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Commentarii Mathematici Helvetici, Vol.44 (1969)

Representative Functions on Topological Groups

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1. Introduction

In this paper we shall study the relations existing between the topological properties of a completely regular topological group G and the algebraic properties of the space of all representative functions $R(G)$ over G.

In the first part we give some results which generalize those of S. Kakutani ([4] pp. 430–431) concerning compactifications of locally compact abelian groups.

For a compact group G the Tannaka duality theorem shows that the algebraic properties of $R(G)$ characterize completely those of G. Using [2], we find algebraic characterizations of the connectedness, local connectedness and arcwise connectedness of G. Similarly, we attempt to generalize, in a certain sense, the well-known result of Pontrjagin ([10] p. 32) about the covering dimension of a compact abelian group. Using these results we obtain some applications to more general topological groups.

I want to express my warmest thanks to Professor K. Chandrasekharan for his guidance in the preparation of this paper, to Professor R. Sridharan for many informative conversations and to Professor G. Hochschild for suggestions and corrections. Finally I want to acknowledge the generous support of the Forschungsinstitut für Mathematik without which it would have been impossible to carry out this work.

2. Compactifications and related questions

Let γ be the map of $R(G)$ into $R(G)\otimes_{\mathbf{C}}R(G)$ induced by the product in G. Following ([6]), one can say that, with the coproduct γ and the pointwise product, $R(G)$ is a Hopf algebra. We consider, as in [2], only Hopf subalgebras of $R(G)$ which are stable under complex conjugation.

Let $\mathscr{H}$ be a Hopf subalgebra of $R(G)$. We denote by $S(\mathscr{H})$ the set of all $\mathbf{C}$-algebra homomorphisms of $\mathscr{H}$ onto $\mathbf{C}$ which commute with complex conjugation. With the finite open topology $S(\mathscr{H})$ is a compact space ([6] p. 28). Let Γ be a non empty subset of $R(G)$; we denote by $\mathscr{H}(\Gamma)$ the least Hopf subalgebra containing Γ. It follows from ([6] p. 29–30) that $S(\mathscr{H}(\Gamma))$ is a compact group and the evaluation map φ_Γ of G into $S(\mathscr{H}(\Gamma))$ is a continuous homomorphism.

PROPOSITION 1. *The group $\varphi_\Gamma(G)$ is dense in $S(\mathscr{H}(\Gamma))$ for every $\Gamma\subset R(G)$.*

Proof. Consider $f\in R(S(\mathscr{H}(\Gamma)))$ with $f=0$ on $\varphi_\Gamma(G)$. By the Tannaka duality theorem ([6] p. 30) there exists $h \in \mathscr{H}(\Gamma)$ such that $s(h) = f(s)$ for every $s \in S(\mathscr{H}(\Gamma))$. In particular $\varphi_\Gamma(x)(h) = h(x) = 0$ for every $x \in G$. This implies that $h = 0$ and therefore $f = 0$. Using ([7] Lemma 5.2.) we obtain $\overline{\varphi_\Gamma(G)} = S(\mathscr{H}(\Gamma))$.

COROLLARY 1. Let $\mathscr{H}$ be any Hopf subalgebra of $R(G)$. Let τ be any element of $S(\mathscr{H})$, let $f_1, \ldots, f_n$ be a finite subset of $\mathscr{H}$ and let ε be any positive number. Then there is a point $x \in G$ such that $|\tau(f_j) - f_j(x)| < \varepsilon$ $(1 \leqslant j \leqslant n)$.

Proof. By definition of the topology of $S(\mathscr{H})$ the set $\{\tau' \in S(\mathscr{H}) \mid |\tau'(f_j) - \tau(f_j)| < \varepsilon$ $1 \leqslant j \leqslant n\}$ is an open neighborhood U of τ. From prop. 1 the existence of $x \in G$ then follows with the required properties.

Remark. This result is proved for characters over a topological group in ([5]). At the end of the same paper, the authors indicate the possibility of generalization.

COROLLARY 2. Let G be an infinite maximally almost periodic group and let $f_1, \ldots, f_n \in R(G)$ and $\varepsilon > 0$. Then there is an element $x \in G$ such that $x \neq e$ and $|f_j(x) - f_j(e)| < \varepsilon$ $(1 \leqslant j \leqslant n)$.

The proof is analogous (using prop. 1) to that in the locally compact abelian case ([4] p. 431).

