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Some Results on the Fourier-Stieltjes Algebra of a Locally Compact Group

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1. Introduction

This paper is concerned with the study of the set $B(G)$ of all finite complex linear combinations of continuous positive definite functions over an arbitrary locally compact group G. We prove (corollary of theorem 6) that on the boundary of the unit sphere of $B(G)$ ([5]), the compact-open topology and the weak topology $\sigma(B(G), L^1(G))$ coincide (we consider $L^1(G)$ with left Haar measure dx). This extends the following theorem due to D. A. Raikov ([2] p. 260–261): the above topologies coincide on the set of all continuous positive definite functions with value 1 at the unit element e of G. We need, for this purpose, some general results on C^*-algebras which are developed in section 2.

As an application we give a new proof in section 4 of a result of H. Leptin ([6]) concerning the existence of an approximate identity in the Fourier algebra $A(G)$ of G ([5]). More precisely, we obtain directly the existence of an approximate identity of the following type: for every $u \in A(G)$ and $\varepsilon > 0$ there exists $v \in A(G)$ with $\|v\| = 1$ such that $\|u - uv\| < \varepsilon$ implies that the group G has property (R). We recall that a locally compact group G is said to have property (R) if the constant 1 on G can be approximated uniformly on compact sets by functions of the form $x \mapsto \int_G k(xy)\overline{k(y)}\,dy$, where k is a continuous complex-valued function on G with compact support.

Finally in section 5 we extend the following multiplier theorem to every locally compact group G, satisfying property (R): if $u \in \mathbf{C}^G$ and if $uf \in B(G)$ for every $f \in A(G)$, then $u \in B(G)$. Moreover, Prof. P. Eymard told me that, under the same assumptions, we have

$$\|u\| = \operatorname{Sup}\{\|uf\| \mid f \in A(G), \|f\| \leq 1\}.$$

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2. Some Remarks on the Polar Decomposition of Linear Continuous Functionals on a C*-Algebra

Let A be a C^*-algebra, A' the set of all linear continuous functionals on A. For

$f \in A'$ define

$$\|f\| = \operatorname{Sup}\{|f(x)| \mid \|x\| \leq 1,\ x \in A\}.$$

By ([2] p. 243) there is only one positive linear functional $|f|$ on A, such that $\|f\| = \||f|\|$, and $|f(x)|^2 \leqq \|f\| \cdot |f|(xx^*)$. Consider on A' the locally convex topology τ defined by the following systems of sets:

$$U_\tau(f_0; \varepsilon, x_1, \ldots, x_n) = \{f \in A' \mid |f(x_j) - f_0(x_j)| < \varepsilon \quad 1 \leq j \leq n, \left|\|f\| - \|f_0\|\right| < \varepsilon\}$$

where $f_0 \in A'$, $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset A$.

PROPOSITION 1. *The map $f \mapsto |f|$ is a continuous map of (A', τ) in $(A', \sigma(A', A))$.*

This proposition was proved in ([3] Lemma 3.5).

Remark. The result remains valid if for the definition of the topology τ one takes the $\{x_1, \ldots, x_n\}$ in a arbitrary dense subspace of A.

PROPOSITION 2. *To every $f \in A'$ there exists a *-representation of A (π, H) with cyclic vector ξ such that $|f|(x) = (\pi(x)\xi, \xi)$. There also exists a unique $\zeta \in H$ with $f(x) = (\pi(x)\zeta, \xi)$. We have*

$$\|f\|^{1/2} = \|\zeta\| = \|\xi\|.$$

Proof. On the dense subspace $\pi(A)\xi$ consider the linear functional defined by

$$F(\pi(x^*)\xi) = \overline{f(x)}.$$

The inequality

$$|f(x)|^2 \leqq \|f\| \ \|\pi(x^*)\xi\|^2$$

implies the continuity of F and therefore the existence and unicity of $\zeta \in H$ with

$$\overline{f(x)} = (\pi(x^*)\xi, \zeta)$$

i.e.

$$f(x) = (\pi(x)\zeta, \xi).$$

From

$$\sup_{\|\pi(x^*)\xi\| \leqq 1} |(\pi(x^*)\xi, \zeta)| = \|\zeta\|$$

there follows $\|\zeta\| \leqq \|f\|^{1/2}$. On the other hand, the inequalities

$$|f(x)| \leqq \|\pi(x^*)\xi\| \ \|\zeta\| \leqq \|x\| \ \|\xi\| \ \|\zeta\|$$

imply

$$|f(x)| \leqq \|x\| \; \|f\|^{1/2} \; \|\zeta\|,$$

that is $\|f\|^{1/2} \leqq \|\zeta\|$, and finally $\|f\|^{1/2} = \|\zeta\|$.

