

NONLINEAR EIGENVALUE PROBLEMS HAVING AN UNBOUNDED BRANCH OF SYMMETRIC BOUND STATES

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Abstract. For a semilinear second order differential equation on \mathbb{R} , the existence of a continuous curve of positive solutions which bifurcates from the lowest eigenvalue of the linearized problem is proved. This curve can be parameterized globally by λ and can be extended to infinity. We establish that all solutions of the equation are even and monotone, and under appropriate conditions, all of them belong to the curve of bifurcation. Our results depend heavily on the combination of symmetry and monotonicity imposed on the equation.

1. Introduction. We consider the nonlinear eigenvalue problem,

$$\begin{aligned} u''(x) + \lambda u(x) + f(x, u(x)) &= 0 \quad \text{for } x \in \mathbb{R} \\ \lim_{|x| \rightarrow \infty} u(x) &= 0, u(x) \geq 0 \quad \text{for all } x \in \mathbb{R} \quad \text{and } u \not\equiv 0, \end{aligned} \tag{D}$$

where $\lambda \in \mathbb{R}$ and the function f has the following properties:

- (A1) (i) $f \in C^1(\mathbb{R}^2)$ with $f(x, 0) = 0$ for all $x \in \mathbb{R}$ and $\partial_2 f(x, s) \rightarrow \partial_2 f(x, 0)$ as $s \rightarrow 0$ uniformly for $x \in \mathbb{R}$.
- (ii) $\partial_2 f(\cdot, 0) \in C^1(\mathbb{R})$ with $\partial_2 f(0, 0) > 0 = \lim_{x \rightarrow \infty} \partial_2 f(x, 0)$.
- (A2) $f(-x, s) = f(x, s)$ for all $x, s \in \mathbb{R}$.
- (A3) (i) $\partial_1 f(x, s) \leq 0$ for all $x, s \geq 0$ and there exists $x_0 > 0$ such that $f(x, s) > f(y, s)$ whenever $0 \leq x < x_0 < y$ and $s > 0$.
- (ii) $\partial_2 f(x, s) > s^{-1} f(x, s) > 0$ for all $x \in \mathbb{R}$ and $s > 0$.

A classical solution to the problem (D) is a pair (λ, u) , where $\lambda \in \mathbb{R}$ and $u \in C^2(\mathbb{R})$ satisfy (D). Under the above hypotheses on f we can give a more or less complete description of the set of all solutions of (D). This is because they constitute a favorable combination of symmetry and monotonicity which is inherited by the solutions of the problem. Note that (A3)(ii) ensures that $s^{-1} f(x, s)$ is a strictly increasing function of s on $(0, \infty)$ for each fixed $x \in \mathbb{R}$, and so

$$\frac{f(x, s)}{s} > \lim_{s \rightarrow 0^+} \frac{f(x, s)}{s} = \partial_2 f(x, 0) \quad \text{for } s > 0 \text{ and } x \in \mathbb{R}.$$

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Furthermore, (A3)(i) means that $f(x, s)$ is a non-increasing, but not constant, function of x on $(0, \infty)$ for each fixed $s > 0$. This implies that

$$\partial_2 f(x, 0) = \lim_{s \rightarrow 0^+} s^{-1} f(x, s)$$

is also a non-increasing function of x on $(0, \infty)$ and so, by (A1)(ii), $\partial_2 f(x, 0) \geq 0$ for all x . Hence,

$$\frac{f(x, s)}{s} > \partial_2 f(x, 0) \geq 0 \quad \text{for } s > 0 \text{ and } x \in \mathbb{R}.$$

Our first results show that, for all solutions of (D), u is an even function which is strictly decreasing on $(0, \infty)$. In particular this implies that the problem (D) is equivalent to the corresponding Neumann problem,

$$\begin{aligned} u''(x) + \lambda u(x) + f(x, u(x)) &= 0 \quad \text{for } x > 0 \\ u'(0) = \lim_{x \rightarrow \infty} u(x) &= 0, \quad u(x) \geq 0 \quad \text{for all } x \geq 0 \text{ and } u \not\equiv 0. \end{aligned} \tag{N}$$

To be more precise, let

$$X = \{u \in H^2((0, \infty)) : u'(0) = 0\}$$

be equipped with the norm

$$\|u\|_X = \{|u|_2^2 + |u''|_2^2\}^{\frac{1}{2}} \quad \text{for } u \in X,$$

where $|u|_p$ denotes the usual norm in $L^p((0, \infty)) = L^p$ for $1 \leq p \leq \infty$. Then $X \subset C^1([0, \infty)) \cap L^p$ for $2 \leq p \leq \infty$ by Sobolev's embedding [4] and $\lim_{x \rightarrow \infty} u(x) = \lim_{x \rightarrow \infty} u'(x) = 0$ for all $u \in X$. Clearly $u \in C^2([0, \infty))$ and (λ, u) is a classical solution to (N) provided that $(\lambda, u) \in \mathbb{R} \times X$ and (N) is satisfied for almost all $x > 0$. Furthermore an even extension of a solution to (N) is a solution of (D). The equivalence between (D) and (N) will be proved in Section 2.

Before proceeding we introduce a reformulation of (D) which is more convenient for much of the ensuing discussion. A function f satisfies the hypotheses (A1) to (A3) if and only if it can be written as

$$f(x, s) = p(x)s + h(x, s),$$

where the functions p and h satisfy the following conditions.

- (H1) $p \in C^1(\mathbb{R})$ is even, $\lim_{|x| \rightarrow \infty} p(x) = 0$ and $p(0) > 0$.
- (H2) $h \in C^1(\mathbb{R}^2)$ with $h(x, s) = h(-x, s)$ for all $(x, s) \in \mathbb{R}^2$, $h(x, 0) = 0$ for all $x \in \mathbb{R}$ and $\lim_{s \rightarrow 0} \partial_2 h(x, s) = 0$ uniformly for $x \in \mathbb{R}$.
- (H3) (i) $\partial_1 [p(x)s + h(x, s)] \leq 0$ for all $s \geq 0$ and $x \geq 0$. Furthermore there exists $x_0 > 0$ such that $0 \leq x < x_0 < x'$ implies $p(x')s + h(x', s) < p(x)s + h(x, s)$ for all $s > 0$.
- (ii) $\partial_2 h(x, s)s > h(x, s) > 0$ for all $s > 0$ and $x \in \mathbb{R}$.

Note that

$$p(x) = \partial_2 f(x, 0) \quad (1)$$

and the equation (D) can be written as

$$u''(x) + \lambda u(x) + p(x)u(x) + h(x, u(x)) = 0.$$

An important constant λ_0 is defined by

$$\lambda_0 = \inf \left\{ \int_{-\infty}^{\infty} (u')^2 - pu^2 dx : u \in H^2(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} u^2 dx = 1 \right\}. \quad (2)$$

It follows that $\lambda_0 < 0$ and, as is shown in Section 2, it is the lowest eigenvalue of the linearization of (N) at $u = 0$. That is,

$$-u''(x) - p(x)u(x) = \lambda u(x), \quad u \in X. \quad (L)$$

The equivalence between (D) and (N) is a consequence of the following result.

Theorem 1. *Let the conditions (H1), (H2) and (H3) be satisfied. Let (λ, u) be a classical solution to (D). Then $\lambda < \lambda_0$ and u is an even function with $u'(x) < 0$ for all $x > 0$.*

Under the hypotheses of this theorem we show that $\lambda_0 < 0$ and that λ_0 is a simple eigenvalue of (L). Then using the Crandall-Rabinowitz Theorem [5] we prove, in Section 3, that a branch of solutions bifurcates from the point $(\lambda_0, 0)$ in $\mathbb{R} \times X$. In fact we are able to prove that the curve of solutions obtained by bifurcation can be parameterized globally by λ and can be extended to infinity. Furthermore the maximum value of the solutions is a strictly decreasing function of λ .

To be more precise, let $K = \{u \in X : u(x) > 0 \text{ and } u'(x) < 0 \text{ for all } x > 0\}$.

Theorem 2. *Let the conditions (H1), (H2) and (H3) be satisfied. Then there exist λ^* and $u \in C^1((\lambda^*, \lambda_0), X)$ such that for all $\lambda \in (\lambda^*, \lambda_0)$, $(\lambda, u(\lambda))$ is a solution to (N) and $u(\lambda) \in K$. Moreover*

$$\frac{d}{d\lambda} u(\lambda)(0) = \frac{d}{d\lambda} |u(\lambda)|_{\infty} < 0$$

for all $\lambda \in (\lambda^*, \lambda_0)$ and

$$\lim_{\lambda \rightarrow \lambda_0} \|u(\lambda)\|_X = 0, \quad \lim_{\lambda \rightarrow \lambda^*} \|u(\lambda)\|_X = \infty.$$

Remarks.

1. Under the assumption (H4) introduced below we also have that

$$\lim_{\lambda \rightarrow \lambda^*} |u(\lambda)|_{\infty} = \infty.$$

2. According to Theorem 2, $u(\lambda)(0) = |u(\lambda)|_\infty$ is a strictly decreasing function of λ . However the solutions $u(\lambda)$ are not ordered. Indeed by Lemma 4 c),

$$\lim_{x \rightarrow \infty} \frac{u(\lambda)(x)}{u(\mu)(x)} = 0 \text{ when } \lambda < \mu,$$

whereas by Theorem 2,

$$\frac{u(\lambda)(0)}{u(\mu)(0)} > 1 \text{ when } \lambda < \mu.$$

Theorem 2 is established by a similar method to that developed in [9]. The two main steps amount to establishing that, given any solution (λ_1, u_1) of (N), the implicit function theorem can be used to show that in a neighborhood of (λ_1, u_1) there exists a unique continuous branch $(\lambda, u(\lambda))$ of solutions satisfying $u(\lambda) \in K$ and that the set of solutions is compact in a certain sense which is made precise in Lemma 5.

Next we concentrate our attention on the value λ^* defined in Theorem 2. For this we now introduce an additional assumption.

- (H4) There exist positive constants σ and A as well as a function $\mathcal{A} \in C^1(\mathbb{R})$ such that

$$\lim_{s \rightarrow 0^+} \frac{h(x, s)2^\sigma}{s^{2\sigma+1}} = \mathcal{A}(x) \geq A > 0.$$

In Section 4 we estimate the value of the parameter at which the norm $\|u\|_X$ becomes infinite. This value λ^* depends on the behavior of $f(x, s)$ when s is large. By (H3)(ii), $s^{-1}f(x, s)$ is an increasing function of s on $(0, \infty)$ for each fixed $x \in \mathbb{R}$ and we distinguish the following cases.

$$(L1) \quad \lim_{s \rightarrow \infty} \lim_{x \rightarrow \infty} s^{-1}f(x, s) = \infty.$$

By (H1) to (H3),

$$0 \leq \lim_{x \rightarrow \infty} s^{-1}f(x, s) \leq s^{-1}f(x, s) \text{ for all } x \geq 0$$

and so it follows from (L1) that $\lim_{s \rightarrow \infty} s^{-1}f(x, s) = \infty$ for all $x \geq 0$.

- (L2) There exists $P \in L^\infty((0, \infty))$ such that $\lim_{s \rightarrow \infty} s^{-1}f(x, s) = P(x)$ uniformly on x in compact intervals.

From (H1) to (H3) it follows that $P(x) > \lim_{s \rightarrow 0^+} s^{-1}f(x, s) = p(x) \geq 0$ for all $x \in \mathbb{R}$ and P is a continuous function which is non-increasing on $(0, \infty)$.

