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C. R. Acad. Sci. Paris, t. 327, Série I, p. 000–000, 1998
Problèmes mathématiques de la mécanique/*Mathematical Problems in Mechanics*

Buckling of a tapered elastica [1]

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(Reçu le 30 juin 2000, accepté ?)

Abstract. The nonlinear eigenvalue problem (1)-(3) below is a model for the buckling of a tapered elastic rod. The coefficient $A \in C([0,1])$ is such that $A(s) > 0$ for $s > 0$ and there exist $p \geq 0$ and $L \in (0,\infty)$ such that $\lim_{s\to 0} \frac{A(s)}{s^p} = L$. For $0 \leq p < 2$, there is bifurcation only at values of μ in a discrete subset of $(0,\infty)$ whereas for $p = 2$ every point in the interval $[L/4,\infty)$ is a bifurcation point. Furthermore, at $p = 2$, one observes a dramatic change in the shape of the equilibrium configurations. Let $u_\mu \not\equiv 0$ be a configuration which minimizes the energy. For $0 \leq p < 2, \lim_{s\to 0} u_\mu(s) \in (-\pi,\pi)$, whereas for $p \geq 2$, $\lim_{s\to 0} u_\mu(s) = \pm\pi$. © Académie des Sciences/Elsevier, Paris

Flambage d'une barre élastique effilée

Résumé. *Le problème aux valeurs propres (1)-(3) ci-dessous est un modèle pour le flambage d'une barre élastique effilée. Le coefficient* $A \in C([0,1])$ *est tel que* $A(s) > 0$ *pour* $s > 0$ *et il existe* $p \geq 0$ *et* $L \in (0,\infty)$ *tels que* $\lim_{s\to 0} \frac{A(s)}{s^p} = L$. *Pour* $0 \leq p < 2$, *il y a bifurcation uniquement pour des valeurs de* μ *d'un sous-ensemble discret de* $(0,\infty)$ *tandis que pour* $p = 2$ *chaque point de l'intervalle* $[L/4,\infty)$ *est un point de bifurcation. De plus, pour* $p = 2$, *on observe un changement spectaculaire de la forme des configurations d'équilibre. Soit* $u_\mu \not\equiv 0$ *une configuration qui minimise l'énergie. Pour* $0 \leq p < 2, \lim_{s\to 0} u_\mu(s) \in (-\pi,\pi)$, *tandis que, pour* $p \geq 2$, $\lim_{s\to 0} u_\mu(s) = \pm\pi$. *© Académie des Sciences/Elsevier, Paris*

Version française abrégée

[1]Note présentée par Philippe G. CIARLET.

L'étude du flambage d'une barre élastique effilée se réduit au problème aux limites (1)-(3) ci-dessous où le coefficient $A \in C([0,1])$ est tel que $A(s) > 0$ pour $s > 0$ et il existe $p \geq 0$ et $L \in (0, \infty)$ tels que $\lim_{s \to 0} \frac{A(s)}{s^p} = L$. Ce coefficient tient compte des variations des sections transverses de la barre élastique. Pour $p = 0$, il s'agit d'un problème classique qui remonte à D. Bernoulli et Euler, [1]. Dans les cas singuliers $p > 0$, l'aire des sections s'annule vers une extrémité libre, l'autre étant encastrée. Pour $0 \leq p < 2$, la situation est semblable au cas $p = 0$ et il y a bifurcation uniquement pour des valeurs de μ d'un sous-ensemble discret de $(0, \infty)$, mais, à partir de $p = 2$, on observe un changement spectaculaire du diagramme de bifurcation ainsi que de la forme des configurations d'équilibre. Il existe un nombre $\Lambda(A) \geq 0$ déterminé par la linéarisation, tel que, pour $\mu \leq \Lambda(A)$, $u \equiv 0$ est l'unique solution du problème et elle minimise l'énergie dans l'espace de toutes les configurations admissibles. Pour $\mu > \Lambda(A)$, l'énergie est minimisée par une solution non triviale.
(1) Pour $0 \leq p < 2$, $\lim_{s \to 0} u(s) \in (-\pi, \pi)$ pour toute solution non triviale tandis que $\lim_{s \to 0} u(s) = \pm\pi$ si $p \geq 2$.
(2) Pour $0 \leq p \leq 2, \Lambda(A) > 0$ tandis que $\Lambda(A) = 0$ pour $p > 2$.
(3) Pour $0 \leq p < 2$, il y a bifurcation de la solution $u \equiv 0$ uniquement aux points μ_i de l'ensemble discret des valeurs propres du problème linéarisé, où $\mu_1 = \Lambda(A)$ et $\lim_{i \to \infty} \mu_i = \infty$. Pour $p = 2$, il y a bifurcation en tous les points $\mu \in [L/4, \infty)$. Les propriétés du problème linéarisé où (1) est remplacé (5) changent également en $p = 2$. Pour $0 \leq p < 2$, le spectre essentiel est vide et toutes les fonctions propres n'ont qu'un nombre fini de zéros dans $[0, 1]$. Pour $p = 2$, L/4 appartient au spectre essentiel et, dans certains cas, il n'y a aucune fonction propre. De plus, pour $p = 2$ and $\mu > L/4$, toute solution de (5) a un nombre infini de zéros dans $[0, 1]$, tandis que les solutions $u \not\equiv 0$ du problème non linéaire (1)-(3) n'ont qu'un nombre fini de zéros dans cet intervalle.

