Guided TM-modes in a self-focusing anisotropic dielectric

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Abstract

We present a new formulation of the equations for planar and cylindrical TM-modes in a uniaxial self-focusing dielectric medium. By using the amplitude of the magnetic field as the basic unknown and by inverting the usual constitutive assumption we obtain a single second order equation. The existence of guided planar TM-modes is then studied.

1 Introduction

The guidance of light is usually achieved by exploiting the effect of variations in the refractive index due to inhomogeneity of the medium through which the beam is propagating. As can be understood from Snell's law, the favorable configuration consists of a region of high refractive index surrounded by layers of material having a lower refractive index. However it is also well-known that nonlinear effects can be used to enhance,or even to produce, guidance. For example, no guidance occurs in a homogeneous linear medium (where the refractive index is constant) whereas it will occur, at least for sufficiently intense beams, in a homogeneous self-focusing medium (where the refractive index is an increasing function of the intensity of the light passing through it).

The mathematical discussion of this phenomenon amounts to the study of special solutions of Maxwell's equations in a homogeneous medium whose constitutive relation expresses the electric displacement field as a nonlinear function of the electric field. In a self-focusing material this dielectric response is an increasing function of the electric field strength. The special solutions which have been studied in this context are either TE (transverse electric field) or TM (transverse magnetic field) modes having a planar or cylindrical symmetry. For isotropic self-focusing materials the mathematical analysis of these problems has been undertaken in [1],[2] ,[3],[4],[5] and [6].

In this paper we deal with TM-modes in an anisotropic self-focusing medium. (For TE-modes in a uniaxial material propagating in the direction of the axis, the equation for the amplitude of the electric field is the same as that for an isotropic material.) It is generally recognized (see [20], [13], [12], [10], [17], [18], [19] for example) that, even for isotropic materials, the discussion of TM-modes is considerably more complicated than the corresponding treatment of TE-modes because the nonlinear effect is a function only of the electric field strength. In a TM-mode the electric field is composed of two orthogonal components which are $\pi/2$ out of phase and the problem is usually formulated as a system of equations for the amplitudes of these two components. In a TE-mode the electric field has a single harmonic component and the problem reduces to a second order differential equation for its amplitude. Using a mixture of approximations and numerics, TM-modes in anisotropic media have been discussed in [11], [12], [13], [15], [14], and [16]. Most of these contributions deal only with the special case of a Kerr nonlinearity and there are a number of other contributions dealing only with isotropic materials [8], [9], [16], [17], [18], [19].

Here we give a rigorous discussion based on a new, but equivalent, formulation of the problem which covers a wide range of constitutive laws for anisotropic materials, with or without saturation. For planar and cylindrical waveguides, we use the amplitude of the magnetic field to reduce the study of TM-modes to a single second order differential equation as in the case of TE-modes. To do so we must first invert the constitutive relation for a self-focusing anisotropic medium and this is the subject of the Section 2. The resulting equations have forms that are more complicated than the semilinear equations obtained for the corresponding TE-modes but, at least for planar modes, the equation does have a first integral. This is exploited in Section 3 to give necessary and sufficient conditions for the existence of planar guided TM-modes. The corresponding equation for a cylindrical waveguide is nonautonomous and it does not have a first integral. The discussion of guided modes in this case is less elementary and is the subject of [6].

2 Equations for TM-modes

We start with Maxwell's equations in the form

$$c(\nabla \wedge E) = -\partial_t B, c(\nabla \wedge H) = \partial_t D, \nabla . D = 0, \nabla . B = 0$$

where c is the speed of light in a vacuum,[7].

To introduce the constitutive assumption for a homogeneous uniaxial material we choose an orthonormal basis $\{e_i : i = 1, 2, 3\}$ with e_3 in the direction of the axis of the medium. All the e-m fields E, D, H and B that we consider can be expressed in the form

$$F(x, y, z, t) = \ F_T(x, y) \cos(kz - \omega t) + F_z(x, y) \sin(kz - \omega t)$$

where $F_T, F_z : \Re^2 \longrightarrow \Re^3$ are such that $F_T.e_3 = 0$ and $F_z.e_i = 0$ for $i = 1, 2$.

