Branches of solutions to semilinear elliptic equations on $\mathbb{R}^N$

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Abstract. For a large class of functions f, we consider the nonlinear elliptic eigenvalue problem

$$\begin{aligned}&-\Delta u(x) + f(x, u(x)) = \lambda u(x) \text{ for } x \in \mathbb{R}^N,\\ &\lim_{|x|\to\infty} u(x) = 0, \ u \not\equiv 0.\end{aligned}$$

We describe the behaviour of the branch of solutions emanating from an eigenvalue of odd multiplicity below the essential spectrum of the linearized problem. A sharper result is obtained in the case of the lowest eigenvalue. The discussion is based on the degree theory for C^2 proper Fredholm maps developed by P.M Fitzpatrick, J. Pejsachowicz and P.J. Rabier.

1 Introduction

We consider a nonlinear elliptic eigenvalue problem of the form

$$\begin{aligned}&-\Delta u(x) + f(x, u(x)) = \lambda u(x) \text{ for } x \in \mathbb{R}^N\\ &\lim_{|x|\to\infty} u(x) = 0, \ u \not\equiv 0\end{aligned} \tag{1.1}$$

where $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a mapping satisfying

(H1) $f(\cdot, 0) \equiv 0$,

(H2) the function f satisfies the conditions of Caratheodory,

(H3) $f(x, \cdot) \in C^2(\mathbb{R})$ for almost all $x \in \mathbb{R}^N$,
for all compact $K \subset \mathbb{R}$, the functions $\{\partial_{22}^2 f(x, \cdot)\colon K \to \mathbb{R} \mid x \in \mathbb{R}^N\}$ are equicontinuous and $\partial_{22}^2 f$ is bounded on $\mathbb{R}^N \times K$,

(H4) $\partial_2 f(\cdot, 0)$ is bounded on $\mathrm{I\!R}^N$. We set $\alpha := \liminf_{|x|\to\infty} \partial_2 f(x,0)$.

For each $C \geq 0$, we introduce the following real number,

$$\beta(C) := \lim_{R\to\infty} \inf_{\substack{|x|\geq R \\ |s|\leq C}} \left\{ \frac{f(x,s)}{s} \right\}.$$

We set

$$\beta = \inf_{C\geq 0} \beta(C),$$

and we make the following hypothesis

(H5) $\beta > -\infty$.

Remarks.

1. When we consider the mapping $(x,s) \mapsto \dfrac{f(x,s)}{s}$, we have always in mind the following one
$$(x,s) \longmapsto \begin{cases} \frac{f(x,s)}{s} & \text{if } s \neq 0 \\ \partial_2 f(x,0) & \text{if } s = 0. \end{cases}$$

2. We have $\beta \leq \alpha$.

3. In Section 3, the condition (H3) is used to ensure that the Nemitsky operator $N : X \to Y$ associated with f is of class C^2 betweeen appropriate function spaces X and Y. This means that we can use the degree theory for proper C^2 Fredholm maps (see [10]). However the authors of [10] have kindly informed us that they can now define a similar degree for proper C^1 Fredholm maps. Anticipating that such a degree will soon be available, we observe that it will then be possible to replace (H3) by

(H3)' $f(x,\cdot) \in C^1(\mathrm{I\!R})$ for almost all $x \in \mathrm{I\!R}^N$, and
for every compact $K \subset \mathrm{I\!R}$, the functions $\{\partial_2 f(x,\cdot) : K \to \mathrm{I\!R} \mid x \in \mathrm{I\!R}^N\}$ are equicontinuous.

In fact, using the notation and arguments of Section 3, the hypotheses (H1),(H2),(H3)' and (H4) are sufficient to ensure the following properties. For every compact $K \subset \mathrm{I\!R}$,

(i) $\partial_2 f$ is bounded on $\mathrm{I\!R}^N \times K$,

(ii) there exists a constant $C = C(K)$ such that for all $(x,s_1),(x,s_2) \in \mathrm{I\!R}^N \times K$, we have $|f(x,s_1) - f(x,s_2)| \leq C|s_1 - s_2|$.

For every $u \in X$,

(iii) there exists a constant $C = C(u)$ such that

$$|f(x, u(x))| \leq C|u(x)| \qquad \text{a.e on } x \in \mathbb{R}^N$$

(iv) $\lim\limits_{|x|\to\infty} \{\partial_2 f(x, u(x)) - \partial_2 f(x, 0)\} = 0.$

Furthermore, $N \in C^1(X, Y)$.

Then, using a degree for proper C^1 Fredholm maps, Theorems 1.1 and 5.1 remain true with (H3) replaced by (H3)'.

Example.
Consider a mapping f of the kind

$$f(x, s) = \Big(p(x) + q(x)r(s)\Big)s$$

with the following properties.

1. $p, q : \mathbb{R}^N \longrightarrow \mathbb{R}$ are mesurable and $r : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.
2. $r(0) = 0$, $r(s)s$ is $C^2(\mathbb{R})$ and $r \geq 0$.
3. $p, q \in L^\infty(\mathbb{R}^N)$.

Set $\alpha = \liminf\limits_{|x|\to\infty} p(x)$.

It is easy to check that under the conditions (1) to (3), the mapping f satisfies all the hypotheses (H1) to (H4).
Furthermore, if $\liminf\limits_{|x|\to\infty} q(x) \geq 0$ then $\alpha = \beta$ and (H5) is satisfied.
However if $-\infty < \liminf\limits_{|x|\to\infty} q(x) < 0$, then

$$\beta \geq \alpha + \Big(\liminf_{|x|\to\infty} q(x)\Big)\Big(\sup_{s\in\mathbb{R}} r(s)\Big)$$

and in this case, (H5) is satisfied if $\sup\limits_{s\in\mathbb{R}} r(s) < \infty$.

For $m \in \mathbb{N}$ and $p \geq 1$, we adopt the standard notation [3] for the Sobolev space $W^{m,p}(\mathbb{R}^N)$. Fixing a value $p \in (N/2, \infty) \cap (1, \infty)$ we set $X := W^{2,p}(\mathbb{R}^N)$, and we recall that the condition $\lim\limits_{|x|\to\infty} u(x) = 0$ is satisfied for all $u \in X$.

We are interested in pairs $(\lambda, u) \in \mathbb{R}\times W^{2,p}(\mathbb{R}^N)$ which are solutions for the problem (1.1). Our purpose is to show the existence of global branches of solutions of (1.1) bifurcating from a trivial solution $(\lambda_0, 0)$ in $\mathbb{R} \times X$.

This kind of problem has been investigated under various assumptions on f in [2], [19] and [7]. For example in [7] it is shown (under appropriate conditions on f) that the branch of positive solutions covers the interval (Λ, α) where Λ is the lowest eigenvalue of the linearization.

The difficulty is that the Laplacian operator does not have a compact inverse on $\mathbb{R}^N$. On a bounded domain, an application of the Global Bifurcation Theorem of Rabinowitz [16] would give us the existence of branches of solutions of problem (1.1). But on unbounded domains, this theorem cannot be used because it is based on Leray-Schauder degree theory for compact operators.

In [2] this difficulty is overcome by approximating (1.1) with Dirichlet boundary value problems on balls, and then by showing that branches of the approximate problems converge to a branch of positive solutions of (1.1).

In this paper, we establish the existence of global branches of solutions (positive or not) directly on $\mathbb{R}^N$.

More precisely, let

$$\mathcal{Z} := \{\, (\lambda, u) \in (-\infty, \beta) \times X \mid (\lambda, u) \text{ is a solution to } (1.1) \,\}.$$

Consider on $\mathcal{Z} \cup \{(\lambda_0, 0)\}$ the topology inherited from $\mathbb{R} \times X$ and let $\mathcal{C}_{\lambda_0}$ be the connected component of $\mathcal{Z} \cup \{(\lambda_0, 0)\}$ containing $(\lambda_0, 0)$.

Moreover consider the linear Schrödinger operator in $L^2(\mathbb{R}^N)$ defined by

$$Su := -\Delta u + \partial_2 f(\cdot, 0)\, u \qquad \text{for} \quad u \in \mathcal{D}(S) := W^{2,2}(\mathbb{R}^N). \tag{1.2}$$

Using these notations, we prove in Section 5, the following theorem

Theorem 1.1 *Suppose that the hypotheses (H1) to (H5) hold, and there exists* $\lambda_0 < \beta$ *such that* $\dim Ker(S - \lambda_0)$ *is odd.*

Then $\mathcal{C}_{\lambda_0}$ *has at least one of the following properties.*

1. $\mathcal{C}_{\lambda_0}$ *is unbounded,*

2. *the closure of* $\mathcal{C}_{\lambda_0}$ *contains a point of the form* $(\lambda^*, 0)$ *with* $\lambda^* \neq \lambda_0$*,*

3. $\displaystyle\sup_{(\lambda,u)\in\mathcal{C}_{\lambda_0}} \lambda = \beta.$

The article is organized as follows.