Remark. Suppose that (π', H') is a *-representation of A with cyclic vector ξ', such that $|f|(x) = (\pi'(x)\xi', \xi')$. Let be ζ' the corresponding vector of H' with $f(x) = (\pi'(x)\zeta', \xi')$. It is well known that there exists only one linear isometry ψ of H onto H' with $\psi\pi(x)\,\xi = \pi'(x)\,\xi'$. It is easy to verify that $\psi(\zeta) = \zeta'$.

PROPOSITION 3. *For every $f \in A'$, and $x, y \in A$, we have*

$$|f(x) - f(y)|^2 \leqq \|f\| \; [\|f\| \, (\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} |f| \, (yx^*)].$$

Proof. Consider π, H, ξ and ζ as in proposition 2. We have

$$f(x) = (\zeta, \pi(x^*)\,\xi);$$

that is

$$|f(x-y)|^2 = |(\zeta, \pi(x^* - y^*)\,\xi)|^2 \leqq \|\zeta\|^2 \; \|\pi(x^* - y^*)\,\xi\|^2.$$

From

$$\|\pi(x^* - y^*)\,\xi\|^2 = \|\pi(x^*)\,\xi\|^2 + \|\pi(y^*)\,\xi\|^2 - 2 \operatorname{Re}(\pi(x^*)\,\xi, \pi(y^*)\,\xi)$$

and

$$(\pi(x^*)\,\xi, \pi(y^*)\,\xi) = |f|(yx^*)$$

we obtain

$$\|\pi(x^* - y^*)\,\xi\|^2 \leqq \|\xi\|^2 (\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} |f| (yx^*)$$

and therefore

$$|f(x-y)|^2 \leqq \|f\| \, [\|f\| \, (\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} |f| (yx^*)].$$

Remark. To every $f \in A'$ there corresponds a unique positive linear functional $p(f)$ on A such that

$$\|f\| = \|p(f)\| \quad \text{and} \quad |f(x)|^2 \leq \|f\| \, p(f)\,(x^*x).$$

We have $p(f) = |f^*|$; for $p(f)$ proposition 1 remains valid. Proposition 2 has to be modified in the following way: Let (π_p, H_p) a *-representation of A with cyclic vector ξ_p such that $p(f)\,(x) = (\pi_p(x)\,\xi_p, \xi_p)$. Then there exists a unique $\zeta_p \in H_p$ with $f(x) = (\pi_p(x)\,\xi_p, \zeta_p)$ etc. ... The inequality of proposition 3 becomes

$$|f(x) - f(y)|^2 \leq \|f\| \, [\|f\| \, (\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} p(f)\,(y^*x)].$$

3. Topologies on the Fourier-Stieltjes Algebra of a Locally Compact Group

In the sequel, G is an arbitrary locally compact group, $C_{00}(G)$ the set of all complex-valued continuous functions on G with compact support, $C^b(G)$ the set of all continuous complex-valued bounded functions on G and $P(G)$ the set of all continuous positive definite functions. For $f \in C^b(G)$ let $\tilde{f}$ denote the function $\tilde{f}(x) = \overline{f(x^{-1})}$. Let $C^*(G)$ be the C^*-algebra of the group G. The isomorphy of $B(G)$ and $C^*(G)'$ permits us (see [5] p. 192–193) to define on $B(G)$ a norm $u \mapsto \|u\|$ and an "absolute value" $u \mapsto |u|$.

PROPOSITION 4. *To every $u \in B(G)$ there corresponds a unitary continuous representation (π, H) with cyclic vector ξ, such that $|u|(x) = (\pi(x)\,\xi, \xi)$ and with a unique vector $\zeta \in H$ such that $u(x) = (\pi(x)\,\zeta, \xi)$. We have, moreover,*

$$\|\zeta\| = \|u\|^{1/2} = \|\xi\| = |u|(e)^{1/2}.$$

This proposition is a direct consequence of prop. 2. A similar result is proved directly in ([5] p. 195).