Thus the transverse and axial components of the fields are out of phase by a quarter of a cycle and the fields constitute monochromatic waves propagating in the direction of the material axis.

For fields of this type the constitutive relation which we adopt is expressed as follows.

(CR) $B = H$ and there exist two continuous functions $\varepsilon_i : [0.\infty)^2 \longrightarrow (0, \infty)$ such that

$$D_T(x, y) = \varepsilon_1(\mid E_T(x, y) \mid^2 /2, \mid E_z(x, y) \mid^2 /2) E_T(x, y)$$

$$D_z(x, y) = \varepsilon_2(\mid E_T(x, y) \mid^2 /2, \mid E_z(x, y) \mid^2 /2) E_z(x, y)$$

where $0 < A < \varepsilon_i(s_1, s_2)$ for $i = 1, 2$ and for $s_1, s_2 \geq 0$.

This means that with respect to the basis $\{e_i : i = 1, 2, 3\}$ the dielectric response tensor is represented by the diagonal matrix $\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix}$ where the elements are functions of the time-averages,$\mid E_T(x, y) \mid^2 /2$ and $\mid E_z(x, y) \mid^2 /2$, of the intensities of the transverse and axial components of the electric field E.

By a TM-mode we mean a solution of the above type for which $H_z \equiv 0$ and for such modes we expect to reduce Maxwell's equations and (CR) to a system of equations for H_T. Indeed since $\partial_t^2 D = -\omega^2 D$ we must have

$-\omega^2 D = c\nabla \wedge \partial_t H$ and hence we set $D = -(c/\omega)\nabla \wedge H_T \sin(kz - \omega t)$.This ensures that $\nabla.D = 0$ and so it suffices to choose H_T in such a way that

$$\nabla.H = 0 \text{ and } c(\nabla \wedge E) = -\omega H_T \sin(kz - \omega t).$$

Setting $H_T.e_i = h_i$ for $i = 1,2$ the first equation becomes $\partial_x h_1 + \partial_y h_2 = 0$ but to express the second equation using H_T we must express E using H_T. Since D has been given in terms of H_T this can be done by inverting the constitutive relation (CR). To show that this is possible we require some additional properties of the dielectric response.

According to (CR) the dielectric response of the medium is determined by a function $\varepsilon = (\varepsilon_1, \varepsilon_2) : [0,\infty)^2 \longrightarrow (0,\infty)^2$ about which we now make some further assumptions.

(V) There is a potential $\varphi \in C^1([0,\infty)^2)$ such that $\varepsilon = \nabla\varphi$.

This property of the dielectric response is used explicitly in [13] where it is assumed that $\partial_2\varepsilon_1 = \partial_1\varepsilon_2$ for all $s_1, s_2 \geq 0$. When (V) holds the potential, normalized so that $\varphi(0) = 0$ is given by

$$\varphi(s) = \int_0^1 \varepsilon(ts).s dt$$

and we define an auxiliary function Φ by

$$\Phi(s_1, s_2) = \varphi(s_1^2/2, s_2^2/2).$$

(C) We suppose that $\Phi \in C^2(\Re^2)$ and that $D^2\Phi(s)$ is positive definite for all $s \in \Re^2$.

