $T(L_A')$ weight vectors in Q_Y/K_{β_3} and in Q_Y/K_{β_5}, we have $x_{-\alpha_2}(t) = x_{-\beta_3}(at^q)x_{-\beta_5}(bt^q)u_1$, $x_{-\alpha_1-\alpha_2}(t) = x_{-13}(a_1t^q)x_{-34}(a_2t^q)x_{-45}(b_1t^q)x_{-56}(b_2t^q)u_2$ and $x_{-3\alpha_1-\alpha_2}(t) = x_{-1234}(ct^q)x_{-2456}(dt^q)u_3$, where $a,b,c,d \in k^*$, $u_i \in K_{\beta_3} \cap K_{\beta_5}$, $a_i,b_i \in k$, a_1 or a_2 nonzero, b_1 or b_2 nonzero. In fact, $a_2 \neq 0$ as a nonidentity element from the root group U_{-34} occurs in the factorization of $[x_{-\alpha_2}(t),x_{-\alpha_1}(u)]$ and $(-\beta_3-\beta_4)|T(L_A') = q\mu_1$. By examining $[x_{-\alpha_2}(t),x_{-3\alpha_1-\alpha_2}(t)]$, we see that $x_{-3\alpha_1-2\alpha_2}(t) = x_{-12345}(f_1(t))x_{-23456}(f_2(t))w$ where $0 \neq f_i(t) \in k[t]$ and $w \in \langle U_{-r} | r = \Sigma c_\gamma \gamma,\ \gamma \in \Pi(Y),\ c_\gamma \in \mathbb{Z}^+,\ c_{\beta_3}+c_{\beta_5} > 2 \text{ or } c_{\beta_7}+c_{\beta_8} > 0\rangle$.

But this gives a nontrivial contribution to the root group $U_{-\beta_1-\beta_2-2\beta_3-2\beta_4-\beta_5}$ in the expression for $[x_{-3\alpha_1-2\alpha_2}(t), x_{-\alpha_1-\alpha_2}(u)]$. Contradiction. Hence $\langle U_{\pm\beta_6}\rangle$ is not a component of L_Y'. (So $Y = E_7$ or E_8.)

If $\langle U_{\pm\beta_6}, U_{\pm\beta_7}\rangle$ is a component of L_Y', then (1.15) implies $Q_A \leq K_{\beta_5}$ and $Q_A \leq K_{\beta_8}$. So $\langle\lambda,\beta_k\rangle = 0$ for $k = 2$ and $k \geq 4$. If $-\beta_5$ is involved in L_A', $\beta_5|Z_A = 0$. If $-\beta_5$ is not involved in L_A', (2.11) implies that there is a nontrivial image of $Q_A{}^{\alpha_2}$ in $Q_Y(\beta_3,\beta_5)$, so the field twist on the embedding of L_A' in $\langle U_{\pm\beta_6}, U_{\pm\beta_7}\rangle$ is also q. Moreover, the L_A' composition factors of $Q_Y(\beta_3,\beta_5)$ have high weights $5q\mu_1$, $3q\mu_1$ and $q\mu_1$, so $(\beta_3+\beta_5)|Z_A = q\alpha_2$. Hence, again $\beta_5|Z_A = 0$. But using the parabolic $P_Y{}^\wedge$ of (2.11), we see that the bound on $\dim V_{\beta_3}(Q_Y{}^\wedge)$ is exceeded.

Thus, $Y = E_8$ and $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_4,\beta_6,\beta_7,\beta_8\}$. Now $-\beta_5$ is not involved in L_A'; else $\beta_5|Z_A = 0$ and we may again produce a contradiction. If the natural module for $\langle U_{\pm\beta_6}, U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$ is a tensor decomposable L_A' module, neither Q_Y/K_{β_5} nor $Q_Y(\beta_3,\beta_5)$ has an L_A' composition factor isomorphic to a twist of $Q_A{}^{\alpha_2}$, contradicting (2.11). So the natural module for $\langle U_{\pm\beta_6}, U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$ is a tensor indecomposable L_A' module; so (1.15) and (2.11) imply $Q_A \nleq K_{\beta_5}$. By (2.17), the field twist on the embedding of L_A' in $\langle U_{\pm\beta_6}, U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$ is also q and one checks that $\beta_5|Z_A = q\alpha_2$. Examining the $T(L_A')$ weight vectors in Q_Y/K_{β_3} and Q_Y/K_{β_5}, we find that $x_{-\alpha_2}(t) = x_{-\beta_3}(at^q)x_{-45}(a_1t^q)x_{-56}(a_2t^q)u_1$ and $x_{-\alpha_1-\alpha_2}(t) = x_{-13}(b_1t^q)x_{-34}(b_2t^q x_{-245}(c_1t^q)x_{-456}(c_2t^q)x_{-567}(c_3t^q)u_2$, where $a \in k^*$, $a_i, b_i, c_i \in k$, $u_i \in K_{\beta_3} \cap K_{\beta_5}$. Also, a_1 or a_2, b_1 or b_2 and some c_i is nonzero. In fact, $a_1 \neq 0 \neq a_2$, else there is a nontrivial contribution to the root group $U_{-\beta_5}$ in the expression for $[x_{-\alpha_2}(t), x_{\alpha_1}(t)]$. Also $b_1 \neq 0$, as a nonidentity element from the root group U_{-13} occurs in the factorization of $[x_{-\alpha_2}(t), x_{-\alpha_1}(t)]$ and $(-\beta_1-\beta_3)|T(L_A') = q\mu_1$. However, we now see that there is a nontrivial contribution to the root group U_{-1345} in the expression for $[x_{-\alpha_2}(t), x_{-\alpha_1-\alpha_2}(t)]$. Contradiction. This completes the proof of Claim 1.

<u>Claim 2:</u> If $L_\ell = \langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$ and $L_m = \langle U_{\pm\beta_2}\rangle$ are components of L_Y' with $\langle U_{\pm\beta_5}\rangle \nleq L_Y'$, then $\dim V^1(Q_Y) = 1$.

Proof: Suppose false. Then (8.6) and (8.7) imply $\langle\lambda,\beta_k\rangle \neq 0$ for a unique $k \in \{1,2,3\}$ and $V^1(Q_Y) \cong M_\ell \otimes M_m$. Let q be the field twist on the embedding of L_A' in L_ℓ and L_m, so $\beta_4|Z_A = q\alpha_2$. Suppose there exists $\delta \in \Pi(Y) - \Pi(L_Y)$, $\delta \neq \beta_4$, with $\langle\lambda,\delta\rangle \neq 0$. Since $Q_A \nleq K_\delta$, $\dim(Q_Y/K_\delta) \geq 4$. By size restrictions and (1.15), $\delta = \beta_5$ and $\langle U_{\pm\beta_6}, U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$ is a component of L_Y' in Y of type E_8. But (2.8) implies $\beta_5|Z_A = q\alpha_2$, so the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_2}$ is exceeded. Thus, no such δ exists and $\langle\lambda,\beta_k\rangle = 0$ for $k \geq 5$.

If $\langle U_{\pm\beta_6}\rangle \nleq L_Y'$, (2.12) implies $\beta_5|Z_A = 0$. Using the parabolic $P_Y\hat{}$ of (2.11), and the bound on $\dim V_{\beta_4}(Q_Y\hat{})$, we have $\lambda|T_Y = \lambda_1$, λ_3 or $c\lambda_2$ for c = 1 or 2. Use (1.23), (6.9), (1.27), (1.32) and [8] to see that in every possible configuration, $\dim V|A \neq \dim V|Y$. Thus $\langle U_{\pm\beta_6}\rangle \leq L_Y'$. If $\langle U_{\pm\beta_6}\rangle$ is a component of L_Y', then $Q_A \leq K_{\beta_5}$ and (2.10) and (2.11) produce a contradiction. So $Y = E_7$ or E_8 and $\langle U_{\pm\beta_6}, U_{\pm\beta_7}\rangle \leq L_Y'$. If $\langle U_{\pm\beta_6}, U_{\pm\beta_7}\rangle$ is a component of L_Y', $Q_A \leq K_{\beta_5}$ and $Q_A \leq K_{\beta_8}$. So $\langle\lambda,\beta_k\rangle = 0$ for $k \geq 5$. If $-\beta_5$ is involved in L_A', $\beta_5|Z_A = 0$. Otherwise, an application of (2.11) implies $\beta_5|Z_A = 0$. So in either case we may use the parabolic $P_Y\hat{}$ of (2.11) to see that the bound on $\dim V_{\beta_4}(Q_Y\hat{})$ implies $\lambda|T_Y = \lambda_1$. Again, use (6.9), (1.27) and [8] to see that $\dim V|A \neq \dim V|Y$.

Therefore, under the hypotheses of Claim 2, if $\dim V^1(Q_Y) > 1$, $Y = E_8$ and $L_Y' = L_\ell \times L_m \times \langle U_{\pm\beta_6}, U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$. We claim that $Q_A \nleq K_{\beta_5}$. For otherwise, since there is no L_A' composition factor of $Q_Y(\beta_4,\beta_5)$ isomorphic to a twist of $Q_A{}^{\alpha_2}$, (2.11) would imply that $-\beta_5$ is involved in L_A. So $\beta_5|Z_A = 0$. But using the parabolic $P_Y\hat{}$ of (2.11) ,the bound on $\dim V_{\beta_4}(Q_Y\hat{})$ implies $\lambda|T_Y = \lambda_1$. Now we can argue as before to produce a contradiction. Since $Q_A \nleq K_{\beta_5}$, (2.4) and (2.8) imply Q_Y/K_{β_5} is a tensor indecomposable L_A' module isomorphic to $(Q_A{}^{\alpha_2})^q$. Examining the $T(L_A')$ weight vectors in Q_Y/K_{β_4} and Q_Y/K_{β_5}, we have $x_{-\alpha_2}(t) = x_{-\beta_4}(at^q)$.

$x_{-\beta_5}(bt^q)u_1$ and $x_{-\alpha_1-\alpha_2}(t) = x_{-\beta_3-\beta_4}(a_1t^q)x_{-\beta_2-\beta_4}(a_2t^q)\cdot$ $x_{-\beta_5-\beta_6}(b_1t^q)u_2$, where a, b, $b_1 \in k^*$, $u_i \in K_{\beta_4} \cap K_{\beta_5}$ and $a_i \in k$ with some a_i nonzero. However, there is a nontrivial contribution to the root group $U_{-\beta_4-\beta_5-\beta_6}$ in the expression for $[x_{-\alpha_2}(t), x_{-\alpha_1-\alpha_2}(t)]$. Contradiction. This completes the proof of Claim 2.

Now suppose L_i has type A_2 and L_j has type A_1 with $\dim(M_i \otimes M_j) > 1$.

Case I: Suppose M_i is nontrivial, so by (8.6) M_j is trivial. Temporarily label as follows: $\Pi(L_i) = \{\gamma_1, \gamma_2\}$, $\Pi(L_j) = \{\gamma_0\}$, $(\gamma_1, \gamma) < 0$. We first note that there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta, \Sigma L_i) \neq 0$ and $(\delta, \Sigma L_k) = 0$ for all $k \neq i$. For otherwise, Q_Y/K_δ is a 3-dimensional irreducible $L_A{}'$-module containing a nontrivial image of $Q_A{}^{\alpha_2}$. These remarks, together with Claim 1, imply $L_i \neq \langle U_{\pm\beta_2}, U_{\pm\beta_4}\rangle$. Also, (8.6) implies that there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta, \Sigma L_j) \neq 0$ and $(\delta, \Sigma L_k) = 0$ for all $k \neq j$. Let q be the field twist on the embeddings of $L_A{}'$ in L_i and in L_j, so $\gamma | Z_A = q\alpha_2$. Examining the $T(L_A{}')$ weight vectors in Q_Y/K_γ, we see that $x_{-\alpha_2}(t) = x_{-\gamma}(at^q)u$, where $a \in k^*$, $u \in K_\gamma$. Thus, $\gamma | T_A = q\alpha_2$.

Consider the possibility that $L_i = \langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$. Then Claim 2 implies $L_j = \langle U_{\pm\beta_5}\rangle$ and previous remarks imply $Y = E_7$ or E_8 and $\langle U_{\pm\beta_7}\rangle \leq L_Y{}'$. Also, (2.12) implies that $\beta_2 | T_A = 0$. Using the parabolic $P_Y{}^\wedge$ of (2.11), we see that $\langle \lambda, \beta_4\rangle = 0$, else the bound on $\dim V_{\beta_4}(Q_Y{}^\wedge)$ is exceeded. Also, $\langle \lambda, \beta_2\rangle = 0$, by (2.3). Suppose $L_Y{}' = L_i \times L_j \times \langle U_{\pm\beta_7}\rangle$. Then (1.15) implies $Q_A \leq K_{\beta_6}$. Thus, $\langle \lambda, \beta_k\rangle = 0$ for k = 5, 6, 7. If $Y = E_8$, $Q_A \leq K_{\beta_8}$, as $\dim(Q_Y/K_{\beta_8}) = 2$. So $\langle \lambda, \beta_8\rangle = 0$, also. Thus, $\lambda | T_Y = \lambda_1$ (with Y of type E_8) or λ_3. By (6.9), $\langle \lambda, \alpha_2\rangle = 0$, q_0 or $2q_0$, for some p-power q_0. However, (1.27), [8] and (1.32) imply $\dim V|A \neq \dim V|Y$.

Thus, if $L_i = \langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$ and $L_j = \langle U_{\pm\beta_5}\rangle$, then $Y = E_8$ and $L_Y{}' = L_i \times L_j \times \langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$. Using standard arguments, we reduce to $\lambda | T_Y = \lambda_1$ or λ_3. But then we proceed as before to produce a contradiction. Thus, $L_i \neq \langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$.

Suppose now $L_i = \langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$ in Y of type E_8. Then, (8.5), (8.6) and the general remarks at the beginning of Case I imply that $\langle U_{\pm\beta_2}\rangle$ and $\langle U_{\pm\beta_8}\rangle$ are components of L_Y; and $\langle U_{\pm\beta_3}\rangle \nleq L_Y'$. Also, the field twists on the embeddings of L_A' in L_i, $\langle U_{\pm\beta_2}\rangle$ and $\langle U_{\pm\beta_8}\rangle$ are q and $\beta_4|Z_A = q\alpha_2 = \beta_7|Z_A$. Thus, the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_2}$ implies $\langle\lambda,\beta_4+\beta_7\rangle = 0$. As usual, $\langle\lambda,\beta_2+\beta_8\rangle = 0$ and $\langle\lambda,\beta_1+\beta_3\rangle = 0$. Thus, $\lambda|T_Y = \lambda_5$ or λ_6 and $\langle\lambda,\alpha_1\rangle = 2q$. By (6.9), $\langle\lambda,\alpha_2\rangle = 0$, $2q_0$ or q_0, for some p-power q_0. However, (1.27) and (1.32) imply $\dim V|A < \dim V|Y$. Thus, $L_i \neq \langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$, in Y of type E_8.

Reviewing the general remarks and the cases considered so far, we see that in Case I it remains to consider $L_i = \langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$ in Y of type E_8; so $L_j = \langle U_{\pm\beta_5}\rangle$. Using (8.5), (8.6), (8.7), (2.17), (2.5) and (2.6), it is easy to see that $V^1(Q_Y) \cong M_i$. Suppose there exists $\delta \in \Pi(Y)-\Pi(L_Y)$, $\delta \neq \beta_6$, with $\langle\lambda,\delta\rangle \neq 0$. Then $Q_A \nleq K_\delta$ implies $\dim(Q_Y/K_\delta) \geq 4$. Thus, $\delta = \beta_4$ and $\langle U_{\pm\beta_2}, U_{\pm\beta_3}\rangle \cap L_Y' \neq 1$. The bound on $\dim V_{\beta_4}(Q_Y)$, of (1.25), implies that $L_Y' = L_i \times L_j \times \langle U_{\pm\beta_k}\rangle$ for k = 2 or 3. But then Q_Y/K_{β_4} has no L_A' composition factor isomorphic to a twist of $Q_A{}^{\alpha_2}$, contradicting (2.3). Thus, there exists no such δ, and $\lambda|T_A = x\lambda_6 + \lambda_7$ or $x\lambda_6 + \lambda_8$, for $p>x\geq 0$. In fact, the bound on $\dim V_{\beta_6}(Q_Y)$ implies that $\lambda|T_Y = x\lambda_6+\lambda_7$, for $p>x\geq 0$. By (6.9), $\langle\lambda,\alpha_2\rangle = 0$, xq_0, q_0 or $2q_0$, for q_0 some p-power. Then, (1.27) and (1.32) imply $\dim V|A < \dim V|Y$ unless $\langle\lambda,\alpha_2\rangle = xq_0$ and $x\neq 0$. Since $\beta_6|T_A = q\alpha_2$, $0 \neq f_{\beta_6}v^+ \in V_{T_A}(\lambda-q\alpha_2)$, so in fact, $q_0 = q$. Then by (1.10), $\lambda|T_A = 2\mu_1+x\mu_2$.

Now, consider the subgroup $D \leq Y$ of type $D_4 \times D_4$ defined by $D = \langle U_{\pm\beta_2}, U_{\pm\beta_3}, U_{\pm(\beta_4+\beta_5+\beta_6)}, U_{\pm\beta_7}\rangle \circ \langle U_{\pm\beta_5}, U_{\pm(\beta_6+\beta_7+\beta_8)}, U_{\pm t_1}, U_{\pm t_2}\rangle$ (commuting product), where $t_1 = 2\beta_1+2\beta_2+3\beta_3+4\beta_4+3\beta_5+2\beta_6+\beta_7$ and $t_2 = \beta_2+\beta_3+2\beta_4+\beta_5$. Then V|D is not irreducible by Theorem (4.1) of [12], so $\langle v^+\rangle$ affords a D-composition factor of V|Y with dimension strictly less than dimV|Y. Restrict this composition factor to the natural subgroup $G < D$ of type $G_2 \times G_2$. (The fixed point subgroup of D_4, under the graph automorphism of order three has type G_2.) Let $\Pi(G) = \Pi_1 \perp \Pi_2$ with $\Pi_1 = \{\gamma_1,\gamma_2\}$, $\Pi_2 = \{\gamma_3,\gamma_4\}$ and $(\gamma_1,\gamma_1) < (\gamma_2,\gamma_2)$, $(\gamma_3,\gamma_3) < (\gamma_4,\gamma_4)$. Let η_i be the fundamental dominant weight corresponding to γ_i. Then, there is a G-composition factor of $V|Y$ afforded by $\langle v^+\rangle$ with high weight $(x\eta_2 + \eta_1) + ((x+1)\eta_4 + (2x+1)\eta_3)$. This composition factor has dimension at least $12\cdot\dim V(x\eta_2 + \eta_1)$. But since $V|A$ occurs as a compostion factor of the tensor product $V(\mu_1 + x\mu_2)\otimes V(\mu_1)$, $\dim V|A \le 7\cdot\dim V(\mu_1 + x\mu_2) < 12\cdot\dim V(x\eta_2 + \eta_1) < \dim V|Y$. Contradiction.