In Section 2, we recall some notions about the degree theory of proper C^2 Fredholm mappings and we state using this degree a quite general version of the Rabinowitz Global

Bifurcation Theorem for C^2 proper Fredholm mappings proved by Fitzpatrick, Pejsachowicz and Rabier in [10].

In Section 3, we develop a functional framework, which will permit us to use this bifurcation theorem in order to handle problem (1.1). Using the hypotheses (H1) to (H3), we find a C^2 mapping $F : \mathbb{R} \times X \longrightarrow L^p(\mathbb{R}^N)$, whose zeros are solutions of the problem (1.1) and such that

$$D_2F_{(\lambda,u)}(v) = -\Delta v + \{\partial_2 f(\cdot, u) - \lambda\}v.$$

We show that the hypothesis (H5) implies that F is boundedly proper for $\lambda < \beta$ in a sense made precise below.

In Section 4, we check that the choice of p in the definition of the space X does not affect the spectrum of the linearization. Furthermore, we show that there exists λ_0 satisfying the hypotheses of Theorem 1.1 if and only if $\Lambda < \beta$ where

$$\Lambda = \inf\left\{ \int_{\mathbb{R}^N} |\nabla\psi|^2 + \partial_2 f(\cdot,0)\psi^2 \;\Big|\; \psi \in C_0^\infty(\mathbb{R}^N), \int_{\mathbb{R}^N} |\psi|^2 = 1\right\}. \tag{1.3}$$

More precisely, using (H4) we conclude that if $\Lambda < \alpha$, then Λ is an isolated eigenvalue of multiplicity one of the Schrödinger operator S defined by (1.2) and, since $\beta \leq \alpha$, the condition $\Lambda < \beta$ implies that $\Lambda < \alpha$. We conclude the discussion of this section by proving, using (H4), that $D_2F_{(\lambda,u)}$ is a linear Fredholm operator with index 0 for every $\lambda < \alpha$.

Finally in Section 5, we complete the proof of Theorem 1.1 and give a more precise result, Theorem 5.1, for the branch $\mathcal{C}_\Lambda$ in which we show that the second property given in Theorem 1.1 cannot occur for this component because u has no zeros when $(\lambda, u) \in \mathcal{C}_\Lambda \setminus \{(\Lambda, 0)\}$.

The more detailed study of $\mathcal{C}_\Lambda$ is pursued in [14] under more restrictive hypotheses which enable us to find a lower bound $b \in (\Lambda, \beta]$ for $\sup\limits_{(\lambda,u)\in\mathcal{C}_\Lambda} \lambda$. This is done by constructing supersolutions which imply that if $\sup\limits_{(\lambda,u)\in\mathcal{C}_\Lambda} \lambda < b$ then $\mathcal{C}_\Lambda$ is bounded in $\mathbb{R}\times X$, contradicting Theorem 5.1.

2 Degree of Fredholm mappings

In this section, we outline the construction of the degree of proper C^2 Fredholm mappings, and then we state the general bifurcation theorem used to prove our result.

Let X and Y be real Banach spaces. Denote by $L(X,Y)$ the Banach space of bounded linear operators from X to Y with the usual norm. An operator in $L(X,Y)$ is called Fredholm with index 0 if its kernel has finite dimension and its image is closed with the same finite codimension in Y. We denote by $\phi_0(X,Y)$ the subset of $L(X,Y)$ consisting of

those operators which are Fredholm with index 0 and by $GL(X,Y)$ the subset of $\phi_0(X,Y)$ consisting of the invertible operators.
If $T \in GL(X)$ is a compact pertubation of the identity, we let $deg_{L.S}(T)$ be the Leray-Schauder degree of $T : U \to X$ with respect to 0, where U is any neighborhood of the origin.
For an interval $I = [a,b]$ and a continuous path $\alpha : I \to \phi_0(X,Y)$ we call a continuous path $\beta : I \to GL(Y,X)$ a parametrix for α if each $\beta(\lambda)\alpha(\lambda)$ is a compact perturbation of the identity. Parametrices always exist [9]. If the ends of the path, $\alpha(a)$ and $\alpha(b)$ are invertible, then the parity of α in I, $\sigma(\alpha, I)$ defined by

$$\sigma(\alpha, I) = deg_{L.S}(\beta(a)\alpha(a))\, deg_{L.S}(\beta(b)\alpha(b))$$

is independent of the choice of parametrix [9].
Note that Leray-Schauder degree is used there only with linear compact perturbations of the identity, and hence $\sigma(\alpha, I) \in \{-1, 1\}$. The parity is an intersection index which, generically, is a mod 2 count of the number of intersections of $\alpha(I)$ with the set of singular operators. It is an additive, homotopy invariant of paths in $\phi_0(X,Y)$ with invertible endpoints. Moreover the parity is 1 if and only if the path is homotopic to a path of invertible operators.

With this in mind, we can describe briefly the construction of the degree.
Let $\mathcal{O}$ be an open, simply connected subset of X and $F : \mathcal{O} \to Y$ be a C^2 Fredholm mapping with index 0 (i.e. such that $DF_{(x)} \in \phi_0(X,Y)$ for $x \in \mathcal{O}$). A base point for the degree of F is any point $x_0 \in \mathcal{O}$ at which $DF_{(x_0)}$ is invertible.
Assume that there exists a base point p for F. Let Ω be a bounded open set with $\Omega \subset \mathcal{O}$ and such that F can be extended by continuity as a proper mapping to the closure $\overline{\Omega}$ of Ω; i.e. such that the preimage $F^{-1}(K) \cap \overline{\Omega}$ of every compact set K in Y is also compact. Then, if $y \notin F(\partial\Omega)$ and if $DF_{(x)}$ is invertible for all $x \in F^{-1}(y) \cap \Omega$ the degree of F on Ω with respect to y and relative to p is defined by

$$d_p(F, \Omega, y) = \sum_{x \in F^{-1}(y) \cap \Omega} \sigma_p(x),$$

where $\sigma_p(x) = \sigma(DF o\gamma, [0,1])$ is the parity of the derivative DF along any curve $\gamma : [0,1] \to \mathcal{O}$ joining p to x. That $\sigma_p(x)$ does not depend on the choice of γ follows immediately from the homotopy invariance of the parity and the simple connectedness of $\mathcal{O}$.
Using the general Sard-Smale theorem [15], the definition of degree is extended by regular value approximation to the case when $DF_{(x)}$ is not necessarily invertible for all $x \in F^{-1}(y) \cap \Omega$.
This base point degree satisfies the usual additivity, excision, and normalization properties. Its most important property is the homotopy property [10].

Definition 2.1 *Let X, Y be Banach spaces and I an open interval of $\mathbb{R}$. We say that a mapping $F : I \times X \to Y$ is "boundedly proper" if the restriction of F to any closed*

bounded subset of $[a,b] \times X$ *is proper for all* a, b *such that* $\inf I < a \leq b < \sup I$.
(i.e for every compact subset K *of* Y *and for every closed bounded subset* B *of* $[a,b] \times X$, $F^{-1}(K) \cap B$ *is compact.)*

Definition 2.2 *Let* X, Y *be Banach spaces and* I *an open interval of* $\mathbb{R}$. *We say that a mapping* $F : I \times X \to Y$ *is "Fredholm with index 0" if* $D_2F_{(\lambda,u)}$ *exists and* $D_2F_{(\lambda,u)} \in \phi_0(X,Y)$ *for all* $(\lambda, u) \in I \times X$.

Now we can recall the global bifurcation theorem for Fredholm mappings [10]. Let $p_1 : \mathbb{R} \times X \to \mathbb{R}$ denote the projection $p_1(\lambda, u) = \lambda$ for $(\lambda, u) \in \mathbb{R} \times X$.