The positive definitiveness of $D^2\Phi(s)$ implies that $\partial_i \left[\varepsilon_i(s_1^2/2, s_2^2/2)s_i\right] > 0$ for $s \in \Re^2$ and $i = 1,2$ and that Φ is strictly convex. Since $\Phi(0) = 0$ and $\nabla\ \Phi(0) = 0$, it follows that $\Phi(s) > 0$ for all $s \in \Re^2 \setminus \{0\}$. By (CR), $\|\nabla\Phi(s_1, s_2)\| \geq A\sqrt{(s_1^2 + s_2^2)}$ for all $(s_1, s_2) \in \Re^2$ and so from the conclusions of Theorem 26.4 to Lemma 26.7 of[22] we find that $\nabla\Phi : \Re^2 \longrightarrow \Re^2$ is a diffeomorphism and $[\nabla\Phi]^{-1} = \nabla\Phi^*$ where Φ^* is the Legendre transform of Φ which can be defined by

$$\Phi^*(\tau) = \sup\left\{s.\tau - \Phi(s) : s \in \Re^2\right\} \text{ for } \tau = (\tau_1, \tau_2) \in \Re^2.$$

From this formula we deduce that $\Phi^*(0,0) = 0$ and $\Phi^*(s_1,s_2) = \Phi^*(| s_1 |, | s_2 |)$. Since $\varepsilon_1$and $\varepsilon_2 > 0$ we also have that $\nabla\Phi$ maps the four quadrants and half-axes onto themselves. Of course $\nabla\Phi(s_1,s_2) = \begin{pmatrix} \varepsilon_1(s_1^2/2, s_2^2/2)s_1 \\ \varepsilon_2(s_1^2/2, s_2^2/2)s_2 \end{pmatrix}$ and we now show that $\nabla\Phi^*$ has a similar structure. From (C) we deduce that $\Phi^* \in C^2(\Re^2)$ and that $D^2\Phi^*(t)$ is positive definite for all $t \in \Re^2$. See Corollary 4.2.10 of [21].

Define $\psi : [0,\infty)^2 \longrightarrow \Re$ by

$$\psi(t_1,t_2) = \Phi^*(\sqrt{2t_1}, \sqrt{2t_2})$$

so that

$$\Phi^*(t_1,t_2) = \psi(t_1^2/2, t_2^2/2).$$

Clearly, $\psi \in C^2((0,\infty)^2)$ with $\partial_i\Phi^*(t_1,t_2) = \partial_i\psi(t_1^2/2, t_2^2/2)t_i$ and we define $\gamma = (\gamma_1,\gamma_2) \in C^1((0,\infty)^2, \Re^2)$ by $\gamma_i = \partial_i\psi$. Then

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_1(s_1^2/2, s_2^2/2)s_1 \\ \varepsilon_2(s_1^2/2, s_2^2/2)s_2 \end{pmatrix}$$

$$\Leftrightarrow t = \nabla\Phi(s) \Leftrightarrow \nabla\Phi^*(t) = s \Leftrightarrow$$

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \gamma_1(t_1^2/2, t_2^2/2)t_1 \\ \gamma_2(t_1^2/2, t_2^2/2)t_2 \end{pmatrix}.$$

It follows that for $i = 1,2$ and $t_1, t_2 > 0, \gamma_i(t_1,t_2) =$

$$1/\varepsilon_i(\gamma_1(t_1,t_2)^2 t_1, \gamma_2(t_1,t_2)^2 t_2) =$$

$$1/\varepsilon_i(\left[\partial_1\Phi^*(\sqrt{2t_1}, \sqrt{2t_2})\right]^2/2, \left[\partial_2\Phi^*(\sqrt{2t_1}, \sqrt{2t_2})\right]^2/2).$$

Using this formula, and recalling that $\nabla\Phi^*$ is continuous on $\Re^2$, we see that γ can be extended by continuity to $[0,\infty)^2$.

Let us note an additional property of γ. Since $D^2\Phi^*(t)$ is positive definite for all $t \in \Re^2$, $0 < \partial_i^2\Phi^*(t_1,t_2) = \partial_i\{\gamma_i(t_1^2/2, t_2^2/2)t_i\}$ for $i = 1,2$ and all $(t_1,t_2) \in \Re^2$.

Summarizing these remarks we see that $\nabla\Phi^*(t_1,t_2) = \begin{pmatrix} \gamma_1(t_1^2/2, t_2^2/2)t_1 \\ \gamma_2(t_1^2/2, t_2^2/2)t_2 \end{pmatrix}$ where γ has the following properties.