We find that under the hypothesis (L1), the curve of solutions $(\lambda, u(\lambda))$ given by Theorem 2 can be extended for all $\lambda \in (-\infty, \lambda_0)$; i.e., $\lambda^* = -\infty$. On the other

hand, the hypothesis (L2) implies that λ^* is finite. Indeed we can characterize λ^* by introducing the “linearization at infinity” of problem (N),

$$-u''(x) - P(x)u(x) = \lambda u(x), \quad u \in X. \quad (\text{N}_\infty)$$

We show that λ^* is the infimum of the spectrum of (N_∞) , but λ^* is not necessarily an eigenvalue of (N_∞) which may have purely continuous spectrum. However we obtain the following estimates

$$\lambda_0 - \sup_{y \geq 0} K(y) \leq \lambda^* \leq \lambda_0 - \inf_{y \geq 0} K(y),$$

where

$$K(x) = P(x) - p(x) > 0 \text{ on } \mathbb{R}.$$

Finally we give conditions on the function f which ensure that all solutions to (N) belong to the curve of solutions obtained in Theorem 2.

Theorem 3. *Let the conditions (H1) to (H4) be satisfied. Let one of the hypotheses (L1) or (L2) hold. Then for the function $u \in C^1((\lambda^*, \lambda_0), X)$ given by Theorem 2 we have that*

$$\{(\lambda, u(\lambda)) : \lambda \in (\lambda^*, \lambda_0)\} = \{(\mu, v) : (\mu, v) \text{ is a solution to (N)}\}.$$

All of our results depend heavily on the combination of symmetry and monotonicity implied by the conditions (A2) and (A3), and Theorem 1 shows that all solutions have similar properties. It should be emphasized that the symmetry of the equation ensured by (A1) and (A2) alone is not sufficient to imply that all solutions of (D) are even. This is illustrated by an important example due to Akhmediev [1] in the context of nonlinear optics. Our results are also relevant to this field and, in the Appendix, we discuss these issues more fully.

2. Properties of the solutions of (D) and (N). In this section we establish a series of properties for the pairs (λ, u) satisfying (D) or (N), one of the main conclusions being that these two problems are in fact equivalent.

We begin by discussing the spectrum of the linearization of the problem (D) under the hypotheses (H1) to (H3). Let $S : \mathcal{D}(S) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ be the self-adjoint operator defined by $Su = -u'' - pu$ for $u \in H^2(\mathbb{R})$ where p is defined by (1). The infimum of the spectrum, $\sigma(S)$, of S is characterized by $\inf \sigma(S) = \lambda_0$ where λ_0 is defined by (2), whereas for the essential spectrum, $\sigma_e(S)$, we have that

$$\inf \sigma_e(S) = \lim_{x \rightarrow \infty} p(x) = 0.$$

Lemma 1. *Let the conditions (H1), (H2) and (H3) be satisfied. Let (λ, u) be a solution of (D). Then $\lambda < \lambda_0 < 0$.*

Proof. It follows easily from (H1) to (H3) that $p'(x) \leq 0$ for all $x \geq 0$ and so $p(x) \geq 0$ for all $x \in \mathbb{R}$. Since $p(0) > 0$ this means that $\lambda_0 < 0$ and that λ_0 is a simple eigenvalue of S with an eigenfunction φ which can be chosen so that $\varphi(x) > 0$ for all $x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} \varphi^2 dx = 1$.

Let (λ, u) be any solution of (D). Then

$$\begin{aligned} \lambda_0 \int_{-\infty}^{\infty} \varphi(x) u(x) dx &= \int_{-\infty}^{\infty} (S\varphi)(x) u(x) dx = \int_{-\infty}^{\infty} (Su)(x) \varphi(x) dx \\ &= \int_{-\infty}^{\infty} (\lambda u(x) + h(x, u(x))) \varphi(x) dx > \lambda \int_{-\infty}^{\infty} u(x) \varphi(x) dx \end{aligned}$$

by (H3)(ii) since $u \geq 0$ but $u \not\equiv 0$. The result follows since we also have that $\int_{-\infty}^{\infty} \varphi u dx > 0$. \square

We now turn to the main part of the proof of Theorem 1 which can be seen as a simplified version of the method of moving planes. See [6] and [7].

Lemma 2. *Let the conditions (H1), (H2) and (H3)(i) be satisfied. Let (λ, u) be a solution of (D) with $\lambda < 0$. Then $u(x) = u(-x)$ and $u'(x) < 0$ for all $x > 0$.*

Proof. To simplify the notation, we reformulate the problem (D). For λ fixed, the problem (D) may be written as

$$\begin{aligned} u''(x) + N(x, u(x)) &= 0 \text{ for } x \in \mathbb{R} \\ \lim_{|x| \rightarrow \infty} u(x) &= 0, u \geq 0 \text{ for all } x \in \mathbb{R} \text{ and } u \not\equiv 0, \end{aligned}$$

where $N \in C^1(\mathbb{R}^2)$ is defined by

$$N(x, s) = \lambda s + f(x, s).$$

Clearly N is symmetric in x on \mathbb{R} and non-increasing in x for $x > 0$. Note some further properties of N which are implied by the hypotheses (H1), (H2) and (H3)(i).

Property 1. $\lim_{s \rightarrow 0} \partial_2 N(x, s) = \lambda + p(x)$ uniformly for $x \in \mathbb{R}$.

Property 2. $N(x, s) \leq N(x', s)$ for $x \leq x'$, $x + x' \leq 0$ and $s > 0$.

Property 3. There exists $x_0 < 0$ such that $N(x, s) < N(x', s)$ if $x < x_0 < x' < -x_0$ and $s > 0$.

We define a function $w : \{(x, \mu) \in \mathbb{R}^2 : x \leq \mu\} \rightarrow \mathbb{R}$ by $w(x, \mu) = u(2\mu - x) - u(x)$. Note that

$$w(\mu, \mu) = 0 \text{ and that } \lim_{x \rightarrow -\infty} w(x, \mu) = 0. \quad (3)$$

Since $p(x) \rightarrow 0$ as $x \rightarrow \infty$ it follows from Property 1 that there exist $\delta > 0$ and $R > 0$ such that $\partial_2 N(x, s) \leq \lambda/2 < 0$ for all $x \leq -R$ and $0 \leq s \leq \delta$. Since $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$, there exists $R_0 \geq R > 0$ such that $0 \leq u(x) \leq \delta$ for all $x \leq -R_0$.

Setting

$$J = \{\mu \leq 0 : w(x, \mu) \geq 0 \text{ for all } x \leq \mu\}.$$

The rest of the proof is split into the following steps.

1. If $x \leq -R_0$ and $w(x, \mu) < 0$ then $\partial_1^2 w(x, \mu) < 0$.
2. For any $\mu \in J$, either a) $w(x, \mu) \equiv 0$ for $x \leq \mu$ or b) $w(x, \mu) > 0$ for all $x < \mu$ and $\partial_1 w(\mu, \mu) < 0$.
3. $(-\infty, -R_0] \subset J$ Then we set,

$$\bar{\mu} = \sup\{l \leq 0 : (-\infty, l] \subset J\}.$$

Since w is continuous on $\{(x, \mu) \in \mathbb{R}^2 : x \leq \mu\}$, we have $\bar{\mu} \in J$.

4. If $\mu \in J \cap (-\infty, 0)$, then 2b) occurs.
5. $\bar{\mu} = 0$.
6. $u(x) = u(-x)$ and $u'(x) > 0$ for all $x < 0$.

Clearly 6 establishes the theorem.

1. By the definition of w ,

$$\begin{aligned} \partial_1^2 w(x, \mu) &= u''(2\mu - x) - u''(x) \\ &= N(x, u(x)) - N(2\mu - x, u(2\mu - x)) \\ &\leq N(x, u(x)) - N(x, u(2\mu - x)) \end{aligned}$$

by Property 2 since $\mu \leq 0$.

But then $\partial_1^2 w(x, \mu) \leq -c(x, \mu)w(x, \mu)$ where

$$c(x, \mu) = \int_0^1 \partial_2 N(x, u(x) + tw(x, \mu)) dt.$$

Now $u(x) + tw(x, \mu) = tu(2\mu - x) + (1 - t)u(x)$ and so for $t \in [0, 1]$, $0 \leq u(x) + tw(x, \mu) \leq u(x) \leq \delta$ since $w(x, \mu) < 0$ and $x \leq -R_0$. It follows that $c(x, \mu) \leq \lambda/2 < 0$ and hence that $\partial_1^2 w(x, \mu) < 0$.

2. As in point 1, for all $x \leq \mu$ we have

$$\begin{aligned} \partial_1^2 w(x, \mu) &\leq -c(x, \mu)w(x, \mu), \quad \text{and so} \\ \partial_1^2 w(x, \mu) - c(x, \mu)^- w(x, \mu) &\leq -c(x, \mu)^+ w(x, \mu) \leq 0, \end{aligned}$$

where $c(x, \mu)^-, c(x, \mu)^+$ and $w(x, \mu)$ are all non-negative. The conclusion now follows from the one-dimensional maximum principle. (See Theorems 3 and 4 in Chapter 1 of [8].)

3. Consider $\mu \leq -R_0$. If there exists $x_0 < \mu$ such that $w(x_0, \mu) < 0$ then there exists $y < -R_0$ such that

$$w(y, \mu) = \min_{x \leq \mu} w(x, \mu) < 0.$$

But then $\partial_1 w(y, \mu) = 0$ and $\partial_1^2 w(y, \mu) \geq 0$. Since this contradicts point 1, we must have that $w(x, \mu) \geq 0$ for all $x \leq \mu$.

4. Let $x_0 < 0$ be the point mentioned in Property 3 of N . Since $\mu < 0$, it follows that $2\mu + x_0 < \min\{x_0, 2\mu - x_0, \mu\}$. We consider a point x such that $2\mu + x_0 < x < \min\{x_0, 2\mu - x_0, \mu\}$. Then $x < x_0 < 2\mu - x < -x_0$ and so by Property 3,

$$N(x, s) < N(2\mu - x, s) \text{ for all } s > 0.$$

But if 2a) holds then

$$\begin{aligned} 0 &= \partial_1^2 w(x, \mu) = N(x, u(x)) - N(2\mu - x, u(2\mu - x)) \\ &= N(x, u(x)) - N(2\mu - x, u(x)) \text{ for all } x \leq \mu. \end{aligned}$$

Thus $u(x) = 0$ for $2\mu + x_0 < x < \min\{x_0, 2\mu - x_0, \mu\}$ if 2a) holds. It follows from (D) that $u \equiv 0$ on \mathbb{R} if $u(x) = u'(x) = 0$ at some point $x \in \mathbb{R}$. Hence we see that 2a) cannot occur.

5. Suppose that $\bar{\mu} \neq 0$. Since $\bar{\mu} \in J$, $w(\cdot, \bar{\mu})$ satisfies 2b). Also there exists a sequence $\{\mu_k\}$ such that $\bar{\mu} < \mu_k < 0$, $\mu_k \rightarrow \bar{\mu}$ and $\mu_k \notin J$. By (3), there exists $y_k < \mu_k$ such that

$$w(y_k, \mu_k) = \min_{x \leq \mu_k} w(x, \mu_k) < 0.$$

In particular, $\partial_1 w(y_k, \mu_k) = 0$ and $\partial_1^2 w(y_k, \mu_k) \geq 0$.

From point 1, we know that $-R_0 \leq y_k < \mu_k$ and so passing to subsequence we can suppose that $y_k \rightarrow y \in [-R_0, \bar{\mu}]$. Then $\partial_1 w(y, \bar{\mu}) = \lim_{k \rightarrow \infty} \partial_1 w(y_k, \mu_k) = 0$ and $w(y, \bar{\mu}) = \lim_{k \rightarrow \infty} w(y_k, \mu_k) \leq 0$. But $w(y, \bar{\mu}) \geq 0$ since $y \in J$. Hence $w(y, \bar{\mu}) = 0$ and $\partial_1 w(y, \bar{\mu}) = 0$ which contradicts 2b).