Theorem 2.1 *Let* X, Y *be real Banach spaces,* $I \subseteq \mathbb{R}$ *be an open interval and* $F : I \times X \to Y$ *be a* C^2 *mapping with* $F(\lambda, 0) = 0$ *for all* $\lambda \in \mathbb{R}$.
Suppose that F *is boundedly proper and Fredholm with index* 0.
Moreover assume that there exist $\lambda_0 \in I$ *and* $\epsilon > 0$ *such that* $0 < |\lambda - \lambda_0| \leq \epsilon$ *implies*

$$\lambda \in I, \quad D_2F_{(\lambda,0)} \in GL(X,Y) \quad \text{and} \quad \sigma\Big(D_2F_{(\lambda,0)}, [\lambda_0 - \epsilon, \lambda_0 + \epsilon]\Big) = -1.$$

Let $\mathcal{Z} = \{\,(\lambda, u) \in I \times X \mid F(\lambda, u) = 0 \text{ and } u \neq 0\,\}$, *and denote by* $\mathcal{C}_{\lambda_o}$ *the connected component of* $\mathcal{Z} \cup \{(\lambda_0, 0)\}$ *containing* $(\lambda_0, 0)$. *Then* $\mathcal{C}_{\lambda_o}$ *has at least one of the following properties.*

1. $\mathcal{C}_{\lambda_o}$ *is unbounded,*
2. *the closure* $\mathcal{C}_{\lambda_o}$ *contains a point of the form* $(\lambda^*, 0)$ *with* $\lambda^* \in I \setminus [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$,
3. *the closure of* $p_1(\mathcal{C}_{\lambda_o})$ *intersects the boundary of* I.

3 A Functional Framework

The aim of this section is to find a mapping $F : \mathbb{R} \times X \to Y$ whose zeros are solutions of the problem (1.1) and which satisfies the hypotheses of Theorem 2.1.

To do this we choose $p \in (\frac{N}{2}, \infty) \cap (1, \infty)$, and we set

$$X = W^{2,p}(\mathbb{R}^N) \quad \text{and} \quad Y = L^p(\mathbb{R}^N) \tag{3.4}$$

with the following usual norms,

$$\| u \|_p = \left\{ \int_{\mathbb{R}^N} |u|^p \right\}^{\frac{1}{p}} \qquad \text{and} \qquad \| u \|_X = \left\{ \sum_{0 \leq |\mu| \leq 2} \| D^\mu u \|_p^p \right\}^{\frac{1}{p}},$$

where μ is a multi-index.

We recall the following properties of the space X (see [3]).

1. $X \hookrightarrow C(\mathbb{R}^N)$, continuously.

   Moreover, the injection $W^{2,p}(B_R) \hookrightarrow C(\overline{B_R})$ is completely continuous for every ball $B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$.

2. $X \hookrightarrow L^q(\mathbb{R}^N)$, continuously, for every $p \le q \le \infty$.

3. $\lim_{|x|\to\infty} u(x) = 0$, for all $u \in X$.

Consider the following mapping

$$\begin{aligned} F: \quad \mathbb{R} \times X \quad &\longrightarrow Y \\ (\lambda, u) \quad &\longmapsto -\triangle u + f(\cdot, u) - \lambda u. \end{aligned} \tag{3.5}$$

Here $\mathbb{R} \times X$ is considered with the norm

$$\| (\lambda, u) \| = |\lambda| + \| u \|_X .$$

The first results show that $F : \mathbb{R} \times X \to Y$ is a well-defined C^2 mapping.

Lemma 3.1 *Let f be a mapping which satisfies the hypotheses (H1) to (H4) and let K be a compact subset of $\mathbb{R}$. Then,*

1. *$\partial_2 f$ is bounded on $\mathbb{R}^N \times K$.*

2. *There exists a constant $C = C(K)$ such that for all $(x, s_1), (x, s_2) \in \mathbb{R}^N \times K$ we have*

$$\begin{aligned} |f(x, s_1) - f(x, s_2)| &\le C|s_1 - s_2|, \\ |\partial_2 f(x, s_1) - \partial_2 f(x, s_2)| &\le C|s_1 - s_2|. \end{aligned}$$

3. *Let $u \in X$. There exists a constant $C = C(u)$ such that*

$$|f(x, u(x))| \le C|u(x)| \qquad a.e. \ on\ \mathbb{R}^N.$$

Proof

1. For all $(x, s) \in \mathbb{R}^N \times K$, we have

$$\begin{aligned} |\,\partial_2 f(x, s)\,| &\le |\,\partial_2 f(x, 0)\,| + \int_0^1 \left| \frac{d}{dt} \Big\{ \partial_2 f(x, ts) \Big\} \right| dt \\ &\le |\,\partial_2 f(x, 0)\,| + |s| \int_0^1 |\,\partial_{22} f(x, ts)\,| \, dt. \end{aligned}$$

   The conclusion is then a consequence of (H3) and (H4).

2. For all $s_1, s_2 \in K$ we have,

$$\begin{aligned}
|\, f(x,s_1) - f(x,s_2)\,| &= \left| \int_0^1 \frac{d}{dt}\Big\{ f\Big(x, s_2 + t(s_1 - s_2)\Big)\Big\}\, dt \,\right| \\
&\le |s_1 - s_2| \int_0^1 \left| \partial_2 f\Big(x, s_2 + t(s_1 - s_2)\Big) \right| dt \\
&\le C|s_1 - s_2|.
\end{aligned}$$

In the same way we prove that

$$|\, \partial_2 f(x,s_1) - \partial_2 f(x,s_2)\,| \le C|s_1 - s_2|.$$

3. Use the injection $X \hookrightarrow L^\infty(\mathrm{I\!R}^N)$, the previous assertion of this lemma and (H1).

□

To study the differentiability of the mapping F, we consider the nonlinear Nemitsky operator

$$\begin{aligned}
N:\ X &\longrightarrow Y \\
u &\longmapsto f(\cdot, u).
\end{aligned}$$

Theorem 3.1 *Let f be a mapping satisfying (H1) to (H4), we have*

1. N is well defined.

2. N is C^2 and for $u \in X$,

$$\begin{aligned}
DN_{(u)}(\xi) &= \partial_2 f(\cdot, u)\,\xi \quad \forall \xi \in X, \\
D^2 N_{(u)}(\xi_1, \xi_2) &= \partial_{22} f(\cdot, u)\,\xi_1\,\xi_2 \quad \forall \xi_1, \xi_2 \in X.
\end{aligned}$$

3. The mapping F (see 3.5) is well defined, C^2 and

$$D_2 F_{(\lambda,u)} = -\triangle + \Big(\partial_2 f(\cdot, u) - \lambda\Big).$$

Proof.

1. The fact that the Nemitsky operator N is well defined is a consequence of (H2), of Thm. 18.3, pp.152 in [20] (which assures that $N(u)$ is measurable for every $u \in X$) and of Lemma 3.1.

2. Let $u \in X$. For every $\xi \in X$, we have

$$\begin{aligned} \| N(u+\xi) - N(u) - \partial_2 f(\cdot,u)\,\xi \|_p &\leq \Big\| \int_0^1 \Big\{ \partial_2 f(\cdot, u+t\xi) - \partial_2 f(\cdot,u) \Big\}\, \xi \, dt \Big\|_p \\ &\leq \sup_{x\in \mathbb{R}^N} \sup_{t\in[0,1]} \Big\{ |\, \partial_2 f(\cdot, u+t\xi) - \partial_2 f(\cdot,u)\,| \Big\} \, \| \xi \|_p \\ &\leq \sup_{x\in \mathbb{R}^N} \sup_{t\in[0,1]} \Big\{ |\, \partial_2 f(\cdot, u+t\xi) - \partial_2 f(\cdot,u)\,| \Big\} \, \| \xi \|_X \,. \end{aligned}$$

Thus, it follows from the second assertion of Lemma 3.1 that

$$\lim_{\|\xi\|_X \to 0} \frac{\| N(u+\xi) - N(u) - \partial_2 f(\cdot,u)\,\xi \|_p}{\| \xi \|_X} = 0.$$

Moreover, by the first assertion of Lemma 3.1, we have that the following linear operator is well defined and bounded,

$$\begin{aligned} X &\longrightarrow Y \\ \xi &\longmapsto \partial_2 f(\cdot,u)\,\xi. \end{aligned}$$

Hence N is Fréchet-differentiable.

Let $u \in X$. For every $\xi_1, \xi_2 \in X$, we have

$$\begin{aligned} & \| DN_{(u+\xi_2)}(\xi_1) - DN_{(u)}(\xi_1) - \partial_{22} f(\cdot,u)\,\xi_1\,\xi_2 \|_p \\ \leq\ & \Big\| \int_0^1 \Big\{ \partial_{22} f(\cdot, u+t\xi_2) - \partial_{22} f(\cdot,u) \Big\}\, \xi_1\, \xi_2 \, dt \Big\|_p \\ \leq\ & \sup_{x\in \mathbb{R}^N} \sup_{t\in[0,1]} \Big\{ |\, \partial_{22} f(\cdot, u+t\xi_2) - \partial_{22} f(\cdot,u)\,| \Big\} \, \| \xi_1 \xi_2 \|_p \\ \leq\ & \sup_{x\in \mathbb{R}^N} \sup_{t\in[0,1]} \Big\{ |\, \partial_{22} f(\cdot, u+t\xi_2) - \partial_{22} f(\cdot,u)\,| \Big\} \, C \, \| \xi_1 \|_X \, \| \xi_2 \|_X \,. \end{aligned}$$

Using now the hypothesis (H3), we see that DN is Fréchet-differentiable.

