The branch of positive solutions to a semilinear elliptic equation on $\mathbb{R}^N$

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Abstract. For a large class of functions f, we consider the nonlinear elliptic eigenvalue problem

$$-\Delta u(x) + f(x, u(x)) = \lambda u(x) \text{ for } x \in \mathbb{R}^N,$$
$$\lim_{|x|\to\infty} u(x) = 0, \; u(x) > 0 \text{ for all } x \in \mathbb{R}^N.$$

If the lowest eigenvalue of the linearization lies below the essential spectrum, it is known [6] that a global branch, $\mathcal{C}$, of solutions bifurcates at this value of λ. The main results of the present paper give properties of f which enable us to provide explicit lower bounds for $\sup\{\lambda \,|\, (\lambda, u) \in \mathcal{C}\}$. Our approach is based on the construction of supersolutions which enable us to refine the alternative established in [6] concerning the global behaviour of $\mathcal{C}$.

1 Introduction

We consider a nonlinear elliptic eigenvalue problem of the form

$$\begin{aligned} &-\Delta u(x) + f(x, u(x)) = \lambda u(x) \text{ for } x \in \mathbb{R}^N \\ &\lim_{|x|\to\infty} u(x) = 0 \text{ and } u(x) > 0 \text{ for } x \in \mathbb{R}^N \end{aligned} \tag{1.1}$$

where $f\colon \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ is a mapping satisfying

(H1) $f(x, s) = -f(x, -s)$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$,

(H2) the function f satisfies the conditions of Caratheodory,

(H3) $f(x, \cdot) \in C^2(\mathbb{R})$ for all $x \in \mathbb{R}^N$,
for all compact $K \subset \mathbb{R}$, the functions $\{\partial^2_{22} f(x, \cdot)\colon K \to \mathbb{R} \mid x \in \mathbb{R}^N\}$ are equicontinuous and $\partial^2_{22} f$ is bounded on $\mathbb{R}^N \times K$,

(H4) $\partial_2 f(\cdot, 0)$ is bounded on $\mathbb{R}^N$. We set $\alpha := \liminf_{|x|\to\infty} \partial_2 f(x, 0)$.

For each $C \geq 0$, we introduce the following real number,

$$\beta(C) := \lim_{R\to\infty} \inf_{\substack{|x|\geq R \\ |s|\leq C}} \left\{ \frac{f(x,s)}{s} \right\}.$$

We set

$$\beta := \inf_{C\geq 0} \beta(C),$$

and we make the following hypothesis

(H5) $\beta > -\infty$.

Remarks.

1. When we shall consider the mapping $(x,s) \mapsto \dfrac{f(x,s)}{s}$, we have always in mind the following one

$$(x,s) \longmapsto \begin{cases} \dfrac{f(x,s)}{s} & \text{if } s \neq 0 \\ \partial_2 f(x,0) & \text{if } s = 0. \end{cases}$$

2. We have $\beta \leq \alpha$.

For $m \in \mathbb{N}$ and $p \geq 1$, we adopt the standard notation [2] for the Sobolev space $W^{m,p}(\mathbb{R}^N)$. Fixing a value $p \in (N/2, \infty) \cap (1, \infty)$ we set $X = W^{2,p}(\mathbb{R}^N)$, and we recall that the condition $\lim_{|x|\to\infty} u(x) = 0$ is satisfied for all $u \in X$.
We are interested in pairs $(\lambda, u) \in \mathbb{R} \times X$ which are solutions of the problem (1.1) and our aim is to obtain some global properties of the branch of solutions which bifurcates from the lowest eigenvalue of the linearization. Similar problems have been investigated under various assumptions on f in [3], [1] and [9]. Our approach is based essentially on the results obtained in [6]. In this previous paper we studied the problem (1.1) without the condition $u(x) > 0$ for all $x \in \mathbb{R}^N$. We showed the existence of global branches of non trivial solutions bifurcating from a trivial solution $(\lambda_0, 0)$ in $\mathbb{R} \times X$, by using the degree theory for proper C^2 Fredholm mappings developped by Fitzpatrick, Pejsachowicz and Rabier in [4]. One of the branches bifurcates from $(\Lambda, 0)$ if $\Lambda < \beta$ where

$$\Lambda := \inf \left\{ \int_{\mathbb{R}^N} |\nabla\psi|^2 + \partial_2 f(\cdot,0)\psi^2 \;\middle|\; \psi \in C_0^\infty(\mathbb{R}^N), \int_{\mathbb{R}^N} |\psi|^2 = 1 \right\}, \tag{1.2}$$

and this branch has a connected component of solutions to the problem (1.1).
More precisely, denote by (I) the problem (1.1) but with the condition $u > 0$ replaced by $u \not\equiv 0$ and let

$$\mathcal{S} = \{ (\lambda, u) \in (-\infty, \beta) \times X \mid (\lambda, u) \text{ is a solution to } (I) \}.$$

On $\mathcal{S} \cup \{(\Lambda, 0)\}$ consider the topology inherited from $\mathbb{R} \times X$ and let $\mathcal{C}_\Lambda$ be the connected component of $\mathcal{S} \cup \{(\Lambda, 0)\}$ containing $(\Lambda, 0)$.

Let

$$\mathcal{C}_\Lambda^+ := \left\{(\lambda, u) \in \mathcal{C}_\Lambda \mid u(x) > 0 \text{ for all } x \in \mathbb{R}^N\right\} \text{ and}$$
$$\mathcal{C}_\Lambda^- := \left\{(\lambda, u) \in \mathcal{C}_\Lambda \mid u(x) < 0 \text{ for all } x \in \mathbb{R}^N\right\}.$$

In Theorem 5.1 of [6], we obtain the following result which is in the spirit of the classic alternative established by Rabinowitz [8].

Theorem 1.1 *Let f be a mapping satisfying the hypotheses (H1) to (H5) and suppose that $\Lambda < \beta$. If $(\lambda, u) \in \mathcal{C}_\Lambda \setminus \{(\Lambda, 0)\}$ then u has no zeros. The component $\mathcal{C}_\Lambda$ has at least one of the following properties.*

1. *$\mathcal{C}_\Lambda$ is unbounded in $\mathbb{R} \times X$,*
2. *$\sup_{(\lambda,u)\in\mathcal{C}_\Lambda} \lambda = \beta$.*

Both $\mathcal{C}_\Lambda^+$ and $\mathcal{C}_\Lambda^-$ are non-empty and connected. Furthermore $\mathcal{C}_\Lambda = \mathcal{C}_\Lambda^+ \cup \mathcal{C}_\Lambda^- \cup \{(\Lambda, 0)\}$.

We can derive the same result for $\mathcal{C}_\Lambda^+$.

Corollary 1.1 *Under the same hypotheses as Theorem 1.1, the component $\mathcal{C}_\Lambda^+$ has at least one of the following properties*

1. *$\mathcal{C}_\Lambda^+$ is unbounded in $\mathbb{R} \times X$,*
2. *$\sup_{(\lambda,u)\in\mathcal{C}_\Lambda^+} \lambda = \beta$.*

Proof. By (H1), $\mathcal{C}_\Lambda^- = \left\{(\lambda, -u) \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\right\}$ and so this follows from Theorem 1.1. □

In particular, Corollary 1.1 ensures the existence of solutions of problem (1.1) and, of course, $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\} \geq \Lambda$. The main purpose of this paper is to formulate conditions under which a better lower bound for $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\}$ can be given. In Section 2 we begin by recalling some useful results from [6] and we show how upper bounds for $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\}$ can be obtained.
The main ingredient in our approach to establishing lower bounds for $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\}$ involves the construction of a supersolution of the problem (1.1) and this can be accomplished in various ways depending on the properties of the function f.
To describe our results we introduce the following notations.

$$\mathcal{S}^+ = \{(\lambda, u) \in (-\infty, \beta) \times X \mid (\lambda, u) \text{ is a solution of (1.1)}\},$$
$$\mathcal{S}_b^+ = \left\{(\lambda, u) \in \mathcal{S}^+ \mid \lambda \leq b\right\},$$
$$\gamma = \inf\left\{\frac{f(x,s)}{s} \mid x \in \mathbb{R}^N \text{ and } s > 0\right\} \text{ and}$$
$$\nu = \lim_{R\to\infty}\left\{\inf_{|x|^2+s^2\geq R} \frac{f(x,s)}{s}\right\}.$$

The first result in Section 3 shows that $\gamma \leq \lambda \leq \alpha$ for all $(\lambda, u) \in \mathcal{S}^+$.
Then supposing that

(H6) $\gamma > -\infty$,

we show that $\mathcal{S}_b^+$ is bounded in $\mathrm{I\!R} \times X$ for any $b < \nu$. Referring to Corollary 1.1, we see that $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\} \geq \nu$ provided that (H1) to (H6) are satisfied and $\Lambda < \beta$. It is easy to find functions f satisfying all these conditions for which $\nu > \Lambda$ and in this case the trivial estimate $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\} \geq \Lambda$ is sharpened. However the lower bound ν is not always optimal and it is equally easy to find cases where $\nu < \Lambda$. To deal with these situations, where the parameter ν does not provide an adequate lower bound for $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\}$, we turn to the construction of supersolutions.

Definition. A pair (b, Ψ) is a supersolution of problem (1.1) if

- $\Psi \in X$,
- $-\Delta\Psi(x) + f(x, \Psi(x)) - b\Psi(x) \geq 0$ a.e . on $\mathrm{I\!R}^N$,
- $\Psi(x) > 0$ for all $x \in \mathrm{I\!R}^N$.

