C.A.Stuart Cylindrical TE and TM modes in a self-focusing dielectric

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INTRODUCTION

This lecture reviews some recent work concerning the existence and behaviour of self-guided beams of light in a nonlinear medium. In order to model propagation in a optical fibre we seek beams having a cylindrical symmetry. The propagation of light is governed by Maxwell's equations together with a nonlinear relationship between the electric field and the displacement field. For a guided beam the light must remain concentrated near the axis of propagation and the amplitudes of the electro-magnetic fields must decay to zero far from this axis.

Mathematically, the problem can be formulated by seeking solutions of Maxwell's equations which have an appropriate cylindrical symmetry and which satisfy certain boundary conditions. There are two types of solution for which Maxwell's equations can be reduced, without approximation, to a much simpler form which is amenable to a rigorous analysis. These are the TE (transverse electric) and the TM (transverse magnetic) field modes described below. In this lecture some of the results concerning these special modes are summarized. They involve the analysis of a boundary-value problem for a second order differential equation on the interval $(0, \infty)$. The equation is of semilinear type in the case of TE-modes, whereas for TM-modes it has a more complicated quasilinear form.

The presentation is organized as follows.

§1.Mathematical formulation of the guidance problem.
§2.Analysis of TE-modes.
§3.Analysis of TM-modes.
§4.Related problems.

1. MATHEMATICAL FORMULATION OF THE GUIDANCE PROBLEM

In a dielectric medium Maxwell's equations can be written as [1],

$$\nabla \wedge E = -\frac{1}{c}\,\partial_t B\,, \nabla \wedge H = \frac{1}{c}\,\partial_t D,$$

$$\nabla \cdot D = 0, \nabla \cdot B = 0 \tag{ME}$$

where $c > 0$ is the speed of light in a vacuum and the electro-magnetic fields are functions of space, (r, θ, z), and time, t.

Here (r, θ, z) denote cylindrical polar co-ordinates and the usual orthonormal basis associated with this system is denoted by

$$i_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad i_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \quad i_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

A field $F : \Re^4 \to \Re^3$ can be resolved into orthogonal components (F_r, F_θ, F_z) by setting

$$F_\alpha\,(r, \theta, z, t) = F\,(r, \theta, z, t) \cdot i_\alpha \text{ for } \alpha = r, \theta, z. \tag{1.1}$$

We seek solutions of Maxwell's equations which have the form of monochromatic cylindrical modes propagating in the direction of the $z-$axis. More precisely, each field F should have the form

$$F_\alpha\,(r, \theta, z, t) = \begin{cases} f_\alpha(r)\cos(kz - \omega t) \text{ for } \alpha = r, \theta \\ f_\alpha(r)\sin(kz - \omega t) \text{ for } \alpha = z \end{cases} \tag{1.2}$$

where $f_\alpha : \; [0, \infty) \to \Re$ is a scalar function and the positive constants k and ω give the wavelength,$\frac{2\pi}{k}$,and the frequency ω of the associated beam.

For such modes the behaviour of the medium is characterized by the following constitutive relations[2,3,4],

$$B(r, \theta, z, t) = H(r, \theta, z, t) \tag{CR}$$
$$D\,(r, \theta, z, t) = \varepsilon\,\left(\frac{1}{2}\,\left[e_r^2 + e_\theta^2 + e_z^2\right]\,(r)\right)\,E\,(r, \theta, z, t)$$

For an isotropic medium, $\varepsilon : [0,\infty) \to (0,\infty)$ is a scalar function called the dielectric response of the medium. The quantity $\frac{1}{2}\left[e_r^2 + e_\theta^2 + e_z^2\right](r)$ is the time-average of $|E(r,\theta,z,t)|^2$ since E has the form (1.2).

It is physically reasonable to assume that ε has the following properties.

(A) $\varepsilon \in C([0,\infty)) \cap C^1((0,\infty))$ with $\varepsilon'(s) \geq 0 \quad \forall s > 0$,
$\varepsilon(0) > 0$, $\varepsilon(\infty) = \lim_{s\to\infty} \varepsilon(s) < \infty$ and $\lim_{s\to 0} s\varepsilon'(s) = 0$. Furthermore, there exist positive constants L and σ such that $\lim_{s\to 0} \frac{\varepsilon(s)-\varepsilon(0)}{s^\sigma} = L$.