6. Since $\bar{\mu} = 0 \in J$ we have that $w(x, 0) \geq 0$ for all $x \leq 0$. Thus $u(-x) \geq u(x)$ for all $x \leq 0$. But setting $v(x) = u(-x)$ we have that (λ, v) also satisfies (D) and so we also have $v(-x) \geq v(x)$ for all $x \leq 0$. Thus

$$u(-x) \geq u(x) = v(-x) \geq v(x) = u(-x) \text{ for all } x \leq 0$$

showing that u is even.

Recalling that $\mu \in J$ for every $\mu < 0$ we have that $w(\cdot, \mu)$ satisfies 2b) for $\mu < 0$ and so $\partial_1 w(\mu, \mu) < 0$. But $\partial_1 w(\mu, \mu) = -2u'(\mu)$. Hence $u'(\mu) > 0$ for all $\mu < 0$. This completes the proof.

Corollary 1. *Let p satisfy (H1) with $p'(x) \leq 0$ for all $x \geq 0$. Then the eigenfunction φ of S , defined in page 6, is even and $\varphi'(x) < 0$ for all $x > 0$.*

Proof. Setting $f(x, s) = p(x)s$, we see that the conditions (H1), (H2) and (H3)(i) are satisfied. Since $\lambda_0 < 0$ and $\varphi \geq 0$, the result follows from Lemma 2. \square

The following is a simple consequence of the fact that all solutions of (N) generate solutions of (D) by even extension.

Lemma 3. *Let the conditions (H1), (H2) and (H3) be satisfied. Let (λ, u) be a solution to (N). Then $\lambda < \lambda_0 < 0$ and $u \in K$.*

The rest of this section is devoted to establishing some important properties of the set of all solutions to (N). To simplify the proofs of most of the results which follow, we introduce a new function $q : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ defined by

$$q(x, s) = \begin{cases} \frac{h(x, \sqrt{2s})}{\sqrt{2s}} & \text{for } s > 0 \\ 0 & \text{for } s = 0, \end{cases}$$

where h satisfies the conditions (H2) and (H3). (As it is shown in the Appendix, this way of expressing the problem arises naturally in the context of guided waves in nonlinear optics.) Then problem (N) can be rewritten as

$$\left. \begin{aligned} u''(x) + \lambda u(x) + p(x)u(x) + q(x, \frac{1}{2}u^2(x))u(x) &= 0 \text{ for all } x > 0 \\ u'(0) = \lim_{x \rightarrow \infty} u(x) = 0, u &\geq 0 \text{ for all } x \geq 0 \text{ and } u \not\equiv 0. \end{aligned} \right\}$$

First let us note some consequences of the assumptions (H1), (H2) and (H3).

1. The function $q(x, s)$ is differentiable on $[0, \infty) \times (0, \infty)$, and $q(x, s) > 0$, $\partial_2 q(x, s) > 0$ for all $s > 0, x \geq 0$.
2. On setting

$$Q(x, s) = \int_0^s q(x, t) dt,$$

we have that $0 < Q(x, s) < q(x, s)s$ for all $(x, s) \in [0, \infty) \times (0, \infty)$.

3. $\partial_1 [p(x) + q(x, s)] \leq 0$ for all $x, s \geq 0$ and $q(x, s) \rightarrow 0$ if $s \rightarrow 0$ uniformly for $x \geq 0$. This implies that $\lim_{s \rightarrow 0} Q(x, s)s^{-1} = 0$ uniformly for $x \geq 0$.
4. Since $\lim_{x \rightarrow \infty} \{p(x) + q(x, s)\} = \lim_{x \rightarrow \infty} q(x, s)$ exists and is a non-decreasing function of s , we can set $g(s) = \lim_{x \rightarrow \infty} q(x, s)$ and $G(s) = \int_0^s g(t) dt$. Furthermore $G(s) \leq g(s)s$ for all $s \geq 0$.
5. Setting

$$B(x, s) = \int_0^s f(x, t) dt,$$

we see that

$$B(x, s) = \frac{1}{2}p(x)s^2 + Q(x, \frac{1}{2}s^2) \quad \text{and} \quad \lim_{x \rightarrow \infty} B(x, s) = G(\frac{1}{2}s^2).$$

Lemma 4. *Let the conditions (H1), (H2) and (H3) be satisfied. Let (λ, u) be a solution of (N).*

- a) $0 < 2G(\frac{1}{2}u^2(x)) \leq -u'(x)^2 - \lambda u^2(x) \leq 2Q(x, \frac{1}{2}u^2(x)) + p(x)u^2(x)$.
- b) $\lim_{x \rightarrow \infty} \frac{u'(x)}{u(x)} = -\sqrt{-\lambda}$ and for all $\epsilon > 0$, $\lim_{x \rightarrow \infty} e^{(\sqrt{-\lambda} - \epsilon)x} u(x) = 0$.
- c) *If (μ, v) is a solution of (N) with $\mu > \lambda$, then*

$$\lim_{x \rightarrow \infty} \frac{u(x)}{v(x)} = 0.$$

Proof. a) From Lemma 3, $u \in K$. By multiplying (N) by u' , we obtain

$$\{u'(x)^2 + \lambda u^2(x)\}' = -2f(x, u(x))u'(x) \quad \text{for } x \geq 0,$$

and then by integrating $u'(x)^2 + \lambda u^2(x) = 2 \int_x^\infty f(y, u(y))u'(y)dy$.

But,

$$\frac{d}{dy}B(y, u(y)) = \int_0^{u(y)} \frac{\partial}{\partial y}f(y, t)dt + f(y, u(y))u'(y)$$

so that

$$\begin{aligned} \int_x^\infty f(y, u(y))u'(y)dy &= -B(x, u(x)) - \int_x^\infty \left\{ \int_0^{u(y)} \frac{\partial}{\partial y}f(y, t)dt \right\} dy \\ &\geq -B(x, u(x)) \end{aligned}$$

by (A3)(i). Hence

$$u'(x)^2 + \lambda u^2(x) \geq -2B(x, u(x)).$$

The other inequality comes from the property $p(x) + q(x, s) \geq g(s)$ for all $x \geq 0$ and $s \geq 0$. Hence

$$\begin{aligned} \int_x^\infty (p(y) + q(y, \frac{1}{2}u^2(y)))u(y)u'(y)dy &\leq \int_x^\infty g(\frac{1}{2}u^2(y))u(y)u'(y)dy \\ &= \int_x^\infty \frac{d}{dy}G(\frac{1}{2}u^2(y))dy = -G(\frac{1}{2}u^2(x)). \end{aligned}$$

b) Using $\lim_{x \rightarrow \infty} p(x) = 0$, $\lim_{s \rightarrow 0} Q(x, s)s^{-1} = 0$ uniformly for $x \geq 0$ and the inequality obtained in a), it follows that $\lim_{x \rightarrow \infty} \frac{u'(x)}{u(x)} = -\sqrt{-\lambda}$.

Moreover, given $\epsilon > 0$, there exists $z \geq 0$ such that

$$u'(x) \leq (-\sqrt{-\lambda} + \epsilon)u(x) \quad \text{for all } x \geq z.$$

This implies that $[e^{(\sqrt{-\lambda}-\epsilon)x}u(x)]$ is a decreasing function of x on $[z, \infty)$.

c) Let $w(x) = \frac{u(x)}{v(x)}$ and $\gamma = \sqrt{-\mu} - \sqrt{-\lambda}$. Then $\gamma < 0$ and $w(x) > 0$ for all $x \geq 0$. But

$$\frac{w'(x)}{w(x)} = \frac{u'(x)}{u(x)} - \frac{v'(x)}{v(x)} \rightarrow \gamma \text{ as } x \rightarrow \infty$$

by part b), and so there exists $x_0 \geq 0$ such that

$$\{\ln w(x)\}' \leq \frac{\gamma}{2} < 0 \text{ for all } x \geq x_0.$$

Hence for $x \geq x_0$,

$$\ln w(x) \leq \frac{\gamma}{2}(x - x_0) + \ln w(x_0)$$

and it follows that $w(x) \rightarrow 0$ as $x \rightarrow \infty$.

Lemma 5. *Let the conditions (H1), (H2) and (H3) be satisfied. Suppose that there is a sequence $\{(\lambda_n, u_n)\}$ of solutions to (N) such that $\lambda_n \rightarrow \lambda$ and $\{\|u_n\|_X\} \leq C$. Then there exist $u \in X$ and a subsequence $\{u_{n_k}\}$, such that $\|u_{n_k} - u\|_X \rightarrow 0$. Furthermore, either $u \equiv 0$ or (λ, u) is a solution of (N).*

Proof. First we note that $u_n \in K$, and so

$$xu_n^2(x) \leq \int_0^x u_n^2(y)dy \leq \|u_n\|_X^2 \leq C.$$

By Lemma 4 a)

$$0 \leq -\lambda_n - \left(\frac{u'_n(x)}{u_n(x)}\right)^2 \leq \frac{Q(x, \frac{1}{2}u_n^2(x))}{\frac{1}{2}u_n^2(x)} + p(x) \leq q(x, \frac{1}{2}u_n^2(x)) + p(x).$$

Since $\partial_2 q(x, s) > 0$ and $u_n^2(x) \leq \frac{C}{x}$ for all $x > 0$

$$-\lambda_n - \left(\frac{u'_n(x)}{u_n(x)}\right)^2 \leq q(x, \frac{C}{2x}) + p(x).$$

Hence, given $\epsilon > 0$, there exists $z > 0$ such that

$$-\lambda_n - \left(\frac{u'_n(x)}{u_n(x)}\right)^2 \leq \epsilon$$

for all n and $x > z$. It follows that there exists m such that

$$\frac{u'_n(x)}{u_n(x)} \leq -\frac{1}{2}\sqrt{-\lambda} \quad \text{for all } x \geq z \quad \text{and } n \geq m,$$

and consequently

$$0 < u_n(x) \leq C \exp\left\{-\frac{1}{2}\sqrt{-\lambda}(x-z)\right\} \quad \text{for all } x \geq z \quad \text{and } n \geq m. \quad (4)$$

Now by using this estimate and the equation (N), let us prove the existence of a subsequence of $\{u_n\}$ converging in X . Since $\|u_n\|_X \leq C$, there exist a subsequence $\{u_{n_k}\}$ and $u \in X$ such that $u_{n_k} \rightharpoonup u$ weakly in $H^2((0, \infty))$. If $u \not\equiv 0$, it is easy to see that (λ, u) satisfies (N). By the estimate (4), for all $\varepsilon > 0$ there exists $X(\varepsilon) > 0$ such that

$$\int_{x \geq X(\varepsilon)} u_{n_k}^2(x) dx < \varepsilon \quad \text{and} \quad \int_{x \geq X(\varepsilon)} u^2(x) dx < \varepsilon.$$

Then we have

$$\begin{aligned} |u_{n_k} - u|_2^2 &\leq \int_{x \leq X(\varepsilon)} |u_{n_k}(x) - u(x)|^2 dx + 2 \int_{x \geq X(\varepsilon)} u_{n_k}^2(x) + u(x)^2 dx \\ &\leq \int_{x \leq X(\varepsilon)} |u_{n_k}(x) - u(x)|^2 dx + 4\varepsilon. \end{aligned}$$

Recalling that $u_{n_k} \rightarrow u$ uniformly on compact subsets of $(0, \infty)$, we see that $|u_{n_k} - u|_2 \rightarrow 0$. Moreover,

$$\begin{aligned} u_n''(x) - u''(x) &= \{-\lambda_n - p(x) - q(x, \frac{1}{2}u_n^2(x))\}u_n(x) \\ &\quad + \{\lambda + p(x) + q(x, \frac{1}{2}u^2(x))\}u(x) \\ &= (\lambda - \lambda_n)u(x) + \{\lambda_n + p(x) + q(x, \frac{1}{2}u_n^2(x))\}(u(x) - u_n(x)) \\ &\quad + \{q(x, \frac{1}{2}u^2(x)) - q(x, \frac{1}{2}u_n^2(x))\}u(x), \end{aligned}$$

where $\lim_{k \rightarrow \infty} q(x, \frac{1}{2}u_{n_k}^2(x)) = q(x, \frac{1}{2}u^2(x))$ on $(0, \infty)$, $\lambda_n \rightarrow \lambda$ and there exists $L > 0$ such that $|p(x)| + |q(x, s)| \leq L$ for all $x \in (0, \infty)$ and $s \leq C$. Hence $|u_{n_k}'' - u''|_2 \rightarrow 0$ and so $u_{n_k} \rightarrow u$ in $H^2((0, \infty))$.