In Section 4 we show that if an appropriate supersolution (b, Ψ) of (1.1) exists then $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\} \geq b$.
The construction of such supersolutions is undertaken in Section 5 where three different methods are used depending on the properties of f. For a function f which has the simple form,

$$f(x, s) = p(x)s + q(x)r(s)s \text{ for } x \in \mathrm{I\!R}^N \text{ and } s \in \mathrm{I\!R}, \tag{1.3}$$

where $q \geq 0$ on $\mathrm{I\!R}^N$ and $\inf_{s \geq 0} r(s) > -\infty$, the differences between these three cases can be roughly described as follows; a more thorough discussion of this kind of example being given in Section 6.
Let $Z = \{x \in \mathrm{I\!R}^N \mid q(x) = 0\}$. The first two constructions require the condition that $p(x) + q(x) \liminf_{s \to \infty} r(s) \geq \beta$ for all $x \in \mathrm{I\!R}^N \setminus Z$. Moreover if $\liminf_{s \to \infty} r(s) = \infty$, we must assume that $p(x) \geq \beta$ for all $x \in \partial Z$. This condition is not necessary for the third construction which however only deals with the cases where $\lim_{s \to \infty} r(s) = \infty$.
The choice of the construction also depends on the characteristics of the sets Z and $\Omega = Z \cap \{x \in \mathrm{I\!R}^N \mid p(x) < \beta\}$. If Ω is an unbounded subset of $\mathrm{I\!R}^N$, only the first construction can be used. The second construction requires Ω to be bounded whereas Z must be bounded for the third one. However these constructions may give a better result in the sense that they may provide a supersolution for a larger value of b, and hence, by Theorem 4.1, a better lower bound for $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\}$. The bounds obtained by the three constructions are in Theorems 5.2, 5.3 and 5.4 respectively.

Remark. In the course of this work the authors of [4] have kindly informed us that they can now define a similar degree for proper C^1 Fredholm maps. In response to this development, we observe in [6] that the existence of such a degree means that Theorem 1.1 and Corollary 1.1 remain valid when (H3) is replaced by

(H3)' $f(x,\cdot) \in C^1(\mathrm{I\!R})$ for almost all $x \in \mathrm{I\!R}^N$, and for every compact $K \subset \mathrm{I\!R}$, the functions $\{\partial_2 f(x,\cdot) : K \to \mathrm{I\!R} \mid x \in \mathrm{I\!R}^N\}$ are equicontinuous.

This observation also holds for all of the results in the present paper.

2 Preliminaries

We choose $p \in (\frac{N}{2}, \infty) \cap (1, \infty)$, and we set

$$X = W^{2,p}(\mathrm{I\!R}^N)$$

with the following norm,

$$\| u \|_X = \left\{ \sum_{0 \le |\mu| \le 2} \| D^\mu u \|_p^p \right\}^{\frac{1}{p}},$$

where μ is a multi-index and $\| \cdot \|_p$ is the usual norm on $L^p(\mathrm{I\!R}^N)$.

We recall the following properties of the space X (see [2]).

1. $X \hookrightarrow C(\mathrm{I\!R}^N)$, continuously.

   Moreover, the injection $W^{2,p}(B_R) \hookrightarrow C(\overline{B_R})$ is completely continuous for every ball $B_R = \{x \in \mathrm{I\!R}^N \mid |x| < R\}$.
2. $X \hookrightarrow L^q(\mathrm{I\!R}^N)$, continuously, for every $p \le q \le \infty$.
3. $\lim_{|x| \to \infty} u(x) = 0$ for all $u \in X$.

Lemma 2.1 *Let f be a mapping which satisfies the hypotheses (H1) to (H4) and let K be a compact subset of* $\mathrm{I\!R}$. *Then,*

1. *$\partial_2 f$ is bounded on $\mathrm{I\!R}^N \times K$.*
2. *There exists a constant $C = C(K)$ such that for all $(x, s_1), (x, s_2) \in \mathrm{I\!R}^N \times K$ we have*

$$\begin{aligned} |f(x,s_1) - f(x,s_2)| &\le C|s_1 - s_2|, \\ |\partial_2 f(x,s_1) - \partial_2 f(x,s_2)| &\le C|s_1 - s_2|. \end{aligned}$$

3. *Let $u \in X$. There exists a constant $C = C(u)$ such that*

$$|f(x,u(x))| \le C|u(x)| \quad \forall x \in \mathrm{I\!R}^N.$$

We have denoted in the introduction

$$\mathcal{S} := \{(\lambda, u) \in (-\infty, \beta) \times X \mid (\lambda, u) \text{ is a solution to } (I)\}.$$

As a particuliar case of Theorem 3.2 of [6], we have the following result.

Theorem 2.1 *Let f be a mapping satisfying the hypotheses (H1) to (H5). Let E be a subset of $\mathcal{S}$ and suppose that there exist $C > 0$ and $k < \beta$ such that for all $(\lambda, u) \in E$,*

$$\| u \|_\infty < C \qquad \text{and} \qquad \lambda \le k < \beta.$$

Then, there exist positive constants a and D such that for all $(\lambda, u) \in E$,

$$|u(x)| \le De^{-a|x|} \quad \forall x \in \mathbb{R}^N.$$

Let us complete these preliminary observations by showing how upper bounds for $\sup\{\lambda \mid (\lambda, u) \in C_\Lambda^+\}$ can be obtained.
If $(\lambda, u) \in \mathcal{S}^+$, it follows easily from Theorem 2.1 that $u \in H^1(\mathbb{R}^N)$ and it can be regarded as a positive solution of the linear eigenvalue problem

$$-\Delta v(x) + \frac{f(x, u(x))}{u(x)} v(x) = \lambda v(x) \text{ a.e. on } \mathbb{R}^N.$$

Consequently λ is characterised by

$$\lambda = \inf \left\{ \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{f(x, u(x))}{u(x)} v^2 \mid v \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} v^2 = 1 \right\}.$$

Setting $m(x) = \sup_{s>0} \left\{ \frac{f(x,s)}{s} \right\}$, we obtain the estimate $\lambda \le \tilde{\lambda}$ where

$$\tilde{\lambda} = \inf \left\{ \int_{\mathbb{R}^N} |\nabla v|^2 + m(x) v^2 \mid v \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} v^2 = 1 \right\}.$$

Clearly $m(x) \ge \lim_{s \to 0} \frac{f(x,s)}{s} = \partial_2 f(x, 0)$ for all $x \in \mathbb{R}^N$ and so $\tilde{\lambda} \ge \Lambda$.
As a first special case, we note that, if $s^{-1} f(x, s) \le \partial_2 f(x, 0)$ for all $x \in \mathbb{R}^N$ and $s > 0$, then $m(x) \equiv \partial_2 f(x, 0)$ on $\mathbb{R}^N$ and $\tilde{\lambda} = \Lambda$. Thus $\sup\{\lambda \mid (\lambda, u) \in C_\Lambda^+\} = \Lambda$ in this case.
More generally, if Ω is a bounded open set in $\mathbb{R}^N$ such that $s^{-1} f(x, s) \le \partial_2 f(x, 0)$ for all $x \in \Omega$ and $s > 0$, then $m(x) = \partial_2 f(x, 0)$ on Ω and $\tilde{\lambda} \le \lambda_1(\Omega)$ where

$$\lambda_1(\Omega) = \inf \left\{ \int_\Omega |\nabla v|^2 + \partial_2 f(x, 0) v^2 \mid v \in H_0^1(\Omega) \text{ and } \int_\Omega v^2 = 1 \right\}$$

is the first eigenvalue of the Dirichlet problem for $-\Delta + \partial_2 f(x, 0)$ on Ω.

3 Properties of the positive solutions.

In this section, the results deal with all positive solutions of (1.1). We will use the following notations

$$\begin{aligned}\mathcal{S}^+ &:= \{(\lambda, u) \in \mathcal{S} \mid (\lambda, u) \text{ is a solution to (1.1)}\}\\ \mathcal{S}_b^+ &:= \{(\lambda, u) \in \mathcal{S}^+ \mid \lambda \leq b\}.\end{aligned}$$

We denote by p_1 (respectively p_2) the natural projection of $\mathbb{R} \times X$ on $\mathbb{R}$ (respectively on X).
First, we show that $p_1(\mathcal{S}^+)$ is bounded below by the constant γ appearing in (H6) .

Remark. The hypothesis (H6) implies (H5) since $-\infty < \gamma \leq \beta$.

Theorem 3.1 *Let the hypotheses (H1) to (H6) be satisfied. Then $p_1(\mathcal{S}^+) \subset [\gamma, \beta)$.*

Proof. For $(\lambda, u) \in S^+$ we have that $\lambda < \beta$, and so as in the discussion following Theorem 2.1 we have

$$\begin{aligned}\lambda &= \inf\left\{\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{f(x, u(x))}{u(x)} v(x)^2 \mid v \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} v^2 = 1\right\}\\ &\geq \inf_{x \in \mathbb{R}^N}\left\{\frac{f(x, u(x))}{u(x)}\right\}\\ &\geq \inf\left\{\frac{f(x, s)}{s} \mid (x, s) \in \mathbb{R}^N \times (0, \infty)\right\}\\ &= \gamma.\end{aligned}$$

□

Theorem 3.2 *Let f be a mapping satisfying (H1) to (H5). Let E be a subset of $\mathbb{R} \times X$ such that $E \subset \mathcal{S}$ and that there exist $b_1 < b_2 < \beta$ such that*

$$p_1(E) \subset [b_1, b_2].$$

Then, the following properties are equivalent

1. *$p_2(E)$ is bounded in X.*

2. *$p_2(E)$ is bounded in L^∞.*

3. *There exists $\eta \in X$ such that for all $(\lambda, u) \in E$*

$$|u(x)| \leq \eta(x) \quad \forall x \in \mathbb{R}^N.$$

Proof. (1) $\Longrightarrow$ (2)
This follows from the continuous injection $X \hookrightarrow L^\infty(\mathbb{R}^N)$.
(2) $\Longrightarrow$ (3)
By hypothesis, there exists a constant C such that for all $(\lambda, u) \in E$,

$$\| u \|_\infty < C.$$

Since $\lambda \leq b_2 < \beta$, it follows from Theorem 2.1 that there exist constants $a, D > 0$ such that

$$|u(x)| \leq De^{-a|x|} \qquad \text{for all } x \in R^N.$$

Hence we can choose η to be any element of X such that $\eta(x) \geq De^{-a|x|}$ on $\mathbb{R}^N$.
(3) $\Longrightarrow$ (1)
Let $(\lambda, u) \in E$. Thus,

$$-\Delta u + f(\cdot, u) - \lambda u = 0.$$

or equivalently

$$(-\Delta + 1)u = -f(\cdot, u) + (\lambda + 1)u.$$

Now, it follows from the hypotheses and Lemma 2.1 that there exists a constant $C = C(|\eta|_\infty)$ such that $|f(x, s)| \leq C(|\eta|_\infty)|s|$ for all $(x, s) \in \mathbb{R}^N \times [-|\eta|_\infty, |\eta|_\infty]$.
Hence

$$|f(\cdot, u)| \leq C|u| \leq C\eta, \text{ and}$$

the set

$$\{(-\Delta + 1)u \ \mid \ (\lambda, u) \in E\}$$

is bounded in $L^p(\mathbb{R}^N)$.
Since, $(-\Delta + 1)$ is an isomorphism of X onto $L^p(\mathbb{R}^N)$, we have that $p_2(E)$ is bounded in X. □