PROPOSITION 5. *Every $u \in B(G)$ satisfies the inequality*

$$|u(x) - u(y)|^2 \leqq 2\,\|u\|\,[|u|(e) - \operatorname{Re}|u|\,(yx^{-1})],$$

for all $x, y \in G$.

Proof. The proof is similar to that of prop. 3. It is enough to choose π, H, ξ and ζ as in prop. 4. From

$$u(x) = (\zeta, \pi(x^{-1})\,\xi)$$

follows

$$|u(x) - u(y)|^2 \leqq \|\zeta\|^2\,\|\pi(x^{-1})\,\xi - \pi(y^{-1})\,\xi\|^2.$$

On the other hand,

$$\|\pi(x^{-1})\,\xi - \pi(y^{-1})\,\xi\|^2 = 2\,\|\xi\|^2 - 2\operatorname{Re}(\pi(x^{-1})\,\xi, \pi(y^{-1})\,\xi)$$

and

$$\|\xi\|^2 = |u|\,(e) = \|\zeta\|^2 = \|u\|$$

imply the required inequality.

Remarks

(1) The same inequality is well known for elements of $P(G)$.

(2) It is possible to consider proposition 5 as a special case of proposition 3.

Consider the topology (denoted τ) on $B(G)$ defined by the following systems of sets:

$$U_\tau(u_0; \varepsilon, f_1, \dots, f_n) = \{u \in B(G) \mid | \int f_j(x)(u(x) - u_0(x))\, dx| < \varepsilon \quad 1 \leqq j \leqq n$$

and

$$|\|u\| - \|u_0\|| < \varepsilon\}$$

where $u_0 \in B(G)$, $\varepsilon > 0$ and $\{f_1, \dots, f_n\} \subset L^1(G)$.

THEOREM 6. *The topology τ on $B(G)$ is stronger than the compact-open topology.*

Proof. Given $u_0 \in B(G)$, $\varepsilon > 0$ and $K \in \mathfrak{K}$ ($\mathfrak{K}$ denotes the set of all compact subsets of G), we have to show the existence of $\lambda \in \mathbf{R}^+$ and $\{h_1, \dots, h_k\} \subset L^1(G)$ such that

$$U_\tau(u_0; \lambda, h_1, \dots, h_k) \subset U(u_0; \varepsilon, K)$$
$$= \{u \in B(G) \mid |u(x) - u_0(x)| < \varepsilon \quad \text{for every } x \in K\}.$$

Consider a compact neighbourhood V of e, such that $x \in V$ implies

$$\big||u_0|(x) - |u_0|(e)\big| < \frac{\varepsilon^2}{54(1 + \|u_0\|)}.$$

Let Φ_V denote the function on G defined as 1 on V and 0 outside. For every $x \in G$ and $u \in B(G)$ we have

$$m(V)^{-1}\, \Phi_V * u(x) - u(x) = m(V)^{-1} \int \Phi_V(t)(u(t^{-1}x) - u(x))\, dt$$

(where $m(V)$ is $\int \Phi_V(t)dt$). From prop. 5 follows

$$|m(V)^{-1}\, \Phi_V * u(x) - u(x)| \leqq 2^{1/2}\, m(V)^{-1} \int_V \|u\|^{1/2} (|u|(e) - \mathrm{Re}\, |u|(t))^{1/2}\, dt$$

and using the Hölder's inequality we obtain

$$|m(V)^{-1}\, \Phi_V * u(x) - u(x)| \leq 2^{1/2}\, m(V)^{-1/2}\, \|u\|^{1/2}\, \Big| \int_V (|u|(e) - u(t))\, dt\Big|^{1/2}.$$

From the remark of prop. 1 follows the possibility of finding $\{f_1, \dots, f_n\} \subset L^1(G)$ and $\eta > 0$ such that $u \in U_\tau(u_0; \eta, f_1, \dots, f_n)$ implies

$$|u| \in U\left(|u_0|, \frac{\varepsilon^2}{54(1 + \|u_0\|)}, \frac{\Phi_V}{m(V)}\right).$$