3. Bifurcation and global continuation. In order to find solutions to (N) we use a theorem about bifurcation from a simple eigenvalue in the form due to Crandall and Rabinowitz [5].

Theorem 4. (Bifurcation from a simple eigenvalue). *Let X and Y be Banach spaces and let $F : \mathbb{R} \times X \rightarrow Y$ have the properties*

1. $F(t, 0) = 0$ for all $t \in \mathbb{R}$.
2. The partial derivatives D_1F, D_2F and $D_{12}F$ exist and are continuous.
3. $\dim N(D_2F(\lambda_0, 0)) = \text{codim} R(D_2F(\lambda_0, 0)) = 1$.
4. $D_{12}F(\lambda_0, 0)x_0 \notin R(D_2F(\lambda_0, 0))$, where $N(D_2F(\lambda_0, 0)) = \text{span}\{x_0\}$.

If Z is any complement of $N(D_2F(\lambda_0, 0))$ in X , then there are a neighborhood U of $(\lambda_0, 0)$ in $\mathbb{R} \times X$, an interval $(-a, a)$ and continuous functions $\lambda : (-a, a) \rightarrow \mathbb{R}$, $\psi : (-a, a) \rightarrow Z$ such that $\lambda(0) = \lambda_0$, $\psi(0) = 0$ and

$$F^{-1}(0) \cap U \subset \{(\lambda(s), sx_0 + s\psi(s)) : |s| < a\} \cup \{(t, 0) : (t, 0) \in U\}.$$

We define the operator $F : \mathbb{R} \times X \rightarrow L^2((0, \infty))$ by

$$F(\lambda, u) = u'' + pu + h(x, u) + \lambda u$$

with p and h satisfying (H1) and (H2) and (H3)(i).

We shall show that the hypotheses of Theorem 4 are satisfied for the operator F at $(\lambda_0, 0) \in \mathbb{R} \times X$ and that pairs (λ, u) which arise from bifurcation at the point $(\lambda_0, 0)$ are solutions of problem (N).

In Section 2 we introduced a self-adjoint operator S associated with the linearization of problem (D). Recall that φ is the eigenfunction associated with the simple eigenvalue λ_0 such that $\varphi(x) > 0$ for all $x \geq 0$ and $|\varphi|_2 = 1$. By Corollary 1 we can suppose that $\varphi \in K$ and we set

$$W = \{u \in X : \int_0^\infty u\varphi dx = 0\}.$$

Lemma 6. *Let the conditions (H1), (H2) and (H3)(i) be satisfied. Then there exist $\delta > 0$, $z \in C((-\delta, \delta), W)$ and $\lambda \in C((-\delta, \delta), \mathbb{R})$ such that $z(0) = 0$ and $\lambda(0) = \lambda_0$ and, for $|s| < \delta$, $(\lambda(s), u(s))$ is a solution of the equation*

$$u''(x) + \lambda u(x) + p(x)u(x) + h(x, u(x)) = 0, \quad (5)$$

where $u(s)(x) = s\{\varphi(x) + z(s)(x)\}$ for all $x \geq 0$. Furthermore, there is an open neighborhood U of $(\lambda_0, 0)$ in $\mathbb{R} \times X$ such that, if $(\lambda, u) \in U$ with $u \neq 0$ and (λ, u) satisfies (5) then

$$(\lambda, u) \in \{(\lambda(s), u(s)) : 0 < |s| < \delta\}.$$

Proof. We check that the operator F satisfies the conditions of Theorem 4. First of all we note that for $v \in X$, $h(x, v(x)) \in L^2((0, \infty))$ and then we show that

$$D_2F(\lambda, u)v = v'' + pv + \partial_2h(x, u)v + \lambda v \text{ for } u, v \in X.$$

For $u, v \in X$,

$$\begin{aligned} & |h(x, u+v) - h(x, u) - \partial_2h(x, u)v|_2 \\ &= \left| \int_0^1 \frac{\partial}{\partial t} h(x, u+tv) dt - \partial_2h(x, u)v \right|_2 \\ &= \left| \left[\int_0^1 \partial_2h(x, u+tv) - \partial_2h(x, u) dt \right] v \right|_2 \\ &\leq \left| \int_0^1 \partial_2h(x, u+tv) - \partial_2h(x, u) dt \right|_\infty \|v\|_2 \\ &\leq \sup_{x \geq 0} \int_0^1 |\partial_2h(x, u(x) + tv(x)) - \partial_2h(x, u(x))| dt \|v\|_X. \end{aligned}$$

Putting $u = 0$, we see that

$$|h(x, v(x))|_2 \leq \sup_{x \geq 0} \int_0^1 |\partial_2 h(x, tv(x))| dt \|v\|_X.$$

By (H2), there exists $\delta > 0$ such that $|\partial_2 h(x, s)| \leq 1$ for all $x \geq 0$ and $|s| \leq \delta$. But $\|v\|_\infty \leq \|v\|_X$ and there exists $R > 0$ such that $|v(x)| \leq \delta$ for all $x \geq R$. Since $\partial_2 h$ is uniformly continuous on $[0, R] \times [0, |v|_\infty]$ it follows that

$$\sup_{x \geq 0} \int_0^1 |\partial_2 h(x, tv(x))| dt \leq \infty$$

and so $|h(x, v(x))|_2 < \infty$. Furthermore, for any $R > 0$,

$$\begin{aligned} & \int_0^1 |\partial_2 h(x, u(x) + tv(x)) - \partial_2 h(x, u(x))| dt \\ & \leq \sup_{x \in (0, R]} \int_0^1 |\partial_2 h(x, u(x) + tv(x)) - \partial_2 h(x, u(x))| dt \\ & + \sup_{x \in (R, \infty)} \int_0^1 |\partial_2 h(x, u(x) + tv(x)) - \partial_2 h(x, u(x))| dt. \end{aligned}$$

Now, for any $R > 0$,

$$\lim_{\|v\|_\infty \rightarrow 0} |\partial_2 h(x, u(x) + tv(x)) - \partial_2 h(x, u(x))| = 0 \quad \text{uniformly on } [0, R].$$

Moreover by (H2), for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$|\partial_2 h(x, s)| \leq \epsilon \quad \text{for all } x \geq 0 \text{ and } |s| \leq \delta(\epsilon)$$

and there exists $R(\epsilon) > 0$ such that $|u(x)| \leq \frac{1}{2}\delta(\epsilon)$ for all $x \geq R(\epsilon)$. If $\|v\|_X \leq \frac{1}{2}\delta(\epsilon)$ it follows that $|u(x) + tv(x)| \leq \delta(\epsilon)$ for all $x \geq R(\epsilon)$ and all $t \in [0, 1]$. Thus

$$\sup_{x \in (R(\epsilon), \infty)} \int_0^1 |\partial_2 h(x, u(x) + tv(x)) - \partial_2 h(x, u(x))| dt \leq 2\epsilon.$$

Hence

$$\lim_{\|v\|_X \rightarrow 0} |h(x, u + v) - h(x, u) - \partial_2 h(x, u)v|_2 \|v\|_X = 0.$$

Consequently,

$$D_2 F(\lambda, u)v = v'' + pv + \partial_2 h(x, u)v + \lambda v$$

and, in particular,

$$D_2F(\lambda, 0)v = -Sv + \lambda v.$$

Furthermore, it is easy to see that $D_1D_2F(\lambda, u)v = v$ and that all these derivatives are continuous. Now $N(D_2F(\lambda_0, 0)) = \text{span}\{\varphi\}$, and since S is self-adjoint, $R(D_2F(\lambda_0, 0)) = \{u \in L^2((0, \infty)) : \int_0^\infty u\varphi dx = 0\}$ and $X = \text{span}\{\varphi\} \oplus W$. Since $D_{12}F(\lambda_0, 0)\varphi = \varphi \notin R(D_2F(\lambda_0, 0))$, the result now follows from Theorem 4. \square

Now we prove that pairs $(\lambda(s), u(s))$ obtained in Lemma 6 have the properties $\lambda(s) < \lambda_0$ and $u(s) \in K$ for small positive s . To establish this we again use the notation $h(x, s) = q(x, \frac{1}{2}s^2)s$ introduced at the end of Section 2.

Lemma 7. *Let the conditions (H1), (H2) and (H3) be satisfied. Let $(\lambda(s), u(s))$ be a pair obtained in Lemma 6. There exists $s_0 > 0$ such that if $0 < s < s_0$ then $(\lambda(s), u(s))$ is a solution to (N), $u(s) \in K$ and $\lambda(s) < \lambda_0 < 0$, whereas for $-s_0 < s < 0$, $u(s) < 0$ on $[0, \infty)$ and $(\lambda(s), u(s))$ does not satisfy (N).*

Proof. By Lemma 3, it suffices to prove that there exists s_0 such that $u(s)(x) > 0$ on $[0, \infty)$ for $0 < s < s_0$, and $u(s)(x) < 0$ on $[0, \infty)$ for $-s_0 < s < 0$. Define

$$v_s(x) = \frac{u(s)(x)}{s} = \varphi(x) + z(s)(x).$$

For fixed $0 < |s| < \delta$, $(\lambda(s), v_s) \in \mathbb{R} \times X$ and satisfies

$$v_s''(x) + p(x)v_s(x) + q(x, \frac{1}{2}u^2(s)(x))v_s(x) + \lambda(s)v_s(x) = 0 \text{ for all } x > 0.$$

By the hypotheses on p and q and since $\lim_{x \rightarrow \infty} u(s)(x) = 0$, there exist $y > 0$ and an open neighborhood U of $(\lambda_0, 0)$ in $\mathbb{R} \times X$ such that

$$p(x) + q(x, \frac{1}{2}u^2(s)(x)) + \lambda(s) \leq \frac{1}{2}\lambda_0 \quad \text{for all } x \geq y$$

provided that $(\lambda(s), u(s)) \in U$. It follows that

$$\begin{aligned} v_s'(x)v_s(x) + \int_x^\infty v_s'(y)^2 dy &= \int_x^\infty \{p(y) + q(y, \frac{1}{2}u^2(s)(y)) + \lambda(s)\}v_s(y)^2 dy \\ &\leq \frac{1}{2}\lambda_0 \int_x^\infty v_s(y)^2 dy < 0 \text{ for } x \geq y. \end{aligned}$$

This proves that $v_s'(x)v_s(x) < 0$ for all $x \geq y$.