This completes the consideration of Case I.

Case II: Suppose M_j is nontrivial and M_i is trivial.

By (8.6), there does not exist $\delta \in \Pi(Y)-\Pi(L_Y)$ with $(\delta,\Sigma L_j) \ne 0$ and $(\delta,\Sigma L_k) = 0$ for all $k\ne i$. Let q be as in Case I.

We now claim that $\Pi(L_j)$ corresponds to an end node of the Dynkin diagram. For, if not, the above remarks imply that there exists $k\ne i$ with $\Pi(L_k)$ separated from $\Pi(L_j)$ by exactly one node of the Dynkin diagram. By (8.5) and (8.6), L_k is unique and has type A_2. Thus, $Y = E_8$ and $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_5,\beta_7,\beta_8\}$. Also, by (8.6) and (2.3), $\langle\lambda,\beta_\ell\rangle = 0$ for $\ell = 1, 2, 3, 7, 8$. As in Case I, $\beta_4|T_A = q\alpha_2 = \beta_6|T_A$. Moreover, by (2.12), $\beta_2|Z_A = 0$. Now, using the parabolic $P_Y\hat{}$ of (2.11), we see that the bound on $\dim V^2(Q_A)_{\lambda-q\alpha_2}$ is exceeded. Thus, $\Pi(L_j)$ must correspond to an end node of the Dynkin diagram, as claimed.

Suppose $L_j = \langle U_{\pm\beta_1}\rangle$. Then Claim 1 implies $L_i = \langle U_{\pm\beta_4},U_{\pm\beta_5}\rangle$ and by (8.6), $\langle U_{\pm\beta_7}\rangle \le L_Y'$. Now $Q_A \le K_{\beta_2}$ and if $-\beta_2$ is involved in L_A', $\beta_2|Z_A = 0$. Otherwise, (2.11) implies that there is a nontrivial image of $Q_A{}^{\alpha_2}$ in $Q_Y(\beta_3,\beta_2)$. But $Q_Y(\beta_3,\beta_2)$ has L_A' composition factors with high weights $3q\mu_1$ and $q\mu_1$, so $(\beta_3 + \beta_2)|Z_A = q\alpha_2$ and again $\beta_2|Z_A = 0$. Using the parabolic $P_Y\hat{}$ of (2.11), and (1.36), we see that the bound on $\dim V_{\beta_3}(Q_Y\hat{})$ is exceeded unless $\langle\lambda,\beta_3\rangle = 0$ and $\langle\lambda,\beta_1\rangle \le 2$.

Suppose $Q_A \nleq K_{\beta_6}$. Then (1.15) implies $\langle U_{\pm\beta_7}\rangle$ is a component of L_Y', and by (2.17) the field twist on the embedding of L_A' in $\langle U_{\pm\beta_7}\rangle$ is also q. Thus, by (2.5) and (2.6), $\langle\lambda,\beta_7\rangle = 0$. Also, the bound on $\dim V_{\beta_6}(Q_Y)$ implies $\langle\lambda,\beta_6\rangle = 0$. Finally, if $Y = E_8$, Q_Y/K_{β_8} a 2-dimensional irreducible L_A'-module implies $Q_A \leq K_{\beta_8}$, so $\langle\lambda,\beta_8\rangle = 0$. Thus, $\lambda|T_Y = c\lambda_1$, for $c\leq 2$. Now, if $Q_A \leq K_{\beta_6}$, (2.3) implies $\langle\lambda,\beta_6+\beta_7\rangle = 0$. If $Y = E_8$, either $\langle U_{\pm\beta_7}\rangle$ or $\langle U_{\pm\beta_7},U_{\pm\beta_8}\rangle$ is a component and we argue as above or apply (2.4) to get $\langle\lambda,\beta_8\rangle = 0$. Thus, if $Q_A \leq K_{\beta_6}$, $\lambda|T_Y = c\lambda_1$ for $c\leq 2$, as above. If $Y = E_7$, then $c \neq 1$. But now we argue as in Claim 1 to produce a contradiction. Thus, $L_j \neq \langle U_{\pm\beta_1}\rangle$.

Consider next the configuration where $L_j = \langle U_{\pm\beta_2}\rangle$. Then by Claim 2, the previous general remarks and symmetry, we may assume $L_i = \langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$ and $Y = E_8$. A straightforward argument, using (2.10) and (2.11), implies that $\langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle \cap L_Y' = 1$, $\langle\lambda,\beta_1+\beta_3\rangle = 0$ and $\beta_1|Z_A = 0 = \beta_3|Z_A$, by (2.12). Using the parabolic $P_Y\hat{}$ of (2.12) and (1.36), we see that the bound on $\dim V_{\beta_4}(Q_Y\hat{})$ is exceeded. Thus, $L_j \neq \langle U_{\pm\beta_2}\rangle$.

The opening remarks of the proof and the cases considered thus far allow us to reduce, finally, to the case where $Y = E_8$ and $L_i = \langle U_{\pm\beta_5},U_{\pm\beta_6}\rangle$, $L_j = \langle U_{\pm\beta_8}\rangle$. We first claim that $V^1(Q_Y) \cong M_j$. For otherwise, (8.5), (8.6) and (8.7) imply that there exists a unique $1\leq k\leq r$, $k \neq i, j$, with L_k of type A_1, $(\Sigma L_k, \beta_4) \neq 0$, and M_k nontrivial. However, (2.17) implies that the field twist on the embedding of L_A' in L_k is also q, contradicting (2.5) and (2.6). Thus, $V^1(Q_Y) \cong M_j$, as claimed. Suppose there exists $\delta \in \Pi(Y)-\Pi(L_Y)$, $\delta \neq \beta_7$ with $\langle\lambda,\delta\rangle \neq 0$. Then $Q_A \nleq K_\delta$ implies $\dim(Q_Y/K_\delta) \geq 4$. Thus $\delta = \beta_4$ and $\langle U_{\pm\beta_2},U_{\pm\beta_3}\rangle \cap L_Y' \neq 1$. However, the bound on $\dim V_{\beta_4}(Q_Y)$, of (1.25), is exceeded. Thus, if $\delta \in \Pi(Y)-\Pi(L_Y)$ with $\langle\lambda,\delta\rangle \neq 0$, $\delta = \beta_7$. So $\lambda|T_Y = x\lambda_7 + c\lambda_8$, for $p>x\geq 0$ and $p>c>0$, and $\langle\lambda,\alpha_1\rangle = c\cdot q$. Recall $\lambda|T_Y \neq \lambda_8$. By (6.9), $\langle\lambda,\alpha_2\rangle = 0$, cq_0 or xq_0 for some p-power q_0. In fact, if $x\neq 0$, $0 \neq f_{\beta_7}v^+ \in V_{T_A}(\lambda-q\alpha_2)$ implies that in the p-adic expansion for $\langle\lambda,\alpha_2\rangle$, q has nonzero coefficient, so by (1.10) $q = q_0 = 1$.

Let $D_0 \leq Y$ be the subgroup of type D_4 defined by $D_0 = \langle U_{\pm\beta_2}, U_{\pm\beta_3}, U_{\pm(\beta_4+\beta_5+\beta_6+\beta_7)}, U_{\pm\beta_8} \rangle$. Then, if $G \leq D_0$ is the fixed point subgroup of the graph automorphism of D_0 of order 3, G has type G_2. Now, D_0 is the Levi factor of a proper parabolic of Y, so the D_0 composition factor of $V|Y$ afforded by v^+ is not all of $V|Y$. Moreover, the G-composition factor afforded by v^+ has high weight $c\mu_1 + x\mu_2$, as G_2-module . Thus, if $\lambda|T_A = c\mu_1 + x\mu_2$, $\dim V|A < \dim V|Y$. Hence, $\langle \lambda, \alpha_2 \rangle = 0$ or cq_0.

We now claim that if $x \neq 0$, $x=c$. Let $P \geq B_A^-$ be the parabolic subgroup of A with Levi factor $L = \langle U_{\pm\alpha_2} \rangle T_A$. Since $V|A$ is basic, (1.9) implies that there is a parabolic subgroup P_0 of Y with $P \leq P_0$, $Q = R_u(P) \leq Q_0 = R_u(P_0)$ and $L \leq L_0$, a Levi factor of P_0. Moreover, since $T_A \leq T_Y$, we may take $T_Y \leq L_0$. Thus, there is a subsystem $\Sigma_0 \subseteq \Sigma(Y)$ with $L_0 = T_Y \langle U_{\pm\gamma} \mid \gamma \in \Sigma_0^+ \rangle$. Write $L_0' = L_{01} \times \cdots \times L_{0d}$, a product of simple algebraic groups, with $L_{01} v^+ \neq v^+$ and $L_{0\ell} v^+ = v^+$ for $\ell \neq 1$. (This is possible since $V|A$ is basic.) Then $V|A$ basic and Theorem (7.1) of [12] imply that L_{01} has type A_k for some k and that if $k>1$, $V^1(Q_0)$ is isomorphic to W_{01}, the natural module for L_{01} (or to W_{01}^*). Note that if $k>1$, there does not exist $K_\gamma \leq Q_0$ such that $Q_0/K_\gamma \cong W_{01}$ or W_{01}^*, and $Q_A \nleq K_\gamma$. For otherwise, Q_0/K_γ is an irreducible L' module on which $Z(L')^\circ$ induces scalars. So $Q_A K_\gamma / K_\gamma$ is an L' submodule of Q_0/K_γ. But $\dim(Q_A K_\gamma/K_\gamma) < \dim(Q_0/K_\gamma)$. Hence, L_{01} has type A_1. So there exists $\tau \in \Sigma_0^+$ with $L_{01} = \langle U_{\pm\tau} \rangle$, $\langle \lambda, \tau \rangle = c$ and $\langle \lambda, \eta \rangle = 0$ for all $\eta \in \Sigma_0$ with $\eta \neq \pm\tau$. But, by the earlier work in this configuration, $x_{-\beta_7}(at^q)$ occurs in the factorization of $x_{-\alpha_2}(t)$, and $\langle \lambda, \beta_7 \rangle \neq 0$. Since $L \leq L_0$, $\beta_7 = \tau$ and $x = \langle \lambda, \beta_7 \rangle = \langle \lambda, \tau \rangle = c$, as claimed. But as in the preceeding paragraph, $\dim V|A < \dim V|Y$. Thus, in fact, $x=0$. Moreover, the preceding argument with D_0 implies that $\langle \lambda, \alpha_2 \rangle \neq 0$.

Suppose $q_0 = q$, where $\langle \lambda, \alpha_2 \rangle = cq_0$. Let $X \leq Y$ be the subgroup of type D_7 defined by $X = \langle U_{\pm t}, U_{\beta_k} \mid 3 \leq k \leq 8,\ t = \beta_1 + 2\beta_2 + 2\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6 \rangle$.

Then X has a natural subgroup, $G \le X$, of type G_2. Moreover, the G-composition factor of $V|Y$ afforded by v^+ has the same dimension, as G_2 module, as does $V(q(c\mu_1+c\mu_2))$. However, X is contained in a proper parabolic of Y and so the X-composition factor of V afforded by v^+ is not all of V. Thus, $\dim V(q(c\mu_1+c\mu_2)) < \dim V|Y$.

Thus, we have $\lambda|T_Y = c\lambda_8$, $c > 1$ and $\lambda|T_A = cq\mu_1+cq_0\mu_2$, with $q_0 \neq q$. Note that $3c = \dim V_{\beta_7}(Q_Y) \le \dim V^2(Q_A)_{\lambda-q\alpha_2} \le 3c$. Hence, $\beta_4|Z_A \neq 0$; in particular, $-\beta_4$ is not involved in L_A'. Previous remarks imply that $\langle U_{\pm\beta_2},U_{\pm\beta_3}\rangle \cap L_Y' \neq \{1\}$. If $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_5,\beta_6,\beta_8\}$ or $\{\beta_1,\beta_3,\beta_5,\beta_6,\beta_8\}$, (1.15) implies $Q_A \le K_{\beta_4}$ and (2.11) implies $\beta_4|Z_A = 0$. If $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_3,\beta_5,\beta_6,\beta_8\}$, (2.11) and (1.15) imply $Q_A \nleq K_{\beta_4}$. Examining the L_A' composition factors and $T(L_A')$ weight vectors in Q_Y/K_{β_4}, we see that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_2}\rangle$ and in $\langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle$ is also q and $\beta_4|T_A = q(\alpha_2-\alpha_1)$. But then $(V_{T_Y}(\lambda-\beta_4-\beta_5-\beta_6-\beta_7-\beta_8) \oplus V_{T_Y}(\lambda-2\beta_7-2\beta_8) + [V,Q_A{}^3])/[V,Q_A{}^3]$ is a 2-dimensional subspace of $(V_{T_A}(\lambda-q(2\alpha_1+2\alpha_2)) + [V,Q_A{}^3]/[V,Q_A{}^3]$. However, using (1.28) and the description of commutator subspaces given in (1.21), we see that the latter weight space has dimension 1. Hence, $L_Y' \ge \langle U_{\pm\beta_2}\rangle$ but $\langle U_{\pm\beta_3}\rangle \nleq L_Y'$, or $L_Y' = L_i \times L_j \times \langle U_{\pm\beta_3}\rangle$. In either case, (2.11) implies $Q_A \nleq K_{\beta_4}$ and arguing as usual, we find that $\beta_4|T_A = q\alpha_2$. Say $\langle U_{\pm\beta_2}\rangle \le L_Y'$ and $\langle U_{\pm\beta_3}\rangle \nleq L_Y'$. (The other case is handled similarly.) Using (1.21) and (1.29), we have $\dim(V_{T_A}(\lambda-q(4\alpha_1+2\alpha_2)) + [V,Q_A{}^3])/[V,Q_A{}^3] \le 5$. However, $(V_{T_Y}(\lambda-2\beta_7-4\beta_8) \oplus V_{T_Y}(\lambda-\beta_6-2\beta_7-3\beta_8) \oplus V_{T_Y}(\lambda-\beta_5-\beta_6-2\beta_7-2\beta_8) \oplus V_{T_Y}(\lambda-2\beta_6-2\beta_7-2\beta_8) \oplus V_{T_Y}(\lambda-\beta_4-\beta_5-\beta_6-\beta_7-2\beta_8) \oplus V_{T_Y}(\lambda-\beta_2-\beta_4-\beta_5-\beta_6-\beta_7-\beta_8) + [V,Q_A{}^3])/[V,Q_A{}^3]$ is a 6-dimensional subspace lying in $(V_{T_A}(\lambda-q(4\alpha_1+2\alpha_2)) + [V,Q_A{}^3])/[V,Q_A{}^3]$, unless $c = 2$ or 3. But if $c = 2$ or 3, (1.38) and (1.27) imply $\dim V|A < \dim V|Y$. Contradiction. This completes the consideration of Case II and the proof of (8.8).□

(8.9). $\langle\lambda,\alpha_1\rangle \neq 0$.

Proof: Suppose false; i.e., suppose $\langle\lambda,\alpha_1\rangle = 0$. Then $\langle\lambda,\alpha_2\rangle \neq 0$. Let $P \geq B_A^-$ be the parabolic subgroup of A with Levi factor $L = \langle U_{\pm\alpha_2}\rangle T_A$. Let P_0 be a parabolic subgroup of Y with $P \leq P_0$, $Q = R_u(P) \leq R_u(P_0) = Q_0$ and such that P_0 is minimal with these properties. Let L_0 be the Levi factor of P_0. Let T_0 be a maximal torus of L_0 with $T_A \leq T_0$. Fix a base $\Pi_0(Y)$ of the root system $\Sigma_0(Y)$, such that $U_A \cap L \leq Q_0(U_0 \cap L_0)$, where U_0 is the product of the T_0 root subgroups corresponding to roots in $\Sigma_0^+(Y)$ and Q_0 is the product of the T_0 root subgroups corresponding to the roots in $\Sigma_0^-(Y) - \Sigma_0(L_0)$. Let $\Pi_0(Y) = \{\gamma_1 \ldots,\gamma_n\}$. Let ν_i be the fundamental dominant weight corresponding to γ_i and $\langle w^+\rangle$ be the unique 1-space of V invariant under $B_0 = \langle U_r \mid r \in \Sigma_0^+(Y)\rangle$. Let $L_0' = D_1 \times \cdots \times D_s$, with D_i a simple algebraic group. Note that each D_i is of classical type by (7.1) of [12] and Remark (6.2), so (1.5) implies $Z = Z(L)^\circ \leq Z_0 = Z(L_0)^\circ$.

Now, referring to (6.9) and using (1.27), (8.1), (1.38), (1.32), [8] and $p>3$, we see that $\dim V|A \neq \dim V|Y$ unless one of the following holds:

(i) $Y = E_8$, $\lambda|T_Y = x\nu_7 + c\nu_8$, for $p>x\geq 0$, $p>c>0$ and $\langle\lambda,\alpha_2\rangle = c\cdot q$, for some p-power q.

(ii) The hypotheses of (6.7) hold.