PROPOSITION 2. *Let G be a topological group. Let H be a compact group. Then the following assertions are equivalent:*

(i) *There is a continuous homomorphism φ of G into H such that $\overline{\varphi(G)} = H$.*

(ii) *H is isomorphic to the compact group $S(\Gamma)$ for some Hopf subalgebra Γ of $R(G)$.*

(iii) *There is a Hopf algebra monomorphism ψ of $R(H)$ into $R(G)$.*

Proof. It is clear that (i) implies (iii) and that (ii) implies (i). Suppose that (iii) holds. The map ψ^* of $S(R(G))$ into $S(R(H))$ defined by $\psi^*(s) = s \circ \psi$ is a continuous group homomorphism. There exists a continuous group homomorphism ψ' of G into H defined by the commutativity of

$$\begin{array}{ccc} S(R(G)) & \xrightarrow{\psi^*} & S(R(H)) \\ {\scriptstyle \varphi_{R(G)}}\uparrow & & \uparrow{\scriptstyle \varphi_{R(H)}} \\ G & \xrightarrow{\psi'} & H \end{array}.$$

The relation $\overline{\psi'(G)} \neq H$ implies the existence of $f \in R(H)$ with $f \neq 0$ and $f(\psi'(x)) = 0$ for any $x \in G$. This contradicts the equality $f \circ \psi' = \psi(f)$. Therefore (iii) implies (i). It remains to prove that (i) implies (ii). Consider the Hopf algebra monomorphism φ^* of $R(H)$ into $R(G)$ defined by $\varphi^*(f) = f \circ \varphi$ and set $\Gamma = \varphi^*(R(H))$. To every $f \in R(H)$ there corresponds a function on $S(\Gamma)$ defined by $s(\varphi^*(f))$ for every

$s \in S(\Gamma)$. This map is a Hopf algebra isomorphism of $R(H)$ onto $R(S(\Gamma))$ and therefore H and $S(\Gamma)$ are isomorphic.

Remark. From the approximation theorem it follows that $S(R(G))$ is isomorphic to the almost periodic compactification of G ([8] p. 168).

3. Some results concerning compact groups

For a compact group G we have $\varphi(G)=S(R(G))$ (we set $\varphi_{R(G)}=\varphi$). This equality permits us to characterize the topological properties of G (as in the abelian case) using the "algebraic" properties of $R(G)$.

First we introduce some notations. If $\mathscr{H}$ is a Hopf subalgebra of $R(G)$, let $\mathscr{H}^{\perp}$ denote the closed normal subgroup of G defined by $\{h \in G \mid {}_hf=f \text{ for every } f \in \mathscr{H}\}$. Conversely, if H is a closed normal subgroup of G, let $H^{\perp}$ be the Hopf subalgebra of $R(G)$ defined by $\{f \in R(G) \mid {}_hf=f \text{ for every } h \in H\}$. In [2] the following result was proved:

THEOREM 1. *For every compact group G, $G_0^{\perp}=\{f \in R(G) \mid f$ is an algebraic element of the* **C***-algebra $R(G)\}$, where G_0 denotes the connected component of the identity in G.*

Proof. We prove at first that the above conditions are sufficient to insure the local connectedness of a compact group G.

THEOREM 2. *A compact group G is locally connected if and only if every finite set of representative functions on G is contained in a finitely generated Hopf subalgebra $\mathscr{H}$ of $R(G)$ such that every non constant element of $R(\mathscr{H}^{\perp})$ is not algebraic.*

Proof. We prove at first that the above conditions are sufficient to insure the local connectedness of G. For every open neighborhood U of e in G there exists an $\varepsilon>0$ and there exists a sequence $\{f_j\}_{j=1}^{n} \subset R(G)$ such that the set $\{x \in G \mid |f_j(x)-f_j(e)| < \varepsilon \ 1 \leqslant j \leqslant n\}$ is contained in U. This implies that $\mathscr{H}(f_1,\ldots,f_n)^{\perp} \subset U$. By hypothesis there exists a finitely generated Hopf subalgebra $\mathscr{E}$ of $R(G)$ with $\mathscr{E} \supset \mathscr{H}(f_1,\ldots,f_n)$ and $\mathscr{E}^{\perp}$ connected. Let π be the canonical map of G onto $G/\mathscr{E}^{\perp}$. The factor group $G/\mathscr{E}^{\perp}$ is a Lie group, since $R(G/\mathscr{E}^{\perp})$ and $\mathscr{E}$ are isomorphic. Let Σ be a fundamental system of open connected neighborhoods of $\pi(e)$ in $G/\mathscr{E}^{\perp}$. It is easy to demonstrate the existence of a subset $O \in \Sigma$ with $\pi^{-1}(O) \subset U$. It suffices to prove that $\pi^{-1}(O)$ is connected. Suppose the contrary. There exist open subsets of G V_1, V_2 such that $V_1, V_2 \neq \emptyset$, $V_1 \cap V_2=\emptyset$ and $V_1 \cup V_2=\pi^{-1}(O)$. The existence of $x \in G$ with $\pi(x) \in \pi(V_1) \cap \pi(V_2)$ contradicts the connectedness of $x\mathscr{E}^{\perp}$. We therefore have $\pi(V_1) \cap \pi(V_2)=\emptyset$ and this implies that O is not connected.