With analogous arguments, we prove that D^2N is continuous.

3. Using the previous assertion of this theorem and the fact that the mapping

$$\mathbb{R} \times X \longrightarrow Y \qquad (\lambda, u) \longmapsto -\triangle u - \lambda u$$

is C^∞, we conclude that F is C^2.

$\square$

The end of this section is devoted to show that the mapping F defined by (3.5) is boundedly proper for $\lambda < \beta$. This will derive principally from the following result.

Theorem 3.2 *Let f be a mapping satisfying the hypotheses (H2) and (H5). Let E be a subset of* $\mathbb{R} \times X$ *such that*

1. *There exist $C > 0$ and $k < \beta$ such that for all $(\lambda, u) \in E$,*

$$\| u \|_\infty < C \qquad \textit{and} \qquad \lambda \le k < \beta.$$

2. *There exists $\tilde{w} \in Y$ such that for all $(\lambda, u) \in E$,*

$$| - \triangle u + f(\cdot, u) - \lambda u \,| \le \tilde{w}.$$

Then, there exist positive constants a and D such that for all $(\lambda, u) \in E$,

$$|u(x)| \le De^{-a|x|} + (-\triangle + a^2)^{-1}(\tilde{w})(x) \quad \forall x \in \mathbb{R}^N.$$

Proof.

Choose $a > 0$ such that

$$0 < a^2 < \frac{1}{2}(\beta - k). \tag{3.6}$$

From the hypothesis (H5), there exists $R > 0$ such that

$$\frac{f(x,s)}{s} \ge \beta - a^2 \quad \forall s \in [-C, C], \forall |x| \ge R.$$

With these constants C, a, R, we define the following functions

$$\begin{aligned} \Psi_1 &= Ce^{-a(|x|-R)} \\ \Psi_2 &= (-\triangle + a^2)^{-1}\tilde{w}. \end{aligned}$$

We derive the following properties.

1. By calculation we see that $-\triangle \Psi_1 + a^2 \Psi_1 \ge 0$ on $\mathbb{R}^N \setminus \{0\}$.

2. As $(-\triangle + a^2) : X \to Y$ is an isomorphism for all $p \in (1, \infty)$, we have $\Psi_2 \in X$ and so $\lim_{|x| \to \infty} \Psi_2(x) = 0$. Moreover, since $(-\triangle + a^2)\Psi_2 = \tilde{w} \ge 0$, the maximum principle implies that
$$\Psi_2 \ge 0.$$

3. For every $x \in \mathbb{R}^N$ with $|x| \le R$, we have

$$u(x) - \Psi_1(x) - \Psi_2(x) \le u(x) - C - \Psi_2(x) \le u(x) - C < 0.$$

4. $\lim\limits_{|x|\to\infty} \{u(x) - \Psi_1(x) - \Psi_2(x)\} = 0.$

We now show that for all $(\lambda, u) \in E$, we have

$$u(x) \leq \Psi_1(x) + \Psi_2(x) \quad \forall |x| \geq R. \tag{3.7}$$

To this end we suppose that there exist $(\lambda, u) \in E$ and $|x_o| \geq R$ such that

$$u(x_o) > \Psi_1(x_o) + \Psi_2(x_o).$$

By property 3, we have that $|x_o| > R$ and setting

$$\Omega = \{\, x \in \mathbb{R}^N \mid |x| > R \text{ and } u(x) > \Psi_1(x) + \Psi_2(x) \,\},$$

we see that $\Omega \neq \emptyset$ and $\overline{\Omega} \subset \mathbb{R}^N \setminus \overline{B(0,R)}$. In particular $\partial\Omega \neq \emptyset$ and

$$u(x) - \Psi_1(x) - \Psi_2(x) = 0 \quad \forall x \in \partial\Omega.$$

Moreover, on Ω we have

1. $u > \Psi_1 + \Psi_2 > 0$,
2. $\dfrac{f(x,u(x))}{u(x)} - \lambda \geq \beta - a^2 - \lambda \geq \beta - a^2 - k$ from (3.6).

Thus, on Ω we have

$$\begin{aligned}
&\triangle(u - \Psi_1 - \Psi_2)\\
&= \left(\frac{f(\cdot,u)}{u} - \lambda\right)u - \Big(-\triangle u + f(\cdot,u) - \lambda u\Big) - \triangle\Psi_1 - \triangle\Psi_2\\
&\geq \left(\frac{f(\cdot,u)}{u} - \lambda\right)u - \widetilde{w} - \triangle\Psi_1 - \triangle\Psi_2\\
&= \left(\frac{f(\cdot,u)}{u} - \lambda\right)u + \triangle\Psi_2 - a^2\Psi_2 - \triangle\Psi_1 - \triangle\Psi_2\\
&\geq \left(\frac{f(\cdot,u)}{u} - \lambda\right)u - a^2(\Psi_2 + \Psi_1)\\
&\geq (\beta - a^2 - k - a^2)u\\
&= (\beta - k - 2a^2)u\\
&> 0,
\end{aligned}$$

this last inequality following from the choice of a in (3.6).

Hence, we have shown that

$$\begin{array}{lll} & \triangle(u-\Psi_1-\Psi_2)>0 & \text{a.e on } \Omega, \\ \text{and} & u-\Psi_1-\Psi_2=0 & \text{on } \partial\Omega. \end{array}$$

Using property 4 and the maximum principle, this implies that $u-\Psi_1-\Psi_2 \leq 0$ on Ω, contradicting the definition of Ω. Thus (3.7) is established.

In the same way, we prove that for all $(\lambda, u) \in E$,

$$-u(x) \leq \Psi_1(x)+\Psi_2(x) \quad \forall |x| \geq R.$$

Thus, since $\Psi_2 \geq 0$, we have

$$|u(x)| \leq De^{-a|x|} + (-\triangle + a^2)^{-1}(\tilde{w})(x) \quad \forall x \in \mathbb{R}^N.$$

where $D = Ce^{aR}$. □

Lemma 3.2 *Let f be a mapping satisfying (H1) to (H5) and $(u_n)_{n=1}^{\infty}$ a sequence in X which satisfies*

1. *$(u_n)_{n=1}^{\infty}$ converges weakly to u in X,*
2. *there exists $\Psi \in Y$ such that $|u_n| \leq \Psi$, $\forall n \in \mathbb{N}$.*

Then,

1. *$(u_n)_{n=1}^{\infty}$ converges strongly to u in Y.*
2. *$\Big(f(\cdot, u_n)\Big)_{n=1}^{\infty}$ converges strongly to $f(\cdot, u)$ in Y.*

Proof.

1. Since $(u_n)_{n=1}^{\infty}$ converges weakly to u, this sequence is bounded in X. Using the fact that $X \hookrightarrow L^{\infty}(\mathbb{R}^N)$ continuously we see that there exists a bounded interval I of $\mathbb{R}$ such that

$$u(x), u_n(x) \in I \quad \forall x \in \mathbb{R}^N.$$

For $R > 0$, we have

$$\begin{aligned} \| u_n - u \|_p &\leq \left\{ \int_{|x|\leq R} |u_n-u|^p \right\}^{\frac{1}{p}} + \left\{ \int_{|x|\geq R} |u_n-u|^p \right\}^{\frac{1}{p}} \\ &\leq \left\{ \int_{|x|\leq R} |u_n-u|^p \right\}^{\frac{1}{p}} + \left\{ \int_{|x|\geq R} |\Psi|^p \right\}^{\frac{1}{p}} + \left\{ \int_{|x|\geq R} |u|^p \right\}^{\frac{1}{p}}. \end{aligned}$$

The proof now follows from the following facts

(a) $\lim_{R\to\infty}\left\{\int_{|x|\geq R}|\Psi|^p\right\}^{\frac{1}{p}} = \lim_{R\to\infty}\left\{\int_{|x|\geq R}|u|^p\right\}^{\frac{1}{p}} = 0,$

(b) the sequence $(u_n)_{n=1}^{\infty}$ converges uniformly on every compact set of $\mathbb{R}^N$.

2. By the second assertion of Lemma 3.1, there exists a constant $C > 0$ such that

$$|\, f(x,u(x)) - f(x,u_n(x))\,| \leq C\,|\,u(x) - u_n(x)\,| \quad \forall x \in \mathbb{R}^N, \forall n \in \mathbb{N}$$

Thus,

$$\|\, f(\cdot,u_n) - f(\cdot,u)\,\|_p \leq C\,\|\,u_n - u\,\|_p \qquad \forall n \in \mathbb{N}.$$

and the result follows from part 1.