The configurations of (i) and (ii) may be described as follows: $\langle\lambda,\alpha_2\rangle = c$, for $0<c<p$ (by (1.10)), D_1 has type A_1, $\dim M_1 = c+1$ and M_i is trivial for $i\neq 1$. Now, one checks that $\dim V^2(Q) = c$ and $\dim V^3(Q) \leq 2c$. Thus, D_1 must be separated by more than one node of the Dynkin diagram from all other components of L_0', else $\dim V^2(Q_0) > c$. However, there do not exist $\delta_1, \delta_2 \in \Pi_0(Y)-\Pi(L_0)$ with $(\delta_1,\Sigma D_1) \neq 0$, $(\delta_1,\delta_2) < 0$ and $(\delta_2,\Sigma L_0) = 0$. For otherwise (2.14) implies $\langle\lambda,\alpha_1\rangle \neq 0$. Thus, there exists $1<i\leq s$ with D_i separated from D_1 by exactly 2 nodes of the Dynkin diagram. Let $\gamma,\delta \in \Pi_0(Y)-\Pi(L_0)$ with $(\gamma,\Sigma D_1) \neq 0 \neq (\delta,\Sigma D_i)$ and $(\gamma,\delta) < 0$. If $-\delta$ is involved in L_0', $\delta|Z = 0$. Otherwise, if $Q_A \leq K_\delta$, (2.11) applies to give $(\gamma+\delta)|Z = \alpha_1$, so again $\delta|Z = 0$. Using the parabolic $P_0\hat{}$ of (2.11), we see that the bound on dim $V^2(Q)$ is exceeded. Thus, $Q_A \nleq K_\delta$ and $-\delta$ is not involved in L_0'. Now $\langle \lambda, \gamma \rangle = 0$, else dim $V^2(Q_0) > c$; but if $\Pi(D_1) = \{\tau_1\}$, $f_{\tau_1+\gamma}w^+$ affords an L_0' composition factor in $V^2(Q_0)$ of dimension c. Hence, $[V, Q_0{}^2] = [V, Q^2]$. Now, $f_{\tau_1+\gamma}{}^2w^+$ and $f_{\tau_1+\gamma+\delta}w^+$ afford distinct L' composition factors in $V^3(Q_0)$ of dimensions c−1 and nc, where $n = \dim V_{L_0'}(-\delta) \geq 2$. But dim $V^3(Q) \leq 2c$ implies c=1 and $\dim V|A = 14 < \dim V|Y$. Contradiction.□

The results (8.2) – (8.9) form a complete proof of Theorem (8.0)(b) in case $p > 3$.□

CHAPTER 9: SPECIAL CASES

In this chapter, we consider the remaining special cases which will complete the proof that the only possible triples (A,Y,V) are those described in the Main Theorem. In particular, we consider the cases where Y has type E_n and rankA = 2 with p=2, or (A,p) has type $(G_2,3)$ or where Y is exceptional and A is a non-simple, semisimple algebraic group. We find that the only possible configuration is that of (8.0)(a).

Section I: Rank(A) = 2 and p = 2.

Before establishing the notation to be used throughout this section of Chapter 9, we prove a preliminary lemma.

(9.1). Let p=2, $X = SL_2$, $Y = SO_8$. If $X < Y$, acting irreducibly on a rational kY-module,V, then X acts irreducibly on two of the three fundamental, restricted 8-dimensional irreducible kY-modules.

Proof: If $X<Y$ is as in Theorem (7.1) (d) of [12], then it is clear that X acts irreducibly on the two fundamental spin modules for Y, since $X < SO_7 < Y$ and each of these representations restrict to the same irreducible representation of SO_7.

Consider now the possibility that V|Y is the "natural" module for Y. Let D be a group of type D_4 and let $X < D$ be as in (7.1)(d) of [12]. In particular, if $\Pi(D) = \{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$ is labelled as throughout and if μ_i is the fundamental dominant weight corresponding to γ_i, X acts irreducibly on the kD-module, W, with high weight μ_3. Also, if $\Pi(X) = \{\alpha\}$, $h_\alpha(c) =$ $h_{\gamma_3}(c^{q_1})h_{\gamma_4}(c^{q_1})h_{\gamma_2+\gamma_3}(c^{q_2})\, h_{\gamma_2+\gamma_4}(c^{q_2})h_{\gamma_1+\gamma_2+\gamma_3}(c^{q_3})h_{\gamma_1+\gamma_2+\gamma_4}(c^{q_3})$, for $c \in k^*$, q_1, q_2, and q_3 distinct p-powers and $U_\alpha \leq \langle U_{\gamma_i} | 1 \leq i \leq 4 \rangle$. Now, by (1.11), D fixes a form on W. Since p=2, we may take the form on W to be symplectic. So if $\varphi: D \to SL(W)$ is the kD-representation corresponding to W, $\varphi(D) \leq Sp(W)$. Then $\varphi(D) < Sp(W)$ is a maximal rank configuration, so by Theorem 4.1 and Table 4.1 of [12], $\varphi(D) = \langle U_r \mid r \in \Sigma(Sp(W)),\ r \text{ short} \rangle$. Now, choose a base $\Pi(\varphi(D)) = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ such that the Dynkin diagram is labelled as throughout and such that $W|\varphi(D)$ has high weight ν_1 (where ν_i is the fundamental dominant weight corresponding to η_i). Since p=2, $X \cong \varphi(X)$ and $D \cong \varphi(D)$ as abstract groups, and $\varphi(X) < \varphi(D) < SL(W)$ is the desired embedding; i.e. $W|\varphi(D)$ is the "natural" module for $\varphi(D)$. Moreover, we may take $\varphi(h_\alpha(c)) = h_{\eta_1}(c^{q_1}) h_{\eta_3}(c^{q_1}) h_{\eta_1+\eta_2}(c^{q_2}) h_{\eta_2+\eta_3}(c^{q_2}) \cdot h_{\eta_1+\eta_2+\eta_4}(c^{q_3}) h_{\eta_2+\eta_3+\eta_4}(c^{q_3})$. Also $\varphi(U_\alpha) \leq \langle U_{\eta_i} | 1 \leq i \leq 4 \rangle$. Hence, $\varphi(X)$ acts irreducibly on the $\varphi(D)$ modules with high weights ν_1 and ν_3.

By Theorem (7.1) of [12] the above considerations are sufficient and the proof of (9.1) is complete.□

For the remainder of this section, we suppose (A,Y,V) is an example in the Main Theorem, with A a rank two simple algebraic group, Y of type E_n and p = 2. Adopt Notation and Hypothesis (2.0); in addition, choose $P_A = Q_A L_A$ with $L_A = \langle U_{\pm\beta} \rangle T_A$ such that $\langle \lambda, \beta \rangle \neq 0$. (Then $\Pi(A) - \Pi(L_A) = \{\alpha\}$ as usual.) Note that (7.1) of [12], the minimality of P_Y and induction imply that if L_j, a component of L_Y', has type A_{k_j} for some k_j, then k_j = 1, 3 or 7. Finally, one checks, using [8] and (1.35) that $\dim V|A = 4^k, 3^k 8^\ell$, or $6^k 14^\ell 64^m$, for $k, \ell, m \in \mathbb{Z}^+$.

<u>(9.2)</u>. Suppose $\dim(M_i) > 1$ for some i.

(i) If L_i has type A_3 and $\dim(M_j) > 1$ for some $j \neq i$, then $\text{rank}(L_j) = 1$.

(ii) L_i has type A_1 or A_3.

<u>Proof:</u> By induction and (7.1) of [12], L_i has type D_k for

$k = 4,5,6$ or 7 or L_i has type A_1, A_3, or A_7. Consider first the case where L_i has type A_3 and suppose there exists $j \neq i$ with M_j nontrivial and $\mathrm{rank}(L_j) > 1$. Rank restrictions imply L_j has type A_3 and by (1.5), $Z_A \leq Z_Y$. Also, (2.5) and (2.7) imply that $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_4,\beta_6,\beta_7,\beta_8\}$. The bound on $\dim V_{\beta_5}(Q_Y)$ implies $\langle \lambda,\beta_j\rangle = 0$ for $j = 5,6,7$. However, $0 \neq w_1 \in V_{T_Y}(\lambda-\beta_5-\beta_6-\beta_7-\beta_8)$ and $0 \neq w_2 \in V_{T_Y}(\lambda-\beta_2-\beta_4-\beta_5) \oplus V_{T_Y}(\lambda-\beta_3-\beta_4-\beta_5)$ afford L_Y' composition factors of $V_{\beta_5}(Q_Y)$ of dimensions 20 and 56, respectively. Hence, the bound is still exceeded and the first statement of the result holds.

Suppose (9.2)(ii) is false. Previous remarks and (1.5) imply that $Z_A \leq Z_Y$. If L_i has type A_7, the bound on $\dim V_{\beta_2}(Q_Y)$ implies that $\lambda|T_Y = \lambda_8$; thus, L_i has type D_k for some k. If $k > 4$, M_i is one of the two irreducible, restricted spin modules for L_i. Thus, $\langle U_{\pm\beta_1}\rangle \leq L_i$ and L_i has type D_5; else Q_Y/K_{β_1} is a 2^{k-1} – dimensional irreducible L_A' module containing a nontrivial image of $Q_A{}^{\alpha}$. Now, the same argument forces Y to be of type E_7 or E_8 with $\langle U_{\pm\beta_7}\rangle$ a component of L_Y'. By (2.5), (2.6) and (2.7), $\langle \lambda,\beta_7\rangle = 0$. However, the bound on $\dim V_{\beta_6}(Q_Y)$ is exceeded in every possible configuration. So we have reduced to L_i having type D_4.

By (9.1), L_A' acts irreducibly on 2 of the three fundamental restricted 8-dimensional irreducible L_i modules. But $Q_A \nleq K_{\beta_1}$, so Q_Y/K_{β_1} is a reducible L_A' module. Using (7.1) of [12] to obtain a precise description of the embedding of L_A' in L_i, we see that L_A' acts on Q_Y/K_{β_1} with composition factors of dimensions 2 and 1. Thus, if $A = G_2$, then β must be long. Also, since $V_{L_i}(-\beta_6)$ is an irreducible L_A' module, $Y = E_7$ or E_8 and $\langle U_{\pm\beta_7}\rangle$ is a component of L_Y'. Moreover, by (2.5), (2.6) and (2.7), $\langle \lambda,\beta_7\rangle = 0$. The above remarks about Q_Y/K_{β_1} imply that $\langle \lambda,\beta_3\rangle = 0$; and by (7.1) of [12], $\langle \lambda,\beta_4\rangle = 0$. Moreover, the bound on $\dim V_{\beta_j}(Q_Y)$, for $j = 1$ or 6, implies that $\langle \lambda,\beta_k\rangle = 0$ for $k = 1,5,6$. (We have used Table 1 of [5] and (1.35).) Then, [8] implies that $Y = E_8$, else $\dim V|A \neq \dim V|Y$. Suppose $\lambda|T_Y = \lambda_2$. The remarks of the previous paragraph imply

$\langle\lambda,\alpha\rangle = 0$, q, $q+q_0$ or $q+q_0+q_1$ for q, q_0 and q_1 distinct p-powers, unless $A = G_2$, in which case $\langle\lambda,\alpha\rangle \neq q+q_0+q_1$. Hence, $\dim V|A \leq 14^3 \cdot 6^2 < \dim V|Y$. A variation of the method described in [8] was run on the VAX at the University of Oregon to determine the multiplicity of the subdominant weight $\lambda-\beta_1-3\beta_2-3\beta_3-5\beta_4-4\beta_5-3\beta_6-2\beta_7-\beta_8$. This computation, together with [8], implies $\dim V|Y > 14^3 \cdot 6^2$. If $\lambda|T_Y = \lambda_2 + \lambda_8$, previous remarks and part (i) imply $\langle\lambda,\alpha\rangle = 0$, q, $q+q_0$ or $q+q_0+q_1$, for q,q_0 and q_1 distinct p-powers. Then $\dim V|A \leq 14^3 \cdot 6^3 < \dim V|Y$, by (1.38). Contradiction.

This completes the proof of (9.2).□

(9.3). Assume β is long if $A = G_2$. If $\dim(M_i) > 1$ then $\mathrm{rank}(L_i) = 1$.

Proof: Suppose false. Then by (9.2), L_i has type A_3, so by (1.5), $Z_A \leq Z_Y$. Since L_A' acts irreducibly on W, the natural module for L_i, (2.3) implies that there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\Sigma L_i) \neq 0$ and $Q_Y/K_\gamma \cong W$ or W^*. Consider the case where $\Pi(L_i) = \{\beta_2,\beta_4,\beta_5\}$, so $Y = E_7$ or E_8 and $\langle U_{\pm\beta_7}\rangle$ is a component of L_Y'. Now, $\langle\lambda,\beta_7\rangle = 0$, by (2.5), (2.6) and (2.7), and $\langle\lambda,\beta_6\rangle = 0$, else the bound on $\dim V_{\beta_6}(Q_Y)$ is exceeded. Also, the bound on $\dim V_{\beta_3}(Q_Y)$ implies $\langle\lambda,\beta_3\rangle = 0$. If $\langle U_{\pm\beta_1}\rangle$ is a component of L_Y', $\langle\lambda,\beta_1\rangle = 0$, else $0 \neq w_1 \in V_{T_Y}(\lambda-\beta_1-\beta_3)$ and $0 \neq w_2 \in V_{T_Y}(\lambda-\beta_2-\beta_3-\beta_4) \oplus V_{T_Y}(\lambda-\beta_3-\beta_4-\beta_5)$ afford distinct L_Y' composition factors of $V_{\beta_3}(Q_Y)$, exceeding the given bound. Now, [8] implies that $\lambda|T_Y \neq \lambda_2$ or λ_5 if $Y = E_7$. Hence, $Y = E_8$. If $\lambda|T_Y = \lambda_2$ or λ_5, (9.2) implies $\langle\lambda,\alpha\rangle = 0$, q or $q + q_0$, for q and q_0 distinct p-powers. So $\dim V|A \leq 14^2 \cdot 6^2 < \dim V|A$ by (1.38). Thus, $\lambda|T_Y = \lambda_2 + \lambda_8$ or $\lambda_5 + \lambda_8$. Then (9.2) implies $\langle\lambda,\alpha\rangle = 0$, q_1, q_1+q_2, or $q_1+q_2+q_3$, for q_1, q_2 ,q_3 distinct p-powers. Hence, $\dim V|A \leq 14^2 \cdot 6^3 < \dim V|Y$, by (1.38). Thus, $\Pi(L_i) \neq \{\beta_2,\beta_4,\beta_5\}$.

Consider the case where $\Pi(L_i) = \{\beta_6,\beta_7,\beta_8\}$. Then $\langle U_{\pm\beta_4}\rangle \leq L_Y'$. If L_Y' has a second component of type A_3, (9.2) implies $V^1(Q_Y) \cong M_i$. But in this case the bound on $\dim V_\gamma(Q_Y)$, for $\gamma \in \Pi(Y) - \Pi(L_Y)$, implies

$\lambda|T_Y = \lambda_8$. Thus, $\langle U_{\pm\beta_4}\rangle$ is a component of L_Y'. By (2.5), (2.6) and (2.7), $\langle\lambda,\beta_4\rangle = 0$ and the bound on $\dim V_{\beta_5}(Q_Y)$ implies $\langle\lambda,\beta_5\rangle = 0$. Suppose L_A' acts on the natural module for L_i with high weight $(q_1+q_2)\mu_\beta$. Then, by (2.6) and (2.7) we may assume that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_4}\rangle$ is q_1. If $\langle U_{\pm\beta_1}\rangle$ is a component of L_Y' with $\langle\lambda,\beta_1\rangle \neq 0$, then $Q_A \nleq K_{\beta_3}$ implies that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_1}\rangle$ is also q_1. But this contradicts (2.5). If $\langle U_{\pm\beta_1}\rangle \nleq L_Y'$, then $\langle\lambda,\beta_1\rangle = 0$ by (2.3). So $\lambda|T_Y = x\lambda_2 + y\lambda_3+\lambda_6$ or $x\lambda_2+y\lambda_3+\lambda_8$, where $x, y\in\{0,1\}$. By (9.2), $\dim V|A \le 14^2\cdot 6^2$ if $x=0=y$ and $\dim V|A \le 14^2\cdot 6^4$ if $x+y\neq 0$. Then (1.38) and (1.32) imply $\dim V|A < \dim V|Y$. Hence, $\Pi(L_i) \neq \{\beta_6,\beta_7,\beta_8\}$.

We have reduced to the configuration where $Y = E_7$ and $\Pi(L_i) = \{\beta_5,\beta_6,\beta_7\}$. If $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_5,\beta_6,\beta_7\}$, (2.5), (2.7) and the bound on $\dim V_{\beta_4}(Q_Y)$ imply that $\langle\lambda,\beta_j\rangle = 0$ for $2 \le j \le 6$. So $\lambda|T_Y = \lambda_7$ or $\lambda_1 + \lambda_7$. But then [8] implies that $\dim V|A \neq \dim V|Y$. The above remarks imply that $\Pi(L_Y) = \Pi(L_i) \cup\{\beta_2\}$, $\Pi(L_i) \cup\{\beta_3\}$, or $\Pi(L_i) \cup\{\beta_1,\beta_2\}$, $V_1(Q_Y) \cong M_i$ in the first two cases, and in each case, $\langle\lambda,\beta_2+\beta_4\rangle = 0$. Now, $\lambda|T_Y \neq \lambda_5$ or λ_7, by [8], so $\langle\lambda,\beta_1+\beta_3\rangle > 0$; in particular, $\Pi(L_Y) \neq \{\beta_2,\beta_5,\beta_6,\beta_7\}$. If $A = A_2$ or B_2, $\dim V|A \le 2^{12} < \dim V|Y$, by (1.32) and [8]. So $A = G_2$. If $L_Y' = L_i \times \langle U_{\pm\beta_3}\rangle$, then previous remarks imply that $\lambda|T_Y = \lambda_1+\lambda_5$ and $\dim V|A \le 6^3\cdot 14^2 < \dim V|Y$, by (1.38). In the remaining case, we consider the action of L_A' on the 56-dimensional irreducible kY-module $V(\lambda_7)$. One checks that there are at least five 4-dimensional L_A' composition factors of $V(\lambda_7)$. However, there is no 56-dimensional kA-module affording such an L_A' composition series. This completes the proof of (9.3).□

<u>(9.4).</u> A does not have type A_2 or B_2.