We can assume that

$$\eta \leqq \min\left(\frac{\varepsilon^2}{54(1+\|u_0\|)}, 1\right).$$

Using the inequality

$$\Big|\int_V (\|u\| - |u|(t))\,dt\Big| \leqq \Big|\int_V (\|u\| - \|u_0\|)\,dt\Big| + \Big|\int_V (\|u_0\| - |u_0|(t))\,dt\Big|$$
$$+ \Big|\int \Phi_V(t)\,(|u_0|(t) - |u|(t))\,dt\Big|,$$

we get for

$$u \in U_\tau(u_0; \eta, f_1, \ldots, f_n) / \Big|\int_V (\|u\| - |u|(t))\,dt\Big| < \frac{\varepsilon^2}{18(1+\|u_0\|)}\, m(V)$$

and

$$\|u\|\,\Big|\int_V (\|u\| - |u|(t))\,dt\Big| < \frac{\varepsilon^2}{18}\, m(V).$$

We have therefore for $u \in U_\tau(u_0; \eta, f_1, \ldots, f_n)$ and every $x \in G$

$$|m(V)^{-1}\,\Phi_V * u(x) - u(x)| < \varepsilon/3\,.$$

Let $S(0, 1+\|u_0\|)$ denote the sphere $\{u \in B(G) \mid \|u\| \leqq 1 + \|u_0\|\}$; the map of $(S(0, 1+\|u_0\|), \sigma(B(G), L^1(G)))$ into $(C^b(G)$, compact-open topology) defined by $u \mapsto m(V)^{-1}\Phi_V * u$ is continuous in u_0. It is therefore possible to find $\eta' > 0$ and $\{g_1, \ldots, g_m\} \subset L^1(G)$ such that

$$u \in U(u_0; \eta', g_1, \ldots, g_m) \cap S(0, 1+\|u_0\|)$$

implies

$$|m(V)^{-1}\,\Phi_V * u(x) - m(V)^{-1}\,\Phi_V * u_0(x)| < \varepsilon/3$$

for every $x \in K$. If we set $\lambda = \min(\eta, \eta')$ and $\{h_1, \ldots, h_k\} = \{f_1, \ldots, f_n, g_1, \ldots, g_m\}$, writing

$$|u(x) - u_0(x)| \leqq |u(x) - m(V)^{-1}\,\Phi_V * u(x)| + |m(V)^{-1}\,\Phi_V * u(x)$$
$$- m(V)^{-1}\,\Phi_V * u_0(x)| + |m(V)^{-1}\,\Phi_V * u_0(x) - u_0(x)|,$$

we obtain finally

$$U_\tau(u_0; \lambda, h_1, \ldots, h_k) \subset U(u_0; \varepsilon, K). \quad \text{q.e.d.}$$

COROLLARY. *On $B_0 = \{u \in B(G) \mid \|u\| = 1\}$ the weak topology $\sigma(B(G), L^1(G))$ and the compact-open topology coincide.*

Proof. On every bounded set in $B(G)$ the compact-open topology is stronger than the weak topology.

Remark. The corollary generalizes the following result due to D. A. Raikov: on $\{u \in P(G) \mid u(e)=1\}$ the above topologies coincide ([2] p. 260–261).

4. Approximate Identity in the Fourier Algebra of a Locally Compact Group

We recall ([5]) some definitions and properties concerning $B(G)$. Let $L^1_\varrho(G)$ denote the vector space $L^1(G)$ with the spectral norm. Then $(L^1_\varrho(G))'$ is isomorphic ([5] p. 192) to a closed subspace $B_\varrho(G)$ of $B(G)$ (for the norm topology). The set $\{k*\tilde{l} \mid k, l \in L^2(G)\}$ is contained in $B_\varrho(G)$ and $\|k*\tilde{l}\| \leqq \|k\|_2 \|l\|_2$ ([5] p. 207). The closure of $B(G) \cap C_{00}(G)$ in $B_\varrho(G)$ (or in $B(G)$) is called the Fourier algebra $A(G)$ of G.

THEOREM 7. *The Fourier algebra $A(G)$ has an approximate identity of the following type: for every $u \in A(G)$ and $\varepsilon > 0$ there exists $v \in A(G)$ such that $\|u-uv\| < \varepsilon$ and $\|v\| = 1$ if and only if G has the property (R).*