(H) $\gamma \in C^1((0,\infty)^2, \Re^2) \cap C([0,\infty)^2, \Re^2)$ and for $i = 1, 2$,

$$0 < \gamma_i(t_1, t_2) < 1/A,$$

$$\gamma_i(t_1^2/2, t_2^2/2) = 1/\varepsilon_i(s_1^2/2, s_2^2/2) \text{ where } t_i = \varepsilon_i(s_1^2/2, s_2^2/2)s_i \text{ and}$$

$$\partial_i \left\{\gamma_i(t_1^2/2, t_2^2/2)t_i\right\} > 0 \text{ for all } (t_1, t_2) \in \Re^2.$$

As an example of the above procedure let us consider the Kerr law for a uniaxial material.

Example 1 *We suppose that there are four positive constants α_T, α_z, P and Q such that*

$\varepsilon_1(s_1, s_2) = \alpha_T + Ps_1 + Qs_2$

$\varepsilon_2(s_1, s_2) = \alpha_z + Qs_1 + Ps_2$ *for all* $s_1, s_2 \geq 0$.

Clearly $\varepsilon = \nabla\varphi$ *where* $\varphi(s_1, s_2) = \alpha_T s_1 + \alpha_z s_2 + Qs_1s_2 + P(s_1^2 + s_2^2)/2$ *and it is easy to check that* $\Phi(s_1, s_2) = \varphi(s_1^2/2, s_2^2/2)$ *satisfies the condition (C) provided that* $Q/P \leq 3$. *Referring to [12] or to page 246 of [11], we see that the ratio* Q/P *is* $\frac{1}{3}$ *when the nonlinearity of the dielectric response is due to electronic distortion, whereas it is 1 when this nonlinearity is caused by electrostriction.*

Returning to the general case we see that, if the function ε in the constitutive relation (CR) has the properties (V) and (C), then (CR) can be expressed in the following way where the function γ has the properties (H).

(CR)* $B = H$ and

$$E_T(x,y) = \gamma_1(|\, D_T(x,y) \,|^2 /2, |\, D_z(x,y) \,|^2 /2) D_T(x,y)$$

$$E_z(x,y) = \gamma_2(|\, D_T(x,y) \,|^2 /2, |\, D_z(x,y) \,|^2 /2) D_z(x,y)$$

where γ is obtained from ε by the above construction.

Returning to the problem of expressing the equations for TM-modes using H_T, and recalling that we have $D = -(c/\omega)\nabla \wedge H_T \sin(kz - \omega t)$ with $H_T = h_1 e_1 + h_2 e_2$, we find that

$$D_T = k(c/\omega)\{-h_2 e_1 + h_1 e_2\} \text{ and that } D_z = (c/\omega)\{\partial_x h_2 - \partial_y h_1\}.$$

Using (CR)* we can now express E in the equation $c(\nabla\wedge E) = -\omega H_T \sin(kz - \omega t)$ in terms of H_T. To write the resulting equations in a concise way it is convenient to introduce the following notation.

Set

$$g_i = \gamma_i\left((kc/\omega)^2\left[h_1^2 + h_2^2\right]/2, (c/\omega)^2\left[\partial_x h_2 - \partial_y h_1\right]^2/2\right) \text{ for } i = 1, 2.$$

The magnetic field

$$H_T(x, y, z, t) = [h_1(x, y)e_1 + h_2(x, y)e_2]\cos(kz - \omega t)$$

satisfies Maxwell's equations and the constitutive relation (CR), where the dielectric response ε is supposed to have the properties (V) and (C), provided that (h_1, h_2) is a solution of the following system of equations.

$$\partial_x h_1 + \partial_y h_2 = 0$$

$$\partial_y\left\{g_2\left[\partial_x h_2 - \partial_y h_1\right]\right\} + k^2 g_1 h_1 - (\omega/c)^2 h_1 = 0$$

$$-\partial_x\left\{g_2\left[\partial_x h_2 - \partial_y h_1\right]\right\} + k^2 g_1 h_2 - (\omega/c)^2 h_2 = 0$$

There are two important cases where this system can be reduced to a single equation.