The embedding of X in $C^1([0, \infty), \mathbb{R})$ is continuous. Since $\varphi(x) > 0$ for all $x \geq 0$ and $\lim_{s \rightarrow 0} z(s) = 0$, on choosing a sufficiently small value $s_0 > 0$, we obtain $v_s(x) > 0$ for all $x \in [0, y]$ provided $0 < s < s_0$. Thus $u(s)(x) = v_s(x)s > 0$ for all $x \geq 0$ and $s \in (0, s_0)$.

Hence, for $0 < s < s_0$, the pair $(\lambda(s), u(s))$ is a solution to (N).

Similarly $v_s(x) < 0$ for all $x \in [0, y]$ provided $-s_0 < s < 0$ and so $u(s)(x) < 0$ on $[0, \infty)$ in this case.

Proof of Theorem 2. We consider again the operator

$$F(\lambda, u) = u'' + pu + h(x, u) + \lambda u.$$

We know from the proof of Lemma 6 that $F \in C^1(\mathbb{R} \times X, L^2((0, \infty)))$ and

$$D_2F(\lambda, u)v = v'' + pv + \partial_2h(x, u)v + \lambda v \quad \text{for } u, v \in X.$$

Let $\tilde{S}v = v'' + \lambda v$ and $R(u)v(x) = [\partial_2f(x, u)]v(x)$. Since $\lambda < 0$, the mapping $\tilde{S} : X \rightarrow L^2((0, \infty))$ is an isomorphism. Furthermore, for $u \in X$ the condition (A1) ensures that $\lim_{x \rightarrow \infty} \partial_2f(x, u(x)) = 0$. Hence $R(u) : X \rightarrow L^2((0, \infty))$ is a compact linear operator. It follows that $D_2F(\lambda, u) = \tilde{S} + R(u) : X \rightarrow L^2((0, \infty))$ is an isomorphism if and only if it is injective. If we show that $D_2F(\lambda, u) : X \rightarrow L^2((0, \infty))$ is injective whenever (λ, u) is a solution to (N), we can apply the implicit function theorem at the solutions $(\lambda(s), u(s))$ obtained in Lemma 7 and construct a curve of solutions to (N) parameterized by λ . If (λ, u) satisfies (N), we have

$$u''(x) + \lambda u(x) + f(x, u(x)) = 0 \quad \text{for all } x > 0, \quad (6)$$

and if $v \in X \setminus \{0\}$ is such that $D_2F(\lambda, u)v = 0$, we have

$$v''(x) + \lambda v(x) + \partial_2f(x, u(x))v(x) = 0 \quad \text{for all } x > 0. \quad (7)$$

It follows that $v(0) \neq 0$ and that $\int_x^\infty v(y)^2 dy > 0$ for all $x \geq 0$.

From (6) and (7) we obtain

$$u''(x)v(x) - u(x)v''(x) + f(x, u(x))v(x) - \partial_2f(x, u(x))u(x)v(x) = 0.$$

Hence, after integration,

$$\int_0^\infty [f(x, u(x)) - \partial_2f(x, u(x))u(x)]v(x)dx = 0.$$

Since $\partial_2f(x, s)s > f(x, s)$ for $s > 0$ and $u(x) > 0$ for $x \geq 0$, it follows that there exists $z > 0$ such that $v(z) = 0$. Furthermore

$$\lim_{x \rightarrow \infty} \partial_2f(x, u(x)) = 0$$

and so since $\lambda < \lambda_0 < 0$, there exists $R > 0$ such that $\lambda + \partial_2 f(x, u(x)) < 0$ for all $x \geq R$. Hence

$$v'(x)v(x) + \int_x^\infty (v'(y))^2 dy = \int_x^\infty \{\lambda + \partial_2 f(y, u(y))\} v(y)^2 dy < 0.$$

In particular $v'(x)v(x) < 0$ for all $x \geq R$. Thus, replacing v by $-v$ if necessary, we can suppose that there exists $x_0 > 0$ such that $v(x_0) = 0$, $v'(x_0) > 0$ and $v(x) > 0$ for all $x > x_0$. Setting $w(x) = u'(x)$, we have that $w(x) < 0$ and

$$w''(x) + \lambda w(x) + \partial_1 f(x, u(x)) + \partial_2 f(x, u(x))w(x) = 0 \text{ for all } x > 0. \quad (8)$$

From (7) and (8), it follows that

$$-w''(x)v(x) + v''(x)w(x) = [\partial_1 f(x, u(x))]v(x).$$

We integrate between x_0 and ∞ to get

$$-w(x_0)v'(x_0) = \int_{x_0}^\infty [\partial_1 f(x, u(x))]v(y)dy.$$

We obtain $w(x_0)v'(x_0) \geq 0$ and so $v'(x_0) \leq 0$. This contradicts the fact that $v'(x_0) > 0$ and we conclude that $D_2 F(\lambda, u) : X \rightarrow L^2((0, \infty))$ must be injective.

Thus the curve, $(\lambda(s), u(s))$ for $0 < s < s_0$, of solutions given by Lemma 7 can be parameterized by λ and extended to a maximal curve, $\mathcal{C} = \{(\lambda, u(\lambda)) : \lambda^* < \lambda < \lambda_0\}$, of solutions of (N) with $u(\lambda) \in K$ where $u \in C^1((\lambda^*, \lambda_0), X)$. Setting

$$\psi_\lambda = \frac{d}{d\lambda} u(\lambda),$$

we have that $\psi_\lambda \in X$ and $D_1 F(\lambda, u(\lambda)) + D_2 F(\lambda, u(\lambda))\psi_\lambda = 0$ for all $\lambda \in (\lambda^*, \lambda_0)$. Hence $\psi'_\lambda(0) = 0$ and

$$\psi''_\lambda(x) + \partial_2 f(x, u(\lambda)(x))\psi_\lambda(x) + \lambda\psi_\lambda(x) = -u(\lambda)(x) \text{ for } x > 0. \quad (9)$$

It is easy to see that

$$\psi_\lambda(0) = \frac{d}{d\lambda} \{u(\lambda)(0)\} = \frac{d}{d\lambda} |u(\lambda)|_\infty$$

and we shall now show that $\psi_\lambda(0) < 0$ for all $\lambda \in (\lambda^*, \lambda_0)$. Since the function $\lambda \rightarrow \psi_\lambda$ is continuous from (λ^*, λ_0) into X , we already know that $\psi_\lambda(0)$ is a continuous function of λ . Furthermore $u(\lambda)(0) \rightarrow 0$ as $\lambda \rightarrow \lambda_0$ since $\lim_{\lambda \rightarrow \lambda_0} \|u(\lambda)\|_X = 0$. Hence there exists $\lambda \in (\lambda^*, \lambda_0)$ such that

$$\frac{d}{d\lambda} u(\lambda)(0) < 0$$

and so to prove that $\psi_\lambda(0) < 0$ for all $\lambda \in (\lambda^*, \lambda_0)$, we need only show that $\psi_\lambda(0) \neq 0$ for all $\lambda \in (\lambda^*, \lambda_0)$. But if $\psi_\lambda(0) = 0$, it follows from (9) that $\psi''_\lambda(0) = -u(\lambda)(0) < 0$ and so ψ_λ has a strict local maximum at $x = 0$. Thus either,

- (i) there exists $x_0 > 0$ such that $\psi_\lambda(x) < 0$ on $(0, x_0)$, $\psi_\lambda(x_0) = 0 \leq \psi'_\lambda(x_0)$, or
- (ii) $\psi_\lambda(x) < 0$ for all $x > 0$.

Setting

$$z_\lambda = u'(\lambda) = \frac{d}{dx}u(\lambda),$$

it follows that $z_\lambda(0) = 0$, $z_\lambda(x) < 0$ for all $x > 0$ and

$$z_\lambda''(x) + \partial_2 f(x, u(\lambda)(x))z_\lambda(x) + \lambda z_\lambda(x) = -\partial_1 f(x, u(\lambda)(x)). \quad (10)$$

Combining (9) and (10), we find that

$$\{z_\lambda' \psi_\lambda - z_\lambda \psi_\lambda'\}(b) = \int_0^b \{u_\lambda(x)z_\lambda(x) - \psi_\lambda(x)\partial_1 f(x, u(\lambda)(x))\}dx \quad (11)$$

since $z_\lambda(0) = 0$ and we are assuming that $\psi_\lambda(0) = 0$.

In case (i), we observe that for $0 < x < x_0$,

$$u_\lambda(x)z_\lambda(x) - \psi_\lambda(x)\partial_1 f(x, u(\lambda)(x)) < 0$$

and so, setting $b = x_0$ in (11), we obtain

$$z_\lambda(x_0)\psi_\lambda'(x_0) > 0$$

which implies that $\psi_\lambda'(x_0) < 0$, a contradiction.

In case (ii), we have that

$$u_\lambda(x)z_\lambda(x) - \psi_\lambda(x)\partial_1 f(x, u(\lambda)(x)) < 0 \text{ for all } x > 0$$

and

$$\lim_{b \rightarrow \infty} \{z_\lambda' \psi_\lambda - z_\lambda \psi_\lambda'\}(b) = 0.$$

Thus (11) again leads to a contradiction.

We conclude that for all $\lambda \in (\lambda^*, \lambda_0)$, we must have $\psi_\lambda(0) \neq 0$ and this proves that

$$\frac{d}{d\lambda}u(\lambda)(0) < 0 \text{ for all } \lambda \in (\lambda^*, \lambda_0).$$

To end the proof, we just need to show that $\lim_{\lambda \rightarrow \lambda^*} \|u(\lambda)\|_X = \infty$. By Lemma 4 a),

$$-\lambda \leq q(0, m(\lambda)) + p(0) \text{ where } m(\lambda) = \frac{u(\lambda)(0)^2}{2} = \frac{|u(\lambda)|_\infty^2}{2}.$$

Hence if $\lambda^* = -\infty$ it follows that $\lim_{\lambda \rightarrow \lambda^*} m(\lambda) = \infty$ and consequently that

$$\lim_{\lambda \rightarrow \lambda^*} \|u(\lambda)\|_X = \infty.$$

On the other hand, if $\lambda^* > -\infty$ and if there exists a sequence $\{(\lambda_n, u(\lambda_n))\} \subset \mathcal{C}$ such that $\lambda_n \rightarrow \lambda^* < \lambda_0$ and $\{\|u(\lambda_n)\|_X\}$ is bounded, then by Lemma 5 there is a subsequence $\{u_{n_k}\}$ and an element $\bar{u} \in X$ such that $\|u_{n_k} - \bar{u}\|_X \rightarrow 0$. Furthermore, either $\bar{u} = 0$, or (λ^*, \bar{u}) satisfies (N). But, if $\bar{u} = 0$, then $(\lambda^*, 0)$ is a bifurcation point for (N) and this implies that $\lambda^* \in \sigma(S)$, contradicting the fact that $\lambda^* < \lambda_0 \equiv \inf \sigma(S)$. If (λ^*, \bar{u}) satisfies (N) we can apply the implicit function theorem at (λ^*, \bar{u}) to contradict the maximality of \mathcal{C} . Thus we also have that $\lim_{\lambda \rightarrow \lambda^*} \|u_\lambda\|_X = \infty$ when $\lambda^* > -\infty$.

4. Characterization of λ^* . In this section we analyze the location of the limit value λ^* for the curve $(\lambda, u(\lambda))$ of solutions given by Theorem 2. This requires additional hypotheses on $h(x, s)$, namely (L1) or (L2). First we derive as a consequence of the hypothesis (H4) that for solutions of (N), $|u|_\infty \rightarrow \infty$ whenever $\|u\|_X \rightarrow \infty$ and λ remains bounded.