Combining Theorem 3.1 and Theorem 3.2, we immediately get

Corollary 3.1 *Let f be a mapping satisfying (H1) to (H6) and $b < \beta$. Then, the following properties are equivalent.*

1. *$\mathcal{S}_b^+$ is bounded in $\mathbb{R} \times X$.*
2. *$\mathcal{S}_b^+$ is bounded in $\mathbb{R} \times L^\infty$.*
3. *There exists $\eta \in X$ such that for all $(\lambda, u) \in \mathcal{S}_b^+$*

$$|u(x)| \leq \eta(x) \quad \forall x \in \mathbb{R}^N.$$

To formulate the last theorem of this section, we introduce the following real number,

$$\nu = \lim_{R \to \infty} \left\{ \inf_{|x|^2 + s^2 \geq R^2} \frac{f(x, s)}{s} \right\}. \tag{3.4}$$

Remark. We have $-\infty < \gamma \leq \nu \leq \beta$.

Theorem 3.3 *Let f be a mapping satisfying (H1) to (H6). Then, $\mathcal{S}_b^+$ is bounded in $\mathbb{R} \times X$ for all $b < \nu$.*

Proof. Let $b < \nu$. Since $\nu \leq \beta$, by Theorem 3.1 and Corollary 3.1 it is enough to prove that $p_2(\mathcal{S}_b^+)$ is bounded in $L^\infty(\mathbb{R}^N)$.
Choose $\epsilon > 0$ such that $\nu - b - \epsilon > 0$.
There exists $R_0 > 0$ such that

$$\frac{f(x,s)}{s} \quad \geq \quad \nu - \epsilon \qquad \text{for all } \; |x|^2 + s^2 \geq R_0^2.$$

Let $(\lambda, u) \in \mathcal{S}_b^+$, we prove

$$\| \, u \, \|_\infty \leq R_0.$$

Set $\Omega = \{x \in \mathbb{R}^N \mid u(x) > R_0\}$ and suppose that $\Omega \neq \emptyset$. Since u is continuous, Ω is open and since $\lim_{|x| \to \infty} u(x) = 0$, we have that Ω is bounded.
Moreover, we have

$$\Delta u(x) = \Big(\frac{f(x,u(x))}{u(x)} - \lambda\Big) u(x) \geq (\nu - \epsilon - b) u(x) > 0 \;\; \text{a.e. } \Omega,$$

and $u(x) = R_0$ on $\partial\Omega$.
Then, from the weak maximum principle (see [5]), it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u = R_0,$$

i.e. $u(x) \leq R_0$ on Ω. This is a contradiction to the definition of Ω and so $\Omega = \emptyset$. $\square$

Corollary 3.2 *Let f be a mapping satisfying (H1) to (H6), let ν be defined by (3.4) and $\Lambda < \beta$. Then*

$$\sup_{(\lambda,u) \in \mathcal{C}_\Lambda^+} \lambda \geq \nu.$$

Proof. If we suppose that

$$\sup_{(\lambda,u) \in \mathcal{C}_\Lambda^+} \lambda < \nu,$$

it then follows from Theorem 3.3 that C_Λ^+ is bounded in $\mathbb{R} \times X$, and this is a contradiction with Corollary 1.1. $\square$

4 Supersolutions and C_Λ^+.

In this section we show how the existence of a supersolution, as defined in the introduction, can be used to estimate $\sup p_1(C_\Lambda^+)$.

Remarks.

1. If (b, Ψ) is a supersolution of the problem (1.1), then (λ, Ψ) is also a supersolution for all $\lambda \leq b$.

2. If (b, Ψ) is a supersolution of the problem (1.1), then

$$\begin{aligned}
\frac{\Delta\Psi(x)}{\Psi(x)} &\leq \frac{f(x,\Psi(x))}{\Psi(x)} - b \\
&= \frac{1}{\Psi(x)} \int_0^1 \frac{d}{dt} f(x, t\Psi(x))\, dt - b \\
&= \frac{1}{\Psi(x)} \left\{ \int_0^1 \{\partial_2 f(x, t\Psi(x)) - \partial_2 f(x,0)\} \Psi(x)\, dt \right\} + \partial_2 f(x,0) - b \\
&\leq C \int_0^1 t\Psi(x)\, dt + \partial_2 f(x,0) - b \ \text{ by Lemma 2.1, part 2} \\
&\leq \frac{1}{2} C\Psi(x) + \partial_2 f(x,0) - b
\end{aligned}$$

Hence, using (H4), we get

$$\limsup_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)} \leq \limsup_{|x|\to\infty} \partial_2 f(x,0) - b < \infty.$$

Theorem 4.1 *Let (H1) to (H6) be satisfied and $\Lambda < \beta$. Suppose that there exists a supersolution (b, Ψ) of the problem (1.1) such that*

$$b \leq \min\{\beta, \alpha - a\} \text{ where } a := \limsup_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)}.$$

Then

$$\sup_{(\lambda,u)\in\mathcal{C}_\Lambda^+} \lambda \geq b.$$

Remark. If $\alpha = \lim_{|x|\to\infty} \partial_2 f(x,0)$, then by the previous remark point 2, necessarily any supersolution (b, Ψ) satisfies $b \leq \alpha - a$. In this case all supersolution of problem (1.1) fulfil the hypotheses of Theorem 4.1.

Proof. Seeking a contradiction, we suppose that $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\} < b$, and we show then that in this case for all $(\lambda, u) \in \mathcal{C}_\Lambda^+$, we have

$$u(x) \leq \Psi(x) \ \text{ for all } x \in \mathbb{R}^N.$$

So by Corollary 3.1, C_Λ^+ is bounded in $\mathbb{R} \times X$ and this contradicts Corollary 1.1.
Let

$$D = \left\{(\lambda, u) \in \mathcal{C}_\Lambda^+ \ \mid \ u(x) \leq \Psi(x) \ \text{ for all } x \in \mathbb{R}^N\right\} \cup \{(\Lambda, 0)\}.$$

We show that D is not empty, is closed and open in $\mathcal{C}_\Lambda^+ \cup \{(\Lambda, 0)\}$, and hence that $D = \mathcal{C}_\Lambda^+ \cup \{(\Lambda, 0)\}$ since $\mathcal{C}_\Lambda^+ \cup \{(\Lambda, 0)\}$ is connected.

1. D is not empty, because $(\Lambda, 0) \in \mathcal{C}_\Lambda^+ \cup \{(\Lambda, 0)\}$.

2. D is closed in $\mathcal{C}_\Lambda^+$.

   Suppose that $(\lambda, u) \in \mathcal{C}_\Lambda^+ \cup \{(\Lambda, 0)\}$ and that there is a sequence $\{(\lambda_n, u_n)\} \subset D$ such that $(\lambda_n, u_n) \to (\lambda, u)$ in $\mathbb{R} \times X$. Clearly $u(x) \leq \Psi(x)$ for all $x \in \mathbb{R}^N$ and so $(\lambda, u) \in D$.

3. D is open in $\mathcal{C}_\Lambda^+ \cup \{(\Lambda, 0)\}$.

   First we establish that if $(\lambda, u) \in D$ then $u(x) < \Psi(x)$ for all $x \in \mathbb{R}^N$.

   Since $\lambda < b$, the function $(\Psi - u) \in X$ satisfies

$$-\Delta(\Psi - u) + f(x, \Psi) - f(x, u) - b(\Psi - u) \geq 0.$$

   We set

$$c(x) = \begin{cases} \dfrac{f(x, \Psi(x)) - f(x, u(x))}{(\Psi(x) - u(x))} - b & \text{if } \Psi(x) > u(x) \\ \\ \partial_2 f(x, u(x)) - b & \text{if } \Psi(x) = u(x). \end{cases}$$

   We have $(\Psi - u)(x) \geq 0$ for all $x \in \mathbb{R}^N$ and

$$-\Delta(\Psi - u) + c^+(x)(\Psi - u)(x) \geq c^-(x)(\Psi - u)(x) \geq 0 \text{ a.e. on } \mathbb{R}^N.$$

   Since $\lim_{|x| \to \infty}(\Psi - u)(x) = 0$, the strong maximum principle (see [5]) implies that either $(\Psi - u)(x) > 0$ for all $x \in \mathbb{R}^N$ or $\Psi \equiv u$.

   But the second possibility gives

$$-\Delta u(x) + f(x, u(x)) - bu(x) \geq 0 \text{ a.e. } \mathbb{R}^N,$$

   which is impossible since $\lambda < b$ and u is a solution of (1.1).

   Hence

$$u(x) < \Psi(x) \text{ for all } x \in \mathbb{R}^N.$$

   Now suppose that $(\tilde{\lambda}, \tilde{u}) \in D$. We must show that there is an open subset U of $\mathbb{R} \times X$ such that $(\tilde{\lambda}, \tilde{u}) \in U$ and $U \cap \left[\mathcal{C}_\Lambda^+ \cup \{(\Lambda, 0)\}\right] \subset D$.

   We have already proved that

$$\tilde{u}(x) < \Psi(x) \text{ for all } x \in \mathbb{R}^N.$$

   Set $\epsilon = b - \sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\}$.