For the second part of the proof, we suppose that $R(G)$ does not satisfy the above conditions, and show that G is not locally connected. In this case there exists an $M \subset R(G)$ with $|M| < \infty$, such that every Hopf subalgebra $\mathscr{E}$ of $R(G)$ with $\mathscr{E} \supset M$ and $\mathscr{E}^{\perp}$ connected is not finitely generated. Let $\mathscr{H}$ be the Hopf subalgebra of $R(G)$ with the property that $\mathscr{H}^{\perp}$ is the connected component of the unit element in the subgroup $\mathscr{H}(M)^{\perp}$ (the connected component of a normal closed subgroup is itself a normal subgroup). Denoting by α the canonical map of $G/\mathscr{H}^{\perp}$ onto $G/\mathscr{H}(M)^{\perp}$, we have $\operatorname{Ker}\alpha = \mathscr{H}(M)^{\perp}/\mathscr{H}^{\perp}$. By a generalization of a wellknown theorem of Hurewicz ([9] theorem 4), $\dim \operatorname{Ker}\alpha = 0$ implies $\dim G/\mathscr{H}^{\perp} \leqslant \dim G/\mathscr{H}(M)^{\perp}$, and then $\dim S(\mathscr{H}) \leqslant \dim S(\mathscr{H}(M))$. It follows that $\dim S(\mathscr{H})$ is finite, because $S(\mathscr{H}(M))$ is a compact Lie group. By hypothesis $\mathscr{H}$ is not finitely generated. This fact implies that $S(\mathscr{H})$ is not locally connected, and therefore (since the natural map of G onto $G/\mathscr{H}^{\perp}$ is open) that G itself is not locally connected.

Remarks.

1) In this proof we have used the two following results: α) A compact group G is a Lie group if and only if the **C**-algebra $R(G)$ is finitely generated; β) Every compact (or locally compact) locally connected group with a finite dimension is a Lie group.

2) The corresponding classical result ([10] p. 33) for compact abelian groups is: G is locally connected if and only if every finite number of continuous characters over G is contained in a finitely generated subgroup H of $\hat{G}$ (group of all continuous characters over G) such that $\hat{G}/H$ is torsion-free.

We denote by $\mathscr{D}(G)$ the set of all **C**-derivations of the **C**-algebra $R(G)$ which commute with complex conjugation and every left translation. Let $D \in \mathscr{D}(G)$. For every $f \in R(G)$ consider the finite dimensional G-module $R(f) = [\{f_x \mid x \in G\}]$. By ([7] prop. 2.5) $R(f)$ is stable under D. This implies that $\sum_{n=1}^{\infty} D^n f/n!$ defines an element $\exp Df$ of $R(f)$ and therefore of $R(G)$.

PROPOSITION 3. *For every $D \in \mathscr{D}(G)$ the map $t \mapsto \varphi^{-1}(\varphi(e)\exp tD)$ is a one-parameter subgroup of G. Conversely every one-parameter subgroup admits such a unique representation.*

Proof. Let $D \in \mathscr{D}(G)$ and $t \in \mathbf{R}$. It is easy to prove that $\exp tD(fg) = \exp tD(f)$ $\operatorname{expt} D(g)$ for every f, $g \in R(G)$. It follows that $\exp tD$ is a **C**-algebra endomorphism of $R(G)$. From the fact that $\exp tD$ commutes with complex conjugation it follows that $\varphi(e)\exp tD \in S(R(G))$. We have therefore that $t \mapsto \varphi^{-1}(\varphi(e)\exp tD)$ is a one-parameter subgroup of G.