Lemma 8. *If the function $h(x, s)$ satisfies (H2) to (H4) then for any constant $C > 0$ there exists $\theta > 0$ such that*

$$\frac{Q(x, s)}{s} - q(x, s) < -\frac{1}{\theta} q(x, s) \quad \text{for all } x \geq 0 \quad (12)$$

provided $s \in (0, C)$.

Proof. By (H4), for all $\epsilon \in (0, A)$, there exists $s_\epsilon > 0$ such that

$$0 < \mathcal{A}(x) - \epsilon < \frac{q(x, s)}{s^\sigma} < \mathcal{A}(x) + \epsilon \quad \text{for all } x \geq 0$$

provided $s \in (0, s_\epsilon)$. Then

$$Q(x, s) = \int_0^s q(x, t) dt \leq \frac{\mathcal{A}(x) + \epsilon}{\sigma + 1} s^{\sigma+1}.$$

Hence

$$\frac{Q(x, s)}{sq(x, s)} < \frac{\mathcal{A}(x) + \epsilon}{(\sigma + 1)(\mathcal{A}(x) - \epsilon)} \quad \text{for all } x \geq 0 \quad \text{and } s \in (0, s_\epsilon).$$

On choosing a sufficiently small $\epsilon > 0$, it follows that there exists $\mu \in (0, 1)$ such that

$$\frac{Q(x, s)}{s} < \mu q(x, s) \quad \text{for all } x \geq 0 \quad \text{and } s \in (0, s_\epsilon),$$

or equivalently

$$\int_0^s q(x, t) dt < \mu sq(x, s) \quad \text{for all } x \geq 0 \quad \text{and } s \in (0, s_\epsilon). \quad (13)$$

Now, for a fixed $C > 0$, we shall prove the inequality (12) for all $s \in (0, C)$. For $s > s_\epsilon$, we have

$$\begin{aligned} \frac{Q(x, s)}{s} - q(x, s) &= \frac{1}{s} \int_0^s \{q(x, t) - q(x, s)\} dt \\ &= \frac{1}{s} \left(\int_0^{s_\epsilon} + \int_{s_\epsilon}^s \right) \{q(x, t) - q(x, s)\} dt \\ &\leq \frac{1}{s} \int_0^{s_\epsilon} \{q(x, t) - q(x, s)\} dt, \end{aligned}$$

since $q(x, t)$ is an increasing functions of t . Using (13) and the fact that $\mu \in (0, 1)$, we get

$$\begin{aligned} \frac{Q(x, s)}{s} - q(x, s) &\leq \frac{1}{s} \{\mu s_\epsilon q(x, s_\epsilon) - s_\epsilon q(x, s)\} \\ &= -\frac{s_\epsilon}{s} \{q(x, s) - \mu q(x, s_\epsilon)\} \leq -\frac{s_\epsilon}{C} (1 - \mu) q(x, s) \end{aligned}$$

provided that $s \in (s_\epsilon, C)$. Hence setting

$$\theta = \max\left\{\frac{1}{1 - \mu}, \frac{C}{(1 - \mu)s_\epsilon}\right\},$$

we obtain the inequality (12) for all $s \in (0, C)$.

Lemma 9. *Let the conditions (H1) to (H4) be satisfied. Given $C > 0$ there exists $\hat{C}(C) > 0$ such that $\|u\|_X \leq \hat{C}$ for all solutions (λ, u) of (N) such that $|u(0)| = |u|_\infty \leq C$ and $|\lambda| < C$.*

Proof. We have $u \in K$ and $\lambda < \lambda_0 < 0$ by Lemma 3. Setting

$$w(x) = \frac{u'(x)}{u(x)},$$

we find that

$$w'(x) = \frac{u''(x)}{u(x)} - \left(\frac{u'(x)}{u(x)}\right)^2.$$

To express $(u'(x))^2$, we multiply the equation (N) by u' and then observe that

$$\begin{aligned} \frac{d}{dx} \{u'(x)^2 + \lambda u^2(x) + p(x)u^2(x) + 2Q(x, \frac{1}{2}u^2(x))\} \\ = 2 \int_0^{\frac{1}{2}u^2(x)} \{p'(x) + \partial_1 q(x, t)\} dt. \end{aligned} \tag{14}$$

Integrating (14), we obtain

$$\begin{aligned} -u'(x)^2 &= \lambda u^2(x) + p(x)u^2(x) + 2Q(x, \frac{1}{2}u^2(x)) \\ &\quad - 2 \int_x^\infty [\int_0^{\frac{1}{2}u^2(y)} \{p'(y) + \partial_1 q(y, t)\} dt] dy. \end{aligned}$$

Then using the fact that $\partial_1[p(x) + q(x, s)] \leq 0$ for $s \geq 0$, it follows that

$$w'(x) \leq -q(x, \frac{1}{2}u^2(x)) + \frac{2}{u^2(x)}Q(x, \frac{1}{2}u^2(x)),$$

and thus by Lemma 8 there exists $\theta > 0$ such that

$$w'(x) \leq -\frac{1}{\theta}q(x, \frac{1}{2}u^2(x)) \quad \text{if } |u|_\infty \leq C.$$

Lemma 4 b) implies that $\lim_{x \rightarrow \infty} w(x) = -\sqrt{-\lambda}$, and consequently

$$\int_0^\infty q(x, \frac{1}{2}u^2(x))dx \leq \theta\sqrt{-\lambda}. \quad (15)$$

We now show that there exists $C_1 > 0$ such that $|u|_1 \leq C_1$. Recall that $\lambda < \lambda_0 < 0$ and hence

$$|\lambda| + \frac{\lambda_0}{2} \geq \frac{|\lambda|}{2} > 0. \quad (16)$$

Moreover, there exists $z > 0$ such that $p(x) + \lambda_0/2 \leq 0$ for all $x \geq z$. From (N),

$$|\lambda| \int_0^\infty u(x)dx = \int_0^\infty \{p(x) + q(x, \frac{1}{2}u^2(x))\}u(x)dx$$

and equivalently

$$(|\lambda| + \frac{\lambda_0}{2}) \int_0^\infty u(x)dx = \int_0^\infty \{p(x) + \frac{\lambda_0}{2}\}u(x) + q(x, \frac{1}{2}u^2(x))u(x)dx.$$

From (15) and (16), it follows that

$$\frac{|\lambda|}{2} \int_0^\infty u(x)dx \leq u(0) \int_0^z |p(x) + \frac{\lambda_0}{2}|dx + \theta\sqrt{-\lambda}u(0).$$

Setting

$$D = \int_0^z |p(x) + \frac{\lambda_0}{2}|dx \geq 0,$$

we have

$$|u|_1 \leq \frac{2}{|\lambda|}(D + \theta\sqrt{-\lambda})u(0),$$

and consequently

$$|u|_2^2 \leq |u|_\infty |u|_1 \leq \frac{2}{|\lambda|}(D + \theta\sqrt{-\lambda})C^2.$$

By the conditions (H1) to (H3) and the equation (N), there exists $K(C) > 0$ such that $|u''(x)| \leq \{|\lambda| + K(C)\}u(x)$ for all $x \geq 0$ and so $|u''|_2 \leq \{|\lambda| + K(C)\}|u|_2$. \square

When s is large two possible behaviors for $h(x, s)$ are given by the hypotheses (L1) and (L2) stated in the introduction. For convenience, we recall these hypotheses using the notation $f(x, s) = p(x) + h(x, s)$. The limits P and K are related by $P(x) = p(x) + K(x)$.

(L1) $\lim_{s \rightarrow \infty} \lim_{x \rightarrow \infty} s^{-1}h(x, s) = \infty$,

(L2) There exists $K \in L^\infty((0, \infty))$ such that $\lim_{s \rightarrow \infty} s^{-1}h(x, s) = K(x)$ uniformly on x in compact intervals.

Under the hypothesis (L1), we establish that $\|u\|_X$ is bounded whenever (λ, u) is a solution to (N) and λ remains bounded.

Theorem 5. *Let the conditions (H1) to (H4) and (L1) be satisfied. Then there exists $u \in C^1((-\infty, \lambda_0), X)$ such that for all $\lambda < \lambda_0$, $(\lambda, u(\lambda))$ is a solution to (N) and $u(\lambda) \in K$.*

Proof. By Lemma 4 a),

$$2G\left(\frac{1}{2}u^2(0)\right) \leq -\lambda u^2(0) \quad (17)$$

for all solutions (λ, u) of (N). Setting $J(s) = s^{-1}G(s)$ for $s > 0$, we have that $\lim_{s \rightarrow 0} J(s) = 0$ and J is increasing on $(0, \infty)$. Moreover $\lim_{s \rightarrow \infty} J(s) = \infty$ by (L1), since

$$\lim_{s \rightarrow \infty} \frac{G(s)}{s} = \lim_{s \rightarrow \infty} G'(s) = \lim_{s \rightarrow \infty} g(s) = \lim_{s \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{h(x, s)}{s} = \infty.$$

Express in terms of J , the inequality (17) becomes

$$J\left(\frac{1}{2}u^2(0)\right) \leq |\lambda| \quad (18)$$

for all solutions (λ, u) of (N).

Consider the function $u \in C^1((\lambda^*, \lambda_0), X)$ given by Theorem 2 and suppose by contradiction that $|\lambda^*| < C$. It follows from (18) that $\{|u(\lambda)(0)|\} = \{|u(\lambda)|_\infty\}$ is bounded for $\lambda^* < \lambda < \lambda_0$ and hence by Lemma 9 and since $|\lambda| < C$ we obtain that $\{\|u(\lambda)\|_X\}$ is bounded. Since $\lim_{\lambda \rightarrow \lambda^*} \|u(\lambda)\|_X = \infty$, we conclude that $\lambda^* = -\infty$. \square

Now we are interested in the value λ^* when $s^{-1}h(x, s)$ satisfies (L2). To state our result, we recall the “linearization at infinity” of problem (N), denoted (N_∞) in the introduction

$$u''(x) + \{P(x) + \lambda\}u(x) = 0, \quad u \in X,$$

where $P(x) = \lim_{s \rightarrow \infty} s^{-1}f(x, s)$ and p, h satisfy (H1) to (H4) and (L2).

Then P is a continuous non-increasing function on $[0, \infty)$. Furthermore, $K(x) = P(x) - p(x)$ and

$$0 < K(x) \leq P(x) \leq P(0) \equiv L \text{ for all } x \geq 0.$$

Using (H1) we see that

$$\lim_{x \rightarrow \infty} P(x) = \lim_{x \rightarrow \infty} K(x) \equiv K_\infty \text{ where } 0 \leq K_\infty \leq L.$$

Next we introduce the self-adjoint operator $T : \mathcal{D}(T) = X \subset L^2((0, \infty)) \rightarrow L^2((0, \infty))$ defined by

$$Tu = -u'' - P(x)u.$$

As for the operator S introduced in Section 2, we have that $\inf \sigma(T)$ is characterized by

$$\Lambda^\infty = \inf \left\{ \int_0^\infty (u'(x))^2 - P(x)u^2(x) dx : u \in X \text{ and } \int_0^\infty u^2(x) dx = 1 \right\}$$

and $\inf \sigma_e(T) = -\lim_{x \rightarrow \infty} P(x) = -K_\infty$.

Remark. Unlike the case of the operator S , our assumptions do not rule out the possibility that $\Lambda^\infty = \inf \sigma(T) = \inf \sigma_e(T)$. Indeed it may happen that $P(x)$ is a constant on $[0, \infty)$. Thus Λ^∞ is not necessarily an eigenvalue of T . To illustrate this possibility, an example is given at the end of this section.