$\square$

Theorem 3.3 *Let f be a mapping satisfying the hypotheses (H1) to (H5), and consider*

$$\begin{aligned} F:\ \mathbb{R}\times X &\longrightarrow Y \\ (\lambda,u) &\longmapsto -\triangle u + f(\cdot,u) - \lambda u. \end{aligned}$$

Then, the restriction of F to $(-\infty,\beta)\times X$ is boundedly proper (see Definition 2.1).

Proof.

Let $[a,b] \subset (-\infty,\beta)$, B be a bounded closed subset of $[a,b]\times X$ and K be a compact subset of Y.

We must prove that every sequence $\{(\lambda_n,u_n)\}$ of $F^{-1}(K)\cap B$ has a convergent subsequence.

We set $F(\lambda_n,u_n) = w_n$.

Without loss of generality, we can suppose that

$$\begin{aligned} \lambda_n &\to \lambda \quad \leq b, \\ u_n &\rightharpoonup u \quad \text{weakly in } X, \\ w_n &\to w \quad \text{strongly in Y with } w \in K. \end{aligned}$$

As is well known, we can choose the subsequence so that

$$w_n \to w \qquad \text{a.e on } \mathbb{R}^N,$$

but we also make use of the fact that it can be choosen in such a way that there exists an element $\tilde{w} \in Y$ such that

$$|w_n| \leq \tilde{w} \text{ a.e on } \mathbb{R}^N \text{ for all } n \in \mathbb{N}.$$

(See Thm. IV.9 in [3].)

It follows from Theorem 3.2 that there exists $\Psi \in Y$ such that

$$|u_n(x)| \leq \Psi(x) \quad \forall x \in \mathbb{R}^N.$$

Now consider the expression

$$\begin{aligned}(-\triangle + 1)u_n &= -\triangle u_n + f(\cdot, u_n) - \lambda_n u_n - f(\cdot, u_n) + \lambda_n u_n + u_n \\ &= F(\lambda_n, u_n) - f(\cdot, u_n) + \lambda_n u_n + u_n.\end{aligned}$$

Applying Lemma 3.2, we deduce that this last expression converges strongly in Y.

The operator $-\triangle + 1$ is an isomorphism of X onto Y, so it follows that $(u_n)_{n=1}^{\infty}$ converges in X, which completes the proof of the theorem.

□

4 The spectrum of the linear problem

In this section we show that the features of the spectrum of the linearized problem, which are important for our results on bifurcation, do not depend on the value of p used in the choice of the spaces $X = W^{2,p}(\mathbb{R}^N)$ and $Y = L^p(\mathbb{R}^N)$.

We present the results for any operator of the form $-\triangle u + Vu$ where $V \in L^\infty(\mathbb{R}^N)$. The linearization of (1.1) is obtained by setting $V = \partial_2 f(\cdot, 0)$.

For each $1 < q < \infty$, we consider the family $(A_{q,\lambda})_{\lambda \in \mathbb{R}}$ of bounded linear operators defined by

$$\begin{aligned}A_{q,\lambda} : W^{2,q}(\mathbb{R}^N) &\longrightarrow L^q(\mathbb{R}^N) \\ u &\longmapsto -\triangle u + (V - \lambda)u\end{aligned}$$

and we set,

$$\Sigma_q = \{\, \lambda \in \mathbb{R} \mid A_{q,\lambda} : W^{2,q}(\mathbb{R}^N) \longrightarrow L^q(\mathbb{R}^N) \text{ is not an isomorphism } \}.$$

Remarks

1. The mapping $(x, s) \mapsto V(x)s$ satisfies the properties (H1) to (H5) with

$$\alpha = \liminf_{|x| \to \infty} V(x) \qquad \text{and} \qquad \beta = \alpha.$$

(Thus we can set, without confusion, $\liminf_{|x| \to \infty} V(x) = \alpha$.)

2. $\Sigma_q = \sigma(S_q)$ where S_q is the Schrödinger operator in $L^q(\mathrm{IR}^N)$ defined by

$$S_q u = -\triangle u + Vu \qquad \text{for } u \in \mathcal{D}(S_q) = W^{2,q}(\mathrm{IR}^N),$$

and $\sigma(S_q)$ is the spectrum of S_q in the usual sense.
In the following the self-adjoint Schrödinger operator S_2 will be denoted by S.

The results below show that, for $q \in (1, \infty)$ and $\lambda < \alpha$,

1. $A_{q,\lambda}$ is a Fredholm operator of index zero,
2. dim Ker $A_{q,\lambda}$ is independent of q,
3. $L^q(\mathrm{IR}^N) =$ Ker $A_{q,\lambda} \oplus$ Range $A_{q,\lambda}$.

In others words, the essential spectrum of S_q is contained in $[\alpha, \infty)$ and, for the eigenvalues in $(-\infty, \alpha)$, the algebraic and geometric multiplicities are equal and independent of q.

As a consequence of Theorem 3.2, we have the following result.

Theorem 4.1 *Let $V \in L^\infty(\mathrm{IR}^N)$ and $q > 1$.*
Consider $\lambda < \alpha$, $h \in \bigcap_{r>1} L^r(\mathrm{IR}^N)$ and $g \in W^{2,q}(\mathrm{IR}^N)$ such that

$$-\triangle g + (V - \lambda)g = h.$$

Then,

1. *there exist positive constants a and D such that*

$$|g(x)| \leq De^{-a|x|} + (-\triangle + a^2)^{-1}(|h|)(x) \quad \forall x \in \mathrm{IR}^N,$$

2. *$g \in \bigcap_{r>1} W^{2,r}(\mathrm{IR}^N)$.*

Proof

Since

$$(-\triangle + 1)g = h - (V - \lambda - 1)g \tag{4.8}$$

it follows that $g \in W^{2,r}(\mathrm{IR}^N)$ for all $r \geq q$. In particular, $g \in W^{2,r}(\mathrm{IR}^N)$ for some $r > \max\{\frac{N}{2}, 1\}$. Moreover we have already noted that the mapping $(x, s) \mapsto V(x)s$ has the properties (H1) to (H5) with $\alpha = \beta$.

We can then apply Theorem 3.2 with

$$f(x,s) = V(x)s, \qquad E = \{\,(\lambda, g)\,\}, \qquad \widetilde{w} = |h|$$

and deduce thus the existence of positive constants a and D such that

$$|g(x)| \leq De^{-a|x|} + (-\triangle + a^2)^{-1}(|h|)(x) \quad \forall x \in \mathbb{R}^N.$$

This implies that $g \in L^r(\mathbb{R}^N)$ for all $r > 1$ and from (4.8), we now deduce that

$$g \in \bigcap_{r>1} W^{2,r}(\mathbb{R}^N).$$

□

As a particular case of the previous theorem, we have

Corollary 4.1 *Let $V \in L^\infty(\mathbb{R}^N)$ and $q > 1$.*
Consider $\lambda < \alpha$ and $g \in W^{2,q}(\mathbb{R}^N)$ such that

$$-\triangle g + (V - \lambda)g = 0.$$

Then,

1. *there exist positive constants a and D such that*

$$|g(x)| \leq De^{-a|x|} \quad \forall x \in \mathbb{R}^N,$$

2. $g \in \bigcap_{r>1} W^{2,r}(\mathbb{R}^N)$.
   In particular, $Ker(A_{q,\lambda}) = Ker(S - \lambda)$.

Remarks

1. For other results in relationship with the property of exponential decay for eigenfunctions of Schrödinger operators we can refer for example to [1] or [11]. See also Theorem C.3.4 in [17].)

2. The spectrum of Schrödinger operators is shown to be independent of the choice of L^p-space in [13], by viewing them as generators of semi-groups. (See also [4].)

Theorem 4.2 *Let $V \in L^\infty(\mathbb{R}^N)$ and $q > 1$.*

1. *If $V \geq 0$ a.e on $\mathbb{R}^N$, $A_{q,\lambda} : W^{2,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ is an isomorphism for all $\lambda < 0$.*

2. If $\lim_{|x|\to\infty} V(x) = 0$*, the multiplication operator defined by*

$$W^{2,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N), \qquad u \mapsto Vu$$

is compact.

Proof.

These statements are in principal well-known but for the reader's convenience we sketch the proofs.

1. For $V \equiv 0$, this is well known and we already used the result several times. (See Proposition 27 of Chapter II in [6] and Proposition 3 of Chapter III in [18].)

   For $V \geq 0$ and $q = 2$, it is also well-known, since S is a positive self-adjoint operator.