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1. Introduction

The study of buckling of a tapered rod reduces to the nonlinear eigenvalue problem

$$\{A(s)u'(s)\}' + \mu \sin u(s) = 0 \text{ for all } s \in (0,1), \tag{1}$$

$$u(1) = \lim_{s \to 0} A(s)u'(s) = 0 \tag{2}$$

$$\text{and} \qquad \int_0^1 A(s)u'(s)^2 ds < \infty \tag{3}$$

where $A \in C([0,1])$ is such that $A(s) > 0$ for all $s > 0$. In the simplest interpretation, the variable s measures arc-length along an inextensible rod whose centre line is a plane curve $\{r(s) : 0 \leq s \leq 1\}$ given by $r(1) = (0,0)$ and $r'(s) = -(\sin u(s), \cos u(s))$. A force $f(0,-1)$ with $f > 0$ is applied at the end $r(0)$. The dependent variable $u(s)$ measures the angle between the tangent at $r(s)$ and the direction $(0,-1)$. We assume that the rod obeys the Bernoulli-Euler bending law with stiffness $E > 0$, [1], and we use $A(s)$ to denote the moment of inertia of the cross-section at $r(s)$ about its centroid, perpendicular to the plane of bending. Then the equation (1) with $\mu = f/E$ expresses the mechanical equilibrium of the rod and (2) means that the lower end $r(1)$ is clamped vertically upwards whereas the upper end $r(0)$ is free. Since the energy of any configuration is $EJ_\mu(u)$ where

$$J_\mu(u) = \int_0^1 [\frac{1}{2}A(s)u'(s)^2 - \mu\{1-\cos u(s)\}]ds \tag{4}$$

the condition (3) is just the requirement that an equilibrium configuration must have finite energy.

The more interesting problem of a heavy rod with variable cross-section buckling under its own weight can also be reduced to the above form by a change of variable. Then no force is applied at $r(0)$ and μ is proportional to the mass density of the rod. In that context we are particularly interested in singular cases where $A(s) \to 0$ as $s \to 0$. With this in mind we introduce the following terminology.

A profile for an elastica with tapering of order $p \geq 0$ is a function $A \in C([0,1])$ such that $A(t) > 0$ for $0 < t \leq 1$ and there exists $L \in (0,\infty)$ such that $\lim_{t\to 0} \frac{A(t)}{t^p} = L$. For such a profile A a solution of Problem P is a function $u \in C^1((0,1])$ such that $Au' \in C^1((0,1])$, and (1), (2) and (3) are satisfied.