Since ε is increasing the dielectric response is of the type called self-focusing. The fact that $\varepsilon(\infty) < \infty$ means that this response saturates as the field strength becomes infinite. For a large class of materials $\sigma = 1$, but other values do occur for some materials, [5,6,7].

Finally we introduce the conditions on the $e-m$ fields corresponding to the requirement that they represent a confined beam of light.

Guidance Conditions

(i) $\int_0^{2\pi} \int_0^{\infty} [E \cdot D + H \cdot B]\; r dr d\theta < \infty$

(ii) $\lim_{r\to\infty} F(r,\theta,z,t) = 0$ where $F = E, D, B$ and H.

The first condition asserts that the total $e-m$ energy in planes transverse to the direction of propagation is finite and the second one means that the fields decay to zero far from the axis of propagation.

In what follows we seek solutions of (ME) and (CR) in the form (1.2) that satisfy these guidance conditions. The strength of these guided beams is measured by the time-average of the energy flux

$$\int_0^{2\pi} \int_0^{\infty} c(E \wedge H) \cdot i_z \; r dr d\theta$$

across planes transverse to the direction of propagation. Using (1.2) this quantity becomes

$$P = \pi c \int_0^\infty (e_r h_\theta - e_\theta h_r) \; r \; dr \tag{1.3}$$

and it is referred to as the power of the beam.

In the following discussion, the frequency ω is fixed and we consider the relation between the power, P, and the wavelength, $\frac{2\pi}{k}$, for guided modes. The results are expressed most conveniently using the variable $\mu = \left(\frac{\omega}{ck}\right)^2$ instead of $\frac{2\pi}{k}$.

Both TE and TM modes have the following common features provided that the dielectric response satisfies (A).

(i)For all guided beams $\frac{1}{\varepsilon(\infty)} < \mu < \frac{1}{\varepsilon(0)}$.

(ii)For all guided beams, the amplitudes of the $e-m$ fields decay exponentially to zero as $r \to \infty$.

(iii)Guided beams exist at sufficiently high powers and, as $P \to \infty$, we have $\mu \to \frac{1}{\varepsilon(\infty)}$.

(iv)If $\sigma \geq 1$ in (A), there are no guided beams with power below a certain threshold, $P_o > 0$.

(v)If $0 < \sigma < 1$ in (A), guided beams exist at arbitrarily small powers and, as $P \to 0$ we have $\mu \to \frac{1}{\varepsilon(0)}$.

Our formulation based on the unknown e_θ for TE-modes is standard. On the other hand, due to the form of the constitutive relation (CR), the basic unknown in the study of cylindrical TM-modes is usually taken to be the couple (e_r, e_z) . (See [15], for example.) Setting $e_r = \varphi$ and $e_z = k\psi$, (ME) with (CR) reduce to the following system of equations,

$$\{\varphi(r) + \psi'(r)\} = \mu\varepsilon\left(\frac{1}{2}\left[\varphi(r)^2 + k^2\psi(r)^2\right]\right)\varphi(r)$$

$$-\frac{1}{r}\;\left[r\left\{\varphi(r) + \psi'(r)\right\}\right]' = \mu\varepsilon\left(\frac{1}{2}\left[\varphi(r)^2 + k^2\psi(r)^2\right]\right)\psi(r)$$

where $\mu = \left(\frac{\omega}{ck}\right)^2$ .However in [9] we have shown that by inverting (CR) (see (CR)* in Section 3) we are able to use the scalar function h_θ as the basic unknown for the TM-modes. In this way we obtain a formulation in the TM case which closely resembles that for TE-modes. In particular the equations (2.2) and (3.2) have exactly the same linearizations at $v \equiv 0$, namely the Bessel equation

$$\left[v'(r) + \frac{v(r)}{r}\right]' + \left\{\frac{\omega^2}{c^2}\,\varepsilon(0) - k^2\right\}\; v(r) = 0$$

since $\gamma(0) = \frac{1}{\varepsilon(0)}$ The low-power modes mentioned in (v) bifurcate from the infimum of the essential spectrum of this linear eigenvalue problem.