<u>Proof:</u> Suppose false. If $\langle\lambda,\alpha\rangle = 0$ or q and $\langle\lambda,\beta\rangle = 0$ or q_0 for some p-powers q and q_0, $\dim V|A = 3,8,9,4$ or $16 < \dim V|Y$. So we may choose a parabolic P_A with Levi factor $L_A = \langle U_{\pm\beta}\rangle T_A$ such that $\langle\lambda,\beta\rangle$ has more than one p-power in its p-adic expansion. Moreover, choose β such

that the number of distinct p-powers in the p-adic expansion of $\langle\lambda,\beta\rangle$ is greater than or equal to the number of p-powers in the p-adic expansion of $\langle\lambda,\alpha\rangle$. Let P_Y be as before. Then (9.3) implies that L_Y' has at least two components of type A_1, so (1.5) implies $Z_A \leq Z_Y$. Also, $\langle\lambda,\beta_i\rangle = 1 = \langle\lambda,\beta_j\rangle$ for some $1 \leq i,j \leq \mathrm{rank}(Y)$, with $(\beta_i,\beta_j) = 0$. Recall, $\dim V|A = 4^k$ or $3^k 8^\ell$. Suppose $\langle\lambda,\beta\rangle = q_1+q_2$, for q_1 and q_2 distinct p-powers. Then, $\dim V|A \leq 2^8$. But counting only the conjugates of $V_{T_Y}(\lambda)$ in $V|Y$, we have $\dim V|Y > \dim V|A$. Hence, $\langle\lambda,\beta\rangle \neq q_1+q_2$. Suppose $\langle\lambda,\beta\rangle = q_1+q_2+q_3$, for q_1, q_2, and q_3 distinct p-powers. Then $\langle\lambda,\beta_i\rangle = 1 = \langle\lambda,\beta_j\rangle = \langle\lambda,\beta_k\rangle$ for some $i\neq j\neq k$ with $(\beta_i,\beta_j) = 0 = (\beta_j,\beta_k) = 0 = (\beta_i,\beta_k)$. Also, $\dim V|A \leq 2^{12}$. Once again, counting only the conjugates of $V_{T_Y}(\lambda)$ in $V|Y$, we have $\dim V|Y > \dim V|A$, unless $Y = E_6$ and $\lambda|T_Y = \lambda_1+\lambda_2+\lambda_6$. However, by (1.38), $\dim V|A < \dim V|Y$ here also. By rank restrictions, it remains to consider the case where $\langle\lambda,\beta\rangle = q_1+q_2+q_3+q_4$. Arguing exactly as above, we find that $\dim V|A < \dim V|Y$. This completes the proof of (9.4).□

(9.5). Let A have type G_2, with β long. Suppose M_i and M_j are nontrivial for some $i \neq j$.

(1) If $(\Sigma L_i,\gamma) \neq 0 \neq (\gamma,\Sigma L_j)$ for some $\gamma \in \Pi(Y) - \Pi(L_Y)$, then $\Pi(L_i \times L_j) \subset \{\beta_2,\beta_3,\beta_5\} \subset \Pi(L_Y)$. Moreover, $\langle\lambda,\beta_2+\beta_3+\beta_5\rangle = 2$.

(2) If $(\Pi(L_i),\gamma_i) \neq 0 \neq (\gamma_i,\gamma_j) \neq 0 \neq (\gamma_j,\Pi(L_j))$ for some $\gamma_i\neq\gamma_j \in \Pi(Y) - \Pi(L_Y)$, then $\beta_4 \in \{\gamma_i,\gamma_j\}$ and $|\{\beta_2,\beta_3,\beta_5\} \cap \Pi(L_Y)| = 2$.

(3) $\langle\lambda,\beta\rangle = 0$, q, $q+q_0$ or $q+q_0+\hat{q}$, for q, q_0 and $\hat{q}$ distinct p-powers. If $Y = E_6$, $\langle\lambda,\beta\rangle \neq q+q_0+\hat{q}$.

Proof: (1) follows from (2.5) and (2.7). For (2), let $\Pi(L_i) = \{\beta_i\}$ and $\Pi(L_j) = \{\beta_j\}$ and take $\ell \in \{i,j\}$. Note that if $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\beta_\ell) \neq 0$ and $(\gamma,\Sigma L_k) = 0$ for all $k\neq\ell$, then $Q_Y/K_\gamma \cong (Q_A{}^\alpha)^q$, where q is the field twist on the embedding of L_A' in L_ℓ. Hence, $x_{-\alpha}(t) = x_{-\gamma}(at^q)u$, for some $a \in k^*$, $u \in K_\gamma$ and $\gamma|T_A = q\alpha$. The result of (2) then follows from (2.5), (2.6) and (2.8). Then (9.3), (9.5)(1) and (9.5)(2) imply (3).□

(9.6). There are no examples in the Main Theorem with A a rank two simple algebraic group, Y of type E_n and p=2.

Proof: Suppose false. Then (9.4) implies that $A = G_2$. Let $\Pi(A) = \{\alpha_1,\alpha_2\}$, with α_1 short. If P_A is as before with $\dim V^1(Q_A) > 1$, then all components of L_Y' have classical type. For otherwise, $Y = E_8$, $\Pi(L_Y) = \{\beta_j \mid j \neq 7\}$ and by induction, $\lambda|T_Y = \lambda_7+\lambda_8$. But (9.2), (9.3) and (1.32) imply $\dim V|A \leq 6^2\cdot 14 < \dim V|Y$. So all components of L_Y' have classical type and by (1.5), $Z_A \leq Z_Y$.

We first make a few notes about the case where $L_A' = \langle U_{\pm\alpha_1}\rangle$ and $\dim(M_i) > 1$ with L_i of type A_3. (1) If $\dim(M_j) > 1$ for some $j\neq i$, (9.2) implies L_j has type A_1. (2) There does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\Sigma L_i) \neq 0$, $(\gamma,\Sigma L_m) = 0$ for all $m\neq i$ and $\dim(Q_Y/K_\gamma) = 6$. For otherwise, the L_A' composition factors of Q_Y/K_γ would have dimensions 1 and 2, contradicting (2.3). (3) By (9.2), (2.5) and (2.7), there does not exist $\gamma \in \Pi(L_Y) - \Pi(L_i)$, $\delta \in \Pi(Y) - \Pi(L_Y)$ with $\langle\lambda,\gamma\rangle \neq 0$, $(\gamma,\delta) \neq 0 \neq (\delta,\Sigma L_i)$ and $\dim(Q_Y/K_\gamma) = 8$. (4) If $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $V_\gamma(Q_Y) \neq 0$, then (2.3) implies $\dim(Q_Y/K_\gamma) \geq 4$.

Note that $\langle\lambda,\alpha_1\rangle \neq q_1+q_2+q_3+q_4$, for q_1, q_2, q_3, and q_4 distinct p-powers. For otherwise, previous remarks imply $Y = E_8$ and $\langle\lambda,\beta_6\rangle = 1 = \langle\lambda,\beta_8\rangle = 1 = \langle\lambda,\beta_2+\beta_3\rangle$. But then (9.5) implies $\dim V|A \leq 6^4\cdot 14^3 < \dim V|Y$, by (1.32). Now, (9.2) implies that if there are r distinct p-powers in the p-adic expansion of $\langle\lambda,\alpha_1\rangle$, there exist $\{\beta_{j_1},\dots,\beta_{j_{r-1}}\} \subset \Pi(Y)$ with $\langle\lambda,\beta_{j_k}\rangle = 1$ and $(\beta_{j_k},\beta_{j_\ell}) = 0$ for $k\neq\ell$. Previous work implies that $\langle\lambda,\alpha_1\rangle = 0$, q_1, q_1+q_2, or $q_1+q_2+q_3$, for distinct p-powers q_1, q_2, and q_3. We next claim that if $\langle\lambda,\alpha_1\rangle = q_1+q_2+q_3$, $Y = E_7$ or E_8. For otherwise, taking P_A as before, with $L_A' = \langle U_{\pm\alpha_1}\rangle$, we may assume $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_4,\beta_6\}$. Also, L_A' acts on the natural module for $\langle U_{\pm\beta_2},U_{\pm\beta_3},U_{\pm\beta_4}\rangle$ with high weight $(q_1+q_2)\mu_\beta$ and the field twist on the embedding of L_A' in $\langle U_{\pm\beta_6}\rangle$ is $q_3 \neq q_i$ for i = 1,2. Consider the action of L_A' on the 27-dimensional irreducible L_A' module $V(\lambda_6)$. There is an L_A' composition factor with high weight $(\lambda-\beta_1-\beta_2-\beta_3-2\beta_4-2\beta_5-\beta_6)|T(L_A') = (q_1+q_2+q_3)\mu_1$. But there is no 27-dimensional kA-module affording such an L_A' composition factor.

Previous remarks, [8] and (1.32) imply that $\lambda|T_Y \neq \lambda_\ell$ for any ℓ, else $\dim V|A \neq \dim V|Y$. Note that (9.3) and (9.5) imply that for each distinct p-power q_j in the p-adic expansion of $\langle\lambda,\alpha_2\rangle$, there exists $\beta_j \in \Pi(Y)$ with $\langle\lambda,\beta_j\rangle = 1$ and $(\beta_j,\beta_k) = 0$ for any β_k corresponding to a different p-power q_k. As well, previous remarks, [8], (1.26), (1.32) and (9.5) imply that $\langle\lambda,\alpha_1\rangle \neq 0 \neq \langle\lambda,\alpha_2\rangle$, else $\dim V|A < \dim V|Y$.

Using the above remarks and the usual dimension arguments, it is straightforward to show that $\langle\lambda,\alpha_2\rangle \neq q$ and if $\langle\lambda,\alpha_2\rangle = q+q_0$, then $Y \neq E_8$. (Here q and q_0 are distinct p-powers.)

Suppose $\langle\lambda,\alpha_2\rangle = q+q_0$ and $Y = E_6$. Then $\dim V|A \leq 6^2\cdot 14^2$. Also there exist $\beta,\eta \in \Pi(Y)$ with $(\beta,\eta) = 0$ and $\langle\lambda,\beta\rangle = 1 = \langle\lambda,\eta\rangle$. Then [8] and (1.32) imply $\langle\lambda,\alpha_1\rangle = q_1+q_2$, else $\dim V|A < \dim V|Y$. Let P_A be as before, with $L_A' = \langle U_{\pm\alpha_1}\rangle$. By previous remarks, symmetry and (1.23), we may assume that either $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_4,\beta_6\}$ with $\lambda|T_Y = \lambda_3+\lambda_5$ or $\Pi(L_Y) = \{\beta_1,\beta_4,\beta_6\}$ with $\langle\lambda,\beta_1\rangle = 1 = \langle\lambda,\beta_6\rangle$ and $\langle\lambda,\beta_4+\beta_2\rangle = 0$. In the first case, the bound on $\dim V_{\beta_5}(Q_Y)$ is exceeded. In the second case, [8] implies that $\lambda|T_Y \neq \lambda_1+\lambda_6$, and (1.38) implies that $\dim V|A < \dim V|Y$ in the remaining cases.

So if $\langle\lambda,\alpha_2\rangle = q+q_0$, $Y = E_7$. Also, $\langle\lambda,\alpha_1\rangle = q_1+q_2+q_3$, else $\dim V|A \leq 6^2\cdot 14^2$ and using (1.32), [8] and (1.38), we see that $\dim V|A < \dim V|Y$. Let P_A be as before with $L_A' = \langle U_{\pm\alpha_1}\rangle$. Then either (a) $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_5,\beta_6,\beta_7\}$, with $\langle\lambda,\beta_1+\beta_3+\beta_6\rangle = 0$ and $\langle\lambda,\beta_2\rangle = 1 = \langle\lambda,\beta_5+\beta_7\rangle$ or (b) L_Y' has components $\langle U_{\pm\beta_1}\rangle$ and $\langle U_{\pm\beta_2}, U_{\pm\beta_4}, U_{\pm\beta_5}\rangle$, with $\langle\lambda,\beta_1\rangle = 1 = \langle\lambda,\beta_2+\beta_5\rangle$ and $\langle\lambda,\beta_4+\beta_7\rangle = 0$ or (c) L_Y' has components $\langle U_{\pm\beta_2}\rangle$, $\langle U_{\pm\beta_5}\rangle$, and $\langle U_{\pm\beta_7}\rangle$, with $\langle\lambda,\beta_j\rangle = 1$, for j = 2,5,7 and $\langle\lambda,\beta_1+\beta_3\rangle = 0$. In the last case, we need only count the conjugates in $V|Y$ of $V_{T_Y}(\lambda)$ to see that $\dim V|A < \dim V|Y$. In case (a), the bound on

$\dim V_{\beta_4}(Q_Y)$ implies that $\langle\lambda,\beta_4+\beta_7\rangle = 0$. So $\lambda|T_Y = \lambda_2+\lambda_5$ and (1.38) implies that $\dim V|A < \dim V|Y$. Finally, to see that the configurations described in (b) do not occur, we consider the action of L_A' on the 56-dimensional irreducible kY module, $V(\lambda_7)$. There is an L_A' composition factor with high weight $\lambda-\beta_3-\beta_4-\beta_5-\beta_6-\beta_7|T(L_A') = (q_1+q_2+q_3)\mu_\beta$. But there is no 56-dimensional kA module which affords such an L_A' composition factor. Thus, in fact, $\langle\lambda,\alpha_2\rangle \neq q+q_0$.

By (9.5), it remains to consider the case where $\langle\lambda,\alpha_2\rangle = q+q_0+q\hat{}$ and $Y = E_7$ or E_8. So there exist $\beta,\eta,\varepsilon\in\Pi(Y)$ such that $\langle\lambda,\beta\rangle = 1 = \langle\lambda,\eta\rangle = \langle\lambda,\varepsilon\rangle$ and $0 = (\beta,\eta) = (\beta,\varepsilon) = (\eta,\varepsilon)$. Also, $\dim V|A \le 6^4\cdot 14^3$. If $Y = E_8$, (1.32) and (1.38) imply $\dim V|A < \dim V|Y$. So $Y = E_7$. Let P_A be as in the previous notation with $L_A' = \langle U_{\pm\alpha_2}\rangle$. Then (9.3) and (9.5) imply that either $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_5,\beta_7\}$, with $\langle\lambda,\beta_1\rangle = 1$ or $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_5,\beta_7\}$, with $\langle\lambda,\beta_k\rangle = 1$, for $k = 2,3,7$ and $\langle\lambda,\beta_5\rangle = 0$. But in the first case (2.7) implies that the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_2}\rangle$, $\langle U_{\pm\beta_5}\rangle$ and $\langle U_{\pm\beta_7}\rangle$ are all equal, contradicting (2.5). In the second case, the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_5}\rangle$ and in $\langle U_{\pm\beta_7}\rangle$ are equal, while the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_2}\rangle$, $\langle U_{\pm\beta_3}\rangle$ and $\langle U_{\pm\beta_7}\rangle$ are distinct, contradicting (2.7) (with $\gamma = \beta_4$).

This completes the proof of (9.6).□

Section II: $A = G_2$ and $p = 3$.

Let $(A,p) = (G_2,3)$ and $\Pi(A) = \{\alpha,\beta\}$. Choose P_A such that $L_A = \langle U_{\pm\beta}\rangle T_A$ and $\langle\lambda,\beta\rangle \neq 0$. Note that I_α is a 2-dimensional irreducible L_A' module. (See (2.2) for the definition of I_α.) Adopt the remaining notation of (2.0). Note that $\dim V|A = 7^k\cdot 27^\ell$, for $k,\ell\in\mathbb{Z}^+$.

<u>(9.7).</u> If M_i is nontrivial, L_i has type A_1 or A_2.

<u>Proof:</u> Suppose false. Then (7.1) of [12] and rank restrictions imply

that all components of L_Y' have classical type. Hence, by (1.5), $Z_A \leq Z_Y$. We first note that L_i has type A_{k_i} for some k_i. For otherwise, (7.1) of [12] implies that L_i has type D_6 and M_i is isomorphic to the natural module for L_i. As well, the bound on $\dim V_{\beta_1}(Q_Y)$ and $\dim V_{\beta_8}(Q_Y)$ implies $Y = E_7$ and $\lambda|T_Y = \lambda_7$. But then $\dim V|A > \dim V|Y$. Thus, L_i has type A_{k_i} as claimed. Also note that L_i does not have type A_7. For otherwise the bound on $\dim V_{\beta_2}(Q_Y)$ implies $\lambda|T_Y = \lambda_8$.

Suppose L_i has type A_5. If $Y = E_6$, the bound on $\dim V_{\beta_2}(Q_Y)$ implies $\lambda|T_Y = \lambda_1$ or λ_6 and $\dim V|Y = 27$. But $\dim V|A > 27$. Hence, $Y = E_7$ or E_8. Note that L_A' acts irreducibly on W, the natural module for L_i, so there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\Sigma L_i) \neq 0$ and $Q_Y/K_\gamma \cong W$ or W^*. Hence, if $Y = E_7$, $\Pi(L_i) = \{\beta_k \mid k \neq 1,3\}$. The bound on $\dim V_{\beta_3}(Q_Y)$ implies $\lambda|T_Y = \lambda_2$ or λ_7. However, then [8] implies $\dim V|A \neq \dim V|Y$. Thus, $Y = E_8$ and $\Pi(L_Y) = \{\beta_k \mid k \neq 2,7\}$. But (1.15) implies $Q_A \leq K_{\beta_7}$, contradicting (2.3).