Planar TM-modes

There is a solution of the form

$$h_1(x, y) = 0 \text{ and } h_2(x, y) = (\omega/kc)u(kx) \tag{1}$$

provided that u satisfies the equation

$$\{g_2 u'(x)\}' - g_1 u(x) + \lambda u(x) = 0 \tag{2}$$

where now $g_i = \gamma_i(u(x)^2/2, u'(x)^2/2)$ for $i = 1, 2$ and $\lambda = (\omega/kc)^2$.

Cylindrical TM-modes

There is a solution of the form

$$h_1(x,y) = -(\omega/kc)u(r)y/r \text{ and } h_2(x,y) = (\omega/kc)u(r)u(r)x/r$$

where $r = \sqrt{x^2+y^2}$, provided that u satisfies the equation

$$\{g_2 v(r)\}' - g_1 u(r) + \lambda u(r) = 0 \tag{3}$$

where now $g_i = \gamma_i(u(r)^2/2, v(r)^2/2)$ for $i = 1,2$, $v(r) = u'(r) + u(r)/r$ and $\lambda = (\omega/kc)^2$.

3 Guided planar TM-modes

We suppose henceforth that the dielectric response function ε in (CR) has the properties (V) and(C).

By a guided planar TM-mode we mean a solution $u \not\equiv 0$ of equation (2) which has the properties that

$$\lim_{x \longrightarrow \pm\infty} u(x) = \lim_{x \longrightarrow \pm\infty} u'(x) = 0. \tag{4}$$

In this section we discuss the existence of such solutions. The analysis is based on the observation that the function I defined by

$$I(p,q) = \Phi^*(p,q) - q\partial_2\Phi^*(p,q) - \lambda p^2/2$$

is a first integral for (2). Indeed, recalling that $\nabla\psi = \gamma = (\gamma_1, \gamma_2)$, we see that

$$I(p,q) = \psi(p^2/2, q^2/2) - \gamma_2(p^2/2, q^2/2)q^2 - \lambda p^2/2$$

and hence it is easy to verify that if u satisfies (2) then

$$\{I(u(x), u'(x))\}' = u'(x)\left[g_1 u(x) - \lambda u(x) - \{g_2 u'(x)\}'\right] = 0.$$

Since $\Phi^*(0,0) = 0$ and $0 < \gamma_2(p,q) < 1/A$, it follows that if u satisfies (2) and(4) then

$$I(u(x), u'(x)) = 0 \text{ for all } x \in \Re.$$

Hence the orbit of a guided TM-mode lies in the set $I^{-1}(0) \setminus \{(0,0)\}$ and, in view of the symmetries of I , it is sufficient to discuss the set

$$C = \{(p,q) : I(p,q) = 0 \text{ with } p \geq 0 \text{ and } q \geq 0\}.$$

For this we introduce some additional hypotheses about the dielectric response.

(S) (a) $\varepsilon_1(s_1,0)$ is a strictly increasing function of s_1 on $[0,\infty)$, and

(b) there exist $B>0$ and $\alpha\geq 0$ such that $\varepsilon_2(s_1,s_2)/s_2^{\alpha}\longrightarrow B$ as $s_2\longrightarrow\infty$, uniformly for s_1 in bounded subsets of $[0,\infty)$.

We observe that, in the example of a Kerr material which was discussed in the previous section, the condition (S) is satisfied with $\alpha=1$. On the other hand for realistic constitutive laws which model saturation, the value $\alpha=0$ in (S)(b) is appropriate. (See [11] for examples of such constitutive relations.)

These properties of ε imply some analogous behaviour of the function γ.