Proof. The property (R) implies ([7] p. 172 and 168) that G has property P_1. By ([4]) it follows (non trivially!) that for every $K \in \mathfrak{K}$ and $\varepsilon > 0$ there exists $U \in \mathfrak{K}$ with $m(U) > 0$ and $m(KU) < (1+\varepsilon)\, m(U)$. Set $\varphi = m(U)^{-1} \Phi_{KU} * \tilde{\Phi}_U$. The function φ is contained in $B(G) \cap C_{00}(G)$ and $\|\varphi\| \leqq m(U)^{-1} \|\Phi_{KU}\|_2 \|\tilde{\Phi}_U\|_2$; using $m(KU) < (1+\varepsilon)\, m(U)$ we obtain $\|\varphi\| < 1+\varepsilon$ and by definition $\mathrm{Res}_K \varphi = 1$. We have therefore proved that for every $K \in \mathfrak{K}$ and every $\varepsilon > 0$ there is $\varphi \in B(G) \cap C_{00}(G)$ with $\|\varphi\| < 1+\varepsilon$ and $\mathrm{Res}_K \varphi = 1$. Let u be an arbitrary element of $A(G)$ and ε an arbitrary positive real number. We can assume $\|u\| > \varepsilon$. There exists by definition $w \in B(G) \cap C_{00}(G)$ with $\|u-w\| < \varepsilon/8$. We can find $\varphi \in B(G) \cap C_{00}(G)$ with $\mathrm{Res}_{\mathrm{supp}\, w}\, \varphi = 1$ ($\mathrm{supp}\, w \neq \emptyset$) and $\|\varphi\| < 1+\eta$ where

$$\eta = \min\left(1, \frac{\varepsilon}{2(1+\|u\|)}\right).$$

From $\|u-u\varphi\| \leqq \|u-w\| + \|\varphi\|\ \|u-w\|$ follows $\|u-u\varphi\| < \varepsilon/2$. If we set

$$v = \frac{\varphi}{\|\varphi\|}$$

we get

$$\|u - uv\| \leqq \|\, \|\varphi\|\, u - u\varphi\| \leqq \|u\varphi - u\| + \|u - \|\varphi\|\, u\| < \varepsilon.$$

To prove the converse, it is enough to find $k \in C_{00}(G)$ for every $\varepsilon > 0$ and $K \in \mathfrak{K}$ such that $|1 - k*\tilde{k}(x)| < \varepsilon$ on K.

By theorem 6 we can find $\varepsilon_1 > 0$ and a finite subset F_1 of $L^1(G)$ such that

$$U(1;\varepsilon_1, F_1)\cap B_0 \subset U(1;\varepsilon/4, K).$$

By the remark of prop. 1 there exists $\varepsilon_2>0$ and F_2 finite subset of $L^1(G)$ for which $u\in U(1;\varepsilon_2, F_2)\cap B_0$ implies $|u|\in U(1;\varepsilon_1, F_1)$. It is also possible to find $\varepsilon_3>0$ and $K'\in\mathfrak{K}$ with

$$U(1;\varepsilon_3, K')\cap B_0 \subset U(1;\varepsilon_2, F_2).$$

Let u be an element of $B(G)\cap C_{00}(G)$ with $\mathrm{Res}_{K'}u=1$. By assumption there exists $v'\in A(G)$ with $\|v'\|=1$ and $\|u-uv'\|<\varepsilon_3/6$; if we take $v''\in B(G)\cap C_{00}(G)$ such that

$$\|v'-v''\| < \min\left(\frac{1}{2}, \frac{\varepsilon_3}{6(1+\|u\|)}\right)$$

and put $v=v''/\|v''\|$ we obtain $\|u-uv\|<\varepsilon_3$ and therefore $|1-v(x)|<\varepsilon_3$ for every $x\in K'$ i.e. $v\in U(1,\varepsilon_3, K')\cap B_0$. From above it follows $|v|\in U(1;\varepsilon/4, K)$. The function v is in $B_\varrho(G)$. This implies that $|v|\in B_\varrho(G)$ (by definition) and then by ([1] Theorem 4.3) $|v|$ must be contained in the closure of $P(G)\cap C_{00}(G)$ (for the compact-open topology) in $P(G)$. Therefore there is $w\in P(G)\cap C_{00}(G)$ such that $w\in U(|v|;\varepsilon/4, K)$. It is well known that we can choose $k\in C_{00}(G)$ with $\mathrm{Sup}_{x\in G}|w(x)-k*\tilde{k}(x)|<\varepsilon/2$, then for every $x\in K$ we have

$$|1-k*\tilde{k}(x)| \leqq |1-|v|(x)| + \big||v|(x)-w(x)| + |w(x)-k*\tilde{k}(x)\big| < \varepsilon.$$

Consequence. For $G=\mathrm{SL}(n,\mathbf{R})$ (with $n\geqq 2$) and more generally for every connected semi-simple non compact Lie group with finite center, $A(G)$ does not have an approximate identity of the above type.