Since $\epsilon > 0$, there exist $R_0 > 0$ and $s_0 > 0$ such that

$$\begin{aligned} \frac{f(x,s)}{s} > \alpha - \frac{\epsilon}{2} \qquad & \forall \ \ |x| \geq R_0, \forall s \leq s_0, \\ \frac{\Delta \Psi(x)}{\Psi(x)} < a + \frac{\epsilon}{2} \qquad & \forall \ \ |x| \geq R_0, \\ 0 < \tilde{u}(x) < \frac{s_0}{2} \qquad & \forall \ \ |x| \geq R_0. \end{aligned}$$

Since $\tilde{u}(x) < \Psi(x)$, we have

$$m(R_0) := \inf_{|x| \leq R_0} \{\Psi(x) - \tilde{u}(x)\} > 0.$$

There exists $\delta > 0$ such that $\| u - \tilde{u} \|_X < \delta$ implies that

$$\| u - \tilde{u} \|_\infty < \min \Big\{ \frac{s_0}{2}, m(R_0) \Big\}.$$

Let $U = \{(\lambda, u) \in \mathrm{I\!R} \times X \mid \|u - \tilde{u}\|_X < \delta\}$.
We prove that $0 \leq u(x) \leq \Psi(x)$ for all $x \in \mathrm{I\!R}^N$ if $(\lambda, u) \in U \cap \Big[\mathcal{C}_\Lambda^+ \cup \{(\Lambda, 0)\}\Big]$.
For $|x| \leq R_0$, we have

$$\begin{aligned} u(x) &= \tilde{u}(x) + (u(x) - \tilde{u}(x)) \\ &\leq \tilde{u}(x) + \| u - \tilde{u} \|_\infty \\ &\leq \Psi(x) - m(R_0) + \| u - \tilde{u} \|_\infty \\ &< \Psi(x). \end{aligned}$$

Now we show by contradiction that $u(x) < \Psi(x)$ for $|x| > R_0$.
Set $\Omega = \{x \in \mathrm{I\!R}^N \mid |x| > R_0 \ \text{ and } u(x) > \Psi(x)\}$ and suppose that $\Omega \neq \emptyset$.
Since $u(x) < \Psi(x)$ for $|x| = R_0$, we have $\overline{\Omega} \subset \mathrm{I\!R}^N \setminus \overline{B(0, R_0)}$ and $u(x) = \Psi(x)$ for all $x \in \partial\Omega$.
Moreover we have on Ω,

$$\begin{aligned} \Delta(u - \Psi) &= \Big(\frac{f(\cdot, u)}{u} - \lambda\Big) u - \Delta\Psi \\ &\geq (\alpha - \frac{\epsilon}{2} - \lambda) u - \Delta\Psi \\ &\geq (\alpha + \frac{\epsilon}{2} - b)\Psi - \Delta\Psi \ \text{ since } b \leq \beta \leq \alpha \\ &\geq (\alpha + \frac{\epsilon}{2} - b)\Psi - (a + \frac{\epsilon}{2})\Psi \\ &= (\alpha - a - b)\Psi \geq 0. \end{aligned}$$

Hence we have shown that for all $(\lambda, u) \in U \cap \left[\mathcal{C}_\Lambda^+ \cup \{(\Lambda, 0)\}\right]$,

$$\begin{aligned} \Delta(u - \Psi)(x) &\geq 0 \text{ a.e. } \Omega \\ \text{and } (u - \Psi)(x) &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Using the fact that $\lim_{|x|\to\infty} u(x) - \Psi(x) = 0$ and the weak maximum principle, we obtain that $(u - \Psi)(x) \leq 0$ on Ω, contradicting the definition of Ω. Hence $\Omega = \emptyset$ and thus $U \cap \left[\mathcal{C}_\Lambda^+ \cup \{(\Lambda, 0)\}\right] \subset D$. □

5 Construction of supersolutions.

We are interested by supersolutions (b, Ψ) with a parameter $b \in \mathbb{R}$ as large as possible, in order to obtain a better lower bound for $\sup\{\lambda \mid (\lambda, u) \in C_\Lambda^+\}$ using Theorem 4.1. To this end we shall construct three supersolutions (b, Ψ) of the problem (1.1) which satisfy the hypotheses of Theorem 4.1 by three different methods. Sometimes one construction is possible whereas the others fail according to the hypotheses imposed on the function $f(x, s)$. Similarly one construction may lead to a sharper lower bound for $\sup\{\lambda \,|\, (\lambda, u) \in C_\Lambda^+\}$ than the others.
Set

$$\tilde{\nu} := \liminf_{|x|\to\infty} \left\{ \inf_{s>0} \frac{f(x,s)}{s} \right\}, \tag{5.5}$$

$$\eta(x) := \min\left\{ \liminf_{s\to\infty} \frac{f(x,s)}{s}, \tilde{\nu} \right\}. \tag{5.6}$$

Remarks.

1. Under the hypotheses (H1) to (H6), we have

$$-\infty < \gamma \leq \nu \leq \tilde{\nu} \leq \beta \leq \alpha,$$

   where ν is defined by (3.4).

2. $\tilde{\nu}$ and β can be different. For example, if we take

$$f(x, s) = -\frac{s^3}{s^2 + |x| + 1},$$

   we have $\tilde{\nu} = -1$ and $\beta = 0$.

3. We can have $\liminf_{s\to\infty} \frac{f(x,s)}{s} = \infty$ for all $x \in \mathbb{R}^N$. This is the case in example 1 of Section 6 if $\lim_{s\to\infty} r(s) = \infty$.

For the construction of the two first supersolutions, we introduce the following hypothesis.

(H7) On every compact K of $\mathbb{R}^N$, and for every $\epsilon > 0$, there exists $s_0 = s_0(\epsilon, K)$ such that

$$\frac{f(x,s)}{s} \geq \eta(x) - \epsilon \quad \forall (x, s) \in K \times (s_0, \infty).$$

First supersolution.

We suppose that the hypotheses (H1) to (H7) hold to construct this supersolution.
Let

$$\mu_0 = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla u|^2 + \Big(\eta - \tilde{\nu}\Big) u^2 \;\Big|\; u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} u^2 = 1 \Big\}. \tag{5.7}$$

Remarks.

1. We have $\mu_0 \leq 0$ since $\eta(x) - \tilde{\nu} \leq 0$ on $\mathbb{R}^N$.
2. $\lim_{|x|\to\infty} \eta(x) - \tilde{\nu} = 0$.

Lemma 5.1 *Let $\sigma < \mu_0$. There exist a mapping $V_\sigma : \mathbb{R}^N \to \mathbb{R}$ and $\mu_\sigma < 0$ such that*

1. $V_\sigma \in L^\infty(\mathbb{R}^N)$.
2. $V_\sigma(x) \leq \eta(x) - \tilde{\nu} \leq 0$ *on* $\mathbb{R}^N$ *and* $\lim_{|x|\to\infty} V_\sigma(x) = 0$.
3. $\sigma < \mu_\sigma \leq \mu_0$ *and*

$$\mu_\sigma = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla u|^2 + V_\sigma u^2 \;\mid\; u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} u^2 = 1 \Big\}.$$

Moreover for such a pair (V_σ, μ_σ) there exists an eigenfunction $\varphi \in X$ such that $\varphi(x) > 0$ for all $x \in \mathbb{R}^N$ and

$$-\Delta\varphi(x) + V_\sigma(x)\varphi(x) \;=\; \mu_\sigma \varphi(x) \text{ a.e. } \mathbb{R}^N.$$

Proof. If $\mu_0 < 0$, we can choose $V_\sigma(x) = \eta(x) - \tilde{\nu}$ and in this case $\mu_\sigma = \mu_0$. In the case $\mu_0 = 0$, the construction of the function V_σ can be done following the proof of Lemma 4.1 in [1]. □

Theorem 5.1 *Let (H1) to (H7) be satisfied. Then, for every $b < \mu_0 + \tilde{\nu}$, there exists a supersolution (b, Ψ) of the problem (1.1) such that*

$$\lim_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)} < \tilde{\nu} - b.$$

Proof. Since $b - \tilde{\nu} < \mu_0$, we can choose $\epsilon > 0$ such that $b - \tilde{\nu} + \epsilon < \mu_0$.
With $\sigma = b - \tilde{\nu} + \epsilon$, let (μ_σ, φ) be defined as in Lemma 5.1. Then set

$$\Psi(x) = C\varphi(x)$$

with C a positive constant to be fixed later on so that (b, Ψ) is a supersolution of problem (1.1). Note that $\Psi \in X$ and $\Psi > 0$.

By the definition (5.5) of $\widetilde{\nu}$, there is $R > 0$ such that

$$\frac{f(x,s)}{s} \geq \widetilde{\nu} - \epsilon \quad \forall |x| \geq R, \forall s > 0. \tag{5.8}$$

We have for almost every $x \in \mathbb{R}^N \setminus B_R$ with $B_R = B(0,R)$,

$$\begin{aligned}
&-\Delta\Psi(x) + f(x,\Psi(x)) - b\Psi(x)\\
&\quad = \mu_\sigma \Psi(x) - V_\sigma(x)\Psi(x) + f(x,\Psi(x)) - b\Psi(x)\\
&\quad \geq \mu_\sigma \Psi(x) + f(x,\Psi(x)) - b\Psi(x) \quad (\text{ since } V_\sigma(x) \leq 0)\\
&\quad = \Big\{\mu_\sigma - b + \frac{f(x,\Psi(x))}{\Psi(x)}\Big\}\Psi(x)\\
&\quad \geq \Big\{\mu_\sigma - b + \widetilde{\nu} - \epsilon\Big\}\Psi(x) \quad (\text{ by } (5.8))\\
&\quad > 0 \quad (\text{ since } \mu_\sigma > \sigma).
\end{aligned}$$

Next, we establish the same inequality for $x \in B_R$.
By (H7), there exists $s_0 = s_0(\epsilon, B_R) > 0$ such that

$$\frac{f(x,s)}{s} \geq \eta(x) - \epsilon \quad \forall (x,s) \in B_R \times (s_0, \infty).$$

Since $\inf\{\varphi(x) \mid x \in B_R\} > 0$, we can choose C such that $C\varphi(x) \geq s_0$, and it follows that for all $x \in B_R$,

$$\frac{f(x,\Psi(x))}{\Psi(x)} = \frac{f(x,C\varphi(x))}{C\varphi(x)} \geq \eta(x) - \epsilon.$$

Hence for every $x \in B_R$, we have

$$\begin{aligned}
&-\Delta\Psi(x) + f(x,\Psi(x)) - b\Psi(x)\\
&\quad = \mu_\sigma \Psi(x) - V_\sigma(x)\Psi(x) + f(x,\Psi(x)) - b\Psi(x)\\
&\quad \geq \mu_\sigma \Psi(x) - \Big(\eta - \widetilde{\nu}\Big)(x)\Psi(x) + f(x,\Psi(x)) - b\Psi(x)\\
&\quad = \Big\{\mu_\sigma - b + \widetilde{\nu} - \eta(x) + \frac{f(x,\Psi(x))}{\Psi(x)}\Big\}\Psi(x)\\
&\quad \geq \Big\{\mu_\sigma - b + \widetilde{\nu} - \eta(x) + \eta(x) - \epsilon\Big\}\Psi(x)\\
&\quad = \Big\{\mu_\sigma - b + \widetilde{\nu} - \epsilon\Big\}\Psi(x)\\
&\quad > 0 \qquad (\text{ again since } \mu_\sigma > \sigma).
\end{aligned}$$

Finally

$$\lim_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)} = \lim_{|x|\to\infty} [V_\sigma(x) - \mu_\sigma] = -\mu_\sigma < -\sigma = \tilde{\nu} - b - \epsilon < \tilde{\nu} - b.$$

□

Remark. We use only the hypotheses (H1), (H2), (H6) and (H7) to prove Theorem 5.1.