2. CYLINDRICAL TE-MODES

We seek solutions for which the electric field E has the additional properties that $e_r \equiv e_z \equiv 0$.

Setting $e_\theta = v$,we have

$$E\left(r,\theta,z,t\right) = v(r)\cos\left(kz-\omega t\right)\, i_\theta \qquad (2.1)$$

In [8] we have shown that (ME) together with (CR) are satisfied in the region $r > 0$ provided that $v \in C^2\left((0,\infty)\right)$and satisfies the equation

$$\left[v'\left(r\right) + \frac{v(r)}{r}\right]' + \frac{\omega^2}{c^2}\varepsilon\left(\frac{1}{2}\, v(r)^2\right) v(r) - k^2 v(r) = 0. \qquad (2.2)$$

From a solution v of this equation the displacement and magnetic fields are defined by

$$D\left(r,\theta,z,t\right) = \varepsilon\left(\frac{1}{2}v(r)^2\right) E\left(r,\theta,z,t\right),$$

$$h_r(r) = -\frac{ck}{\omega}v(r), \quad h_\theta(r) = 0, \quad h_z(r) = \frac{c}{\omega}\left[v\left(r\right) + \frac{v(r)}{r}\right] \text{ and}$$

$$B\left(r,\theta,z,t\right) = H\left(r,\theta,z,t\right).$$

To extend these fields smoothly onto the axis $r = 0$, we require that

$$\lim_{r\to 0} v(r) = \lim_{r\to 0}\left[v'(r) + \frac{v(r)}{r}\right]' = 0 \text{ and that } \lim_{r\to 0} v'(r) \text{ exists.}$$

Furthermore the guidance conditions reduce to

(i)$\int_0^\infty \left\{ \left[\frac{c^2k^2}{\omega^2} + \varepsilon\ \left(\frac{1}{2}\ v(r)^2\right)\right]\ v(r)^2 + \frac{c^2}{\omega^2}\ \left[v(r) + \frac{v(r)}{r}\right]^2 \right\}\ r\, dr < \infty$
and
(ii)$\lim_{r\to\infty}\ v(r) = \lim_{r\to\infty}\ v'(r) = 0.$

The power (1.3) of such a guided beam is given by

$$P = \frac{\pi c^2 k}{\omega} \int_0^\infty v(r)^2 \, r \, dr.$$

For the analysis of this problem it is convenient to introduce the new variables defined by

$$u(r) = r^{1/2} v(r) \quad \text{and} \quad \eta = -k^2$$

Then we have shown [8] that the guidance problem is equivalent to finding $(\eta, u) \in (-\infty, 0) \times H_0^1(0, \infty)$ such that

$$u \not\equiv 0 \text{ and } J'(u) = \eta u \tag{GE}$$

where $J : H_0^1(0, \infty) \to \Re$ is defined by

$$J(u) = \int_0^\infty \frac{1}{2} \left[u'(r) + \frac{u(r)}{2r} \right]^2 - r \, E\left(\frac{1}{2r} u(r)^2 \right) \, dr$$

with $E(s) = \left(\frac{\omega}{c}\right)^2 \int_0^s \varepsilon(t) dt$.

Using Hardy's inequality it is easy to check that J is $C^1$and, for any solution (η, u) of (GE), we see that $u \in C^2((0, \infty))$ and $-\frac{\omega^2}{c^2} \varepsilon(\infty) < \eta < 2J(u)$ since $E(s) < \left(\frac{\omega}{c}\right)^2 \varepsilon(s) s \; \forall s > 0$.

One way of obtaining solutions of (GE) is to solve the following minimization problem.

Given $d > 0$,set

$$S(d) = \left\{ u \in H_0^1(0, \infty) \; : \; \int_0^\infty u(r)^2 dr = d^2 \right\} \text{ and} \tag{ME}$$

$$m(d) = \inf \{ J(u) \; : \; u \in S(d) \}.$$

Then find $u \in S(d)$ such that $J(u) = m(d)$.