(a) Recalling that $\gamma_1(t_1,0)=1/\varepsilon_1(\gamma_1(t_1,0)^2t_1,0)$, it follows from (H) that $\gamma_1(t_1,0)$ is a strictly decreasing function of t_1 on $[0,\infty)$. Setting $\gamma_1(\infty,0)=\lim_{t\longrightarrow\infty}\gamma_1(t,0)$, we have $0\leq\gamma_1(\infty,0)=1/\lim_{s\longrightarrow\infty}\varepsilon_1(s,0)=1/\varepsilon_1(\infty,0)$.

(b) For $t_1,t_2\in[0,\infty)$ let $\begin{pmatrix} s_1\\ s_2\end{pmatrix}=\begin{pmatrix}\gamma_1(t_1^2/2,t_2^2/2)t_1\\ \gamma_2(t_1^2/2,t_2^2/2)t_2\end{pmatrix}$.

Then $\begin{pmatrix} t_1\\ t_2\end{pmatrix}=\begin{pmatrix}\varepsilon_1(s_1^2/2,s_2^2/2)s_1\\ \varepsilon_2(s_1^2/2,s_2^2/2)s_2\end{pmatrix}$ and so $t_1\geq As_1$. Hence if t_1 varies over a bounded subset of $[0,\infty)$ so does s_1. Also, setting $\beta=\alpha/(1+2\alpha)$, we have that

$$\gamma_2(t_1^2/2,t_2^2/2)[t_2^2/2]^{\beta}=2^{-\beta}\left\{\varepsilon_2(s_1^2/2,s_2^2/2)s_2\right\}^{2\beta}/\varepsilon_2(s_1^2/2,s_2^2/2)$$

$$=1/\left\{\varepsilon_2(s_1^2/2,s_2^2/2)/(s_2^2/2)^{\alpha}\right\}^{1/(1+2\alpha)}\longrightarrow L\ \text{ as }t_2\longrightarrow\infty,$$

uniformly for t_1 in bounded subsets of $[0,\infty)$, where $L=1/B^{1/(1+2\alpha)}$.

Lemma 2 *Let the dielectric response have the properties (CR),(V),(C) and (S).*

If $\lambda\notin(\gamma_1(\infty,0),\gamma_1(0,0))$, *then* $C\cap\{(p,0):p>0\}=\emptyset$.

If $\lambda\in(\gamma_1(\infty,0),\gamma_1(0,0))$, *then there exist* $p_\lambda>0$ *and* $f\in C([0,p_\lambda])\cap C^1((0,p_\lambda))$ *such that*

$f(0)=0, f(\ p_\lambda)=0, f(p)>0$ *for all* $p\in(0,p_\lambda)$

and $C=\{(p,f(p)):0\leq p\leq p_\lambda\}$.

Furthermore $\lim_{p\longrightarrow 0}f'(p)=\sqrt{\frac{\gamma_1(0,0)-\lambda}{\gamma_2(0,0)}}$ *and* $\lim_{p\longrightarrow p_\lambda}f'(p)=-\infty$.

Also, $p_\lambda\longrightarrow\begin{cases}0\\ \infty\end{cases}$ *as* $\lambda\longrightarrow\begin{cases}\gamma_1(0,0)\\ \gamma_1(\infty,0)\end{cases}$.

Proof. *For $p > 0$,*

$I(p,0) = p^2 \{\Phi^*(p,0)/p^2 - \lambda/2\}$ *and*

$$\Phi^*(p,0)/p^2 = \int_0^1 \gamma_1(t^2p^2/2,0)tdt \longrightarrow \begin{cases} \gamma_1(0,0)/2 \\ \gamma_1(\infty,0)/2 \end{cases} \text{as } p \longrightarrow \begin{cases} 0 \\ \infty \end{cases} .$$