The above remarks imply L_i has type A_3. If W is the natural module for L_i, $W|L_A'$ is a 4-dimensional tensor decomposable irreducible and $M_i \cong W$ or W^*. If $M_i|L_A'$ has high weight $(q_1+q_2)\mu_\beta$, the L_A' composition factors of $W\wedge W$ have high weights $2q_1\mu_\beta$ and $2q_2\mu_\beta$. Hence, there does not exist $\delta \in \Pi(Y) - \Pi(L_Y)$ with $(\delta,\Sigma L_i) \neq 0$ and $Q_Y/K_\delta \cong W$, W^*, or $W\wedge W$. So if $\Pi(L_i) = \{\beta_2,\beta_4,\beta_5\}$, then $Y = E_7$ or E_8 and $\langle U_{\pm\beta_1},U_{\pm\beta_7}\rangle \leq L_Y$. The above remarks about $W\wedge W$, (2.5) and (2.7) imply $\langle\lambda,\beta_1+\beta_7\rangle = 0$. Also, if $\langle U_{\pm\beta_8}\rangle \leq L_Y$, $\langle\lambda,\beta_8\rangle = 0$. The bound on $\dim V_{\beta_3}(Q_Y)$ and $\dim V_{\beta_6}(Q_Y)$ implies $\lambda|T_Y = \lambda_2+x\lambda_8$ or $\lambda_5+x\lambda_8$, where $x \in \{0,1,2\}$. Now, [8] implies $\dim V|A \neq \dim V|Y$ if $Y = E_7$. If $x = 0$, the work of this result and (1.38) imply $\dim V|A \leq 7^4 < \dim V|Y$. Hence $x \neq 0$. But then previous work and (1.32) imply $\dim V|A \leq 7^6 < \dim V|Y$. So, $\Pi(L_i) \neq \{\beta_2,\beta_4,\beta_5\}$.

Consider now the case where $Y = E_8$ and $\Pi(L_i) = \{\beta_6,\beta_7,\beta_8\}$. Then (1.15) and previous remarks imply $\langle U_{\pm\beta_4}\rangle$ is a component of L_Y'. By (2.5) and (2.7), $\langle\lambda,\beta_4\rangle = 0$. The bound on $\dim V_{\beta_5}(Q_Y)$ implies $\langle\lambda,\beta_5\rangle = 0$.

If $\langle U_{\pm\beta_1}\rangle \nleq L_Y$, then $\langle \lambda, \beta_1+\beta_3\rangle = 0$, by (2.3) and (2.13). If $\langle U_{\pm\beta_1}\rangle \leq L_Y$, (1.15) implies $Q_A \leq K_{\beta_3}$ and again $\langle \lambda, \beta_1+\beta_3\rangle = 0$. Hence, $\lambda|T_Y = x\lambda_2+\lambda_j$, where $x \in \{0,1,2\}$ and j = 6 or 8. If $\lambda|T_Y = \lambda_2+\lambda_6$, $\lambda_2+\lambda_8$, $2\lambda_2+\lambda_6$ or $2\lambda_2+\lambda_8$, $\dim V|A \leq 7^6 < \dim V|Y$, by (1.38). If $\lambda|T_Y = \lambda_6$, $\dim V|A \leq 7^4 < \dim V|Y$ by (1.38). Hence $\Pi(L_i) \neq \{\beta_6,\beta_7,\beta_8\}$.

Finally, consider the case where $Y = E_7$ and $\Pi(L_i) = \{\beta_5,\beta_6,\beta_7\}$. Applying (1.15), we see that $\Pi(L_Y) = \Pi(L_i) \cup \mathcal{S}$, where $\mathcal{S}$ is (a) $\{\beta_3\}$, (b) $\{\beta_2\}$, (c) $\{\beta_2,\beta_1\}$, or (d) $\{\beta_2,\beta_1,\beta_3\}$. In (d), (2.5), (2.7) and the bound on $\dim V_{\beta_4}(Q_Y)$ imply $\lambda|T_Y = \lambda_7$ or $\lambda_1+\lambda_7$. But then [8] implies $\dim V|A \neq \dim V|Y$. In each of the remaining cases, we consider the action of L_A' on the 56-dimensional irreducible kY module, $V(\lambda_7)$. With q_1 and q_2 as above, we may assume that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_k}\rangle$ is q_1, where k = 3 in (a) and k = 2 in (b) and (c). In case (c), we find that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_1}\rangle$ is either q_1 or q_2, else there is an 8-dimensional L_A' composition factor of $V(\lambda_7)$ with high weight $\lambda-\beta_2-\beta_3-2\beta_4-\beta_5-\beta_6-\beta_7|T(L_A')$. But there is no 56-dimensional kA module affording such an L_A' compositon factor. Now, in cases (a) - (c), there is a 6-dimensional L_A' composition factor of $V(\lambda_7)$. This implies that there is a 49-dimensional A composition factor of $V(\lambda_7)$. Also, there are less than seven 1-dimensional L_A' composition factors, so there must also be a 7-dimensional A composition factor of $V(\lambda_7)$. But this contradicts the following information about $V(\lambda_7)$: If (a) holds, there are four trivial L_A' composition factors of $V(\lambda_7)$; if (b) holds, there are eight 4-dimensional L_A' composition factors; if (c) holds, there are either exactly 2 trivial L_A' composition factors or exactly six 4-dimensional L_A' composition factors.

This completes the proof of (9.7).□

Remarks (9.8). Assume $Z_A \leq Z_Y$. (1) If $i \neq j$ with M_i and M_j nontrivial and $(\Sigma L_i, \gamma) \neq 0 \neq (\Sigma L_j, \gamma)$ for some $\gamma \in \Pi(Y) - \Pi(L_Y)$, then (9.7) and (2.5) – (2.7) imply $\gamma = \beta_4$ and $\{\beta_2,\beta_3,\beta_5\} \subset \Pi(L_Y)$.

(2) Suppose $\langle U_{\pm\beta_\ell}\rangle$ and $\langle U_{\pm\beta_m}\rangle$ are components of L_Y' and $\gamma, \delta \in \Pi(Y) - \Pi(L_Y)$ with $(\beta_\ell,\gamma) \neq 0 \neq (\gamma,\delta) \neq 0 \neq (\delta,\beta_m)$ and $\dim(Q_Y/K_\gamma) = 2 = \dim(Q_Y/K_\delta)$. If $Q_A \nleq K_\gamma$ and $Q_A \nleq K_\delta$, then the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_\ell}\rangle$ and $\langle U_{\pm\beta_m}\rangle$ are equal. For one may check that there is a nontrivial contribution to some root group of $Q_Y(\gamma,\delta)$ in the factorization of nonidentity elements of Q_A'. Since $Z_A \leq Z_Y$ and $Q_A' \leq K_\gamma \cap K_\delta$, the image of Q_A' in $Q_Y(\gamma,\delta)$ is a 1-dimensional L_A' submodule of $Q_Y(\gamma,\delta)$. Such a submodule can exist only if the twists are equal.

(3) If L_ℓ and L_m have type A_1 with $(\Sigma L_\ell,\gamma) \neq 0 \neq (\Sigma L_m,\gamma)$ for some $\gamma \in \Pi(Y) - \Pi(L_Y)$ and $\dim(Q_Y/K_\gamma) = 4$, then (1.15) implies that $Q_A \leq K_\gamma$.

<u>(9.9).</u> Suppose M_i is nontrivial and L_i has type A_2. Then Y has type E_6, $\lambda|T_Y = \lambda_1$ or λ_6 and $\lambda|T_A = 2\mu_1$.

<u>Proof:</u> Since all components of L_Y' are necessarily of classical type, $Z_A \leq Z_Y$. Also, there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\Sigma L_i) \neq 0$ and $(\gamma,\Sigma L_k) = 0$ for $k\neq i$. For otherwise, Q_Y/K_γ is a 3-dimensional irreducible L_A' module, contradicting (2.4).

Consider first the case where $\Pi(L)_i = \{\beta_2,\beta_4\}$. The above remarks imply $\langle U_{\pm\beta_1}\rangle$ is a component of L_Y' and $\langle U_{\pm\beta_6}\rangle \leq L_Y$. In fact, p=3 and (1.15) imply $\langle U_{\pm\beta_6}\rangle$ is a component of L_Y'. By (2.6) and (2.7), the field twists on the embeddings of L_A' in L_i, $\langle U_{\pm\beta_1}\rangle$ and $\langle U_{\pm\beta_6}\rangle$ are all equal. Hence, (2.5) implies $\langle\lambda,\beta_1+\beta_6\rangle = 0$. If $\langle U_{\pm\beta_8}\rangle$ is a component of L_Y', (9.8) (3) implies that $\langle\lambda,\beta_7+\beta_8\rangle = 0$. Otherwise, (2.3) and (2.13) imply the same conclusion. Note that $\langle\lambda,\beta_3+\beta_5\rangle \neq 0$. For otherwise, (9.7), the work of this result so far, [8] and (1.38) imply $\dim V|A \neq \dim V|Y$. Since $\langle\lambda,\beta_3\rangle \neq 0$ or $\langle\lambda,\beta_5\rangle \neq 0$, the bound on $\dim V_{\beta_k}(Q_Y)$, for k = 3 or 5, implies $\langle\lambda,\beta_4\rangle = 1$ and $\langle\lambda,\beta_2\rangle = 0$. Moreover, (1.35) implies $\langle\lambda,\beta_k\rangle = 1$ when $\langle\lambda,\beta_k\rangle \neq 0$, for k = 3 or 5. If $\lambda|T_Y = \lambda_3+\lambda_4$ or $\lambda_4+\lambda_5$,

$\dim V|A \le 27^2$. Counting only the conjugates of $V_{T_Y}(\lambda)$ in $V|Y$, we have $\dim V|A < \dim V|Y$. A similar argument rules out $\lambda|T_Y = \lambda_3+\lambda_4+x\lambda_7$, $\lambda_4+\lambda_5+x\lambda_7$ and $\lambda_3+\lambda_4+\lambda_5+x\lambda_7$ when $x \ne 0$ and $\lambda|T_Y = \lambda_3+\lambda_4+\lambda_5$ when $Y = E_7$ or E_8. Finally, if $Y = E_6$ and $\lambda|T_Y = \lambda_3+\lambda_4+\lambda_5$, $\dim V|A \le 27^3 < \dim V|Y$, by (1.38). Hence, $\Pi(L_i) \ne \{\beta_2,\beta_4\}$. Similar arguments show that $\Pi(L_i) \ne \{\beta_6,\beta_7\}$.

Claim: If $\langle U_{\pm\beta_2}\rangle$, $\langle U_{\pm\beta_1},U_{\pm\beta_3}\rangle$ and $\langle U_{\pm\beta_5},U_{\pm\beta_6}\rangle$ are components of L_Y', then the statement of (9.9) holds.

Reason: First note that $Y \ne E_7$, else previous remarks and the bound on $\dim V_{\beta_4}(Q_Y)$ imply $\lambda|T_Y = \lambda_1$. Now $\Pi(L_i) = \{\beta_1,\beta_3\}$ or $\{\beta_5,\beta_6\}$. Suppose M_j is nontrivial for some $j \ne i$ with $(\Sigma L_j,\beta_4) \ne 0$. Then (2.5), (2.7) and the bound on $\dim V_{\beta_4}(Q_Y)$ imply $\langle\lambda,\beta_1\rangle = 1 = \langle\lambda,\beta_6\rangle$ and $\langle\lambda,\beta_k\rangle = 0$ for $2 \le k \le 5$. If $Y = E_6$, $\dim V|A \ne \dim V|Y$, by [8]. Previous remarks then imply $Y = E_8$ and $\langle U_{\pm\beta_8}\rangle$ is a component of L_Y'. By (2.7), $\langle\lambda,\beta_8\rangle = 0$. The bound on $\dim V_{\beta_7}(Q_Y)$ and (1.35) imply that if $\langle\lambda,\beta_7\rangle \ne 0$, then $\langle\lambda,\beta_7\rangle = 1$. In any case, $\dim V|A \le 27^4 < \dim V|Y$, by (1.32). Thus, $\langle\lambda,\beta_1+\beta_2+\beta_3+\beta_5+\beta_6\rangle = 1$. The bound on $\dim V_{\beta_4}(Q_Y)$ implies that $\langle\lambda,\beta_4\rangle = 0$ and $V_{L_i}(-\beta_4) \cong M_i^*$.

Suppose $Y = E_8$. If $\langle\lambda,\beta_6\rangle = 1$, $\langle U_{\pm\beta_8}\rangle$ is a component of L_Y', and by (2.7), $\langle\lambda,\beta_8\rangle = 0$. So $\dim V|A \le 27^2 < \dim V|Y$ by (1.32). Thus, $\langle\lambda,\beta_1\rangle = 1$. If $\langle\lambda,\beta_7+\beta_8\rangle \ne 0$, previous remarks, the bound on $\dim V_{\beta_7}(Q_Y)$ and (1.35) imply $\langle U_{\pm\beta_8}\rangle$ is a component of L_Y' and $\langle\lambda,\beta_8\rangle \ne 0$. If $\langle\lambda,\beta_8\rangle = 2$, $\dim V|A \le 27^4 < \dim V|Y$ by (1.38). If $\langle\lambda,\beta_8\rangle = 1$, the bound on $\dim V_{\beta_7}(Q_Y)$ implies $\langle\lambda,\beta_7\rangle = 0$, and then $\dim V|A \le 7\cdot 27^3 < \dim V|Y$, by (1.38).

Thus, $Y = E_6$, $\dim V|Y = 27$; so $\langle\lambda,\alpha\rangle = 0$. Hence by (1.10), $\lambda|T_A = 2\mu_1$ or $2q\mu_2$. If $\lambda|T_A = 2q\mu_2$, (4.1) of [12] implies that there is a closed subgroup $B < A$ of type A_2 such that $V|B$ is irreducible. Moreover, (1.10) and knowledge of the embedding of B in A implies that $q = 1$. Thus, V is a restricted kA module and hence irreducible for $L(A)$, by (1.1) But now, the proof of (11.1) in [15] shows that the span of the e_r and f_r, for short roots $r \in \Sigma(A)$, is a noncommutative ideal which lies in the kernel of the action of

L(A) on V. But the action of L(Y) on V has no such ideal in its kernel. Hence, $\lambda|T_A = 2\mu_1$, and the Claim holds.

Consider now the case where $\Pi(L_i) = \{\beta_1,\beta_3\}$. By the preceding Claim, previous remarks and (1.15), we may assume that either L_Y' has component $\langle U_{\pm\beta_j}\rangle$ and $\langle U_{\pm\beta_k}\rangle \nleq L_Y'$ for {j,k} = {2,5} or $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_2,\beta_5,\beta_6,\beta_7\}$. In either case, $\langle\lambda,\beta_2\rangle = 0$; as well, in the first case $\langle\lambda,\beta_j\rangle = 0$ and in the second case $\langle\lambda,\beta_5+\beta_6+\beta_7\rangle = 0$. (We have used (9.7) and the bound on $\dim V_{\beta_4}(Q_Y)$.) Now $Y \neq E_6$; else (1.23) and previous remarks imply that $\lambda|T_Y = \lambda_1+\lambda_6$, $\lambda_3+\lambda_5$, or $\lambda_3+\lambda_4+\lambda_5$. But by [8], $\dim V|A \neq \dim V|Y$ in the first case and in the latter cases, $\dim V|A \leq 27\cdot 7^2 < \dim V|Y$, by (1.38).

Suppose $\Pi(L_i) = \{\beta_1,\beta_3\}$ and $Y = E_7$. We first note that $\langle\lambda,\beta_5+\beta_6+\beta_7\rangle \neq 0$. For otherwise, the previous work of this result implies $\lambda|T_Y = \lambda_1$, λ_3 or $\lambda_3+\lambda_4$. But [8] rules out the first two possibilities and in the last case, $\dim V|A \leq 27^2 < \dim V|Y$ by (1.38). Now examining all possible configurations, using previous work, (2.3), (9.8) and (2.13), we see that either (a) $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_2,\beta_6\}$, or (b) $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_2,\beta_7\}$ and $\langle\lambda,\beta_7\rangle \neq 0$. Consider the action of L_A' on the 56-dimensional irreducible kY module, $V(\lambda_7)$. In case (a), if the field twist on the embedding of L_A' in $\langle U_{\pm\beta_6}\rangle$ is not equal to the twist on the embedding in $\langle U_{\pm\beta_2}\rangle$ (and L_i), there are exactly two 6-dimensional and two 4-dimensional L_A' composition factors of $V(\lambda_7)$. In case (b), the L_A' composition series of $V(\lambda_7)$ has the same properties. But there is no 56-dimensional kA module affording such an L_A' composition series. Hence, (2.5), (2.6), and (2.7) imply that L_Y' is as in (a) with $\langle\lambda,\beta_6\rangle = 0$. So $\langle\lambda,\beta_5+\beta_7\rangle \neq 0$ and $\dim V|A \leq 7\cdot 27^2$. (We have used (9.8).) But [8], (1.32) and (1.38) then imply $\dim V|A \neq \dim V|Y$.

Now suppose $\Pi(L_i) = \{\beta_1,\beta_3\}$ and $Y = E_8$. Note that if L_Y' has component $\langle U_{\pm\beta_5},U_{\pm\beta_6},U_{\pm\beta_7}\rangle$ (respectively, $\langle U_{\pm\beta_6},U_{\pm\beta_7},U_{\pm\beta_8}\rangle$, $\langle U_{\pm\beta_6},U_{\pm\beta_7}\rangle$), then $\langle\lambda,\beta_5+\beta_6+\beta_7+\beta_8\rangle = 0$, as Q_Y/K_{β_8} (respectively,

Q_Y/K_{β_5}, Q_Y/K_{β_8}) is a 3- or 4-dimensional irreducible L_A' module. If $\langle U_{\pm\beta_7}, U_{\pm\beta_8}\rangle$ is a component of L_Y', $\langle\lambda,\beta_5+\beta_6+\beta_7+\beta_8\rangle = 0$, also. For if $\langle U_{\pm\beta_5}\rangle \nleq L_Y'$, we may argue as above. Otherwise, use (2.5) – (2.7) and the bound on $\dim V_{\beta_6}(Q_Y)$. But if $\langle\lambda,\beta_5+\beta_6+\beta_7+\beta_8\rangle = 0$, $\dim V|A \le 27^2 < \dim V|Y$, by [8] and (1.32).