This note summaries some of the results in [6] which show that in several respects (shape of the buckled configurations, nature of the bifurcation diagrams) tapering of order 2 plays a critical role, in the sense that the situation when $p < 2$ is very different from what occurs when $p \geq 2$. I am grateful to Professor S.J. Cox for drawing my attention to the absence of results about bifurcation for tapered rods.

2. Energy space and linearized problem

For $p \in [0,\infty)$, let

$$H_p = \left\{ u \in L^1_{loc}((0,1]) : \int_0^1 s^p u'(s)^2 ds < \infty \text{ and } u(1) = 0 \right\}.$$

If A is a profile for an elastica with tapering of order $p \geq 0$, then

$$\int_0^1 s^p u'(s)v'(s)ds \text{ and } \langle u,v\rangle_A = \int_0^1 A(s)u'(s)v'(s)ds$$

are scalar products on $H_A = H_p$ and they induce equivalent norms. Thus the space $(H_A, \langle\cdot,\cdot\rangle_A)$ of all configurations with finite energy is a Hilbert space. Solutions of Problem P are stationary points of the energy (4) in the following sense. A function u is a solution of Problem P if and only if $u \in H_A$ and

$$\int_0^1 A(s)u'(s)v'(s)ds = \mu \int_0^1 v(s)\sin u(s)ds \text{ for all v} \in H_A \cap L^1(0,1).$$

For large p, the energy J_μ is not differentiable. However, for $p \in [0,2], J_\mu \in C^1(H_A)$ and there exists $G_A \in C(H_A, H_A)$ such that

$$J'_\mu(u)v = \langle u, v\rangle_A - \mu \int_0^1 v(s) \sin u(s) ds = \langle u - \mu G_A(u), v\rangle_A \quad \text{for all } u, v \in H_A.$$

Furthermore $G_A : H_A \to H_A$ is compact and the Problem P can be written as

$$F(\mu, u) = 0 \text{ where } F = I - \mu G_A : \mathbb{R} \times H_A \to H_A.$$

We now consider the differentiability of G_A for profiles with tapering of order $p \in [0,2]$. For each $u \in H_A$, there is a unique bounded linear operator, denoted by $L_A(u) : H_A \to H_A$ such that

$$\langle L_A(u)w, v\rangle_A = \int_0^1 w(s)v(s) \cos u(s) ds \text{ for all } v, w \in H_A.$$

Clearly $L_A(u) : H_A \to H_A$ is self-adjoint.
(i) For $0 \le p < 2$, $L_A(u)$ is the Fréchet derivative of G_A at u, $J_\mu \in C^2(H_A)$ and $G'_A(u) = L_A(u) : H_A \to H_A$ is a compact linear operator.
(ii) For $p = 2$ and all $u \in H_A, L_A(u)$ is the Gâteaux derivative of G_A at u, but $L_A(0)$ is not a compact linear operator and G_A is not Fréchet differentiable at $u = 0$. Thus $J_\mu \notin C^2(H_A)$.

Along with (1) we consider its linearization

$$\{A(s)u'(s)\}' + \mu u(s) = 0 \text{ for all } s \in (0,1] \tag{5}$$

and the infimum of its Rayleigh quotient

$$\Lambda(A) = \inf \left\{ \int_0^1 A(s)u'(s)^2 ds \Big/ \int_0^1 u(s)^2 ds : u \in H_A \text{ with } u \not\equiv 0 \right\}.$$

We find that $\Lambda(A) = 0$ if $p > 2$ and $\Lambda(A) \ge L/4 > 0$ if $p \in [0,2]$.

Indeed, for $p \in [0,2]$, we set $T = L_A(0)$ and observe that $1/\Lambda(A)$ is the infimum of the spectrum of T. Furthermore the non-trivial solutions of (5), (2) and (3) are precisely the eigenfunctions of the bounded self-adjoint operator $T : H_A \to H_A$. Let $\sigma_e(T)$ denote its essential spectrum.

For $p \in [0,2]$, u is a solution of the linear problem (5), (2) and (3) if and only if $u \in H_p$ and $u = \mu T u$. Furthermore, all eigenvalues of T are simple.