Let $\lambda \in \operatorname{Hom}_{\text{cont}}(\mathbf{R}, G)$. For every $f \in R(G)$ and $t \in \mathbf{R}$ set $U_t f = f_{\lambda(t)}$. The operator U_t is unitary under the scalar-product of $R(G)$ defined by the normalized Haar measure of G. We denote by U_t' the extension of U_t to $L^2(G)$. There exists an operator D of $L^2(G)$ with iD selfadjoint and such that $\lim_{t \to 0} \|(U_t' f - f)t^{-1} - Df\|_2 = 0$ for every $f \in R(G)$. The operator $-iD$ has the spectral representation $\int_{-\infty}^{+\infty} \mu \, dE_\mu$ and U'_t is equal to $\int_{-\infty}^{+\infty} e^{i\mu t} \, dE_\mu$. For every f in $R(G)$ and $t \neq 0$ we have $(U_t f - f)\, t^{-1} \in R(f)$ and therefore $Df \in R(f)$, i.e. $D(R(G)) \subset R(G)$. It is easy to verify that the restriction of D to $R(G)$ is contained in $\mathscr{D}(G)$. As above we can define $\exp tD$. It is clear that the **C**-algebra endomorphism $\exp tD$ commutes with complex conjugation and left translations and invoking ([7] Lemma 5.4) we obtain that $\exp tD$ is a unitary operator of $R(G)$. For every f of $R(G)$ we have $\lim_{t\to 0} \|(\exp tDf - f)\, t^{-1} - Df\|_2 = 0$. Let U''_t be the extension of $\exp tD$ to $L^2(G)$. As above there exists an operator D' of $L^2(G)$ with iD' self-adjoint and $\lim_{t\to 0} \|(U''_t h - h)\, t^{-1} - D'h\|_2 = 0$ for every $h \in R(G)$. We have therefore $D = D'$ and $U''_t = U'_t$ i.e. $\exp tDf = f_{\lambda(t)}$ for every $f \in R(G)$.

COROLLARY. For a compact Lie group G, the Lie algebra g of G is isomorphic to $\mathscr{D}(G)$.

Remarks.

1) Proposition 3 gives a characterisation of the Lie algebra of a compact group. The corollary has been already proved for more general Lie groups than compact Lie groups ([7] Theorem 11.1).

2) For the second part of the proof of proposition 3 Professor G. Hochschild has suggested a method which avoids the use of operator theory in $L^2(G)$. If V is any finite dimensional right-submodule of $R(G)$ the map $t \mapsto U_t$ (where $U_t f = f_{\lambda(t)}$) defines a continuous homomorphism of **R** into the full linear group of V. This homomorphism is therefore of the form $t \mapsto \exp tD_V$, where D_V is some linear endomorphism of V. Since $R(G)$ is the union of such V's, the D_V's match up to give a linear endomorphism D of $R(G)$ with the required properties.

We set for $\Gamma \subset R(G)$ and $M \subset \mathscr{D}(G)$:

(i) $\mathrm{Ann}(\Gamma) = \{D \in \mathscr{D}(G) \mid Df = 0 \text{ for every } f \in \Gamma\}$,

(ii) $\mathscr{H}_l(\Gamma)$ = the least subalgebra of $R(G)$ invariant under the left-translations and the complex conjugation containing Γ.

(iii) $\mathrm{Ann}(M) = \{f \in R(G) \mid Df = 0 \text{ for every } D \in M\}$.

It is easy to see that $\mathrm{Ann}(\Gamma)$ is a Lie subalgebra of $\mathscr{D}(G)$, and that $\mathrm{Ann}(M) = \mathscr{H}_l(\mathrm{Ann}(M))$.

PROPOSITION 4. *For every subset Γ of $R(G)$, we have $\mathscr{H}_l(\Gamma \cup \mathscr{A}) =$ Ann $(\mathrm{Ann}(\Gamma))$, where $\mathscr{A}$ is the subset of all algebraic elements of $R(G)$.*

Proof. Denote by $\lambda(D)$ the element of $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{R}, G)$ corresponding to $D \in \mathscr{D}(G)$. From $f \in \lambda(D)(\mathbf{R})_r^{\perp}$ [1]) it follows that $\exp tDf = f$ for every $t \in \mathbf{R}$ i.e. $f \in \mathrm{Ker}\, D$ and