   Using these special cases, we can establishing the general case in the following way.

   For $q \in (1, \infty)$ set $k(t) = |t|^{q-2}t$ for $t \in \mathbb{R}$. By the basic result for Nemitsky operators, the mapping

$$L^q(\mathbb{R}^N) \to L^{q'}(\mathbb{R}^N) \qquad u \mapsto k(u)$$

   is bounded and continuous, where $\frac{1}{q} + \frac{1}{q'} = 1$.

   For any $u \in W^{2,q}(\mathbb{R}^N)$, there exists $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{2,q}(\mathbb{R}^N)$. Then for $q \geq 2$,

$$\begin{aligned}
\int_{\mathbb{R}^N} \{-\triangle u + Vu\}k(u) &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \{-\triangle u_n + Vu_n\}k(u_n) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} (q-1)|\nabla u_n|^2 |u_n|^{q-2} + V|u_n|^q \\
&\geq 0
\end{aligned}$$

   For $q \in (1,2)$, let $\{h_m\}$ be a sequence of odd, increasing C^1-functions such that $h_m \to k$ uniformly on $\mathbb{R}$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \{-\Delta u_n + Vu_n\}\, k(u_n) \\
&= \lim_{m\to\infty} \int_{\mathbb{R}^N} \{-\Delta u_n + Vu_n\}\, h_m(u_n) \\
&= \lim_{m\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 h_m'(u_n) + Vu_n h_m(u_n) \\
&\geq \int_{\mathbb{R}^N} Vu_n k(u_n) = \int_{\mathbb{R}^N} V|u_n|^q \geq 0
\end{aligned}$$

   and so, also in this case,

$$\int_{\mathbb{R}^N} \{-\triangle u + Vu\}k(u) \geq 0.$$

Now fix $\lambda < 0$. Then,

$$\int_{\mathbb{R}^N} \{-\triangle u + Vu - \lambda u\} k(u) \geq -\lambda \int_{\mathbb{R}^N} uk(u) = |\lambda| \parallel u \parallel_q^q$$

and

$$\int_{\mathbb{R}^N} \{-\triangle u + Vu - \lambda u\} k(u) \leq \parallel -\triangle u + Vu - \lambda u \parallel_q \parallel k(u) \parallel_{q'} = \parallel -\triangle u + Vu - \lambda u \parallel_q \parallel u \parallel_q^{q/q'}$$

Thus, for $u \in W^{2,q}(\mathbb{R}^N)$,

$$\parallel -\triangle u + Vu - \lambda u \parallel_q \geq |\lambda| \parallel u \parallel_q,$$

showing that $A_{q,\lambda}$ is injective and that its inverse maps Range $A_{q,\lambda}$ continuously into $L^q(\mathbb{R}^N)$. But we know that Range $A_{2,\lambda} = L^2(\mathbb{R}^N)$ and so from Theorem 4.1 (with $q = 2$) we deduce that $C_0^\infty(\mathbb{R}^N) \subset$ Range $A_{q,\lambda}$ for all $q \in (1, \infty)$. But given $w \in L^q(\mathbb{R}^N)$, there exists $\{w_n\} \subset C_0^\infty$ such that $w_n \to w$ in $L^q(\mathbb{R}^N)$. Setting $u_n = (A_{q,\lambda})^{-1}(w_n)$, we know that $\{u_n\}$ is a Cauchy sequence in $L^q(\mathbb{R}^N)$ and so $\exists u \in L^q(\mathbb{R}^N)$ such that $u_n \to u$ in $L^q(\mathbb{R}^N)$. But then

$$\begin{aligned} (-\triangle + 1)u_n &= w_n + (\lambda + 1)u_n - Vu_n \\ &\to w + (\lambda + 1)u - Vu \text{ (in } L^q(\mathbb{R}^N) \text{)} \end{aligned}$$

and we deduce that $\{u_n\}$ converges to u in $W^{2,q}(\mathbb{R}^N)$. Thus $u \in W^{2,q}(\mathbb{R}^N)$ and $A_{q,\lambda}(u) = w$ showing that Range $A_{q,\lambda} = L^q(\mathbb{R}^N)$. Thus $A_{q,\lambda}$ is an isomorphism.

2. Let

$$\chi_\rho(x) = \begin{cases} 1 & \text{for } |x| \leq \rho \\ 0 & \text{for } |x| > \rho. \end{cases}$$

By the compactness of the Sobolev embedding on bounded domains it follows that the operator

$$W^{2,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N) \qquad u \mapsto \chi_p V u$$

is compact. But for $u \in W^{2,q}(\mathbb{R}^N)$,

$$\begin{aligned} \parallel Vu - \chi_\rho Vu \parallel_q^q &= \int_{|x|>\rho} |V||u|^q \\ &\leq \sup_{|x|>\rho} \{|V(x)|\} \parallel u \parallel_q \\ &\leq \sup_{|x|>\rho} \{|V(x)|\} \parallel u \parallel_{W^{2,q}(\mathbb{R}^N)} \end{aligned}$$

Since $\lim_{\rho\to\infty} \{\sup_{|x|>\rho} |V(x)|\} = 0$, it follows that $u \mapsto Vu$ can be approximated by compact operators and so is itself compact.

$\square$

Theorem 4.3 *Let $V \in L^\infty(\mathbb{R}^N)$ and $q > 1$. We have the following assertions.*

1. *For $\lambda < \alpha$, $A_{q,\lambda} : W^{2,q}(\mathbb{R}^N) \longrightarrow L^q(\mathbb{R}^N)$ is Fredholm with index zero.*
2. *For all $\lambda \in \Sigma_q \cap (-\infty, \alpha)$, we have*
$$L^q(\mathbb{R}^N) = \ Ker(A_{q,\lambda}) \oplus \ Range(A_{q,\lambda}).$$
3. *Let Λ be as defined in Section 1 by (1.3). If $\Lambda < \alpha$, we have $dim\ Ker(A_{q,\Lambda}) = 1$.*

Proof.

1. We write
$$A_{q,\lambda} = -\triangle + (V-\alpha)^+ + (\alpha - \lambda) - (V-\alpha)^-.$$
From Theorem 4.2 (i) we have that
$$W^{2,q}(\mathbb{R}^N) \longrightarrow L^q(\mathbb{R}^N), \quad u \longmapsto -\triangle u + (V-\alpha)^+ u + (\alpha-\lambda)u$$
is an isomorphism for all $\lambda < \alpha$ and is so a Fredholm operator with index 0.
Moreover, we have $\lim\limits_{|x|\to\infty} (V-\alpha)^-(x) = 0$. It follows from Theorem 4.2 (ii) that the multiplication operator
$$W^{2,q}(\mathbb{R}^N) \longrightarrow L^q(\mathbb{R}^N), \quad u \longmapsto (V-\alpha)^- u$$
is a compact operator.
Recall that if $T : X \longrightarrow Y$ is a bounded linear Fredholm operator and $K : X \longrightarrow Y$ is a compact operator, then $T + K$ is Fredholm and $\mathrm{ind}(T) = \mathrm{ind}(T+K)$ (see for example [12], Thm 4.2, pp. 189).
Thus, the operator $A_{q,\lambda}$ is a Fredholm operator with index 0 for all $\lambda < \alpha$.

2. Let $\lambda \in \Sigma_q \cap (-\infty, \alpha)$, and consider $h \in \ \mathrm{Ker}(A_{q,\lambda}) \cap \ \mathrm{Range}(A_{q,\lambda})$.
Thus, on the one hand we have by Corollary 4.1 that $h \in \bigcap\limits_{r>1} W^{2,r}(\mathbb{R}^N)$.
On the other hand there exists $g \in W^{2,q}(\mathbb{R}^N)$ such that $h = A_{q,\lambda}(g)$, and from Theorem 4.1 we have that $g \in \bigcap\limits_{r>1} W^{2,r}(\mathbb{R}^N)$.
Hence,
$$h \in \ \mathrm{Ker}(A_{2,\lambda}) \cap \ \mathrm{Range}(A_{2,\lambda}).$$
Since the Schrödinger operator S is self-adjoint, the above intersection is reduce to $\{0\}$ and then $h = 0$.
Moreover, since $A_{q,\lambda}$ is Fredholm with index 0, we deduce that
$$\mathrm{Ker}(A_{q,\lambda}) \oplus \ \mathrm{Range}(A_{q,\lambda}) = L^q(\mathbb{R}^N), \quad \forall \lambda < \alpha.$$

3. It follows from Corollary 4.1 that $\mathrm{Ker}\,(A_{q,\Lambda}) = \mathrm{Ker}\,(S-\Lambda)$.

   Since S is self-adjoint, $\Lambda \in \sigma(S)$ and is equal to the infimum of $\sigma(S)$. It follows that $A_{q,\Lambda}$ is Fredholm with index 0, and applying the open mapping theorem, we see that
$$\mathrm{Ker}(A_{q,\Lambda}) \neq \{0\}.$$
   It follows then that $\dim\ \mathrm{Ker}\,(S-\Lambda) = 1$ (see for example Theorem 10.33 in [21]).