For $0 \le p < 2$, $\sigma(T) = \{\lambda_i : i \in \mathbb{N}\} \cup \{0\}$ and $\sigma_e(T) = \{0\}$ where $\lambda_{i+1} < \lambda_i, \lambda_1 = \Lambda(A)^{-1}$ and $\lim_{i\to\infty} \lambda_i = 0$. If φ_i is an eigenfunction of T associated with λ_i then $\varphi_i \in C^1((0,1]) \cap C([0,1]), \varphi_i(0) \neq 0$ and φ_i has exactly i zeros in $(0,1]$ and all of which are simple.

For $p = 2$, $\max \sigma_e(T) = \frac{4}{L}$ and, for some profiles, T has no eigenfunctions in H_A.

3. Energy minimizing solutions

The energy minimizing solutions u_μ, defined below, form a continuous curve in H_A which bifurcates to the right from the solution $u \equiv 0$ at $\mu = \Lambda(A)$. For $0 \le p < 2$, this curve also bifurcates from 0 in $L^\infty(0,1)$, but this is no longer true for $p \ge 2$. Indeed for $p \ge 2$, there is a boundary layer phenomenon at $s = 0$ as $\mu \to \Lambda(A)+$ since $\lim_{s\to 0} u_\mu(s) = \pi$ for all $\mu > \Lambda(A)$ but $u_\mu(s) \to 0$ as $\mu \to \Lambda(A)+$ uniformly on $[t,1]$ for all $t \in (0,1)$.

However for all $p \ge 0$, we observe that the curve of centroids of the sections, $r_\mu(s)$, generated by the solution u_μ converges uniformly on $[0,1]$ to the straight configuration $(0, 1-s)$ as $\mu \to \Lambda(A)+$.

THEOREM 3.1. – *Let A be a profile with tapering of order $p \ge 0$ and, for $\mu > 0$, set*

$$m(\mu) = \inf\{J_\mu(u) : u \in H_A\}.$$

(a) $m : (0,\infty) \to \mathbb{R}$ is a non-increasing Lipschitz continuous function with $\lim_{\mu\to 0} m(\mu) = 0$ and $\lim_{\mu\to\infty} m(\mu) = -\infty$.

(b) There is an element $u_\mu \in H_A$ such that $J_\mu(u_\mu) = m(\mu)$ and $u_\mu(s) \ge 0$ for all $s \in (0,1]$.

(c) u_μ is a solution of Problem P.

(d) For $0 < \mu \le \Lambda(A)$, $m(\mu) = 0$, $u_\mu \equiv 0$ and $J_\mu(u) > 0$ for all $u \in H_A\backslash\{0\}$. For $\mu > \Lambda(A)$, $m(\mu) < 0$, $0 < u_\mu(s) < \pi$ for all $s \in (0,1)$, $u'_\mu(s) < 0$ for all $s \in (0,1]$ and $\{u \in H_A : J_\mu(u) = m(\mu)\} = \{\pm u_\mu\}$. Thus in both cases, u_μ is uniquely determined by μ for all $\mu > 0$.

(e) For $\lambda > \mu > \Lambda(A)$, $u_\lambda(s) > u_\mu(s)$ for all $s \in (0.1)$.

(f) Setting $U(\mu) = u_\mu$, the function $U : (0,\infty) \to H_A$ is continuous.

(g) If $\{v_n\}$ is a solution of Problem P for $\mu_n > \Lambda(A)$ where $J_{\mu_n}(v_n) \le 0$ and $\lim_{n\to\infty} \mu_n = \Lambda(A)$, then $\lim_{n\to\infty} J_{\mu_n}(v_n) = \lim_{n\to\infty} \|v_n\|_A = 0$ and $v_n \to 0$ uniformly on compact subsets of $(0,1]$. In particular, $\lim_{\mu\to\Lambda(A)+} \|u_\mu\|_A = 0$.

(h) If $p \ge 2$, then $\lim_{s\to 0} u_\mu(s) = \pi$ for all $\mu > \Lambda(A)$.