Theorem 5.2 *Let the hypotheses (H1) to (H7) be satisfied and $\Lambda < \beta$. Then*

$$\sup_{(\lambda,u)\in C_\Lambda^+} \lambda \geq \tilde{\nu} + \mu_0,$$

where μ_0 and $\tilde{\mu}$ are defined by (5.5) and (5.7) respectively.

Proof. Fix any $b < \tilde{\nu} + \mu_0$. By Theorem 5.1, we have obtained a supersolution (b, Ψ) such that

$$\limsup_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)} < \tilde{\nu} - b.$$

Setting

$$a := \limsup_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)},$$

we verify that $b < \min(\beta, \alpha - a)$.
Since $a < \tilde{\nu} - b$ and $\tilde{\nu} \leq \beta \leq \alpha$, we have $b < \tilde{\nu} - a \leq \alpha - a$. Moreover $\mu_0 \leq 0$ implies $b < \tilde{\nu} + \mu_0 \leq \beta$.
Hence by Theorem 4.1 we can conclude that $\sup\{\lambda \mid (\lambda, u) \in C_\Lambda^+\} \geq b$. □

Second supersolution.

Let

$$\Omega \ := \ \{x \in \mathbb{R}^N \ \mid \ \eta(x) < \tilde{\nu}\}, \tag{5.9}$$

where $\tilde{\nu}$ and η are defined by (5.5) and (5.6).
We suppose that Ω is bounded and we construct an another supersolution in order to improve the lower bound for $\{\sup \lambda \mid \lambda \in C_\Lambda^+\}$.
The idea is to construct a supersolution on Ω, another one on $\mathbb{R}^N \setminus \Omega$ and afterwards to obtain the final supersolution on all $\mathbb{R}^N$ by connecting the two. But to do that, we may need to make the domain Ω larger in order to have good properties on the boundary.

Definition 5.1 *The domain $\Omega^\star$ is called an admissible domain for Ω if $\Omega \subset \Omega^\star$, $\Omega^\star$ is bounded, the boundary $\partial\Omega^\star$ is C^1 and there exist $x_0 \in \Omega^\star$ and $\delta > 0$ such that $n(x).(x - x_0) \geq \delta$ a.e. on $\partial\Omega^\star$ where $n(x)$ is the outward unit normal.*
In particular $\Omega^\star$ is star-shaped and we may suppose without loss of generality that $x_0 = 0 \in \Omega^\star$.

Lemma 5.2 *Let $\sigma < 0$. Then there exist a continuous increasing mapping $W_\sigma : [0, \infty) \to (-\infty, 0)$ with $\lim_{r\to\infty} W_\sigma(r) = 0$, and a function $\varphi \in X$ such that*

$$-\Delta\varphi(x) + W_\sigma(|x|)\varphi(x) = \sigma\varphi(x).$$

Moreover $\varphi(x) > 0$ for all $x \in \mathbb{R}^N$ and φ is radially symmetric, $\partial_r\varphi < 0$ for $r > 0$ where r is the radial coordinate. Furthermore φ decays exponentially as $|x| \to \infty$.

Proof. Choose a continuous differentiable function $W : [0,\infty) \to (-\infty, 0)$ such that $W'(r) > 0$ for all $r > 0$, $W'(0) = 0$, $\lim_{r\to\infty} W(r) = 0$ and where

$$\mu := \inf \left\{ \int |\nabla v|^2 + W(|x|)v^2 \mid v \in H^1 \text{ and } \int v^2 = 1 \right\} < 0.$$

Since $\mu < 0$, there exists a function $v \in X$ such that $v(x) > 0$ for all $x \in \mathbb{R}^N$ and

$$-\Delta v(x) + W(|x|)v(x) \;=\; \mu v(x) \text{ on } \mathbb{R}^N$$

Furthermore v decays exponentially as $|x| \to \infty$.
We shall prove that when $\sigma = \mu$ the functions W and v as the properties required by the lemma. This will end the proof since to obtain the desired functions W_σ and φ at any $\sigma < 0$, it is sufficient to set

$$t = \sqrt{\frac{\sigma}{\mu}},\ W_\sigma = t^2 W(tr) \text{ and } \varphi = v(t|x|).$$

Let ξ be the Schwarz symmetrisation of v. Then by the basic properties of symmetrisation (see properties (C), (P1) and (G1) of [7]) we have that

$$\int \xi^2 = \int v^2 \text{ and } \int |\nabla \xi|^2 + W(\xi)^2 \leq \int |\nabla v|^2 + Wv^2$$

since $-W$ is Schwarz symmetric.
By identification we get that $\mu = \int |\nabla \xi|^2 + W\xi^2$ and, since μ is a simple eigenvalue, we conclude that $v = \xi$. Thus v is radial and satisfies $\partial_r v \leq 0$ for all $r \geq 0$.
Set $\psi = \partial_r v$. We shall show that $\psi(r) < 0$ for all $r > 0$.
Suppose by contradiction that $\psi(r_0) = 0$ for some $r_0 > 0$, then we must have $\psi'(r_0) = 0$ and $\psi''(r_0) \leq 0$ since $\psi \leq 0$ for all $r \in [0,\infty)$.
Remark that ψ satisfies

$$-\psi'' + \frac{(N-1)}{r^2}\psi - \frac{(N-1)}{r}\psi' + \frac{\partial W}{\partial r}\xi + W\psi = \mu\psi \text{ for } r > 0,$$

then when $r = r_0$, we obtain

$$-\psi''(r_0) + \frac{\partial W}{\partial r}(r_0)\xi(r_0) = 0$$

with $\partial_r W(r_0) > 0$ and $\xi(r_0) > 0$. Thus we get a contradiction and W and v have all the required properties for $\sigma = \mu$. □

Theorem 5.3 *Let the hypotheses (H1) to (H7) be satisfied. Suppose that* Ω *is bounded and that* $\lim_{|x|\to\infty} \partial_2 f(x,0) = \alpha$. *Let* $\lambda^\star$ *be the first eigenvalue of* $-\Delta + \eta$ *on* $W_0^{1,2}(\Omega^\star)$ *where* $\Omega^\star$ *is an admissible domain for* Ω.
Then for all $b < \min\{\widetilde{\nu}, \lambda^\star\}$ *there exists a supersolution* (b, Ψ) *for (1.1) with*

$$\limsup_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)} < \alpha - b,$$

and so, if $\Lambda < \beta$,

$$\sup_{(\lambda,u)\in\mathcal{C}_\Lambda^+} \lambda \geq \min\{\tilde{\nu}, \lambda^\star\}.$$

Remark. If $\Omega_1 \subset \Omega_2$, the first eigenvalue of the operator $-\Delta + \eta$ on $H_0^1(\Omega_1)$ is bigger than on $H_0^1(\Omega_2)$. Hence our interest is to choose a domain $\Omega^\star$ as small as possible. For some functions f one can have $\Omega^\star = \Omega$.

Proof. We fix $b < \min\{\tilde{\nu}, \lambda^\star\}$.
Let $\Omega_\delta^\star$ denote the δ- neighborhood of $\Omega^\star$. Taking $\delta > 0$ sufficiently small, we have

$$b < \lambda_\delta < \lambda^\star$$

where λ_δ is the first eigenvalue of $-\Delta - \eta$ on $H_0^1(\Omega_\delta^\star)$. We denote by Φ the associated positive eigenfunction.
Choose $\epsilon > 0$ such that $\epsilon < \min\{\tilde{\nu} - b, \lambda_\delta - b\}$.
By (H7), there exists $s_0 > 0$ such that

$$\frac{f(x,s)}{s} \geq \eta(x) - \epsilon \quad \forall (x,s) \in \Omega^\star \times (s_0, \infty).$$

We claim that $\varphi_0(x) = C\Phi(x)$, where C is chosen sufficiently large in order that $C\Phi(x) > s_0$ for all $x \in \Omega^\star$, is a supersolution for (1.1) on $\Omega^\star$ associated to b.
Indeed a.e. $\Omega^\star$, we have

$$\begin{aligned}
&-\Delta\varphi_0(x) + f(x,\varphi_0(x)) - b\varphi_0(x) \\
&= -\eta(x)\varphi_0(x) + \lambda_\delta\varphi_0(x) + \frac{f(x,\varphi_0(x))}{\varphi_0(x)}\varphi_0(x) - b\varphi_0(x) \\
&= \left(-\eta(x) + \frac{f(x,\varphi_0(x))}{\varphi_0(x)}\right)\varphi_0(x) + (\lambda_\delta - b)\varphi_0(x) \\
&\geq 0.
\end{aligned}$$