Now, if $\lambda|T_Y = \lambda_1+\lambda_8$ or $\lambda_3+x\lambda_4+\lambda_8$, for $x\in\{0,1\}$, $\dim V|A \le 27^2\cdot 7^2 < \dim V|Y$, by (1.38) and (1.32). If $\lambda|T_Y = \lambda_1+2\lambda_8$ or $\lambda_3+x\lambda_4 +2\lambda_8$, x as above, then $\dim V|A \le 27^4 < \dim V|Y$, by (1.38). Also, if $\lambda|T_Y = \lambda_1+z\lambda_5$ or $\lambda_3+x\lambda_4+z\lambda_5$ with $0 < z \le 2$ and x as above, $\dim V|A \le 27^2\cdot 7 < \dim V|Y$, by (1.32). Applying the restrictions imposed on $\lambda|T_Y$ by (2.3), (2.13), (9.8) and the above remarks, we find that one the following holds:

(a) $L_Y' = L_i \times \langle U_{\pm\beta_5}\rangle \times \langle U_{\pm\beta_8}\rangle$ and $\langle\lambda,\beta_6+\beta_7\rangle \ne 0$,

(b) $L_Y' = L_i \times \langle U_{\pm\beta_2}\rangle \times \langle U_{\pm\beta_6}\rangle$ and $\langle\lambda,\beta_6\rangle \ne 0 = \langle\lambda,\beta_7+\beta_8\rangle$, or

(c) $L_Y' = L_i \times \langle U_{\pm\beta_2}\rangle \times \langle U_{\pm\beta_7}\rangle$ and $\langle\lambda,\beta_7\rangle \ne 0 = \langle\lambda,\beta_5+\beta_6\rangle$.

In case (a), (9.8), (2.5) and (2.7) imply that at most one of $\langle\lambda,\beta_6\rangle$ and $\langle\lambda,\beta_8\rangle$ is nonzero. If $\langle\lambda,\beta_8\rangle \ne 0$, (so $\langle\lambda,\beta_7\rangle \ne 0$) $\dim V|A \le 27^4$. But counting only the conjugates of $V_{T_Y}(\lambda)$ and $V_{T_Y}(\lambda-\beta_7-\beta_8)$ in $V|Y$, we have $\dim V|Y > 27^4$. So $\langle\lambda,\beta_8\rangle = 0$ and $\dim V|A \le 27^3 < \dim V|Y$ by (1.32). In case (b), $\dim V|A \le 27^4 < \dim V|Y$ by (1.32). In case (c), if $\langle\lambda,\beta_7\rangle = 1$, $\dim V|A \le 27^3\cdot 7$; if $\langle\lambda,\beta_7\rangle = 2$, $\dim V|A \le 27^4$. But we need only count the conjugates of $V_{T_Y}(\lambda)$ when $\langle\lambda,\beta_7\rangle = 1$, and in addition, the conjugates of $V_{T_Y}(\lambda-\beta_7)$ when $\langle\lambda,\beta_7\rangle = 2$, to see that $\dim V|Y > \dim V|A$. Hence, we have shown that if $L_i = \langle U_{\pm\beta_1}, U_{\pm\beta_3}\rangle$, then the result holds.

Consider the case where $\Pi(L_i) = \{\beta_5,\beta_6\}$. By symmetry and previous work of this result, we may assume $Y = E_8$ and $\langle U_{\pm\beta_j}\rangle$ is a component of L_Y' with $\langle\lambda,\beta_j\rangle = 0$, for j = 2 or 3, and $\langle U_{\pm\beta_2}, U_{\pm\beta_3}\rangle \nleq L_Y'$. Also, $\langle U_{\pm\beta_8}\rangle \le L_Y'$ and $\langle\lambda,\beta_8\rangle = 0$. If $\langle U_{\pm\beta_3}\rangle$ is a component of L_Y', $\langle\lambda,\beta_2\rangle = 0$. Moreover, earlier remarks which apply to the bound on $\dim V_{\beta_k}(Q_Y)$ for k = 4 or 7, imply $\lambda|T_Y = y\lambda_1+x\lambda_4+\lambda_5$ or $y\lambda_1+\lambda_6+x\lambda_7$, where $x\in\{0,1\}$ and $0\le y\le 2$.

If $y = 0$, $\dim V|A \le 27^2$ and if $y \ne 0$, $\dim V|A \le 27^3$. But counting only the conjugates of $V_{T_Y}(\lambda)$ in $V|Y$, we have $\dim V|Y > \dim V|A$. Thus, $\langle U_{\pm\beta_2}\rangle$ is a component of L_Y'. Also, $\lambda|T_Y = y\lambda_1+z\lambda_3+x\lambda_4+\lambda_5$ or $y\lambda_1+z\lambda_3+\lambda_6+x\lambda_7$, where x and y are as above and $0 \le z \le 2$. If $y = 0 = z$, $\dim V|A \le 27^2$, if $y = 0 \ne z$, $\dim V|A \le 27^3$ and otherwise, $\dim V|A \le 27^4$. But in each case (1.32) implies that $\dim V|A < \dim V|Y$. Hence, $L_i \ne \langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$.

Finally, consider the case where $\Pi(L_i) = \{\beta_7,\beta_8\}$. Previous remarks, the bound on $\dim V_{\beta_6}(Q_Y)$ and (1.15) imply $\langle U_{\pm\beta_5}\rangle$ is a component of L_Y' with $\langle\lambda,\beta_5\rangle = 0$. We first claim that there exists $j \ne i$ with M_j nontrivial. For suppose not. Then, (a) if $\langle\lambda,\beta_k\rangle = 0$ for $1 \le k \le 4$, $\dim V|A \le 27^2$, (b) if $\langle\lambda,\beta_1+\beta_3+\beta_4\rangle > 0$ and $\langle\lambda,\beta_2\rangle = 0$ or if $\langle\lambda,\beta_2\rangle \ne 0$ and $\langle\lambda,\beta_1+\beta_3+\beta_4\rangle = 0$, $\dim V|A \le 27^3$, and (c) $\dim V|A \le 27^4$ otherwise. (We have used (9.8) and the previous work of this result.) Recall that $\lambda|T_Y \ne \lambda_8$. In (a) and (b), we need only count the conjugates of $V_{T_Y}(\lambda)$ in $V|Y$ to see that $\dim V|Y > \dim V|A$. For (c), we may assume $\langle\lambda,\beta_1+\beta_3+\beta_4\rangle > 0$ and $\langle\lambda,\beta_2\rangle > 0$. But again, as in (a) and (b) we show that $\dim V|A < \dim V|Y$. Thus, there exists a $j \ne i$ with M_j nontrivial.

The previous work of this result and (9.8) imply that either (a) $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_5,\beta_7,\beta_8\}$, (b) $\Pi(L_Y) = \{\beta_1,\beta_5,\beta_7,\beta_8\}$ or (c) $\Pi(L_Y) = \{\beta_2,\beta_3,\beta_5,\beta_7,\beta_8\}$. If (a) or (b) holds, (2.3), (2.13) and (9.8) imply that $\langle\lambda,\beta_2+\beta_4\rangle = 0$. So $\lambda|T_Y = c\lambda_1+x\lambda_3+\lambda_8$ or $c\lambda_1+x\lambda_3+y\lambda_6+\lambda_7$ for $c \in \{1,2\}$, $0 \le x \le 2$ and $y \in \{0,1\}$. Moreover, $\dim V|A \le 7\cdot 27^3$ if $c=1$ and $\dim V|A \le 27^4$ if $c=2$. In each case, (1.38) and (1.32) imply $\dim V|A < \dim V|Y$. Suppose L_Y' is as in (c). By (2.7), the set of field twists on the embeddings of L_A' in $\langle U_{\pm\beta_k}\rangle$ for $k=2,3,5$ consists of at most 2 distinct primes. Recall that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_5}\rangle$ equals that on the embedding in L_i. And since $\langle\lambda,\beta_2+\beta_3\rangle \ne 0$, (2.5) implies that there are exactly two distinct field twists on the embeddings of L_A' in the triple of A_1's, $\langle U_{\pm\beta_k}\rangle$, $k=2,3,5$, and exactly one of $\langle\lambda,\beta_2\rangle$ and $\langle\lambda,\beta_3\rangle$ is nonzero. Moreover, the bound on $\dim V_{\beta_4}(Q_Y)$ implies $\langle\lambda,\beta_4\rangle = 0$.

So in case (c), $\lambda|T_Y = x\lambda_1+c\lambda_k+\lambda_8$ or $x\lambda_1+c\lambda_k+y\lambda_6+\lambda_7$ for $k = 2$ or 3, $0\leq x\leq 2$, $c\in\{1,2\}$ and $y\in\{0,1\}$. Referring to cases (a) and (b), (9.8) and the previous work of this result, we see that $\dim V|A \leq 7^3\cdot 27^2$ if $c=1$ and $\dim V|A \leq 27^4\cdot 7$ if $c=2$. Again (1.32) and (1.38) imply $\dim V|A < \dim V|Y$. Hence, $L_i \neq \langle U_{\pm\beta_7},U_{\pm\beta_8}\rangle$.

This completes the proof of (9.9).□

<u>(9.10).</u> Let $\gamma\in\Pi(Y) - \Pi(L_Y)$ and $i\neq j$ such that $(\Sigma L_i,\gamma) \neq 0 \neq (\gamma,\Sigma L_j)$. Then M_i or M_j is trivial.

<u>Proof:</u> Suppose false; i.e., suppose M_i and M_j are both nontrivial. Then (9.7) – (9.9) imply that L_i and L_j each have type A_1 and there exists $k \neq i,j$ with $(\Sigma L_k,\gamma) \neq 0$. Moreover, (1.15), p=3 and the minimality of P_Y imply that L_k has type A_1 also. Let β_i, β_j, β_k be such that $L_\ell = \langle U_{\pm\beta_\ell}\rangle$ for $\ell = i,j,k$. So $\langle\lambda,\beta_i\rangle \neq 0 \neq \langle\lambda,\beta_j\rangle$. Let q_ℓ be the field twist on the embedding of $L_A{}'$ in L_ℓ for $\ell = i,j,k$. Then we may assume $q_k = q_j \neq q_i$. The $L_A{}'$ composition factors of Q_Y/K_γ have high weights $(2q_j+q_i)\mu_\beta$ and $q_i\mu_\beta$. Hence, $\dim(Q_AK_\gamma/K_\gamma) = 2$. Now since $V_{T_Y}(\lambda-\beta_i-\gamma) \neq 0$, a nonidentity element from the set $U_{-\gamma}\cdot U_{-\gamma-\beta_i}$ must occur in the factorization of some element of Q_AK_γ/K_γ. However, γ (respectively, $-\gamma-\beta_i$) affords $T(L_A{}')$ weight $(q_i+2q_j)\mu_\beta$ (respectively, $(2q_j-q)\mu_\beta$). While the weights in Q_AK_γ/K_γ are $q_i\mu_\beta$ and $-q_i\mu_\beta$, contradicting (2.4). This completes the proof of (9.10).□

<u>(9.11).</u> If (A,Y,V) is an example in the Main Theorem, with $(A,p) = (G_2,3)$ and Y of type E_n, then $\lambda|T_Y = \lambda_1$ or λ_6 and $\lambda|T_A = 2\mu_1$.

<u>Proof:</u> Suppose $\langle\lambda,\gamma\rangle = 0$ for some $\gamma\in\Pi(A)$. Then (4.1) of [12] implies that there exists $B < A$ of type A_2 such that $V|B$ is irreducible. Then (6.0) implies that $Y = E_6$, $\lambda|T_Y = \lambda_1$ or λ_6 and $\lambda|T_A = 2q\mu_1$ or $2q\mu_2$ for some p-power q. Hence, the hypotheses of (9.9) hold (for some choice of $\beta\in\Pi(A)$), and we have the result of (9.11). Thus, we may assume

$\langle \lambda, \gamma \rangle \neq 0$ for all $\gamma \in \Pi(A)$ and that the hypotheses of (9.9) do not hold. Thus, for any choice of $\beta \in \Pi(A)$ with P_A and P_Y as before, if M_i is nontrivial then L_i has type A_1. So we will choose $\beta \in \Pi(A)$ such that the number of p-powers in the p-adic expansion of $\langle \lambda, \beta \rangle$ is as large as possible. And we note that there must be at least two such p-powers; i.e., $V^1(Q_A)$ is tensor decomposable. Otherwise, $\langle \lambda, \beta \rangle = q$ and $\dim V|A = 49$ or 189 or $\langle \lambda, \beta \rangle = 2q$, $\langle \lambda, \beta_\ell \rangle = 2$ for some $1 \leq \ell \leq \mathrm{rank} Y$ and $\dim V|A = 189$ or 729. But in each case, [8] and (1.32) imply $\dim V|A \neq \dim V|Y$. Thus, there exists $1 \leq k \neq \ell \leq \mathrm{rank} Y$ with $(\beta_k, \beta_\ell) = 0$ and $\langle \lambda, \beta_k \rangle \neq 0 \neq \langle \lambda, \beta_\ell \rangle$.

Suppose $Y = E_6$. The opening remarks of the proof, (9.8), (9.10) and symmetry imply that either $\Pi(L_Y) = \{\beta_1, \beta_2, \beta_5, \beta_6\}$ with $\langle \lambda, \beta_5 + \beta_6 \rangle = 0$ and $\langle \lambda, \beta_1 \rangle \neq 0 \neq \langle \lambda, \beta_2 \rangle$, or L_Y' has components $\langle U_{\pm\beta_1} \rangle$ and $\langle U_{\pm\beta_6} \rangle$ with $\langle \lambda, \beta_k \rangle \neq 0$ for $k = 1$ and 6. The first configuration is ruled out by (1.23). So suppose L_Y' has components $\langle U_{\pm\beta_k} \rangle$ with $\langle \lambda, \beta_k \rangle \neq 0$ for $k = 1$ and 6. If $\langle U_{\pm\beta_2} \rangle$ is also a component of L_Y', (9.8) implies that $Q_A \leq K_{\beta_4}$; else the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_1} \rangle$, $\langle U_{\pm\beta_2} \rangle$ and $\langle U_{\pm\beta_6} \rangle$ are all equal, contradicting (2.5) and (2.6). But $-\beta_4$ is not involved in L_A', by (2.10). So there is a nontrivial image of $Q_A{}^\alpha$ in $Q_Y(\beta_3, \beta_4)$ and in $Q_Y(\beta_5, \beta_4)$ again implying that the field twists are equal. Hence, (2.7) implies $\Pi(L_Y) = \{\beta_1, \beta_6\}$. Let q_k be the field twist on the embedding of L_A' in $\langle U_{\pm\beta_k} \rangle$. Now, there are two 4-dimensional L_A' composition factors of the 27-dimensional kY- module $V(\lambda_1)$. However, there is no 27-dimensional kA module affording such an L_A' composition series. Hence $Y \neq E_6$.

Suppose $Y = E_8$. Choose i and j such that M_i and M_j are nontrivial. So L_i and L_j each have type A_1 and one of the following holds:

(a) ΠL_i and ΠL_j are separated by more than 2 nodes of the Dynkin diagram.

(b) $\{\Pi L_i, \Pi L_j\} = \{\beta_1, \beta_2\}$ and $\langle U_{\pm\beta_5}, U_{\pm\beta_6} \rangle \leq L_Y'$.

(c) $\{\Pi L_i, \Pi L_j\} = \{\beta_2, \beta_6\}$ and $\langle U_{\pm\beta_1}, U_{\pm\beta_3} \rangle \leq L_Y'$.

This follows from (9.8), (9.10) and (1.15). Rank restrictions, (9.8) and (9.10) imply that if there exists $k \neq i, j$ with M_k nontrivial, then (b) holds and $L_Y' = L_i \times L_j \times \langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle \times \langle U_{\pm\beta_8}\rangle$, with $\langle \lambda, \beta_8 \rangle \neq 0$. But (2.7) implies that the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_2}\rangle$, $\langle U_{\pm\beta_5}, U_{\pm\beta_6}\rangle$ and $\langle U_{\pm\beta_8}\rangle$ must all be equal, contradicting (2.5). Hence, $V^1(Q_Y) \cong M_i \otimes M_j$, and by the choice of β, $\dim V|A \leq 27^4$.

Now we may assume $\lambda|T_Y = c\lambda_r + d\lambda_t$ for $0 < c,d < 3$. For if W_0, the stabilizer of λ in W has rank at most 5, the number of distinct conjugates of $V_{T_Y}(\lambda)$ in $V|Y$ exceeds 27^4, unless W_0 has type D_5. But if we count also the conjugates of a maximal subdominant weight we find that $\dim V|Y > 27^4$. So $\lambda|T_Y = c\lambda_r + d\lambda_t$ and by (9.10), $(\beta_r, \beta_t) = 0$. Now, $|W:W_0| > 27^4$ unless W_0 has type $D_5 \times A_1$, D_6, $A_5 \times A_1$ or A_6. If $2 \in \{c,d\}$, we count the conjugates of $V_{T_Y}(\lambda - \beta_r)$ or $V_{T_Y}(\lambda - \beta_t)$, whichever is dominant, in addition to the conjugates of $V_{T_Y}(\lambda)$, in order to see that $\dim V|Y > \dim V|A$ in most cases. For the remaining cases, refer to (1.38) for the same conclusion.

Finally, consider the case where $Y = E_7$. Applying (9.8), (9.10) and previous work of this result, we see that L_Y' has exactly 2 nontrivially acting components, $\langle U_{\pm\beta_k}\rangle$, $\langle U_{\pm\beta_\ell}\rangle$ where $\{k,\ell\} = \{7,j\}$, j = 1, 2 or 3, $\{1,2\}$, $\{1,6\}$ or $\{2,6\}$. If $\{k,\ell\} = \{3,7\}$, (9.8) implies (i) $\Pi(L_Y) = \{\beta_3, \beta_7\}$. If $\{k,\ell\} = \{2,7\}$, (9.8) implies that either (ii) $\Pi(L_Y) = \{\beta_2, \beta_7\}$ or (iii) $\Pi(L_Y) = \{\beta_1, \beta_2, \beta_7\}$ or (iv) $\Pi(L_Y) = \{\beta_1, \beta_3, \beta_2, \beta_7\}$. If $\{k,\ell\} = \{1,2\}$, (9.8) implies (v) $\Pi(L_Y) = \{\beta_1, \beta_2, \beta_5, \beta_6\}$ or (vi) $\Pi(L_Y) = \{\beta_1, \beta_2, \beta_5, \beta_6, \beta_7\}$. If $\{k,\ell\} = \{2,6\}$, (1.15) and (9.8) imply (vii) $\Pi(L_Y) = \{\beta_1, \beta_3, \beta_2, \beta_6\}$. If $\{k,\ell\} = \{1,6\}$, previous remarks about Y of type E_6 imply (viii) $\Pi(L_Y) = \{\beta_1, \beta_6\}$. Finally, suppose $\{k,\ell\} = \{1,7\}$. Then (1.15) implies $\langle U_{\pm\beta_k}\rangle$ is not a component of L_Y', for k = 4 or 5. Also, $\langle U_{\pm\beta_4}, U_{\pm\beta_5}\rangle$ is not a component of L_Y', else (2.7) implies that the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_1}\rangle$, $\langle U_{\pm\beta_4}, U_{\pm\beta_5}\rangle$ and $\langle U_{\pm\beta_7}\rangle$ are all equal, contradicting (2.5). So (ix) $\Pi(L_Y) = \{\beta_1, \beta_2, \beta_4, \beta_5, \beta_7\}$ or (x) $\Pi(L_Y) = \{\beta_1, \beta_2, \beta_4, \beta_7\}$, or (xi) $\Pi(L_Y) = \{\beta_1, \beta_7\}$, or L_Y' is as in (iii).