□

Theorem 4.4 *Let f be a mapping satisfying (H1) to (H4). Consider the mapping F defined by (3.5). Then, the restriction of F to $(-\infty,\alpha)\times X$ is Fredholm with index 0 (in the sense of Definition 2.2).*

Proof.

Let $u \in X$. We have by Theorem 3.1 that $D_2F_{(\lambda,u)} = -\triangle + \partial_2 f(\cdot,u) - \lambda$.

Since $\lim_{|x|\to\infty} u(x) = 0$, it follows from Lemma 3.1 that
$$\lim_{|x|\to\infty} \Big\{\partial_2 f(x,u(x)) - \partial_2 f(x,0)\Big\} = 0.$$
Hence,
$$\begin{aligned}\liminf_{|x|\to\infty} \{\partial_2 f(x,u(x))\} &\geq \liminf_{|x|\to\infty} \Big\{\partial_2 f(x,u(x)) - \partial_2 f(x,0)\Big\} + \liminf_{|x|\to\infty} \{\partial_2 f(x,0)\} \\ &= \alpha.\end{aligned}$$
The conclusion now follows from the first assertion of Theorem 4.3.

□

Remark
Let $\lambda \in \Sigma_q \cap (-\infty,\alpha)$. From the second assertion of Theorem 4.3, we know that the condition (6.17) of [8] is satisfied and thus, from Theorem 6.18 of [8], λ is an isolated point in Σ_q.

5 Results on global bifurcation

Throughout this section, we consider a mapping f satisfying the hypotheses (H1) to (H5) and we use the spaces X and Y defined by (3.4). Referring to Theorem 4.4, we consider the following smooth mapping

$$\begin{aligned} A : (-\infty, \alpha) &\longrightarrow \phi_0(X, Y) \\ \lambda &\longmapsto A_\lambda := D_2 F_{(\lambda,0)} \end{aligned}$$

where $\phi_0(X,Y)$ has been defined in Section 2 and $D_2F_{(\lambda,0)}(u) = -\triangle u + \partial_2 f(\cdot,0)\, u - \lambda u$ (see Theorem 3.1).

Notations.

1. We set $\Sigma = \{\, \lambda \in \mathbb{R} \mid D_2F_{(\lambda,0)}$ is not an isomorphism $\}$. We have verified in the previous section that this set is independent of the choice of p in the definition of X and Y. (See the remark at the beginning of Section 4.)
2. Let $\lambda \in \Sigma \cap (-\infty, \alpha)$. We have remarked at the end of the previous section that λ is an isolated point in Σ. So there exists a closed interval $J \subset (-\infty, \alpha)$ such that $J \cap \Sigma = \{\lambda\}$ and $\lambda \in J^0$. The parity (see Section 2) of the restriction of A to J will be denoted by $\sigma(A, \lambda)$.

From Theorem 4.3, we see that the hypotheses of Theorem 6.18 in [8] are satisfied, thus we conclude that for every $\lambda \in \Sigma \cap (-\infty, \alpha)$,

$$\sigma(A,\lambda) = -1 \Longleftrightarrow \dim \operatorname{Ker}(A_\lambda) \text{ is odd }.$$

From Corollary 4.1, we have $\operatorname{Ker}(A_\lambda) = \operatorname{Ker}(S - \lambda)$ and then for every $\lambda \in \Sigma \cap (-\infty, \alpha)$,

$$\sigma(A,\lambda) = -1 \Longleftrightarrow \dim \operatorname{Ker}(S-\lambda) \text{ is odd }.$$

We are now able to prove the theorem on global bifurcation which was anounced in the introduction.

Proof of Theorem 1.1

To prove this theorem, we verify the hypotheses of Theorem 2.1.

By (H1) $F(\lambda, 0) \equiv 0$.
From Theorem 3.1 we see that F is a C^2 mapping.
From Theorem 3.3, the restriction of F to $(-\infty, \beta) \times X$ is boundedly proper.
From Theorem 4.4, and since $\beta \leq \alpha$, we deduce that the restriction of F to $(-\infty, \beta) \times X$ is Fredholm with index 0.

Moreover, since $\lambda_0 < \beta \leq \alpha$, from the previous considerations we deduce

$$\sigma(A, \lambda_0) = -1.$$

Hence all the hypotheses of Theorem 2.1 are fullfiled.

$\square$

Theorem 1.1 can be improved for the branch $\mathcal{C}_\Lambda$ as follows.

Theorem 5.1 *Let f be a mapping satisfying the hypotheses (H1) to (H5) and suppose that $\Lambda < \beta$. If $(\lambda, u) \in \mathcal{C}_\Lambda \setminus \{(\Lambda, 0)\}$ then u has no zeros. Futhermore the component $\mathcal{C}_\Lambda$ has at least one of the following properties.*

1. *$\mathcal{C}_\Lambda$ is unbounded in $\mathbb{R} \times X$,*
2. *$\sup\limits_{(\lambda,u)\in\mathcal{C}_\Lambda} \lambda = \beta$.*

Proof.

Let $\mathcal{D} = \{\, (\lambda, u) \in \mathcal{C}_\Lambda \mid u^2(x) > 0 \;\forall x \in \mathbb{R}^N\}$ and set $Q = \mathcal{D} \cup \{(\Lambda, 0)\}$.

The first step is to show that Q is (i) closed and (ii) open in $\mathcal{C}_\Lambda$, and hence that $Q = \mathcal{C}_\Lambda$.

i) Suppose that $(\lambda, u) \in \mathcal{C}_\Lambda$ and that there is a sequence $\{(\lambda_n, u_n)\} \subset Q$ such that $(\lambda_n, u_n) \to (\lambda, u)$ in $\mathbb{R} \times X$.

   Then $\lambda < \beta$ and, passing to a subsequence we can suppose that either $u_n(x) > 0$ for all $x \in \mathbb{R}^N$ and all $n \in \mathbb{N}$ or $u_n(x) < 0$ for all $x \in \mathbb{R}^N$ and all $n \in \mathbb{N}$. We shall deal with the first case, the second being similar.

   Clearly, $u(x) \geq 0$ for all $x \in \mathbb{R}^N$ and

$$-\Delta u(x) + c^+(x)u(x) = c^-(x)u(x) \geq 0 \text{ a.e. on } \mathbb{R}^N$$

   where

$$c(x) = \begin{cases} \frac{f(x,u(x))}{u(x)} - \lambda & \text{if } u(x) > 0 \\ \partial_2 f(x, 0) - \lambda & \text{if } u(x) = 0. \end{cases}$$

   Since $u \in X$ we also have that $\lim\limits_{|x|\to\infty} u(x) = 0$, and so the maximum principle implies that either

   a) $u \equiv 0$ or

   b) $u(x) > 0$ for all $x \in \mathbb{R}^N$.

   In case (b), we already have that $(\lambda, u) \in \mathcal{D} \subset Q$.

   If $u \equiv 0$, then $\|u_n\|_X \to 0$. By Theorem 3.2, $u_n \in W^{2,q}(\mathbb{R}^N)$ for all $q > 1$, since we may assume that $\lambda_n < \beta$ for all $n \in \mathbb{N}$. Thus u_n is a positive eigenfunction of the Schrödinger operator $-\Delta + V_n$ where

$$V_n(x) = \frac{f(x, u_n(x))}{u_n(x)}.$$

It follows that

$$\lambda_n = \inf\left\{\int_{\mathbb{R}^N} |\nabla v|^2 + V_n v^2 \mid v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} v^2 = 1\right\}.$$

Since $\|u_n\|_\infty \to 0$, it follows from (H1) to (H3) that V_n converges to $\partial_2 f(\cdot, 0)$ uniformly on $\mathbb{R}^N$.

Hence

$$\begin{aligned}\lim_{n\to\infty} \lambda_n &= \inf\left\{\int_{\mathbb{R}^N} |\nabla v|^2 + \partial_2 f(\cdot,0) v^2 \mid v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} v^2 = 1\right\} \\ &= \Lambda.\end{aligned}$$

Thus in case (a), $(\lambda, u) = (\Lambda, 0) \in Q$.

ii) Suppose that $(\lambda, u) \in Q$. We must show that there is an open subset U of $\mathbb{R} \times X$ such that $(\lambda, u) \in U$ and $U \cap \mathcal{C}_\Lambda \subset Q$.