Clearly if u is a solution of (ME) then there is a Lagrange multiplier η such that (η, u) satisfies (GE) provided that $\eta < 0$. However (ME) does not always have a solution. Indeed we have shown in Theorem 4.6(ii) of [8] that,if $\sigma \geq 1$ in (A), then there exists $d_1 > 0$ such that $\int_0^\infty u(r)^2 dr \geq d_1^2$ for all solutions (η, u) of (GE).

Consequently (ME) has no solution for $d \in (0, d_1)$. This happens because in such cases minimizing sequences on $S(d)$ converge weakly to zero, and this in turn implies that $m(d) \geq -\frac{1}{2} \frac{\omega^2}{c^2} \varepsilon(0) d^2$. But, for any $\sigma > 0$ in (A),

$$\lim_{d\to\infty} \frac{m(d)}{d^2} \leq -\frac{1}{2} \frac{\omega^2}{c^2} \varepsilon(\infty) < -\frac{1}{2} \frac{\omega^2}{c^2} \varepsilon(0)$$

and so $\exists\, d_o \geq 0$ such that

$$m(d) < -\frac{1}{2} \frac{\omega^2}{c^2} \varepsilon(0) d^2 \text{ for all } d > d_o.$$

Furthermore, if $0 < \sigma < 1$ in (A), we can set $d_o = 0$. Hence for $d > d_o$, we are able to show that (ME) has a solution, u_d, with $u_d > 0$ on $(0, \infty)$. The corresponding Lagrange multiplier satisfies the following inequalities,

$$-\frac{\omega^2}{c^2} \varepsilon(\infty) < \eta_d < 2m(d)/d^2 < -\frac{\omega^2}{c^2} \varepsilon(0)$$

where $\lim_{d\to\infty} \eta_d = -\frac{\omega^2}{c^2} \varepsilon(\infty)$ and, if $0 < \sigma < 1$ in (A), $\lim_{d\to 0} \eta_d = -\frac{\omega^2}{c^2} \varepsilon(0)$.

In proving these results in [8] we found it convenient to replace J by

$$\hat{J}(u) = J(u) + \frac{1}{2} \left(\frac{\omega}{c}\right)^2 \varepsilon(0) \int_0^\infty u(r)^2 dr.$$

The main steps in proving that (ME) has a solution are

(a) to show that $\hat{J} : H_o^1(0, \infty) \to \Re$ is weakly sequentially lower semi-continuous, and

(b) to show that $\hat{m}(d) < 0$ where

$$\hat{m}(d) = \inf\left\{ \hat{J}(u) : u \in S(d) \right\} = m(d) + \frac{1}{2} \left(\frac{\omega}{c}\right)^2 \varepsilon(0)\, d^2.$$

It is in the analysis of part (b) that the difference between the cases $0 < \sigma < 1$ and $\sigma \geq 1$ occurs.

3. CYLINDRICAL TM-MODES

We seek solutions for which the magnetic fields have the additional properties that $h_r \equiv b_r \equiv h_z \equiv b_z \equiv 0$.

Setting $h_\theta(r) = b_\theta(r) = \frac{\omega}{c} v(r)$, this means that

$$H(r,\theta,z,t) = B(r,\theta,z,t) = \frac{\omega}{c}\, v(r) \cos(kz - \omega t)\; i_\theta. \tag{3.1}$$

Our initial aim is to reduce Maxwell's equations (ME) with the constitutive relations (CR) to an equation for v.The first step involves an inversion the constitutive relation. This is done as follows.

Supposing that ε satisfies (A), we set

$$f(s) = \varepsilon(s)^2 s \text{ for } s \geq 0$$

and then define a new function γ by

$$\gamma(\tau) = \begin{cases} 1/\varepsilon(0) \text{ for } \tau = 0 \\ \sqrt{f^{-1}(\tau)/\tau} \text{ for } \tau > 0 \end{cases}.$$

It is shown in [9] that γ is well-defined and has the following properties.

(**H**) $\gamma \in C([0,\infty)) \cap C^1((0,\infty))$ with $\gamma'(\tau) \leq 0 < \gamma(\tau) + 2\tau\gamma'(\tau)$ for all $\tau > 0, \gamma(\infty) = \lim_{\tau\to\infty} \gamma(\tau) = 1/\varepsilon(\infty) > 0$ and $\lim_{\tau\to 0} \tau\gamma'(\tau) = 0$. Furthermore, $\lim_{\tau\to 0} \frac{\gamma(\tau)-\gamma(0)}{\tau^\sigma} = -K$ where $K = L/\varepsilon(0)^{2(1+\sigma)}$.