Now we shall construct a supersolution on $\mathbb{R}^N \setminus \Omega^\star$. For this we apply Lemma 5.2 with $\sigma = b - \tilde{\nu} + \epsilon$.
This gives two functions W_σ and $\tilde{\varphi}$ satisfying

$$-\Delta\tilde{\varphi} + W_\sigma(|x|)\tilde{\varphi}(x) = \sigma\tilde{\varphi}(x) \text{ a.e. } \mathbb{R}^N$$

with $\tilde{\varphi} > 0$ and $\partial_r\tilde{\varphi} < 0$ for $r = |x| > 0$.
Set $\varphi_1(x) = D\tilde{\varphi}(x)$. As in the proof of Theorem 5.1, by the definition of $\tilde{\nu}$, there exists $R > 0$ such that

$$\frac{f(x,s)}{s} \geq \tilde{\nu} - \epsilon \quad \forall |x| \geq R, \forall s > 0.$$

Using the hypothesis (H7) with $K = B_R$ and fixing $D > 0$ sufficiently large, we have that

$$\frac{f(x,\varphi_1(x))}{\varphi_1(x)} \geq \eta(x) - \epsilon \quad \forall |x| < R.$$

Consequently, since $\eta(x) = \tilde{\nu}$ for $x \notin \Omega$ and $\Omega \subset \Omega^\star$, we obtain

$$\frac{f(x,\varphi_1(x))}{\varphi_1(x)} \geq \tilde{\nu} - \epsilon \quad \forall x \in \mathbb{R}^N \setminus \Omega^\star.$$

Thus on $\mathbb{R}^N \setminus \Omega^\star$,

$$\begin{aligned} &-\Delta\varphi_1(x) + f(x,\varphi_1(x)) - b\varphi_1(x) \\ &\quad = \; -W(|x|)\varphi_1(x) + \sigma\varphi_1(x) + f(x,\varphi_1(x)) - b\varphi_1(x) \\ &\quad \geq \; (\sigma - b + \tilde{\nu} - \epsilon)\varphi_1(x) \\ &\quad = \; 0. \end{aligned}$$

Now we construct a supersolution of (1.1) on $\mathbb{R}^N$ from the two we have already constructed.

Set $\varphi(x) = \begin{cases} \varphi_0(x) & \text{for all } x \in \Omega^\star \\ \varphi_1(x) & \text{for all } x \in \mathbb{R}^N \setminus \Omega^\star. \end{cases}$

Denote by n the outward normal of the boundary of $\Omega^\star$. Recall that $\varphi_1(x) = D\tilde{\varphi}(x)$ and that $\partial_r\tilde{\varphi} < 0$ for $r > 0$. We shall show that choosing D sufficiently large, we have

$$\frac{\partial\varphi_0}{\partial n} \geq \frac{\partial\varphi_1}{\partial n} \quad \text{a.e. } \partial\Omega^\star. \tag{5.10}$$

This is done by proving that $\dfrac{\partial\varphi_0}{\partial n}$ is bounded from below.

First note that $\varphi_0(x) = C\Phi(x)$ where $\Phi \in W^{2,q}_{loc}(\Omega^\star_\delta)$ for all $q > 1$ by the choice of $\Omega^\star_\delta$. Hence $\Phi \in C^1(\Omega^\star_\delta)$ and thus $\nabla\Phi \in C(\overline{\Omega^\star})$.

We deduce that

$$\frac{\partial\varphi_0}{\partial n} = n(x).\nabla\varphi_0(x) \geq -|\nabla\varphi_0(x)| \geq -|\nabla\varphi_0(x)|_{C(\overline{\Omega^\star})}.$$

Next recall that $x_0 = 0$ and set $\overline{r} := \max\{r : x \in \partial\Omega^\star\}$. Thus we have

$$\frac{\partial\tilde{\varphi}}{\partial n} = \frac{n(x).x}{r}\frac{\partial\tilde{\varphi}}{\partial r} \leq \frac{\delta}{\overline{r}}\max\left\{\frac{\partial\tilde{\varphi}}{\partial r} : x \in \partial\Omega^\star\right\} < 0,$$

where $\delta > 0$ is such that $n(x).x \geq \delta$ a.e. $\partial\Omega^\star$.

Hence the choice of D gives the inequality (5.10).

Note that we have that for all $p \geq 1$ and $a \in \mathbb{R}$

$$f(x,\varphi(x)) + a\varphi(x) \in L^p(\mathbb{R}^N) \text{ since } \varphi_1 \text{ decays exponentially as } |x| \to \infty.$$

Then we can define $\Psi \in X$ such that

$$-\Delta\Psi(x) + c\Psi(x) + f(x,\varphi(x)) - c\varphi(x) - b\varphi(x) = 0 \text{ on } \mathbb{R}^N$$

where the constant c is choosen such that

$$c > \max\left(0, \max_{\substack{x\in \mathbb{R}^N \\ |s|\leq K}} \frac{\partial}{\partial s} f(x,s) - b\right)$$

and $K = |\varphi|_\infty$.
We now show that (b, Ψ) is a supersolution.
The function $s \mapsto f(x,s) - bs - cs$ is decreasing in $0 \leq s \leq K$ and for all $x \in \mathbb{R}^N$ by the choice of c. Hence $f(x,\varphi(x)) - (c+b)\varphi(x) \leq f(x,0) - (c+b)0 = 0$ from which it follows easily that $\Psi(x) > 0$ for all $x \in \mathbb{R}^N$.
Next we show that $\Psi(x) \leq \varphi(x)$ for all $x \in \mathbb{R}^N$.
Set $w(x) = \varphi(x) - \Psi(x)$.
For all $v \in H^1(\mathbb{R}^N)$, $v \geq 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla w(x)\nabla v(x) + cw(x)v(x)dx \\
&= \int_{\partial\Omega^\star} \left\{\frac{\partial\varphi_0(x)}{\partial n} - \frac{\partial\varphi_1(x)}{\partial n}\right\} v(x)dx + \int_{\Omega^\star} (-\Delta w(x) + cw(x))v(x)dx \\
&\quad + \int_{\mathbb{R}^N\setminus\Omega^\star} (-\Delta w(x) + cw(x))v(x)dx \\
&= \int_{\partial\Omega^\star} \left\{\frac{\partial\varphi_0(x)}{\partial n} - \frac{\partial\varphi_1(x)}{\partial n}\right\} v(x)dx + \int_{\Omega^\star} \left(-\Delta\varphi_0(x) + f(x,\varphi_0(x)) - b\varphi_0(x)\right) v(x)dx \\
&\quad + \int_{\mathbb{R}^N\setminus\Omega^\star} \left(-\Delta\varphi_1(x) + f(x,\varphi_1(x)) - b\varphi_1(x)\right) v(x)dx \\
&\geq 0
\end{aligned}$$

by definitions of φ_0 and φ_1.
Hence $w(x) \geq 0$ on $\mathbb{R}^N$. This implies that $0 < \Psi(x) \leq \varphi(x) \leq K$ for all $x \in \mathbb{R}^N$.
Thus by definition of Ψ,

$$\begin{aligned}
&-\Delta\Psi(x) + f(x,\Psi(x)) - b\Psi(x) \\
&\quad = -\Delta\Psi(x) + c\Psi(x) + f(x,\Psi(x)) - (b+c)\Psi(x) \\
&\quad \geq -\Delta\Psi(x) + c\Psi(x) + f(x,\varphi(x)) - (b+c)\varphi(x) \\
&\quad = 0.
\end{aligned}$$

We have now shown that (b, Ψ) is a supersolution. From the remark in page 10, it follows that,

$$\limsup_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)} \leq \limsup_{|x|\to\infty} \partial_2 f(x,0) - b = \alpha - b,$$

since $\lim_{|x|\to\infty} \partial_2 f(x,0) = \alpha$.
The last result is obtained using Theorem 4.1.

Setting $a := \limsup_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)}$, we check that $b \le \min\{\beta, \alpha - a\}$.
This is obvious since $a \le \alpha - b$ and $b < \tilde{\nu} \le \beta$. Thus we can conclude that

$$\sup_{(\lambda,u)\in\mathcal{C}_\Lambda^+} \lambda \ge b \text{ for all } b < \min\{\tilde{\nu}, \lambda^\star\}.$$

□

Third supersolution.

For constructing this supersolution, we do not require the property (H7) of f.

Theorem 5.4 *Let (H1) to (H6) be satisfied. Set*

$$\omega = \{x \in \mathrm{I\!R}^N \mid \lim_{s\to\infty} \frac{f(x,s)}{s} = \infty\} \qquad \textit{and} \qquad \Omega_0 = \mathrm{I\!R}^N \setminus \overline{\omega}.$$

Suppose that

1. Ω_0 *is bounded.*
2. *there exist* $\delta_0 > 0$ *and* $s_0 > 0$ *such that* $s^{-1}f(x,s) \ge \partial_2 f(x,0)$ *for all* $x \in \Omega_{\delta_0}$ *and all* $s \ge s_0$, *where* Ω_{δ_0} *denotes the* δ_0*- neighborhood of* Ω_0.
3. $\lim_{s\to\infty} \frac{f(x,s)}{s} = \infty$, *uniformly on compact subsets of* w.

Let

$$\lambda_0 = \inf\left\{\int_{\Omega_0} |\nabla u|^2 + \partial_2 f(x,0)u^2 \mid u \in H_0^1(\Omega_0) \text{ and } \int_{\Omega_0} u^2 = 1\right\}.$$

Then for any $b < \min\{\tilde{\nu}, \lambda_0\}$ *there exists a supersolution* (b, Ψ) *of problem (1.1) such that*

$$\limsup_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)} < \tilde{\nu} - b.$$

If, in addition, $\Lambda < \beta$, *then*

$$\sup_{(\lambda,u)\in\mathcal{C}_\Lambda^+} \lambda \ge \min\{\tilde{\nu}, \lambda_0\}.$$

Remarks.

1. We have $\Omega \subset \Omega_0$ where Ω is defined by (5.9). This means that Ω is also bounded
2. The construction used in Theorem 5.4 is a generalization of the one found in [1] (see example 2 in Section 6).