Theorem 6. *Let the conditions (H1) to (H4) and (L2) be satisfied. Let $u \in C^1((\lambda^*, \lambda_0), X)$ be the function given by Theorem 2. Then $\lambda^* = \Lambda^\infty \equiv \inf \sigma(T)$ and furthermore λ^* satisfies the estimates*

$$\lambda_0 - \sup_{y \geq 0} K(y) \leq \lambda^* \leq \lambda_0 - \inf_{y \geq 0} K(y),$$

where K is the function introduced in (L2).

Proof. First we remark that under (L2), there cannot exist solutions (λ, u) to (N) with $\lambda < -L$. This is a consequence of Lemma 4 a)

$$-\lambda u^2(0) \leq 2Q(0, \frac{1}{2}u^2(0)) + p(0)u^2(0).$$

Since $s^{-1}Q(0, s) \leq q(0, s) \leq K(0)$, it follows that $|\lambda| \leq K(0) + p(0) = L$. In particular $\lambda^* \geq -L$. Let $\{(\lambda_n, u_n)\}$ be a sequence of solutions to (N) such that

$$\begin{aligned}\lambda_n &\rightarrow \lambda^* \\ |u_n(0)| &= |u_n|_\infty \rightarrow \infty \quad \text{as } n \rightarrow \infty.\end{aligned}$$

Setting

$$w_n(x) = \frac{u_n(x)}{|u_n(0)|}$$

for all $x \geq 0$, we have that $|w_n|_\infty = 1$, $0 \leq w_n(x) \leq 1$ and

$$w_n''(x) + w_n(x)\{\lambda_n + p(x) + q(x, \frac{1}{2}|u_n(0)|^2 w_n^2(x))\} = 0 \text{ for all } x > 0. \quad (19)$$

The sequence $\{w_n\}$ is bounded in $L^\infty((0, \infty))$. Then $\{w_n\}$ is bounded in $L^2((0, I))$ for all $I < \infty$ and by (L2), using (19), $\{w_n\}$ is also bounded in $H^2((0, I))$ for all $I < \infty$. Thus there exists a function $w \in H^2((0, I))$ for all $I < \infty$ such that $\{w_n\}$ is weakly convergent to w in $H^2((0, I))$ and $0 \leq w(x) \leq 1$. From the compact embedding of $H^2((0, I))$ in $C^1([0, I])$, it follows that $w_n \rightarrow w$ in $C^1([0, I])$ for all $I < \infty$ and $w \in C^1([0, \infty))$. Thus $w(0) = 1$, $w'(0) = 0$ and $w'(x) \leq 0$ on $[0, \infty)$. By the conditions (H3) and (L2)

$$0 < q(x, s) \leq K(x) \leq L \text{ for all } x \geq 0 \text{ and } s > 0.$$

Hence $|q(x, \frac{1}{2}|u_n(0)|^2 w_n^2(x))|_\infty \leq L$ and so there exists a function $z \in L^2((0, I))$ for all $I < \infty$ such that $|z|_\infty \leq L$ and $\{q(x, \frac{1}{2}|u_n(0)|^2 w_n^2(x))\} \rightharpoonup z(x)$ weakly in $L^2((0, I))$ for all $I < \infty$. By (19), for all $v \in C_0^\infty((0, \infty))$,

$$\int_0^\infty w_n'(x)v'(x)dx = \int_0^\infty w_n(x)v(x)\{\lambda_n + p(x) + q(x, \frac{1}{2}|u_n(0)|^2 w_n^2(x))\}dx.$$

Now since

$$\begin{aligned}\int_0^\infty w_n(x)v(x)q(x, \frac{1}{2}u_n^2(x))dx &= \int_0^\infty \{w_n(x) - w(x)\}v(x)q(x, \frac{1}{2}u_n^2(x)) \\ &\quad + w(x)v(x)\{q(x, \frac{1}{2}u_n^2(x)) - z(x)\} + w(x)v(x)z(x)dx\end{aligned}$$

and using the fact that $w_n \rightharpoonup w$ in $H^2((0, I))$ and $q(x, \frac{1}{2}u_n^2(x)) \rightharpoonup z(x)$ in $L^2((0, I))$ for all $I < \infty$, we obtain for all $v \in C_0^\infty((0, \infty))$

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_0^\infty w_n(x)v(x)\{\lambda_n + p(x) + q(x, \frac{1}{2}|u_n(0)|^2 w_n^2(x))\}dx \\ = \int_0^\infty w(x)v(x)\{\lambda^* + p(x) + z(x)\}dx\end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \int_0^\infty w'_n(x) v'(x) dx = \int_0^\infty w'(x) v'(x) dx.$$

Hence for all $v \in C_0^\infty((0, \infty))$

$$\int_0^\infty w'(x) v'(x) dx = \int_0^\infty w(x) v(x) \{\lambda^* + p(x) + z(x)\} dx.$$

We obtain the existence of the second weak derivative of $w(x)$ which moreover satisfies

$$w''(x) = -w(x) \{\lambda^* + p(x) + z(x)\} \text{ for almost all } x > 0.$$

Now we prove that $w(x) > 0$ for all $x \in (0, \infty)$ and thus we shall obtain the equality between $z(x)$ and the function $K(x)$. We know that $w(x) \geq 0$ on $[0, \infty)$ and $w \in C^1([0, \infty))$. Suppose that there exists x_0 such that $w(x_0) = 0$ and $w'(x_0) = 0$. On integrating the weak derivative, we have

$$w(x) = - \int_{x_0}^x \int_{x_0}^\tau w(y) \{\lambda^* + p(y) + z(y)\} dy d\tau.$$

By Fubini's theorem, we can permute the order of integration to get

$$\begin{aligned} w(x) &= - \int_{x_0}^x \left\{ \int_y^x 1 d\tau \right\} w(y) \{\lambda^* + p(y) + z(y)\} dy \\ &= - \int_{x_0}^x (x - y) w(y) \{\lambda^* + p(y) + z(y)\} dy. \end{aligned}$$

Set

$$m(x_1) = \max_{x \in [x_1, x_0]} |w(x)| \leq 1$$

and

$$C = |\lambda^*| + |p|_\infty + |z|_\infty.$$

Then, for all $x \in [x_1, x_0]$,

$$|w(x)| \leq C m(x_1) \int_x^{x_0} |x - y| dy = \frac{1}{2} C m(x_1) (x_0 - x)^2 \leq \frac{1}{2} C m(x_1) (x_0 - x_1)^2.$$

We choose x_1 such that $C(x_0 - x_1)^2 < 1$ and so

$$m(x_1) \leq \frac{1}{2} C m(x_1) (x_0 - x_1)^2 < \frac{1}{2} m(x_1).$$

Thus, we have $m(x_1) = 0$ and this implies $w(x) = 0$ for all $x \in [x_0 - (1/\sqrt{C}), x_0]$. On repeating the same computation for $x_2 < x_1$, we obtain $w \equiv 0$ on $(0, x_0)$. This

contradicts the fact that $w(0) = 1$ and $w \in C^1([0, \infty))$. Hence $w(x) > 0$ for all $x \in [0, \infty)$ and

$$\lim_{n \rightarrow \infty} q(x, \frac{1}{2}|u_n(0)|^2 w_n^2(x)) = K(x).$$

Hence the limit value λ^* of the sequence $\{\lambda_n\}$ satisfies an equation of the form

$$w''(x) + w(x)\{\lambda^* + p(x) + K(x)\} = 0 \text{ for } x > 0, \quad (20)$$

where $w \in C^2([0, \infty))$ is such that $0 < w(x) \leq 1$ for $x \geq 0$, $w(0) = 1$, $w'(0) = 0$ and $w'(x) \leq 0$ on $[0, \infty)$. We also have that $w'(x)^2 \leq L$ on $[0, \infty)$. In fact,

$$\begin{aligned} \{w'(x)^2\}' &= -\{w(x)^2\}'\{\lambda^* + P(x)\} \\ &\leq -L\{w(x)^2\}' \text{ since } w(x)w'(x)' \leq 0. \end{aligned}$$

Hence $\{w'(x)^2 + Lw(x)^2\}$ is non-increasing and, in particular, $w'(x)^2 + Lw(x)^2 \leq w'(0)^2 + Lw(0)^2 = L$ for all $x \geq 0$. These observations will allow us to establish that λ^* is equal to the infimum of the spectrum of T , namely Λ^∞ . In this direction, first note that for any (λ, u) satisfying (N),

$$\begin{aligned} \lambda|u|_2^2 &= \int_0^\infty (u'(x))^2 - p(x)u^2(x) - h(x, u(x))u(x)dx \\ &\geq \int_0^\infty (u'(x))^2 - p(x)u^2(x) - K(x)u^2(x)dx \\ &\geq \Lambda^\infty|u|_2^2. \end{aligned}$$

Hence $\lambda \geq \Lambda^\infty$ and consequently $\lambda^* \geq \Lambda^\infty$.

To establish the opposite inequality we distinguish two cases.

- i) $P(x) \equiv K_\infty$ for all $x \geq 0$,
- ii) $P(0) > \lim_{x \rightarrow \infty} P(x) = K_\infty$.

In case i), we have that $\inf \sigma(T) = \inf \sigma_e(T) = -K_\infty$, and that the function w satisfies the equation

$$w''(x) = -w(x)\{\lambda^* + K_\infty\} \quad \text{for all } x \geq 0.$$

Since $w(x) > 0$ for all $x \geq 0$, we must have the condition $\lambda^* + K_\infty \leq 0$ otherwise all solutions have an infinite number of zeros. Hence $\lambda^* \leq -K_\infty = \inf \sigma_e(T)$ and since we have already observe that $\lambda^* \geq \inf \sigma(T)$ we can conclude that $\lambda^* = \Lambda^\infty$.

Consider case ii). In this case there exists $\psi \in X$ such that

$$\psi''(x) + [p(x) + K(x)]\psi(x) + \Lambda^\infty\psi(x) = 0, \quad (21)$$

where $\psi(x) > 0$ for all $x \geq 0$. We have

$$\begin{aligned} \int_0^z [\psi''(x) + (p(x) + K(x))\psi(x)]w(x)dx \\ = \psi'(z)w(z) - \psi(z)w'(z) + \int_0^z \{w''(x) + (p(x) + K(x))w(x)\}\psi(x)dx. \end{aligned}$$

This implies, using (21) and the fact that (λ^*, w) satisfies the equation (20),

$$-\Lambda^\infty \int_0^z \psi(x)w(x)dx = \psi'(z)w(z) - \psi(z)w'(z) - \lambda^* \int_0^z \psi(x)w(x)dx,$$

and hence

$$\psi(z)w'(z) - \psi'(z)w(z) = (\Lambda^\infty - \lambda^*) \int_0^z \psi(x)w(x)dx.$$

When z tends to infinity,

$$\psi'(z)w(z) \rightarrow 0 \quad \text{and} \quad \psi(z)w'(z) \rightarrow 0$$

since $w, w' \in L^\infty((0, \infty))$ and $\psi \in X$. If $\Lambda^\infty \neq \lambda^*$, this implies that $\int_0^\infty \psi(x)w(x)dx$ exists and is zero which is false since $\psi w \in C([0, \infty))$, $\psi w \geq 0$ on $[0, \infty)$ and $\psi(0)w(0) = \psi(0) > 0$. Hence $\Lambda^\infty = \lambda^*$.