According to Proposition 1.1 of [9] the constitutive relationship (CR) is equivalent to

$$\begin{aligned} H(r,\theta,z,t) &= B(r,\theta,z,t) \\ E(r,\theta,z,t) &= \gamma\left(\frac{1}{2}\left[d_r^2 + d_\theta^2 + d_z^2\right](r)\right) D(r,\theta,z,t). \end{aligned} \tag{CR*}$$

Using this it can be shown [9] that Maxwell's equations with the constitutive relation (CR) are satisfied in the region $r > 0$ provided that $v \in$

$C^2((0,\infty))$ and v satisfies the differential equation

$$\left\{\gamma\left(\frac{1}{2}\left[k^2v(r)^2+w(r)^2\right]\right)w(r)\right\}'-k^2\gamma\left(\frac{1}{2}\ \left[k^2v(r)^2+w(r)^2\right]\right)v(r)+\frac{\omega^2}{c^2}\ v(r)=0 \tag{3.2}$$

where $w(r)=v'(r)+\frac{v(r)}{r}$.

Eliminating w, we see that this is a single second order quasilinear differential equation for v.

From a solution v of this equation the electric fields are defined by

$$d_r(r)=kv(r),\ \ d_\theta(r)\equiv 0,\ \ d_z(r)=-w(r)\quad \text{and}$$

$$E(r,\theta,z,t)=\gamma\left(\frac{1}{2}\ \left[k^2v(r)^2+w(r)^2\right]\right)\ D(r,\theta,z,t)\,.$$

In order to extend these fields onto the axis $r=0$ we require that

$$\lim_{r\to 0}v(r)=0 \text{ and } \lim_{r\to 0}v'(r)=0.$$

(See Proposition 5.1 of [9].)

The guidance conditions reduce to

(i)$\int\limits_0^\infty \left\{\frac{\omega^2}{c^2}v(r)+\gamma\left(\frac{1}{2}\left[k^2v(r)^2+w(r)^2\right]\right)\left[k^2v(r)^2+w(r)^2\right]\right\} r\ dr<\infty$
and
(ii)$\lim_{r\to\infty}v(r)=0$ and $\lim_{r\to\infty}v'(r)=0$.

The power (1.3) of such guided beams is given by

$$P=\pi\omega k^2\int\limits_0^\infty \gamma\left(\frac{1}{2}\ \left[k^2v(r)^2+w(r)^2\right]\right)v(r)^2r\ dr.$$

For the analysis of this problem it is convenient to introduce new variables as follows,

$$u(r)=k\sqrt{r}\ v(r/k) \text{ and } \mu=\left(\frac{\omega}{ck}\right)^2.$$

Then as is shown in [10] the guidance problem is equivalent to finding $(\mu, u) \in (0,\infty) \times H_0^1((0,\infty))$ such that

$$u \not\equiv 0 \text{ and } j(u) = \mu u \tag{GM}$$

where $j : H_0^1(0,\infty) \to \Re$ is defined by

$$j(u) = \int_0^\infty \Gamma\left(\frac{1}{2r}\left[u(r)^2 + (Tu(r))^2\right]\right) r\, dr$$

with $Tu(r) = u'(r) + \frac{u(r)}{2r}$ and $\Gamma(\tau) = \int_0^\tau \gamma(t)dt$

By (H), $\Gamma \in C^1([0,\infty))$ and $0 < \gamma(\infty)\tau \le \Gamma(\tau) \le \gamma(0)\tau \quad \forall \tau > 0$.

Hardy's inequality implies that $T : H_0^1(0,\infty) \to L^2(0,\infty)$ is a bounded linear operator. Hence j is C^1 and for any solution of (GM) we find that $\mu > \gamma(\infty)$. By applying regularity theorems due to Tonelli we have shown in Theorem 3.3 of [10] that $u \in C^2((0,\infty))$ whenever (μ, u) is a solution of (GM).

One way of obtaining solutions of (GM) is to solve the following minimization problem.