Furthermore, by (S)(a), $\gamma_1(t_1,0)$ is a strictly decreasing function of t_1 on $[0,\infty)$ and so $\Phi^*(p,0)/p^2$ is a strictly decreasing function of p on $[0,\infty)$. Thus,
if $\lambda \notin (\gamma_1(\infty,0),\gamma_1(0,0))$, then $C \cap \{(p,0) : p > 0\} = \emptyset$ and
if $\lambda \in (\gamma_1(\infty,0),\gamma_1(0,0))$, then there exists a unique $p_\lambda > 0$ such that $I(p_\lambda,0) = 0$.
Clearly $p_\lambda \longrightarrow \begin{cases} 0 \\ \infty \end{cases}$ as $\lambda \longrightarrow \begin{cases} \gamma_1(0,0) \\ \gamma_1(\infty,0) \end{cases}$.

From now on we suppose that $\lambda \in (\gamma_1(\infty,0),\gamma_1(0,0))$.
For all (p,q), $\partial_2 I(p,q) = -q\partial_2^2\Phi^*(p,q)$ and so $\partial_2 I(p,q) < 0$ if $q > 0$. Hence for $p = 0$ and $p \geq p_\lambda, I(p,q) < 0$ for all $q > 0$.

Using (S)(b) we now show that $I(p,q) \longrightarrow -\infty$ as $q \longrightarrow \infty$. In fact,

$$\Phi^*(p,q) - q\partial_2\Phi^*(p,q) = \int_0^1 \partial_1\Phi^*(tp,tq)p + \partial_2\Phi^*(tp,tq)qdt - q\partial_2\Phi^*(p,q)$$

$$= \int_0^1 \gamma_1(t^2p^2/2,t^2q^2/2)tp^2 + \gamma_2(t^2p^2/2,t^2q^2/2)tq^2dt - \gamma_2(p^2/2,q^2/2)q^2 .$$

Hence

$$q^{-2(1+\alpha)/(1+2\alpha)}\{\Phi^*(p,q) - q\partial_2\Phi^*(p,q)\} =$$

$$q^{-2(1+\alpha)/(1+2\alpha)}\int_0^1 \gamma_1(t^2p^2/2,t^2q^2/2)tp^2dt + \int_0^1 \gamma_2(t^2p^2/2,t^2q^2/2)[t^2q^2/2]^\beta[t^2/2]^{-\beta}tdt$$

$$-\gamma_2(p^2/2,q^2/2)[q^2/2]^\beta 2^\beta$$

$\longrightarrow 0 + 2^\beta L\left\{\int_0^1 t^{1-2\beta}dt - 1\right\} = -2^\beta L/2(1+\alpha) < 0$ as $q \longrightarrow \infty$, uniformly for p in bounded intervals.

It follows that, given $P > 0$, there exists $Q(P) > 0$ such that $I(p,q) < 0$ for all $p \in [0,P]$ and $q \geq Q(P)$.

Thus, for $p \in (0,p_\lambda)$, there is a unique $q = f(p)$ such that $I(p,f(p)) = 0$. Since $\partial_2 I(p,f(p)) < 0$, the implicit function theorem yields $f \in C^1((0,p_\lambda))$.

Furthermore, if $p_n \longrightarrow 0$ and $f(p_n) \longrightarrow q$ we have that $q \in [0,Q(p_\lambda)]$ and so $I(0,q) = 0$. This implies that $q = 0$ and we may conclude that $\lim_{p\longrightarrow 0} f(p) = 0$.

Similarly, $\lim_{p\longrightarrow p_\lambda} f(p) = 0$.

For $p \in (0, p_\lambda)$,

$$f'(p) = -\partial_1 I(p, f(p))/\partial_2 I(p, f(p)) = \partial_1 I(p, f(p))/f(p)\partial_2^2 \Phi^*(p, f(p))$$

and

$$\partial_1 I(p, q) = \gamma_1(p^2/2, q^2/2)p - \partial_1\gamma_2(p^2/2, q^2/2)pq^2 - \lambda p.$$

Since $\partial_1 I(p, f(p))/p \longrightarrow \gamma_1(p_\lambda^2/2, 0) - \lambda < \gamma_1(0,0) - \lambda < 0$ as $p \longrightarrow p_\lambda$, we see that $f'(p) \longrightarrow -\infty$ as $p \longrightarrow p_\lambda$..