[1]) For every subset H of G, $H_r{}^{\perp}$ denotes the set $\{f \in R(G) \mid f_x = f \text{ for every } x \in H\}$ and for any subalgebra Γ of $R(G)$ with $\mathscr{H}_l(\Gamma) = \Gamma$ $\Gamma_r{}^{\perp}$ is the closed subgroup $\{x \in G \mid f_x = f \text{ for every } f \in \Gamma\}$.

conversely, we have therefore $\lambda(D)(\mathbf{R})_r^{\perp}=\operatorname{Ker}D$. Using the fact that every one-parameter subgroup is contained in G_0 we obtain $\operatorname{Ker}D\supset\mathscr{A}$ and in particular $\operatorname{Ann}(\operatorname{Ann}(\Gamma))\supset\mathscr{H}_l(\Gamma\cup\mathscr{A})$. It is easy to verify that $\operatorname{Ann}(\Gamma)=\{D\in\mathscr{D}(G)\mid\lambda(D)(\mathbf{R})\subset\mathscr{H}_l(\Gamma\cup\mathscr{A})_r^{\perp}\}$. Since the closed subgroup $\mathscr{H}_l(\Gamma\cup\mathscr{A})_r^{\perp}$ is connected, we have $\mathscr{H}_l(\Gamma\cup\mathscr{A})=\{\lambda(D)(\mathbf{R})\mid D\in\mathscr{D}(G),\lambda(D)(\mathbf{R})\subset\mathscr{H}_l(\Gamma\cup\mathscr{A})_r^{\perp}\}_r^{\perp}$ and therefore $\mathscr{H}_l(\Gamma\cup\mathscr{A})=\{\lambda(D)(\mathbf{R})\mid D\in\operatorname{Ann}(\Gamma)\}_r^{\perp}=\bigcap\{\operatorname{Ker}D\mid D\in\operatorname{Ann}(\Gamma)\}=\operatorname{Ann}(\operatorname{Ann}(\Gamma))$.

Remarks.

1) For $\Gamma=\emptyset$ we obtain $\mathscr{A}=\operatorname{Ann}(\mathscr{D}(G))$ which gives another characterisation of the set of all algebraic elements of $R(G)$.

2) The group G is solenoïdal if and only if there is $D\in\mathscr{D}(G)$ with $\operatorname{Ker}D=\mathbf{C}\cdot 1_G$.

3) There is a bijective map between the closed subgroups of G_0 and the Lie subalgebras M of $\mathscr{D}(G)$ such that $M=\operatorname{Ann}(\operatorname{Ann}(M))$. That is, to every closed subgroup H of G_0 we associate $M=\operatorname{Ann}(H_r^{\perp})$. The subgroup H is normal in G if and only if M is an ideal of $\mathscr{D}(G)$.

THEOREM 3. *A compact group G is arcwise connected if and only if for every $x\in G$ there is an element D of $\mathscr{D}(G)$ such that the following diagram commutes:*

$$\begin{array}{ccc} R(G) & \xrightarrow{\varphi(x)} & \mathbf{C} \\ & {\scriptstyle \exp D}\searrow \quad \nearrow{\scriptstyle \varphi(e)} & \\ & R(G) & \end{array}$$

LEMMA. *A compact group is arcwise connected if and only if it is the union of all one-parameter subgroups.*

Proof. By ([11] Theorem 1) every arc beginning at the unit element is homotopic to the restriction to $[0,1]$ of a one-parameter subgroup.

Proof of theorem 3. Suppose first that G is arcwise connected. In this case for every $x\in G$ there exists $\lambda\in\operatorname{Hom}_{\text{cont}}(\mathbf{R},G)$ and $a\in\mathbf{R}$ with $\lambda(a)=x$. There exists $D\in\mathscr{D}(G)$ such that $\lambda(at)=\varphi^{-1}(\varphi(e)\exp tD)$ and therefore $\varphi(e)\exp D=\varphi(x)$.

Conversely suppose that for every $x\in G$ there exists $D\in\mathscr{D}(G)$ such that $\varphi(x)=\varphi(e)\exp D$. If we set $\lambda(t)=\varphi^{-1}(\varphi(e)\exp tD)$ we obtain $\lambda\in\operatorname{Hom}_{\text{cont}}(\mathbf{R},G)$ and $\lambda(1)=x$.

Remarks.

1) The classical result for compact abelian groups ([3]) is: G is arcwise connected if and only if for every $x\in G$ there exists $\lambda\in\operatorname{Hom}(\hat{G},\mathbf{R})$ such that

$$\begin{array}{ccc} \hat{G} & \xrightarrow{x} & S^1 \\ & {\scriptstyle\lambda}\searrow & \uparrow{\scriptstyle e^{2\pi i}} \\ & & \mathbf{R} \end{array}$$

commutes.