Suppose first that $u(x) > 0$ for all $x \in \mathbb{R}^N$.

Since $\lambda < \beta \leq \alpha$ there exist s_0 and $R > 0$ such that

$$\frac{f(x,s)}{s} \geq \alpha - \frac{1}{2}(\alpha - \lambda)$$

for all $|s| \leq s_0$ and $|x| \geq R$.

Futhermore, there exists $R_1 \geq 0$ such that $0 < u(x) < \frac{1}{2}s_0$ for all $|x| \geq R_1$. Thus we can choose $\delta > 0$ such that $|w(x)| \leq s_0$ for all $|x| \geq R_1$ and

$$w(x) \geq \frac{1}{2}\min\{u(x) \mid |x| = R_1\} > 0$$

for all $w \in B_X(u, \delta) \equiv \{w \in X : \|u - w\|_X < \delta\}$.

Let

$$U = \{(\mu, w) \in \mathbb{R} \times X \mid |\mu - \lambda| < \frac{1}{2}(\alpha - \lambda) \text{ and } \|u - w\|_X < \delta\}.$$

Then for $(\mu, w) \in U \cap \mathcal{C}_\Lambda$ we have that,

$$-\Delta w(x) + c(x)w(x) = 0$$

where

$$\begin{aligned}c(x) &\equiv \frac{f(x, w(x))}{w(x)} - \mu \\ &\geq \alpha - \frac{1}{2}(\alpha - \lambda) - \mu \\ &\geq \alpha - \frac{1}{2}(\alpha - \lambda) - \lambda - \frac{1}{2}(\alpha - \lambda) = 0\end{aligned}$$

for all $|x| \geq R_1$.

The maximum principle now implies that $w(x) > 0$ for all $(\mu, w) \in U \cap \mathcal{C}_\Lambda$ and for all $|x| \geq R_1$. Since $u(x) > 0$ for all $x \in \mathbb{R}^N$, by making δ small enough, we also have that

$$w(x) \geq \frac{1}{2} \min\{u(y) \mid |y| \leq R_1\}$$

for all $(\mu, w) \in U$ and all $|x| \leq R_1$.

Thus $U \cap \mathcal{C}_\Lambda \subset \mathcal{D} \subset Q$.

The case where $u(x) < 0$ for all $x \in \mathbb{R}^N$ being similar to the one just discussed, we now turn to the only remaining possibility, namely $(\lambda, u) = (\Lambda, 0)$.

Using the theorem of Crandall and Rabinowitz [5] concerning bifurcation from a simple eigenvalue, we know that there exist an open subset V in $\mathbb{R} \times X$ containing $(\Lambda, 0)$, $\delta > 0$ and two functions $\mu \in C^1((-\delta, \delta), \mathbb{R})$, $\eta \in C^1((-\delta, \delta), X)$ such that $(\mu(0), \eta(0)) = (\Lambda, 0)$ and

$$V \cap \mathcal{Z} = \{(\mu(s), s[\phi + \eta(s)]) \mid 0 < |s| < \delta\}$$

where $\operatorname{Ker}(A_{p,\Lambda}) = span\{\phi\}$ in the notation of Section 4 and $\mathcal{Z}$ is defined in the introduction. As is well-known, we can suppose that $\phi(x) > 0$ for all $x \in \mathbb{R}^N$.

There exist $\delta_1 \in (0, \delta)$ and $R > 0$ such that

$$\frac{f(x, s[\phi + \eta(s)](x))}{s[\phi + \eta(s)](x)} \geq \alpha - \frac{1}{2}(\alpha - \Lambda)$$

for all $|s| < \delta_1$ and all $|x| \geq R$.

(Recall that if $s[\phi + \eta(s)](x) = 0$, the left hand side is interpreted as $\partial_2 f(x, 0)$.)

Thus

$$-\Delta[\phi + \eta(s)](x) + c(x)[\phi + \eta(s)](x) \geq 0$$

for all $|s| < \delta_1$ and $|x| \geq R$ where

$$c(x) \geq \alpha - \frac{1}{2}(\alpha - \Lambda) - \mu(s) > 0$$

provided that δ_1 is small enough.

Futhermore, for δ_1 small enough, we also have that

$$[\phi + \eta(s)](x) \geq \frac{1}{2} \min\{\phi(y) \mid |y| = R\} > 0$$

for all $|x| = R$.

Since $\lim\limits_{x \to \infty} [\phi + \eta(s)](x) = 0$, the maximum principle implies that

$$[\phi + \eta(s)](x) > 0 \text{ for all } |s| < \delta_1 \text{ and } |x| \geq R.$$

Finally by choosing δ_1 sufficiently small we also have that

$$[\phi+\eta(s)](x) \geq \frac{1}{2}\min\{\phi(y) \mid |y| \leq R\}$$

for all $|x| \leq R$.

Hence for $0 < |s| < \delta_1$, we have that $(\mu(s), s[\phi+\eta(s)]) \in \mathcal{D}$ and so there is an open subset U of $\mathbb{R} \times X$ such that $(\Lambda, 0) \in U$ and $U \cap \mathcal{Z} \subset \mathcal{D}$.

Having established (i) and (ii) we conclude that $\mathcal{C}_\Lambda = Q$.

To complete the proof we must show that $\mathcal{C}_\Lambda$ cannot have the property 2 of Theorem 1.1. To see this it is sufficient to consider a sequence

$$\{(\lambda_n, u_n)\} \in \mathcal{C}_\Lambda \setminus \{(\Lambda, 0)\} = \mathcal{D}$$

such that $\|u_n\|_X \to 0$. As in the proof of case (b) in (i), this implies that $\lambda_n \to \Lambda$. □

In fact from this result and its proof we can deduce some extra information about $\mathcal{C}_\Lambda$.

Let

$$\mathcal{C}_\Lambda^+ = \left\{(\lambda, u) \in \mathcal{C}_\Lambda \mid u(x) > 0 \text{ for all } x \in \mathbb{R}^N\right\}$$

and

$$\mathcal{C}_\Lambda^- = \left\{(\lambda, u) \in \mathcal{C}_\Lambda \mid u(x) < 0 \text{ for all } x \in \mathbb{R}^N\right\}.$$

Corollary 5.1 *Under the hypotheses of Theorem 5.1, both* $\mathcal{C}_\Lambda^+$ *and* $\mathcal{C}_\Lambda^-$ *are non-empty and connected. Futhermore* $\mathcal{C}_\Lambda = \mathcal{C}_\Lambda^+ \cup \mathcal{C}_\Lambda^- \cup \{(\Lambda, 0)\}$.

Proof.

In the proof of Theorem 5.1, we see that

$$(\mu(s), s[\phi+\eta(s)]) \in \mathcal{C}_\Lambda^+ \text{ for } 0 < s < \delta_1$$

and that

$$(\mu(s), s[\phi+\eta(s)]) \in \mathcal{C}_\Lambda^- \text{ for } -\delta_1 < s < 0.$$

Since $\mathcal{C}_\Lambda = Q = \mathcal{C}_\Lambda^+ \cup \mathcal{C}_\Lambda^- \cup \{(\Lambda, 0)\}$, we need only to establish the connectedness of $\mathcal{C}_\Lambda^+$ and $\mathcal{C}_\Lambda^-$.

To show that $\mathcal{C}_\Lambda^+$ is connected, it suffices to prove that

$$P = \mathcal{C}_\Lambda^+ \setminus \left\{(\mu(s), s[\phi+\eta(s)]) \mid 0 < s \leq \frac{1}{2}\delta_1\right\}$$

is connected.

Since we showed in the proof of Theorem 5.1 that

$$\overline{\mathcal{C}_\Lambda} \cap \{(\lambda, 0) \mid \lambda \in \mathbb{R}\} = (\Lambda, 0),$$

it follows that there exists $\epsilon > 0$ such that $\|u - v\|_X \geq \epsilon$ whenever $(\lambda, u) \in P$ and $(\mu, v) \in \mathcal{C}_\Lambda^- \cup \{(\Lambda, 0)\}$.

If P is not connected there must be a non-empty subset A of

$$P \setminus \left\{(\mu(s), s[\phi + \eta(s)]) \mid \frac{1}{2}\delta_1 < s < \frac{3}{4}\delta_1\right\}$$

which is both open and closed in P. But then we see that A is open and closed in $\mathcal{C}_\Lambda$ contradicting the connectedness of $\mathcal{C}_\Lambda$.

Hence P and consequently $\mathcal{C}_\Lambda^+$ is connected. The connectedness of $\mathcal{C}_\Lambda^-$ follows from a similar argument. $\square$

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