Proof. Fix $b < \min\{\tilde{\nu}, \lambda_0\}$.
Choose $\delta \in (0, \delta_0)$ so small that $b < \lambda_\delta < \lambda_0$ where

$$\lambda_\delta = \inf\left\{ \int_{\Omega_\delta} |\nabla u|^2 + \partial_2 f(x,0)u^2 \mid u \in H_0^1(\Omega_\delta) \text{ and } \int_{\Omega_\delta} u^2 = 1 \right\}.$$

We can choose an eigenfunction $\eta \in H_0^1(\Omega_\delta)$ such that

$$-\Delta\eta(x) + \partial_2 f(x,0)\eta(x) = \lambda_\delta \eta(x) \qquad \text{on } \Omega_\delta$$

and, for $0 < \gamma < \delta$, the function η has the following properties

1. $\eta \in W^{2,p}(\Omega_\gamma)$ for all $p \in (1, \infty)$ by (H4),
2. $\eta(x) > 0$ for all $x \in \Omega_\gamma$,
3. $\Delta\eta \in L^\infty(\Omega_\gamma)$.

Next we choose $\epsilon > 0$ such that $b - \tilde{\nu} + \epsilon < 0$ and then for $\sigma = b - \tilde{\nu} + \epsilon$ we denote by W_σ and φ the functions given by Lemma 5.2. By definition of $\tilde{\nu}$ there exists $R > 0$ such that $\overline{\Omega}_{\delta_0} \subset B(0,R)$ and

$$\frac{f(x,s)}{s} \geq \tilde{\nu} - \epsilon \text{ for all } s > 0 \text{ and } x \in \mathbb{R}^N \setminus B(0,R).$$

It is easy to see that there is a function Φ having the following properties,
$\Phi \in X$ and $\Delta\Phi \in L^\infty(\mathbb{R}^N)$, $\Phi(x) > 0$ for all $x \in \mathbb{R}^N$,

$$\Phi(x) = \begin{cases} \eta(x) & \text{for all } \ x \in \Omega_{\delta/2}, \\ \varphi(x) & \text{for all } \ x \in \mathbb{R}^N \setminus B(0,R). \end{cases}$$

Set $m = \min\{\Phi(x) : x \in \overline{\Omega_{\delta/2}}\}$ and $M = \inf\{\dfrac{-\Delta\Phi(x)}{\Phi(x)} : x \in \overline{B(0,R)} \setminus \Omega_{\delta/2}\}$, we have that $m > 0$ and $M > -\infty$.
By the hypothesis 3, there exists $s_1 > 0$ such that

$$\frac{f(x,s)}{s} \geq -M + b \text{ for all } x \in \overline{B(0,R)} \setminus \Omega_{\delta/2} \text{ and all } s \geq s_1.$$

Now set $\Psi = C\Phi$ where C is so large that $\Psi(x) \geq s_0$ for all $x \in \Omega_\delta$ and $\Psi(x) \geq s_1$ for all $x \in \overline{B(0,R)}$.
We now show that (b, Ψ) is a supersolution.
For $x \in \Omega_{\delta/2}$,

$$-\Delta\Psi(x) + f(x,\Psi(x)) - b\Psi(x) = \left\{ \frac{f(x,\Psi(x)}{\Psi(x)} - \partial_2 f(x,0) + \lambda_\delta - b \right\} \Psi(x) \geq 0$$

by the hypothesis 2.

For $x \in \overline{B(0,R)} \setminus \Omega_{\delta/2}$,

$$-\Delta\Psi(x) + f(x,\Psi(x)) - b\Psi(x) \geq \left\{ M + \frac{f(x,\Psi(x))}{\Psi(x)} - b \right\} \Psi(x) \geq 0$$

since $\Psi(x) \geq s_1$.
For $x \in \mathbb{R}^N \setminus B(0,R)$,

$$\begin{aligned} & -\Delta\Psi(x) + f(x,\Psi(x)) - b\Psi(x) \\ = & \left\{ -W_\sigma(|x|) + \sigma + \frac{f(x,\Psi(x))}{\Psi(x)} - b \right\} \Psi(x) \\ \geq & \ \{\sigma + \tilde{\nu} - \epsilon - b\}\Psi(x) = 0 \end{aligned}$$

since $W_\sigma \leq 0$ and $|x| \geq R$.
Thus (b, Ψ) is a supersolution and furthermore

$$\limsup_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)} = \limsup_{|x|\to\infty} \{W(|x|) - \sigma\} = -\sigma = \tilde{\nu} - b - \epsilon < \tilde{\nu} - b.$$

Setting $a := \limsup_{|x|\to\infty} \frac{\Delta\Psi(x)}{\Psi(x)}$, we easily check from $a < \tilde{\nu} - b$ and $\tilde{\nu} \leq \beta \leq \alpha$ that $b \leq \min\{\beta, \alpha - a\}$.
Hence by Theorem 4.1, we obtain $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\} \geq b$. □

6 Special cases

In order to show more clearly the situations in which Theorem 1.1 can be applied and in which we are able to construct a supersolution of the problem (1.1), let us consider the special case where f can be written as,

$$f(x,s) = (p(x) + q(x)r(s))\, s \ \text{ for } x \in \mathbb{R}^N \ \text{ and } s \in \mathbb{R}. \tag{6.11}$$

We assume that p, q and r have the following properties.

(A1) $p,\, q \in L^\infty(\mathbb{R})$.

(A2) $r \in C(\mathbb{R})$, $r(s)s \in C^2(\mathbb{R})$, r is even and $r(0) = 0$.

Since only the product $q(x)r(s)$ appears in (6.11), we can assume henceforth that

$$\liminf_{|x|\to\infty} q(x) \geq 0. \tag{6.12}$$

It is easy to verify that (A1) and (A2) ensure that f satisfies the conditions (H1) to (H4) and that

$$\alpha = \liminf_{|x|\to\infty} p(x) \ \text{ since } \partial_2 f(x,0) = p(x).$$

To see if (H5) is satisfied, we begin by calculating $\beta(C)$. By (6.12), we have that

$$\lim_{|x|\to\infty}\{q(x)-q^+(x)\}=0$$

and so for any $C\geq 0$,

$$\begin{aligned}\beta(C) &= \lim_{z\to\infty}\inf_{\substack{|x|\geq z\\ 0\leq s\leq C}}\Big\{p(x)+q(x)r(s)\Big\}\\ &= \lim_{z\to\infty}\inf_{\substack{|x|\geq z\\ 0\leq s\leq C}}\Big\{p(x)+q^+(x)r(s)\Big\}\\ &= \liminf_{|x|\to\infty}\Big\{p(x)+q^+(x)R(C)\Big\}\end{aligned}$$

where $R(C):=\inf_{0\leq s\leq C} r(s)$. Since $r(0)=0$, $R(C)\leq 0$ for all $C\geq 0$.
Thus, if for instance, $\liminf_{|x|\to\infty} q(x)>0$ and $\lim_{s\to\infty} r(s)=-\infty$, it follows that

$$\beta=\inf_{C\geq 0}\beta(C)=-\infty$$

and the condition (H5) is not satisfied.
To avoid this kind of situation we assume henceforth that

(A3) either (a) $\limsup_{|x|\to\infty} q(x)=0$ or (b) $\inf_{s\geq 0} r(s)>-\infty$.

In case (a), $\beta(C)=\liminf_{|x|\to\infty} p(x)$ for all $C\geq 0$ and so $\beta=\alpha$.
If (A3)(b) holds, $\beta=\liminf_{|x|\to\infty}\Big\{p(x)+q^+(x)\inf_{s\geq 0} r(s)\Big\}\leq\alpha$ by (A2), and we have the following estimates

$$\alpha+\Big\{\limsup_{|x|\to\infty} q(x)\Big\}\inf_{s\geq 0} r(s)\leq\beta\leq\alpha+\Big\{\liminf_{|x|\to\infty} q(x)\Big\}\inf_{s\geq 0} r(s)\leq\alpha.$$

Clearly $\beta>-\infty$ so (H5) is satisfied when (A1) to (A3) hold and we can apply Theorem 1.1 provided that $\Lambda<\alpha$ in the case (A3)(a), and $\Lambda<\liminf_{|x|\to\infty}\{p(x)+q(x)\inf_{s\geq 0} r(s)\}$ in the case (A3)(b).
The results in this paper aim to sharpen the conclusion of Corollary 1.1 under various additional assumptions.
In the present context the hypothesis (H6) becomes

$$\gamma:=\inf\Big\{p(x)+q(x)r(s)\mid x\in\mathbb{R}^N \text{ and } s\in\mathbb{R}\Big\}>-\infty$$

and this is ensured by the following properties of p,q and r.

(A4) One of the following cases occurs

(a) $|r|_\infty < \infty$,

(b) $\sup_{s\geq 0} r(s) = \infty$ but $\inf_{s\geq 0} r(s) > -\infty$ and $q(x) \geq 0$ for all $x \in \mathbb{R}^N$.

(c) $\inf_{s\geq 0} r(s) = -\infty$ but $\sup_{s\geq 0} r(s) < \infty$ and $q(x) \leq 0$ for all $x \in \mathbb{R}^N$.

Note that if (A4)(c) occurs then by (6.12) we have that (A3)(a) is also satisfied.
The lower bounds for $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\}$ involve the quantities ν and $\tilde{\nu}$ which can be expressed as

$$\nu = \lim_{z\to\infty} \left[\inf_{|x|^2+s^2\geq z} p(x) + q(x)r(s) \right] \text{ and } \tilde{\nu} = \liminf_{|x|\to\infty} \left[\inf_{s\geq 0}\{p(x) + q(x)r(s)\} \right].$$

To limit the number of different cases which can occur we restrict our attention to the case where q and r satisfy the following conditions.

(A5) $q(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $\inf_{s\geq 0} r(s) > -\infty$.