Given $d > 0$,set

$$S(d) = \left\{u \in H_0^1(0,\infty) : \int_o^\infty u(r)^2 dr = d^2\right\} \text{ and} \tag{MM}$$
$$M(d) = \inf\{j(u) : u \in S(d)\}.$$

Then find $u \in S(d)$ such that $j(u) = M(d)$.

Clearly if u is a solution of (MM) then there exists a Lagrange multiplier μ such that (μ, u) satisfies (GM). However (MM) does not always have a solution.

Indeed we have shown in Theorem 4.6 of [10] that if $\sigma \ge 1$in (A) then there exists $d_1 > 0$ such that $\int_0^\infty u(r)^2 dr \ge d_1^2$ for all solutions (μ, u) of (GM). Consequently (MM) has no solution for $d \in (0, d_1)$. In such cases minimizing sequences on $S(d)$ converge weakly to zero. On the other hand, if a minimizing sequence on $S(d)$ converges weakly to zero then we find that $M(d) \ge \gamma(0)d^2/2$.

But for any $\sigma > 0$ in (A),

$$\lim_{d \to \infty} M(d)/d^2 \leq \gamma(\infty)/2 < \gamma(0)$$

and so there exists $d_0 \geq 0$ such that

$$M(d) < \gamma(0)d^2/2 \text{ for all } d > d_0.$$

Furthermore, if $0 < \sigma < 1$ in (A), we can set $d_0 = 0$.

In such cases, minimizing sequences on $S(d)$ cannot converge weakly to zero and we are able to show that (MM) has a solution, u_d, with $u_d > 0$ on $(0, \infty)$. The corresponding Lagrange multiplier satisfies the following inequalities

$$\gamma(\infty) < \mu_d < \gamma(0) \text{ for all } d > d_0$$

where

$$\lim_{d \to \infty} \mu_d = \gamma(\infty) \text{ and, if } 0 < \sigma < 1, \ \lim_{d \to 0} \mu_d = \gamma(0).$$

In proving these results in [10] we found it convenient to replace j by

$$\hat{j}(u) = j(u) - \frac{1}{2}\gamma(0) \int_0^\infty u(r)^2 dr.$$

The main steps in proving that (MM) has a solution are

(a)to show that $\hat{j} : H_0^1(0, \infty) \to \Re$ is weakly sequentially lower semicontinuous, and

(b)to show that $\hat{M}(d) < 0$ where

$$\hat{M}(d) = \inf \left\{ \hat{j}(u) : u \in S(d) \right\} = M(d) - \frac{1}{2}\gamma(0)d^2.$$

It is in the analysis of part (b) that the difference between the cases $0 < \sigma < 1$ and $\sigma \geq 1$ occurs.

4. RELATED RESULTS

We end with some remarks about variants of the problems discussed in Sections 2 and 3.

Higher modes

In the variational principles (ME) and (MM) we have only considered the existence of fundamental TE and TM modes. However higher modes can also be found by studying all critical points of J and j on $S(d)$, rather than just looking for minima. This has been done by H.-J. Ruppen in a series of interesting papers [11,12].

Inhomogeneity of the medium

The assumption (A) requires the medium to be homogeneous. In the context of optical fibres it is natural to allow the material composition to vary with distance from the axis $r = 0$. This amounts to considering ε to be a function of two variables, $\varepsilon(r, s)$, where $s = \frac{1}{2}\left[e_r^2 + e_\theta^2 + e_z^2\right](r)$. The discussion of TE-modes in [8] covers this case. For TM-modes work in this direction in underway.

Defocusing media

The assumption (A) requires the medium to be self-focusing since $\varepsilon' \geq 0$. However some materials have a dielectric response for which $\varepsilon' \leq 0$. The existence of cylindrical guided modes has also been studied in such defocusing media, but it is essential to allow appropriate inhomogeneity of the fibre to counteract the dispersive effect of the nonlinearity. The case of cylindrical TE-modes is discussed in [13], and the study of TM-modes is in progress.

Planar waveguides

Analogous results are available for the case of planar rather than cylindrical symmetry. See [12] and [16] for the case of planar TE-modes. For planar TM-modes work is in progress.

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