On the other hand, $I(p, q) = 0$
$\Rightarrow \int_0^1 \gamma_1(t^2p^2/2, t^2q^2/2)tp^2 + \gamma_2(t^2p^2/2, t^2q^2/2)tq^2 dt - \gamma_2(p^2/2, q^2/2)q^2 - \lambda p^2/2 = 0$
$\Rightarrow \{\int_0^1 \gamma_2(t^2p^2/2, t^2q^2/2)tdt - \gamma_2(p^2/2, q^2/2)\}q^2 = -\{\int_0^1 \gamma_1(t^2p^2/2, t^2q^2/2)tdt - \lambda/2\}p^2$
$\Rightarrow \{f(p)/p\}^2 \longrightarrow \frac{\lambda/2 - \gamma_1(0,0)/2}{\gamma_2(0,0)/2 - \gamma_2(0,0)} = \frac{\gamma_1(0,0) - \lambda}{\gamma_2(0,0)}$ as $p \longrightarrow 0$.

Hence $f'(p) \longrightarrow \sqrt{\frac{\gamma_1(0,0) - \lambda}{\gamma_2(0,0)}}$ as $p \longrightarrow 0$. ∎

From these properties of C we can deduce the following information about guided TM-modes in a straightforward way so we only give a brief indication of the arguments.

Recall that $\gamma_1(0,0) = 1/\varepsilon_1(0,0)$ and that $\gamma_1(\infty, 0) = 1/\varepsilon_1(\infty, 0)$.

Theorem 3 *Let the dielectric response have the properties (CR),(V),(C) and (S).*

If $\lambda \notin (\gamma_1(\infty, 0), \gamma_1(0,0))$, *then there is no guided TM-mode.*

If $\lambda \in (\gamma_1(\infty, 0), \gamma_1(0,0))$, *then there exists a unique guided TM-mode* u_λ *such that* $u_\lambda(0) = p_\lambda$ *and* $u'_\lambda(0) = 0$.

Furthermore, $u_\lambda(x) = u_\lambda(-x) > 0$ *for all* $x \in \Re$ *and, for any* $\mu < \sqrt{\frac{\gamma_1(0,0)-\lambda}{\gamma_2(0,0)}}, \lim_{x \longrightarrow \infty} \exp(\mu x)\ u_\lambda(x) = 0$.

All guided TM-modes are of the form $\pm u_\lambda(x + \delta)$ *for some* $\delta \in \Re$ *and some* $\lambda \in (\gamma_1(\infty, 0), \gamma_1(0,0))$.

Proof. If u is a guided TM-mode so is $\pm u(x + \delta)$ for all $\delta \in \Re$. Hence we may assume that $u(0) = \max u(x) > 0$ and that $u'(0) = 0$. Since $(u(0), u'(0)) \in C$, this shows that $\lambda \in (\gamma_1(\infty, 0), \gamma_1(0,0))$.

For λ in this range we denote by u_λ the solution of (2) which satisfies the initial conditions $u(0) = p_\lambda$ and $u'(0) = 0$ where p_λ is given by the lemma. Note that in (2) we have that

$$\{g_2 u'(x)\}' = \partial_1\gamma_2(u(x)^2/2, u'(x)^2/2)u(x)u'(x)^2$$

$$+\{\partial_2\gamma_2(u(x)^2/2,u'(x)^2/2)u'(x)^2+\gamma_2(u(x)^2/2,u'(x)^2/2)\}u''(x)$$

Recalling that $\partial_2\left\{\gamma_2(t_1^2/2,t_2^2/2)t_2\right\}>0$, we see that the coefficient of $u''(x)$ is a C^1-function of u and u' which cannot vanish. It is now easy to show that u_λ is unique, is defined on all of $\Re$ and is a guided TM-mode. ■

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