Clearly (A1), (A2) and (A5) ensure that (A3) and (A4) are also satisfied. More precisely if (A1), (A2) and (A5) are satisfied then (A3)(b) is satisfied and either (A4)(a) or (A4)(b) must also be satisfied. Note also that (A3)(a) is not excluded. Furthermore assuming (A1), (A2) and (A5) the quantities $\tilde{\nu}$ and ν become

$$\tilde{\nu} = \liminf_{|x|\to\infty} \left\{ p(x) + q(x) \inf_{s\geq 0} r(s) \right\} = \beta$$

and

$$\nu = \min\left\{ \beta, \inf_{x\in\mathbb{R}^N} [p(x) + q(x) \liminf_{s\to\infty} r(s)] \right\}.$$

Notice that $\nu \leq \inf\{p(x) \mid x \in \mathbb{R}^N \text{ such that } q(x) = 0\}$.
By Corollary 3.2 we know that $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\} \geq \nu$. If $\nu = \beta$ (which occurs, for example, when $q(x) > 0$ for all $x \in \mathbb{R}^N$ and $\lim_{s\to\infty} r(s) = \infty$) this gives the best possible lower bound for $p_1(\mathcal{C}_\Lambda^+)$. On the other hand, if $\nu \leq \Lambda$ (which occurs, for example, if $\inf\{p(x) \mid x \in \mathbb{R}^N \text{ such that } q(x) = 0\} \leq \Lambda$) Corollary 3.2 yields no information and we must turn to the results in Section 5. Even when $\Lambda < \nu$ but $\nu < \beta$, the results in Section 5 may give a better lower bound for $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_\Lambda^+\}$ than Corollary 3.2.
In Section 5, the first two methods of constructing supersolutions require the hypothesis (H7) which we now discuss in the context of (6.11).
Letting $\omega = \{x \in \mathbb{R}^N \mid q(x) > 0\}$, we formulate the following condition.

(A6) For all $x \in \omega$, $(p(x) + q(x) \liminf_{s\to\infty} r(s)) \geq \beta$ and if $\lim_{s\to\infty} r(s) = \infty$, we assume in addition that for any $\epsilon > 0$ and any compact subset K of $\mathbb{R}^N$, there exists $\delta = \delta(\epsilon, K) > 0$ such that $p(x) \geq \beta - \epsilon$ whenever $x \in K$ and $0 < q(x) < \delta$.

Under the conditions (A1), (A2), (A5) and (A6), the hypothesis (H7) is satisfied and we have that

$$\begin{aligned}\eta(x) &= \min\{p(x)+q(x)\liminf_{s\to\infty} r(s),\beta\} \\ &= \begin{cases}\beta & \text{if } x\in\omega \\ \min\{p(x),\beta\} & \text{if } x\notin\omega\end{cases}\end{aligned}$$

and thus the region Ω which occurs in the second construction becomes

$$\Omega \equiv \{x\in \mathbb{R}^N \mid \eta(x)<\tilde{\nu}\} = \{x\in\mathbb{R}^N \mid q(x)=0 \text{ and } p(x)<\beta\} \text{ since } \tilde{\nu}=\beta \text{ by } (A5).$$

Thus $\sup\{\lambda \mid (\lambda,u)\in \mathcal{C}_\Lambda^+\}$ can be estimated using Theorems 5.2 and 5.3 provided that (A1),(A2),(A5) and (A6) are satisfied, but Theorem 5.3 requires Ω to be bounded.
In particular, if (A1), (A2), (A5) and (A6) hold and if $\Omega=\emptyset$, then Theorem 5.2 gives $\sup\{\lambda \mid (\lambda,u)\in\mathcal{C}_\Lambda^+\}=\beta$ since $\eta(x)\equiv\tilde{\nu}=\beta$ and $\mu_0=0$ in this case.
On the other hand Theorem 5.4 does not require the hypothesis (H7) and it can be used provided that, in addition to (A1), (A2) and (A5), we impose the following condition.

(A7) The set $\mathbb{R}^N\setminus\bar{\omega}$ is bounded, $\lim_{s\to\infty} r(s)=\infty$ and for any compact set K of ω, $\inf_{x\in K} q(x)>0$.

Note however that (H7) can be satisfied, and so Theorems 5.2 and 5.3 can be used in the cases where $\mathbb{R}^N\setminus\bar{\omega}$ is unbounded and $\limsup_{s\to\infty} r(s)<\infty$.
For $q\in C(\mathbb{R}^N)$ the sets Ω and Ω_0 appearing in Theorem 5.3 and 5.4 are given by

$$\Omega=\Big\{x\in\mathbb{R}^N \mid q(x)=0 \text{ and } p(x)<\beta\Big\}$$

and

$$\Omega_0 = \text{ interior}\Big\{x\in\mathbb{R}^N \mid q(x)=0\Big\}$$

so Ω can be small (even empty) in cases where $\Omega_0=\mathbb{R}^N\setminus\overline{\omega}$ is large (even unbounded).

Example 1. In addition to the assumptions (A1) and (A2), we suppose that

i) $q\in C(\mathbb{R}^N)$ with $q(x)>0$ for all $x\in\mathbb{R}^N$,

ii) $\inf_{s\geq 0} r(s)=0$,

iii) $\liminf_{s\to\infty} r(s)\geq \dfrac{\alpha-p(x)}{q(x)}$ for all $x\in\mathbb{R}^N$.

Clearly (A5) is satisfied and $\alpha=\beta=\tilde{\nu}$. Furthermore $\omega=\mathbb{R}^N$ and the property (iii) implies that (A6) is satisfied. In fact, if $\liminf_{s\to\infty} r(s)<\infty$, then

$$p(x)+q(x)\liminf_{s\to\infty} r(s)\geq\beta=\alpha \text{ for all } x\in\mathbb{R}^N.$$

Also given any compact subset K of $\mathbb{R}^N$, $\inf_{x\in K} q(x) > 0$ and so, setting

$$\delta(\epsilon, K) = \frac{1}{2}\inf_{x\in K} q(x),$$

we have $\{x \in K \mid 0 < q(x) < \delta\} = \emptyset$.
Finally we observe that $\eta(x) = \tilde{\nu} = \alpha$ for all $x \in \mathbb{R}^N$, which implies that $\mu_0 = 0$. Theorem 5.2 can now be invoked to conclude that $\sup\{\lambda \mid (\lambda, u) \in C_\Lambda^+\} = \alpha$ provided that $\Lambda < \alpha$. If $\lim_{s\to\infty} r(s) = \infty$ this also follows from Corollary 3.2 since $\nu = \alpha$ in this case.
The situation considered in this example is comparable to that treated in Theorem 10 of [3], where under stronger assumptions it is shown that C_Λ^+ is a continuous curve and its boundedness as λ approaches α is also analysed.

Example 2. In addition to the assumptions (A1), (A2) and (A5), we suppose that

i) $q \in C(\mathbb{R}^N)$ and $Z \equiv \{x \in \mathbb{R}^N \mid q(x) = 0\}$ is non-empty and bounded,

ii) $\lim\limits_{s\to\infty} r(s) = \infty$.

Again we have that $\tilde{\nu} = \beta$.
In the notation of Section 5.3, $\omega = \mathbb{R}^N \setminus Z$ and $\Omega_0 = \text{int}Z$. There exists $s_0 > 0$ such that $r(s) \geq 0$ for all $s \geq s_0$ and so $p(x) + q(x)r(s) \geq p(x) = \partial_2 f(x, 0)$ for all $x \in \mathbb{R}^N$ and $s \geq s_0$.
If K is a compact subset of ω, $\inf_{x\in K} q(x) > 0$ and so given any $N > 0$ there exists $s_1 = s(N, K) > 0$ such that $p(x) + q(x)r(s) \geq N$ for all $x \in K$ and $s \geq s_1$.
Thus we see that Theorem 5.4 can be applied to deduce that

$$\sup_{(\lambda,u)\in C_\Lambda^+} \lambda \geq \min\{\lambda_0, \beta\}$$

provided that $\Lambda < \beta$ where

$$\lambda_0 = \inf\left\{\int_{\Omega_0} |\nabla v|^2 + pv^2 \mid v \in H_0^1(\Omega_0) \text{ and } \int_{\Omega_0} v^2 = 1\right\}.$$

But in the notation of Section 2, $\lambda_0 = \lambda_1(\Omega_0)$ and so, by the discussion at the end of Section 2, we also have that

$$\sup_{(\lambda,u)\in C_\Lambda^+} \lambda \leq \lambda_0.$$

Thus to obtain the sharp conclusion that $\sup\{\lambda \mid (\lambda, u) \in C_\Lambda^+\} = \lambda_0$, it is sufficient to ensure that $\lambda_0 \leq \beta$. This can be done by choosing q so that Ω_0 is large enough since, given any $\epsilon > 0$, there exists $R > 0$ such that $\lambda_1(\Omega_0) < \Lambda + \epsilon$ provided that $B(0, R) \subset \Omega_0$.
The situation treated in this example is comparable to that covered by Theorem 4.4 of [1] where (in our notation) it is shown that

$$\sup_{(\lambda,u)\in C_\Lambda^+} \lambda \geq \min\left\{\lambda_0, \alpha + |q|_\infty \inf_{s\geq 0} r(s)\right\}.$$

However, our conclusion is somewhat sharper since

$$\beta \geq \alpha + \left\{ \limsup_{|x|\to\infty} q(x) \right\} \inf_{s\geq 0} r(s).$$

Furthermore we have not required any smoothness of q similar to that in (A1) of [1].

Example 3. In addition to the assumptions (A1), (A2) and (A5), we suppose that

i) $q \in C(\mathbb{R}^N)$ and $Z = \left\{ x \in \mathbb{R}^N \mid q(x) = 0 \right\}$ is non-empty,

ii) $p \in C(\mathbb{R}^N)$ and $p(x) \geq \alpha = \lim_{|x|\to\infty} p(x)$ for all $x \in \partial Z$,

iii) $\liminf_{s\to\infty} r(s) \geq \dfrac{\alpha - p(x)}{q(x)}$ for all $x \in \omega = \mathbb{R}^N \setminus Z$,

iv) $\{x \in Z \mid p(x) < \alpha\} \subset B(0, R)$.

The set $\Omega_0 = \text{int } Z$ need not be bounded in this example. Nonetheless $\tilde{\nu} = \beta$ by (A5) and (A6) is still satisfied with

$$\eta(x) = \begin{cases} \beta & \text{for } x \notin \Omega \\ \min\{p(x), \beta\} & \text{for } x \in Z. \end{cases}$$

Thus $B(0, R)$ is an admissible region containing $\Omega = Z \cap \{x \in \mathbb{R}^N \mid \eta(x) < \beta\}$ (see the definition 5.1). From Theorem 5.3, we can conclude that

$$\sup_{(\lambda,u)\in C_\Lambda^+} \lambda \geq \min\{\beta, \lambda_1(B(0, R))\}$$

provided that $\Lambda < \beta$.
We observe that $B(0, R)$ can be much smaller that Z and that $\lambda_1(B(0, R)) \to \infty$ as $R \to 0$.

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