2) It is not necessary to give conditions which imply the local arcwise connectedness of G because a compact connected group is locally arcwise connected if and only if it is arcwise connected ([11]).

The dimension of a compact abelian group is equal by ([10] p. 32) to the rank of its character group. The next theorem is to be considered as a possible generalization to the non abelian case.

THEOREM 4. *The dimension of a compact group G is equal to the dimension of the real vector space $\mathscr{D}(G)$.*

Proof. There exists an inverse system $(G_\alpha, u_{\alpha\beta})$ consisting of compact Lie groups G_α and continuous epimorphisms $u_{\beta\alpha}: G_\beta \to G_\alpha (\alpha < \beta)$ such that $G \cong \overleftarrow{\lim}(G_\alpha, u_{\alpha\beta})$. We denote by π_α the projection of G onto G_α; by R_α, the Hopf subalgebra of $R(G)$, $(\mathrm{Ker}\,\pi_\alpha)^\perp$; by $\mathscr{D}_\alpha$ the set of all **C**-derivations of R_α which commute with complex conjugation and all left translations and finally by $i_{\alpha\beta}$ $(\alpha < \beta)$ the natural injection of R_α into R_β. It follows from ([2]) that $R(G)$ and $\overrightarrow{\lim}(R_\alpha, i_{\alpha\beta})$ are isomorphic. The restriction $R_{\beta\alpha}$ $(\alpha < \beta)$ of an element of $\mathscr{D}_\beta$ to R_α belongs to $\mathscr{D}_\alpha$. The differential $\dot{u}_{\beta\alpha}$ of $u_{\beta\alpha}$ is a linear map of the Lie algebra g_β of G_β onto g_α. It is easy to verify that the projective systems $(\mathscr{D}(G), \mathrm{id})$, $(\mathscr{D}_\alpha, \mathrm{Res}_{\alpha\beta})$ and $(g_\alpha, \dot{u}_{\alpha\beta})$ are isomorphic. From $\dim \mathscr{D}_\alpha = \dim G_\alpha$ (corollary of prop. 3), $\dim G = \sup_\alpha \dim G_\alpha$ and $\dim \mathscr{D}(G) = \sup_\alpha \dim \mathscr{D}_\alpha$ the theorem follows.

4. Applications

For non-compact groups the relations between the properties of G and those of $R(G)$ are more complicated.

If the **C**-algebra $R(G)$ of a locally compact maximally almost periodic group G is finitely generated, then G is a Lie group. The condition is not necessary. However, if G is a Lie group such that G/G_0 is finite then $R(G)$ is finitely generated if and only if the factor group of G modulo the closure of the commutator of G_0 is compact ([7] theorem 11.1).

PROPOSITION 5. *If a topological group G is connected, then every non constant representative function over G is non algebraic. If every representative function over a maximally almost periodic group is algebraic then the group is totally disconnected.*

Proof. The connectedness of G implies the same property for $S(R(G))$. From Theorem 1 the first part of proposition 5 follows. The proof of the second part is completely analogous.

THEOREM 5. *Every locally countably compact torsion group with a maximally almost periodic connected component of the identity is totally disconnected.*

Proof. Suppose that G is a compact torsion group. For every $f \in R(G)$ consider $R(f)$ and the corresponding continuous finite dimensional representation ϱ_f; $\varrho_f(G)$ is a compact torsion Lie group and therefore is a finite group. It follows $\operatorname{Ker} \varrho_f \supset G_0$ i.e. $f \in G_0^{\perp}$. Using theorem 1, we have that G is totally disconnected. For the general case consider, the continuous map $\alpha_n : G \to G$ defined by $\alpha_n(x) = x^n$ for every positive integer n. By assumption we have $G = \bigcup_{n=1}^{\infty} \operatorname{Ker} \alpha_n$, the category theorem of Baire implies the existence of n_0 such that $\operatorname{Ker} \alpha_{n_0}$ is open and therefore $\operatorname{Ker} \alpha_{n_0} \supset G_0$. From this it follows that $S(R(G_0))$ is a torsion group. Using the first part of the proof, theorem 1 and proposition 5 we have the desired result.

Remark. This theorem generalizes a result proved by Braconnier ([1] p. 51) for the case of a locally compact abelian group.

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Eingegangen den 18.2.69