Recall that the field twists on the embedding of L_A' in $\langle U_{\pm\beta_k}\rangle$ and in $\langle U_{\pm\beta_\ell}\rangle$ are not equal. In case (iii), (9.8), (2.10) and (2.11) imply that the field twists on the embeddings of L_A' in $\langle U_{\pm\beta_1}\rangle$ and in $\langle U_{\pm\beta_2}\rangle$ must be equal. In cases (iv), (v), (vii) and (x), if L_m is the component of type A_2, (2.7) implies that the field twist on the embedding of L_A' in L_m is equal to the twist on the embedding of L_A' in $\langle U_{\pm\beta_k}\rangle$, where $(\beta_k,\gamma) \neq 0 \neq (\gamma,\Sigma L_m)$, for some $\gamma \in \Pi(Y) - \Pi(L_Y)$. In case (vi) (respectively, case (ix)), if L_m is the component of type A_3 with natural module W, say $W|L_A'$ has high weight $(q_1+q_2)\mu_\beta$. By (2.7), we may assume that the field twist on the embedding of L_A' in $\langle U_{\pm\beta_2}\rangle$ (respectively, $\langle U_{\pm\beta_7}\rangle$) is q_1. Moreover, in case (ix), the field twist on the embedding of L_A' in $\langle U_{\pm\beta_1}\rangle$ is q_2, else Q_Y/K_{β_3} has no 2-dimensional L_A' composition factor.

We now consider the action of L_A' on the 56-dimensional irreducible kY module, $V(\lambda_7)$. We note that there is no 56-dimensional kA module affording an 8-dimensional L_A' composition factor, nor exactly three 6-dimensional L_A' composition factors. As well, any 56-dimensional kA module affording exactly two 4-dimensional L_A' composition factors must also afford no 6-dimensional and six 3-dimensional L_A' composition factors. One checks that these restrictions rule out all configurations except that of case (iii). In this case, $V(\lambda_7)$ has no 6-dimensional, exactly four 4-dimensional, and exactly two 3-dimensional L_A' composition factors. However again, there is no 56-dimensional kA module affording such an L_A' composition series. This completes the proof of (9.11).□

Section III: A < Y, A non-simple

In this section, we consider the case where (A,Y,V) is an example in the main theorem with A a non-simple, semisimple algebraic group and Y a simple algebraic group of exceptional type. Theorem 4.1 of [12] implies that rankA < rankY. Let $A = H_1 \circ H_2 \circ \cdots \circ H_m$ be a commuting product of simple algebraic groups H_i, with a fixed maximal torus T_A. Let $P_A = L_AQ_A$ be a parabolic subgroup of A with Levi factor $L_A = H_1 \cdot T_A$ and $R_u(P_A) = Q_A$. Adopt the remaining notation of (2.0). We first make a few general

Remarks (9.12): (1) If (A,Y,V) is as above, we may assume, after a suitable reordering that $\text{rank}(H_1) > 1$. For if $\text{rank}(H_i) = 1$ for all i, there exists $B \leq A$, B a simple algebraic group of type A_1 such that $V|B$ is irreducible. But this contradicts (7.1) of [12]. In particular, rank(Y) > rank(A) > 2. As well, since A is an actual subgroup of Y, $\dim V^1(Q_A) > 1$.

(2) Since $Z(L_A') \leq Z(A) \leq Z(Y)$, if $Z(Y) = 1$, then $Z(L_A') = 1$.

(3) Note that all L_A' composition factors of V are isomorphic to $V^1(Q_A)$. So if μ is a weight in $V|Y$ which affords the high weight of an L_Y' (and hence of an L_A') composition factor, then $\mu|T(L_A') = \lambda|T(L_A')$. In particular, for $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma, \Sigma L_Y) \neq 0$ and $\gamma|T(L_A') \neq 0$, $\langle \lambda, \gamma \rangle = 0$. Otherwise, $\mu = \lambda - \gamma$ fails to satisfy the above condition.

(4) Given $\alpha \in \Pi(A) - \Pi(L_A)$, Q_A/K_α is a 1-dimensional, irreducible L_A' module. Assume $Z_A \leq Z_Y$ (as will be the case under the hypotheses of (1.5)). Let $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $V_\gamma(Q_Y) \neq 0$. Then the proof of (3.3)(ii) in [12] implies that there exists $\alpha_0 \in \Pi(A) - \Pi(L_A)$ such that $U_{-\alpha_0} \nleq K_\gamma$. Hence, Q_Y/K_γ has a 1-dimensional L_A' composition factor. Also, if $U_{-\alpha_0} \nleq K_\gamma$, $U_{-\alpha_0} \leq \langle U_{-r} \leq Q_Y \mid r|T(L_A') = 0 \rangle K_\gamma$. See (2.4).

(9.13): Y has type E_n for some n.

Proof: Suppose false. Then (9.12) (1) implies that $Y = F_4$, rank(A) = 3 and $\text{rank}(L_A') = 2$. As well, (1.5) implies $Z_A \leq Z_Y$. If L_A' has type B_2, the Main Theorem of [12] implies that $L_Y' = \langle U_{\pm\beta_2}, U_{\pm\beta_3} \rangle$. Since Q_Y/K_{β_4} is an irreducible L_A' module, (9.12) (4) implies that $V_{\beta_4}(Q_Y) = 0$. Hence, p = 2. By (1.7) we may assume that $V|Y$ is either a basic or p-basic module. So [8] implies that $\dim V|A = 26, 246$ or 4096. As well, $\dim V^1(Q_A) | \dim V|A$. So $\dim V|A = 4096$, $\lambda|T_Y = \lambda_1 + \lambda_2$ or $\lambda_3 + \lambda_4$ and

$\dim V^1(Q_A) = 4$. But $A = B_2 \times A_1$, so by induction, $\dim V|A \le 32 < \dim V|Y$. Thus, $L_A' \neq B_2$.

Suppose L_A' has type A_2. The Main Theorem of [12] implies $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_3\}$ and $p = 3$, or L_Y' has type A_2. If L_Y' has type A_2, (9.12)(4) implies that $p = 2$ and $L_Y' = \langle U_{\pm\beta_1},U_{\pm\beta_2}\rangle$. As well, we may assume that $V|Y$ is a p-basic module, so $\lambda|T_A = \lambda_1$, λ_2 or $\lambda_1+\lambda_2$, $\dim V|Y = 26$, 246 or 4096, respectively, and $\dim V^1(Q_A) = 3$,3 or 8, respectively. Since $\dim V^1(Q_A) | \dim V|A$, $\lambda|T_Y = \lambda_2$ or $\lambda_1+\lambda_2$. But by induction, $\dim V|A \le 64 < \dim V|Y$. So if L_A' has type A_2, $\Pi(L_Y) = \{\beta_1,\beta_2,\beta_3\}$ and the Main Theorem of [12] implies $\lambda|T_Y = \lambda_1+x\lambda_4$ or $2\lambda_1+x\lambda_4$. By (9.12) (3), $x = 0$ in either case. (It is necessary to compute the embedding of $T(L_A')$ in $T(L_Y')$ to see this.) Since $\lambda|T_Y \neq \lambda_1$, we have $\lambda|T_Y = 2\lambda_1$. However, $\mu = \lambda-\beta_1-\beta_2-\beta_3-\beta_4$ contradicts (9.12) (3). Thus, L_A' does not have type A_2.

It remains to consider the case where $L_A' = G_2$. By the Main Theorem of [12], L_Y' has type B_3 or $p = 2$ and L_Y' has type C_3. If $p = 2$, $\dim V|A = 6^k \cdot 14^\ell \cdot 64^m \cdot 2^n$, for $k, \ell, m, n \in \mathbb{Z}^+$. Since we may assume $V|Y$ is tensor indecomposable, $V|Y$ is either basic or p-basic. (See (1.7).) Thus, [8] implies $\lambda|T_Y = \lambda_1+\lambda_2$ or $\lambda_3+\lambda_4$ and $\dim V|Y = 2^{12}$. By induction, $\lambda|T_Y = \lambda_3+\lambda_4$ and $\dim V^1(Q_A) = 64$. If $P_0 \ge B_A^-$ is the parabolic of A with Levi factor $H_2 \cdot T_A$, H_2 of type A_1, then $\dim V^1(R_u(P_0)) = 64$. But this contradicts (1.19) and (7.1) of [12]. Thus, if $L_A' = G_2$, L_Y' has type B_3 and $p \neq 2$. One checks that if $\Pi(L_A) = \{\alpha_1,\alpha_2\}$, with α_1 short, then $h_{\alpha_1}(c) = h_{\beta_1}(c^q) \cdot h_{\beta_3}(c^q)$ and $h_{\alpha_2}(c) = h_{\beta_2}(c^q)$, where q is the field twist on the embedding of L_A' in L_Y'. So $\beta_4|T(L_A') \neq 0$ and by (9.12) (3), $\langle\lambda,\beta_4\rangle = 0$. For $\lambda|T_Y = k\lambda_1$, let $\mu = \lambda=\beta_1-\beta_2-2\beta_3-2\beta_4$, for $\lambda|T_Y = x\lambda_2+y\lambda_3$, let $\mu = \lambda-\beta_3-\beta_4$ and for $\lambda|T_Y = y\lambda_1+x\lambda_2$, let $\mu = \lambda-\beta_2-\beta_3-\beta_4$. In each case, μ contradicts (9.12) (3). But by the Main Theorem of [12], these are the only possible configurations.

This completes the proof of (9.13).□

(9.14). If Y has type E_n, then L_Y' is not a quasisimple algebraic group.

Proof: Suppose false. We first note that $L_A' \not\cong L_Y'$, otherwise there exists some $\gamma \in \Pi(Y) - \Pi(L_Y)$ which contradicts (9.12) (4). Suppose L_Y' has type A_k for some k. Then by (1.5), $Z_A \leq Z_Y$. Since L_A' acts irreducibly on W, the natural module for L_Y', (9.12) (4) implies that there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $Q_Y/K_\gamma \cong W$ or W^*. Hence, $k \geq 4$. In fact, $k > 4$, else $p > 2$, $L_A' = B_2$ and there exists $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $V_\gamma(Q_Y) \neq 0$ and Q_Y/K_γ a nontrivial irreducible L_A' module ($\cong W \wedge W$ or $W^* \wedge W^*$), contradicting (9.12) (4).

Consider the case where $L_Y' = A_5$. By induction, L_A' has type A_2, A_3, C_3, or $p = 2$ and L_A' has type G_2 or B_3. Previous remarks imply that either $Y = E_6$ or $Y = E_7$ and $\Pi(L_Y) = \{\beta_k \mid k \neq 1,3\}$. If $Y = E_6$, $V_{\beta_2}(Q_Y) \neq 0$ and $Q_Y/K_{\beta_2} \cong W \wedge W \wedge W$. But there is no 1-dimensional L_A' composition factor of $W \wedge W \wedge W$, contradicting (9.12)(4). Hence, the second configuration holds. Then $V_{\beta_3}(Q_Y) \neq 0$ and $Q_Y/K_{\beta_3} \cong W \wedge W$ or $W^* \wedge W^*$, which has a 1-dimensional L_A' composition factor only if L_A' has type C_3 or $p = 2$ and L_A' has type G_2, A_3 or B_3. Also, one checks that $\beta_3 | T(L_A') \neq 0$, so (9.12) (3) implies $\langle \lambda, \beta_3 \rangle = 0$. In fact, $\langle \lambda, \beta_1 \rangle = 0$, as well. (Consider the L_A' composition factor afforded by $\lambda - \beta_1 - \beta_3$.) In the cases where $p = 2$, $\lambda | T_Y = \lambda_7$, else $\mu = \lambda - \beta_2 - \beta_3 - \beta_4$ contradicts (9.12) (3). But $6 = \dim V^1(Q_A) \nmid \dim V = 56$. Contradiction. Hence, L_A' has type C_3. Examining the $T(L_A')$ weight vector decomposition of Q_Y/K_{β_3}, we find that for $\alpha \in \Pi(A) - \Pi(L_A)$ such that $U_{-\alpha} \not\leq K_{\beta_3}$, $U_{-\alpha} \leq (U_{-34567} \cdot U_{-23456} \cdot U_{(0,1,1,2,1,0,0)})K_{\beta_3}$. This fact, together with the equality $[V, Q_A] = [V, Q_Y]$, restricts the possible weights in $[V, Q_Y]$, and hence the labellings of $V^1(Q_Y)$. In fact, referring to the Main Theorem of [12], we find that $\langle \lambda, \beta_m \rangle = 0$ for $m = 2,4,5,6$ and $\langle \lambda, \beta_7 \rangle = c$. So $\lambda | T_Y = c\lambda_7$. But then $\mu = \lambda - \beta_1 - \beta_2 - 2\beta_3 - 2\beta_4 - \beta_5 - \beta_6 - \beta_7$ contradicts (9.12) (3). So L_Y' does not have type A_5.

Consider now the case where $L_Y{}' = A_6$; so $L_A{}'$ has type G_2 or B_3 or $p = 3$ and $L_A{}'$ has type A_2. If $\Pi(L_Y) = \{\beta_j \mid j = 1, 3 \le j \le 7\}$, so $Y = E_7$, then $\beta_2|T(L_A) \ne 0$. So $\langle\lambda,\beta_2\rangle = 0$. Also, Q_Y/K_{β_2} has a 1-dimensional $L_A{}'$ composition factor only if $L_A{}'$ has type G_2 or A_2. In each of these cases $V^1(Q_A) \cong W$ or W^*. Now, $\lambda|T_Y \ne \lambda_1$, so $\lambda|T_Y = \lambda_7$. However, $\mu = \lambda-\beta_2-\beta_4-\beta_5-\beta_6-\beta_7$ contradicts (9.12) (3). Thus, if $L_Y{}' = A_6$, then $Y = E_8$ and $\Pi(L_Y) = \{\beta_j \mid j \ne 1,3\}$. One checks that $W \wedge W$ has a 1-dimensional $L_A{}'$ composition factor only if $L_A{}' = A_3$. So $V^1(Q_A) \cong W$ or W^*. Now, $\langle\lambda,\beta_1+\beta_3\rangle = 0$ as $\beta_3|T(L_A{}') \ne 0$. So $\lambda|T_Y = \lambda_2$. But $\mu = \lambda-\beta_2-\beta_3-\beta_4$ contradicts (9.12) (3).

Thus, for $L_Y{}'$ of type A_k, we have reduced to $L_Y{}'$ of type A_7 in E_8. So $L_A{}'$ has type A_2, B_3, C_4 or D_4 or $p = 2$ and $L_A{}'$ has type C_3 or B_4. If $L_A{}'$ has type C_4, B_4 or D_4, Q_Y/K_{β_2} has no 1-dimensional $L_A{}'$ composition factor. In the remaining cases, $V^1(Q_Y) \cong W$ or W^*. Now, $\beta_2|T(L_A{}') \ne 0$, so by (9.12) (3), $\langle\lambda,\beta_2\rangle = 0$ and $\lambda|T_Y = \lambda_1$. But $8 = \dim V^1(Q_A) \mid \dim V|A$, contradicting [8]. Hence, $L_Y{}'$ does not have type A_k.

Suppose $L_Y{}'$ has type D_k, for some $k \ge 4$. Again (1.5) implies $Z_A \le Z_Y$. Now $L_A{}'$ must act reducibly on the 2 fundamental spin modules for $L_Y{}'$, as there exists $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $V_\gamma(Q_Y) \ne 0$ and Q_Y/K_γ isomorphic to one of these. (See (9.12)(4).) The Main Theorem of [12] then implies that $L_A{}'$ must act irreducibly on W, the natural module for $L_Y{}'$. Hence, there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $Q_Y/K_\gamma \cong W$. Thus, the triple $(L_A{}',L_Y{}',p)$ is one of (A_2,D_4,p), $(B_2,D_5,5)$, $(B_2,D_7,3)$, $(C_3,D_7,3)$, or $(C_3,D_7,7)$. In the first case, $L_A{}'$ acts irreducibly on all three of the fundamental 8-dimensional irreducible $L_Y{}'$ modules, so there is no 1-dimensional $L_A{}'$ composition factor of Q_Y/K_{β_1}. If $L_Y{}' = D_7$, (9.12) (3) implies $\langle\lambda,\beta_1\rangle = 0$, but then $\lambda|T_Y = \lambda_8$. Thus, $(L_A{}',L_Y{}',p)$ has type $(B_2,D_5,5)$. Also, there does not exist $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $Q_Y/K_\gamma \cong W$, so we may assume $\Pi(L_Y) = \{\beta_j \mid 1 \le j \le 5\}$. However, there is no 1-dimensional composition factor of Q_Y/K_{β_6}.

Hence, if $L_Y{}'$ is quasisimple, $L_Y{}'$ has type E_m for some m. By induction and the previous work of this paper, the pair $(L_A{}', L_Y{}')$ is one of the following: (A_2,E_6), (G_2,E_6), (C_4,E_6) or (F_4,E_6). In the first three cases, $L_A{}'$ acts irreducibly on Q_Y/K_{β_7}; so Z_A induces scalars on Q_Y/K_{β_7}. But this forces $Q_AK_{\beta_7}/K_{\beta_7}$ to be an $L_A{}'$ submodule of Q_Y/K_{β_7}, contradicting (9.12) (4). In the last case, (9.12) (3) implies $\langle\lambda,\beta_7\rangle = 0$. It is now a check to see that in every configuration afforded by induction, there exists a weight μ (of Q_Y level 1) which contradicts (9.12) (3).□

<u>(9.15).</u> There are no examples (A,Y,V) in the main theorem with A non-simple and Y of type E_n.