To establish the estimates of Theorem 6, we recall that there exists a function $\varphi \in K$ such that $|\varphi|_2 = 1$ and

$$\varphi''(x) + \varphi(x)(p(x) + \lambda_0) = 0 \quad \text{for all } x > 0.$$

Then

$$\begin{aligned} w'(x)\varphi(x) - w(x)\varphi'(x) &= \int_0^x w''(y)\varphi(y) - w(y)\varphi''(y)dy \\ &= \int_0^x \{\lambda_0 - \lambda^* - K(y)\}w(y)\varphi(y)dy. \end{aligned}$$

But $w, w' \in L^\infty((0, \infty))$ and $\varphi \in X$ so

$$\lim_{x \rightarrow \infty} \int_0^x \{\lambda_0 - \lambda^* - K(y)\}w(y)\varphi(y)dy \text{ exists and is equal to } 0.$$

This implies that either $\lambda_0 - \lambda^* - K(y) \equiv 0$ on $(0, \infty)$ or $\lambda_0 - \lambda^* - K(y)$ changes sign on $(0, \infty)$. In any case

$$\inf_{y \geq 0} (-\lambda_0 + \lambda^* + K(y)) \leq 0 \leq \sup_{y \geq 0} (-\lambda_0 + \lambda^* + K(y))$$

as required.

Proof of Theorem 3. Let \mathcal{C} denote the continuous branch of solutions to (N) defined in Theorem 2. Suppose that there exists a solution (μ, v) of (N) such that $(\mu, v) \notin \mathcal{C}$. We have already found that $\mu < \lambda_0$ and that $v \in K$; now we prove that $\mu > \lambda^*$. When $q(x, s)$ satisfies (L1), there is nothing to show. Under the second hypothesis (L2), if (μ, v) is a solution to (N), we have

$$v''(x) + v(x)\{\mu + p(x) + q(x, \frac{1}{2}v^2(x))\} = 0 \text{ for } x > 0. \quad (22)$$

Recall that we have established in Theorem 6 that $\lambda^* = \Lambda^\infty$ the infimum of the spectrum of problem (N_∞) and we have shown in the proof of Theorem 6 that there exists a function $w \in C^2([0, \infty)) \cap L^\infty((0, \infty))$ such that $w(x) > 0$ for all $x \geq 0$, $w' \in L^\infty((0, \infty))$, $w'(0) = 0$ and

$$w''(x) + w(x)\{\lambda^* + p(x) + K(x)\} = 0 \text{ for } x > 0. \quad (23)$$

Combining (22) and (23), it follows that

$$w'(x)v(x) - w(x)v'(x) = \int_0^x [\mu - \lambda^* + q(y, \frac{1}{2}v^2(y)) - K(y)]w(y)v(y)dy.$$

Hence

$$\int_0^\infty [\mu - \lambda^* + q(y, \frac{1}{2}v^2(y)) - K(y)]w(y)v(y)dy = 0.$$

Since $q(y, s) < K(y)$ for all $y \geq 0$ and $s \geq 0$, this implies that

$$(\lambda^* - \mu) \int_0^\infty w(y)v(y)dy = \int_0^\infty \{q(y, \frac{1}{2}v^2(y)) - K(y)\}w(y)v(y)dy < 0$$

and so $\mu > \lambda^*$.

Using the same arguments as in Theorem 2, we can find a maximal interval $(s^*, s_0) \subset (\lambda^*, \lambda_0)$ and a function $v \in C^1((s^*, s_0), X)$ such that for each $s \in (s^*, s_0)$, $(s, v(s))$ is a solution to (N) and $v(s) \notin \mathcal{C}$.

It is impossible to have $\lim_{s \rightarrow s_0} \|v(s)\|_X = 0$ or $\lim_{s \rightarrow s^*} \|v(s)\|_X = 0$ since λ_0 is the smallest eigenvalue of the linearization of (N) at $u = 0$ and all solutions to (N) bifurcating from $(\lambda_0, 0)$ belong to \mathcal{C} .

If $\lim_{s \rightarrow s_0} \|v(s)\|_X < \infty$ or $\lim_{s \rightarrow s^*} \|v(s)\|_X < \infty$, we can extend the interval (s^*, s_0) by Lemma 5, contradicting the assumption that (s^*, s_0) is maximal. Hence we must have $\lim_{s \rightarrow s_0} \|v(s)\|_X = \infty$. But if (L1) is satisfied, the proof of Theorem 5 shows that this is impossible. On the other hand if (L2) is satisfied then $\lim_{s \rightarrow s^*} \|v(s)\|_X = \infty$ too and the proof of Theorem 6 shows that $s_0 = s^* = \lambda^*$,

contradicting the fact that $s^* < \mu < s_0$. Hence there is no solution (μ, v) to (N) such that $(\mu, v) \notin \mathcal{C}$.

Example. This example illustrates the fact that $h(x, s)$ can be increasing in x and we show that in this case we can obtain a value Λ^∞ which is not an eigenvalue of T . Consider a function $r \in C^1(\mathbb{R}, \mathbb{R})$ with the properties

1. $r'(s) > 0$ for all $s > 0$,
2. $\lim_{s \rightarrow \infty} r(s) = R < \infty$,
3. there exist positive constants σ and A such that $\lim_{s \rightarrow 0} r(s)s^{-2\sigma} = A > 0$.

Then for each function $p \in C^1(\mathbb{R})$ satisfying (H1) with $p'(x) \leq 0$ for all $x \geq 0$, we set

$$h(x, s) = \left\{C - \frac{p(x)}{R}\right\} r(s)s \quad \text{where } C > \frac{p(0)}{R}.$$

This function h satisfies the conditions (H2) to (H4) and (L2).

Now if we examine the function $P(x)$ appearing in the problem (N_∞) , we find that $P(x) = p(x) + K(x) = p(x) + \lim_{s \rightarrow \infty} \left\{C - \frac{p(x)}{R}\right\} r(s) = CR$ for all $x \geq 0$. Hence $\sigma_e(T) = [-CR, \infty)$ and T has no eigenvalues.

5. Appendix. In the theory of nonlinear optical waveguides (see [10]) the differential equation

$$u''(x) + \left(\frac{\omega}{c}\right)^2 \varepsilon(x, \frac{1}{2}u(x)^2)u(x) - k^2u(x) = 0 \quad \text{for } x \in \mathbb{R}$$

governs the propagation of transverse electric fields modes in a planar waveguide. In Cartesian coordinates the electric field is given by

$$E(x, y, z, t) = u(x) \cos(kz - \omega t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

so it is a wave propagating in the direction of the z -axis with frequency ω and wave length $\frac{2\pi}{k}$. The polarization is parallel to the y -axis and the dielectric response of the medium through which the electromagnetic fields are propagating is determined by the function $\varepsilon : \mathbb{R} \times [0, \infty) \rightarrow (0, \infty)$ via the constitutive relations

$$\begin{aligned} H(x, y, z, t) &= B(x, y, z, t) \quad \text{and} \\ D(x, y, z, t) &= \varepsilon(x, \frac{1}{2}u(x)^2)E(x, y, z, t). \end{aligned}$$

Thus the medium is isotropic and its refractive index at position (x, y, z) depends only on x and on the time average, $\frac{1}{2}u(x)^2$, of the intensity, $|E(x, y, z, t)|^2$, of the electric field at (x, y, z) . In a *self-focusing medium* the dielectric response, $\varepsilon(x, s)$,

is an increasing function of s . In a *symmetric planar waveguide* the origin of the coordinate system can be placed so that $\varepsilon(-x, s) = \varepsilon(x, s)$ for all $x \in \mathbb{R}$ and $s \geq 0$. It is customary to split the dielectric response into two parts,

$$\varepsilon_L(x) = \varepsilon(x, 0) \quad \text{and} \quad \varepsilon_{NL}(x, s) = \varepsilon(x, s) - \varepsilon(x, 0)$$

representing its linear and nonlinear contributions respectively.

In the context of planar waveguides, a *guided TE-mode* is an electromagnetic field of the above type which has the additional property that $u \in H^1(\mathbb{R})$. This ensures that the fields decay to zero far from the axis of symmetry and that the total intensity,

$$I = \frac{c^2 \omega}{2k} \int_{-\infty}^{\infty} u(x)^2 dx,$$

of the associated beam of light is finite.

Setting

$$f(x, s) = \left(\frac{\omega}{c}\right)^2 \varepsilon(x, \frac{1}{2}s^2)s \quad \text{and} \quad \lambda = -k^2,$$

we see that the problem (D) corresponds to the study of the fundamental guided TE-mode in a symmetric planar self-focusing waveguide at fixed frequency as a function of the wavelength. In particular,

$$\partial_2 f(x, s) = \left(\frac{\omega}{c}\right)^2 \left\{ \partial_2 \varepsilon(x, \frac{1}{2}s^2)s^2 + \varepsilon(x, \frac{1}{2}s^2) \right\} > \frac{f(x, s)}{s}$$

for $s > 0$ since $\varepsilon(x, s)$ is an increasing function of s in a self-focusing medium. Furthermore

$$f(x, 0) = 0 \quad \text{and} \quad \partial_2 f(x, 0) = \left(\frac{\omega}{c}\right)^2 \varepsilon_L(x).$$

Thus the conditions (A1)(i), (A2) and (A3)(ii) are appropriate for any symmetric, planar waveguide made from self-focusing materials. The assumptions (A1)(ii) and (A3)(i) mean that, at fixed intensity s , the refractive index decreases as a function of distance from the axis of symmetry. As suggested by Snell's law in geometric optics, this configuration favors guidance. Redefining f and λ as

$$f(x, s) = \left(\frac{\omega}{c}\right)^2 \left\{ \varepsilon(x, \frac{1}{2}s^2) - \varepsilon_L(\infty) \right\} s \quad \text{and} \quad \lambda = \left(\frac{\omega}{c}\right)^2 \varepsilon_L(\infty) - k^2,$$

we see that the assumption $\lim_{x \rightarrow \infty} \partial_2 f(x, 0) = 0$ does not involve any further loss of generality.

As the intensity of the electric field becomes infinite the dielectric response $\varepsilon(x, s)$ approaches a finite saturation value $\varepsilon(x, \infty) = \lim_{s \rightarrow \infty} \varepsilon(x, s)$ and our assumption (L2) deals with this situation showing that guidance occurs for wavelengths corresponding to values of λ in the finite interval (λ^*, λ_0) . However, much

of the physics literature treats simplified models (such as the Kerr response where $\varepsilon(x, s) = \varepsilon_L(x) + a(x)s$ with $a(x) > 0$) in which $\lim_{s \rightarrow \infty} \varepsilon(x, s) = \infty$, corresponding to our case (L1). Finally we note that most materials are well approximated by the Kerr response at small intensities of the electrical field so that our assumption (H4) is satisfied with $\sigma = 1$ in such cases. Other values of σ are appropriate for the so-called power law responses.

Our results show that the all fundamental TE-modes have the symmetry of the waveguide and are in one-to-one correspondence with the wavelengths in the guidance interval. Furthermore the intensity of the beam decreases with distance from the axis of symmetry. This desirable situation is not present in all symmetric waveguides as was first discovered by Akhmediev who presented an example of a symmetric planar waveguide in which at some wavelengths both symmetric and asymmetric fundamental TE-modes occur (see [1], see also the recent work by Ambrosetti, Arcoya and Gámez [2]).

A much more detailed description of the physical problem discussed here is contained in [10], where results concerning guidance through defocusing as well as self-focusing media are obtained. However in [10] the discussion of self-focusing waveguides concentrates on the case where the linearization has no eigenvalues below the continuous spectrum. This situation is interesting because it offers the possibility of low power cut-off. The present contribution deals with the complementary case where there is an eigenvalue below the continuous spectrum at which there is bifurcation of guided TE-modes. Thus there cannot be low power cut-off and the interesting questions concern the uniqueness and symmetry of the solutions and the global behavior of the branch of fundamental TE-modes.

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