(i) If $0 \le p < 2$, then $\lim_{s\to 0} u_\mu(s) < \pi$ for all $\mu > \Lambda(A)$, and $\|u_\mu\|_{L^\infty(0,1)} \to 0$ as $\mu \to \Lambda(A)+$.

(j) For any $t \in (0,1)$, $u_\mu(s) \to \pi$ as $\mu \to \infty$, uniformly on $(0,t]$.

4. Subcritical tapering

We discuss all bifurcations from the trivial solution $u \equiv 0$ in the case $p < 2$. In this case, $\sigma(T) = \{\lambda_i : i \in \mathbb{N}\} \cup \{0\}$ and the singularity at $s = 0$ is sufficiently weak so that we obtain global branches (in the sense of [5]) emanating from all of the eigenvalues $\left\{\mu_i = \frac{1}{\lambda_i} : i \in \mathbb{N}\right\}$ of the linearized problem.

Let

$$E = \{(\mu, u) \in \mathbb{R} \times H_A : u \ne 0 \text{ and } F(\mu, u) = 0\}.$$

and for $i \in \mathbb{N}$, consider $E_i = E \cup \{(\mu_i, 0)\}$ with the metric inherited from $\mathbb{R} \times H_A$. Let C_i denote the maximal connected subset of this space that contains the point

$(\mu_i, 0)$. Let

$$C_i^+ = \{(\mu, u) \in C_i : u'(1) < 0\} \text{ and } C_i^- = \{(\mu, u) \in C_i : u'(1) > 0\}.$$

Clearly $C_i^+ \cap C_i^- = \emptyset$ and $C_i = C_i^+ \cup C_i^- \cup \{(\mu_i, 0)\}$. We set $P(\mu, u) = \mu$.

THEOREM 4.1. – *Let A be a profile with tapering of order $p < 2$ and let $i \in \mathbb{N}$. Then*
(a) C_i^+ is a connected subset of $\mathbb{R} \times H_A$ and of $\mathbb{R} \times C([0,1])$. Furthermore

$$C_i^- = \left\{(\mu, -u) : (\mu, u) \in C_i^+\right\} \text{ and } C_i^+ = \left\{(\mu, -u) : (\mu, u) \in C_i^-\right\}.$$

(b) $u(0) \neq 0$ and u has exactly i zeros in $[0,1]$ for all $(\mu, u) \in C_i^+$.
(c) $PC_i^+ = (\mu_i, \infty)$.
(d) $\|u\|_{L^\infty(0,1)} \in (0, \pi)$ for all $(\mu, u) \in C_i^+$.
(e) There is a function $U \in C^1((\mu_1, \infty), H_A)$ such that $C_1^+ = \{(\mu, U(\mu)) : \mu_i < \mu < \infty\}$. Furthermore $U(\mu)$ is the energy minimizing solution u_μ discussed in Theorem 3.1.

5. Critical tapering

For profiles with tapering of order $p = 2$ the relationship between Problem P and its linearization is weaker than when $p < 2$ since the function $F : \mathbb{R} \times H_A \to H_A$ is not Fréchet differentiable at $(\mu, 0)$ for any $\mu > 0$. Using a well-known result of Clark [4] concerning the genus of sets of critical points we find that every point in $[L/4, \infty)$ is a bifurcation point. This kind of conclusion was first observed in [2,3] for a semilinear elliptic equation on $\mathbb{R}^N$.

THEOREM 5.1. – *Let A be a profile with tapering of order 2 and consider $\mu > L/4$. For this value of μ, there are infinitely many solutions $\{u_k\}$ of Problem P with the property that $|u_k(s)| < \pi$ for all $s \in (0,1]$. Furthermore, $\|u_k\|_A \to 0$ as $k \to \infty$ and the number of zeros of u_k tends to infinity as $k \to \infty$. If the profile A is differentiable near 0 and $\lim_{s\to 0} A'(s)/s$ exists, all non-trivial solutions of Problem P have only a finite number of zeros. On the other hand, for $\mu > L/4$ all solutions of the linearized equation (5) have infinitely many zeros in $[0,1]$.*

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