<u>Proof:</u> Suppose false. Let P_A, P_Y be as before. Then (9.14) implies that $L_Y{}'$ has more than one component. In particular, $\mathrm{rank}(L_A{}') \le 3$. Also rank $L_A{}' > 1$ and (1.5) imply $Z_A \le Z_Y$. If $\mathrm{rank}(L_A{}') = 3$, rank restrictions imply $L_A{}' = A_3$. Since A_3 has no 5-dimensional irreducible representation, $L_Y{}'$ has type $A_3 \times A_3$ in E_8. Now, Q_Y/K_{β_5} has a trivial $L_A{}'$ composition factor only if $\Pi(L_Y) = \{\beta_j \mid j \ne 2,5\}$. Moreover, if $L_A{}' = \langle U_{\pm\alpha_i} \mid 1 \le i \le 3\rangle$, labelled as throughout, we may assume $h_{\alpha_1}(c) = h_{\beta_1}(c^q)h_{\beta_6}(c^q)$, $h_{\alpha_2}(c) = h_{\beta_3}(c^q)h_{\beta_7}(c^q)$ and $h_{\alpha_3}(c) = h_{\beta_4}(c^q)h_{\beta_8}(c^q)$, for some p-power q. Also, since Q_Y/K_{β_2} is a 4-dimensional irreducible $L_A{}'$ module, (9.12) (4) implies $\langle\lambda,\beta_i\rangle = 0$ for $1 \le i \le 4$. The $T(L_A{}')$ 0-weight space of Q_Y/K_{β_5} is spanned by the root groups U_{-1345}, U_{-5678}, U_{-4567} and U_{-3456}. Hence, if $\alpha \in \Pi(A) - \Pi(L_A)$ with $U_{-\alpha} \nleq K_{\beta_5}$,
$U_{-\alpha} \le \langle U_{-1345} \cdot U_{-5678} \cdot U_{-4567} \cdot U_{-3456}\rangle K_{\beta_5}$. This restricts the possible T_Y weights in $[V,Q_Y] = [V,Q_A]$. In particular, $\langle\lambda,\beta_5+\beta_6+\beta_7\rangle = 0$, so $\lambda|T_Y = x\lambda_8$, for some $x > 1$. But this contradicts (9.12) (2). Thus, $\mathrm{rank}(L_A{}') < 3$. Note that rank restrictions imply $L_A{}' \ne G_2$.

Suppose $L_A{}' = B_2$. Then $L_Y{}'$ has a component of type A_3 and (9.12) (2) implies $p = 2$. Since $L_A{}'$ has no 5-dimensional irreducible representation, $L_Y{}'$ has type $A_3 \times A_3$. As in the previous case, we reduce to

$\Pi(L_Y) = \{\beta_j \mid j \neq 2,5\}$, and $\langle\lambda,\beta_k\rangle = 0$ for $1 \le k \le 4$. Also, $\beta_5|T(L_A') \neq 0$, so $\langle\lambda,\beta_5\rangle = 0$. The Main Theorem of [12] implies $\lambda|T_Y = \lambda_6$. But then $\mu = \lambda-\beta_5-\beta_6$ contradicts (9.12) (3). Thus, $L_A' \neq B_2$.

It remains to consider the case where $L_A' = A_2$. The minimality of P_Y, the Main Theorem of [12] and rank restrictions imply that $L_Y' = L_1 \times L_2$, where L_i has type A_2 or D_4. Suppose $L_i = A_2$ for $i = 1,2$. If there exists $\gamma \in \Pi(Y) - \Pi(L_Y)$ with $(\gamma,\Sigma L_j) \neq 0$ and $(\gamma,\Sigma L_m) = 0$ for $m \neq j$, then Q_Y/K_γ is a 3-dimensional irreducible L_A' module. So (9.12) (4) implies $V_\gamma(Q_Y) = 0$. Thus, one of the following holds:

(a) $\Pi(L_Y) = \{\beta_1,\beta_3,\beta_5,\beta_6\}$ and $\langle\lambda,\beta_5+\beta_6+\beta_7\rangle = 0$ if $Y = E_7$ or E_8.

(b) $\Pi(L_Y) = \{\beta_2,\beta_4,\beta_6,\beta_7\}$ in Y of type E_7 and $\langle\lambda,\beta_k\rangle = 0$ for $k = 2,3,4$.

(c) $\Pi(L_Y) = \{\beta_3,\beta_4\beta_6,\beta_7\}$ in Y of type E_7 and $\langle\lambda,\beta_k\rangle = 0$ for $1 \le k \le 4$.

(d) $\Pi(L_Y) = \{\beta_4,\beta_5,\beta_7,\beta_8\}$ and $\langle\lambda,\beta_k\rangle = 0$ for $2 \le k \le 5$.

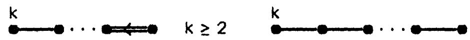
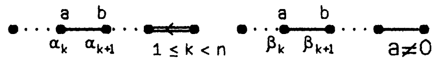
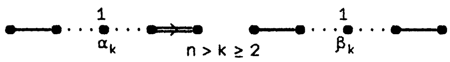
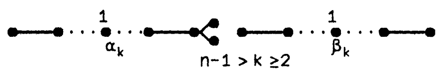
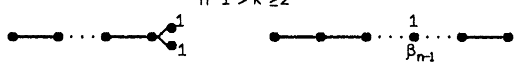
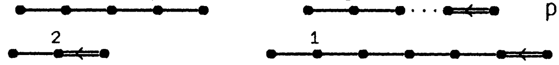
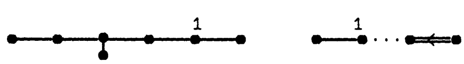
Now for $\gamma \in \Pi(Y) - \Pi(L_Y)$ such that $(\gamma,\Sigma L_1) \neq 0 \neq (\gamma,\Sigma L_2)$, $\gamma|T(L_A) \neq 0$, so (9.12) (3) implies $\langle\lambda,\gamma\rangle = 0$. In fact, (9.12) (3) implies $\langle\lambda,\beta_2+\beta_8\rangle = 0$, in case (a) and $\langle\lambda,\beta_1\rangle = 0$ in cases (b) and (d).

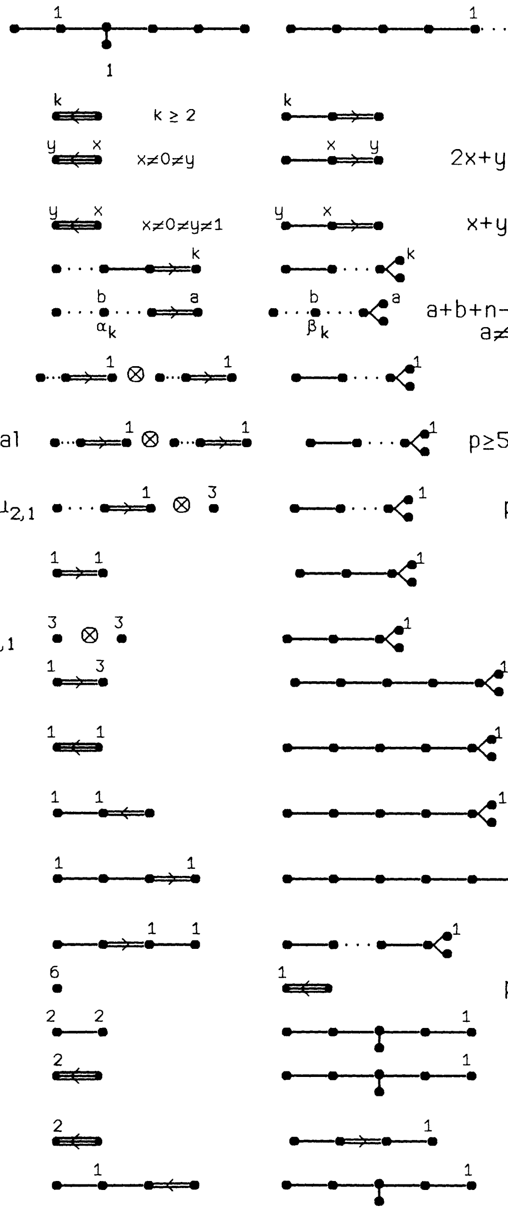
Temporarily label as follows: $\Pi(L_Y) = \{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$, $\gamma \in \Pi(Y) - \Pi(L_Y)$ and $\Pi(L_A) = \{\alpha_1,\alpha_2\}$, where $\Pi(L_1) = \{\gamma_1,\gamma_2\}$, $\Pi(L_2) = \{\gamma_3,\gamma_4\}$ and $(\gamma_2,\gamma) \neq 0 \neq (\gamma,\gamma_3)$. Since Q_Y/K_γ must have a 1-dimensional L_A' composition factor, the field twists on the embeddings of L_A' in L_1 and L_2 must be equal, so $V^1(Q_Y)$ is tensor indecomposable. As well, we may assume that $h_{\alpha_1}(c) = h_{\gamma_1}(c^q)h_{\gamma_3}(c^q)$ and $h_{\alpha_2}(c) = h_{\gamma_2}(c^q)h_{\gamma_4}(c^q)$, for some p-power q. Then, the $T(L_A')$ 0 - weight space in Q_Y/K_γ is spanned by $U_{-\gamma_1-\gamma_2-\gamma}$, $U_{-\gamma_2-\gamma-\gamma_3}$ and $U_{-\gamma-\gamma_3-\gamma_4}$. This restricts the possible T_Y weights in $[V,Q_Y] = [V,Q_A]$. For instance, suppose $\langle\lambda,\gamma_3+\gamma_4\rangle = 0$. Then, the factorization of elements in Q_A implies that $\langle\lambda,\gamma_2\rangle = 0$, so $\langle\lambda,\gamma_1\rangle \neq 0$. Similarly, if $\langle\lambda,\gamma_1+\gamma_2\rangle = 0$, then $\langle\lambda,\gamma_3\rangle = 0$. In case (a) (respectively, (b), (c), (d)),

$\mu = \lambda-\beta_1-\beta_2-\beta_3-2\beta_4-\beta_5$ (respectively, $\mu = \lambda-\beta_3-\beta_4-\beta_5-\beta_6-\beta_7$, $\mu = \lambda-\beta_2-\beta_4-\beta_5-\beta_6-\beta_7$, $\mu = \lambda-\beta_2-\beta_4-\beta_5-\beta_6-\beta_7-\beta_8$) contradicts (9.12) (3).

Thus, it remains to consider the case where $L_A' = A_2$ and $L_Y' = L_1 \times L_2$ with L_1 of type D_4 and L_2 of type A_2; so $Y = E_8$. One checks that L_A' acts on each of the three fundamental 8-dimensional representations of L_1 with composition factors of dimensions 1 and 7, when $p = 3$, and otherwise L_A' acts irreducibly. It is then easy to see that there is no 1-dimensional L_A' composition factor of Q_Y/K_{β_6}, contradicting (9.12) (4). This completes the proof of (9.15).□

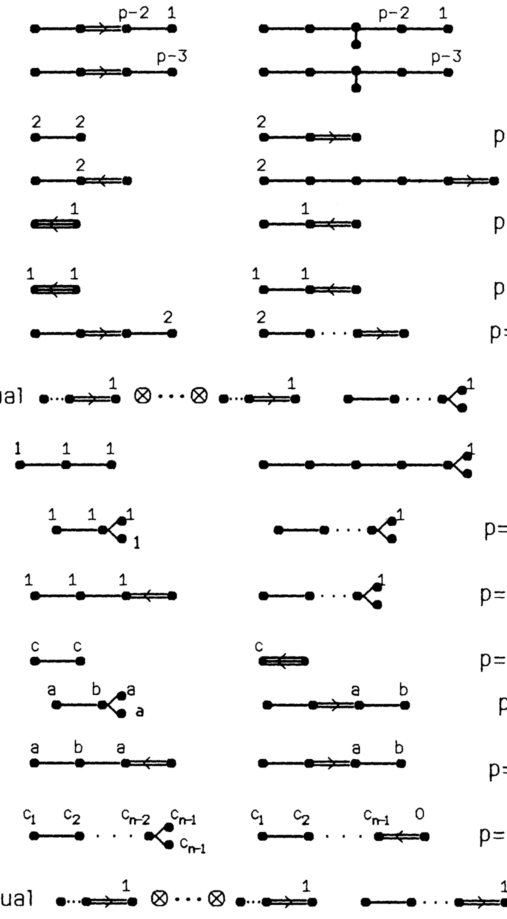
TABLE 1

no.	A < Y	W|A	V|A	V|Y	p
I_1	$C_n < A_{2n-1}$, $n \geq 2$	μ_1			$k > 1$
I_1'	$C_n < A_{2n-1}$, $n \geq 2$	μ_1			$a+b = p-1 > 1$, $a \neq 0$ if $k=n-1$
I_2	$B_n < A_{2n}$, $n \geq 3$	μ_1			$p \neq 2$
I_3	$B_n < A_{2n}$, $n \geq 2$	μ_1			$p \neq 2$
I_4	$D_n < A_{2n-1}$, $n \geq 4$	μ_1			$p \neq 2$
I_5	$D_n < A_{2n-1}$, $n \geq 4$	μ_1			$p \neq 2$
I_6	$A_n < A_{n^2+n-2/2}$, $n \geq 3$	μ_2			$p \neq 2$
I_7	$A_n < A_{n^2+3n/2}$, $n \geq 2$	$2\mu_1$			$p \neq 2$
I_8	$D_5 < A_{15}$	μ_5			$p \neq 2$
I_9	$D_5 < A_{15}$	μ_5			$p \neq 2,3$
I_{10}	$E_6 < A_{26}$	μ_1			$p \neq 2$
I_{11}	$E_6 < A_{26}$	μ_1			$p \neq 2,3$
I_{12}	$E_6 < A_{26}$	μ_1			$p \neq 2,3$
II_1	$A_5 < C_{10}$	μ_3			$p \neq 2$
II_2	$C_3 < C_7$	μ_3			$p \neq 2,7$
II_3	$C_3 < C_7$	μ_3			$p \neq 2,3$
II_4	$D_6 < C_{16}$	μ_6			$p \neq 2$
II_5	$D_6 < C_{16}$	μ_6			$p \neq 2,3$
II_6	$E_7 < C_{28}$	μ_7			$p \neq 2$
II_7	$E_7 < C_{28}$	μ_7			$p \neq 2,3$
II_8	$E_7 < C_{28}$	μ_7			$p \neq 2,3$



II_9	$E_7 < C_{28}$	μ_7		$p \neq 2,3,5$
III_1	$G_2 < B_3$	μ_1	$k \geq 2$	$p \neq 2$
III_1'	$G_2 < B_3$	μ_1	$x \neq 0 \neq y$	$2x+y+2 \equiv 0\ (p)$
III_1''	$G_2 < B_3$	μ_1	$x \neq 0 \neq y \neq 1$	$x+y+1 \equiv 0\ (p)$
IV_1	$B_n < D_{n+1}$, $n \geq 3$	usual		
IV_1'	$B_n < D_{n+1}$, $n \geq 3$	usual		$a+b+n-k \equiv 0\ (p)$, $a \neq 0 \neq b$
IV_2	$B_{n-k} \cdot B_k < D_{n+1}$, $n \geq 2$	usual		
IV_2'	$X \to' B_{n-k} \cdot B_k < D_{n+1}$	usual		$p \geq 5$ if $\pi_i(X) = A_1$
IV_3	$B_n \cdot A_1 < D_{n+3}$, $n \geq 2$	$\mu_{1,1} \oplus 4\mu_{2,1}$		$p \geq 5$
IV_4	$B_2 < D_5$	$2\mu_2$		$p \neq 2,5$
IV_5	$A_1 \cdot A_1 < D_5$	$4\mu_{1,1} \oplus 4\mu_{2,1}$		$p \geq 5$
IV_6	$B_2 < D_7$	$2\mu_1$		$p > 7$
IV_7	$G_2 < D_7$	μ_2		$p \neq 3,7$
IV_8	$C_3 < D_7$	μ_2		$p \neq 3,7$
IV_9	$B_4 < D_8$	μ_4		$p \neq 3$
IV_{10}	$F_4 < D_{13}$	μ_4		$p \neq 3,7,13$
V_1	$A_1 < G_2$	$6\mu_1$		$p \geq 7$
VI_1	$A_2 < E_6$	$2\mu_1 + 2\mu_2$		$p \neq 2,5$
VI_2	$G_2 < E_6$	$2\mu_1$		$p \neq 2,7$
VI_2'	$G_2 < F_4$	$2\mu_1$		$p = 7$
VI_3	$C_4 < E_6$	μ_2		$p \neq 2$

T_1	$F_4 < E_6$	$\mu_4 \oplus 0$	$p \neq 2$
T_2	$F_4 < E_6$	$\mu_4 \oplus 0$	$p \neq 2,3$
S_1	$A_2 < B_3$	$\mu_1 + \mu_2$	$p=3$
S_2	$C_3 < B_6$	μ_2	$p=3$
S_3	$G_2 < C_3$	μ_1	$p=2$
S_4	$G_2 < C_3$	μ_1	$p=2$
S_5	$F_4 < B_{12}$	μ_4	$p=3$
S_6	$X \to' B_{n_1} \circ \dots \circ B_{n_k} < D_{n_1+\dots+n_k+1}$	usual	$p=2$
S_7	$A_3 < D_7$	$\mu_1 + \mu_3$	$p=2$
S_8	$D_4 < D_{13}$	μ_2	$p=2$
S_9	$C_4 < D_{13}$	μ_2	$p=2$
MR_1	$A_2 < G_2$	$\mu_1 + \mu_2$	$p=3$
MR_2	$D_4 < F_4$	μ_2	$p=2$
MR_3	$C_4 < F_4$	μ_2	$p=2$
MR_4	$D_n < C_n$	μ_1	$p=2$
MR_5	$X \to' B_{n_1} \circ \dots \circ B_{n_k} < B_{n_1+\dots+n_k}$	usual	$p=2$



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