PROPOSITION 8. *A sufficient condition for G to satisfy the property (R) is that $A(G)$ has an approximate identity with $v\in A(G)\cap P(G)$.*

Proof. For every $K\in\mathfrak{K}$ there exists $u\in B(G)\cap C_{00}(G)$ with $\mathrm{Res}_K u=1$. By assumption for every $\varepsilon>0$ one can find $v'\in A(G)\cap P(G)$ such that $\|u-uv'\|<\varepsilon/4$. For the same reason as above v' must be contained in the closure of $P(G)\cap C_{00}(G)$ (for the compact-open topology) in $P(G)$. We can therefore find $v\in P(G)\cap C_{00}(G)$ with $v\in U(v';\varepsilon/4, K)$. Then choosing $k\in C_{00}(G)$ such that $\mathrm{Sup}_{x\in G}|v(x)-k*\tilde{k}(x)|<\varepsilon/2$ we conclude as at the end of proof of theorem 6.

5. A Multiplier Theorem

Let Σ be the set of all unitary continuous representations of G. For $\pi\in\Sigma$ and an arbitrary bounded Radon measure μ, we denote by $\|\pi(\mu)\|$ the norm of the operator $\int\pi(x)\,d\mu(x)$, and by $\|\mu\|_\Sigma$ the $\sup\{\|\pi(\mu)\| \mid \pi\in\Sigma\}$.

From the proof of theorem 7 it follows that property (R) is equivalent to the following condition: for every $K\in\mathfrak{K}$ and $\varepsilon>0$, there exists $f\in B(G)\cap C_{00}(G)$ with $\operatorname{Res}_K f=1$ and $\|f\|<1+\varepsilon$. This remark permits us to prove:

THEOREM 9. *If group G has property (R), and if $u\in\mathbf{C}^G$ and $uf\in B(G)$ for every $f\in A(G)$, then $u\in B(G)$. Moreover, $\|u\|=\operatorname{Sup}\{\|uf\| \mid f\in A(G),\ \|f\|\leqq 1\}$.*

Proof. As in the abelian case ([8] p. 74) we can find a positive constant C such that $\|uf\|\leqq C\|f\|$ for every $f\in A(G)$. Given $\{x_1,\dots,x_n\}\subset G$ and $\varepsilon>0$ there exists $f\in B(G)\cap C_{00}(G)$ such that $\|f\|<1+\varepsilon$ and $f(x_j)=1$ for $1\leqq j\leqq n$. For arbitrary $\{c_1,\dots,c_n\}\subset\mathbf{C}$ we have

$$\Big|\sum_j c_j u(x_j)\Big| \leq \|fu\|\ \Big\|\sum_j c_j\delta_{x_j}\Big\|_\Sigma$$

(δ_a denotes the evaluation at point a); thus

$$\Big|\sum_j c_j u(x_j)\Big| < C(1+\varepsilon)\ \Big\|\sum_j c_j\delta_{x_j}\Big\|_\Sigma$$

and therefore

$$\Big|\sum_j c_j u(x_j)\Big| \leq C\ \Big\|\sum_j c_j\delta_{x_j}\Big\|_\Sigma.$$

From the continuity of u and from ([5] p. 202) it follows that $u\in B(G)$.

Let $|||u|||$ be the $\sup\{\|uf\| \mid f\in A(G),\ \|f\|\leqq 1\}$. We have to show that $\|u\|\leqq|||u|||$ (the converse inequality is clear). For arbitrary $h\in C_{00}(G)$ with $\|h\|_\Sigma\leqq 1$ and arbitrary $\varepsilon>0$ there exists $w\in B(G)\cap C_{00}(G)$ with $\operatorname{Res}_{\operatorname{supp}h} w=1$ and $\|w\|<1+\varepsilon$ (if $\operatorname{supp}h$ is empty we choose $w=0$). We obtain

$$\Big|\int u(x)\,h(x)\,dx\Big| = \Big|\int u(x)\,w(x)\,h(x)\,dx\Big| < |||u|||(1+\varepsilon)$$

and therefore $|\int u(x)\,h(x)\,dx|\leqq|||u|||$ for every $h\in C_{00}(G)$ with $\|h\|_\Sigma\leqq 1$, that is $\|u\|\leqq|||u|||$.

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