Bifurcation into spectral gaps.

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1 Introduction

These notes present a unified treatment, together with some refinements, of recent work on the existence and bifurcation of solutions of equations having the form

$$Su - R(u) = \lambda u \text{ for } (\lambda, u) \in R \times [\mathcal{D}(S) \cap \mathcal{D}(R)]; \qquad (1.1)$$

Here S is an unbounded self-adjoint operator acting in a real Hilbert space $\mathcal{H}$ and R is a potential operator such that $R(u) = o(u)$ for u near 0 in a sense that is made precise later. This discussion concerns the existence of solutions of (1.1) with $u \neq 0$ and the existence of bifurcation points for such solutions from the line $\{(\lambda, 0) : \lambda \in R\}$ of trivial solutions. Of course bifurcation points for (1.1) must belong to the spectrum, $\sigma(S)$, of S. Since all the results presented here concern non-trivial solutions for which $\lambda \notin \sigma(S)$ we must suppose that $\sigma(S) \neq R$. Then, without further loss of generality, we may assume that $0 \notin \sigma(S)$. Our aim is to obtain results about (1.1) without making additional assumptions about the nature of the spectrum of S. Thus a bifurcation point may be an eigenvalue (of finite or infinite multiplicity) or a point of the continuous spectrum. In the applications to differential equations that we consider it is the latter case which occurs.

The exposition of the material is organised in the following way.

2. Spectral decomposition of the space.
3. Global Lyapunov-Schmidt reduction.
4. Applying the Mountain Pass Theorem.
5. Existence results for asymptotically definite nonlinearities.
6. Existence results for semi-definite nonlinearities.
7. Bifurcation results.
8. Nonlinear perturbation of an unbounded self-adjoint operator.
9. Bound states for nonlinear Schrdinger equations.

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10. Homoclinic solutions of Hamiltonian systems.

11. References.

The study of (1.1) by variational methods is based on a associated equation having the form

$$(A - \lambda L)u - N(u) = 0 \text{ for } (\lambda, u) \in R \times H. \tag{1.2}$$

Here H is a real Hilbert space, A and L are bounded self-adjoint operators on H and $N \in C^1(H, H)$ with $N(0) = 0$ and $N'(0) = 0$. He precise relationship between (1.1) and (1.2) is set out in section 8. For the moment we simply remark that $\mathcal{D}(S) \subset H = \mathcal{D}(|S|^{1/2}) \subset \mathcal{H}$ and that solutions of (1.2) may legitimately be considered to be generalised solutions of (1.1). Furthermore the assumption that $0 \notin \sigma(S)$ implies that $0 \notin A$.

The results in sections 2 to 7 deal directly with equations of the form (1.2), sections 2 to 4 being devoted to preparatory work and sections 5 to 7 containing the main conclusions. The basic hypotheses on A, L and N are lard down in section 2. The space H is then split into orthogonal subspaces V and W corresponding to the positive and negative parts of $\sigma(S)$. We also define the open interval (a, b) about 0 which contains the values of λ for which non-trivial solutions of (1.2) are sought. In the case where (1.2) is derived from (1.1) it is shown in section 8 that (a, b) is the maximal interval in $R \backslash \sigma(S)$ containing 0.

It is assumed that the function N has a convex potential $\varphi \in C^2(H, R)$ and this means that the equation (1.2) can be reduced to an equivalent problem on the subspace V. This is carred out in section 3 and underlies the fact that in section 7 bifurcation is investigated at the point b. However, by changing the signs of S and λ we can treat the case where the potential φ is concave and in that case the results would give criteria for bifurcation at a rather than b.

The reduced equation on V is studied using the method g the Mountain Pass. It amounts to finding non-zero critical points v of a functional $F(\lambda, v)$. The three main ingrediants are showing that $F(\lambda, \cdot)$ has a strict local minimum at $v = 0$, showing that $F(A, \cdot)$ has some kind of compactness and showing that $F(\lambda, v) < F(\lambda, 0)$ for some choice of v. The first two steps are carried out in section 4. Here we note that it is important to treat cases where the relevant compactness cannot be obtained from the Palais-Smale condition. We show how symmetry with respect to a group acting on H can play an important role. To obtain $\inf F(\lambda, \cdot) < F(\lambda, 0)$ we require additional properties of φ. In section 5 we show how this is established when $\varphi(u) > \varphi(0)$ for all large u in H. The case where we simply have $\varphi(u) \geq \varphi(0)$ is treated in section 6 by postulating the existence of suitable test-functions through the condition $T(\delta)$ which is also used in section 7 to formulate the criterion for bifurcation.

In section 9 the results for (1.2) are applied to the scalar nonlinear eigenvalue problem

$$-\Delta u(x) + V(x)u(x) - p(x)|u(x)|^{p-2}u(x) = \lambda u(x) \tag{1.3}$$
$$u \in H^2(R^N)$$

where $2 < p < 2^*$ with $2^* = \infty$ for $N = 1, 2$ and $2^* = \frac{2N}{N-2}$ for $N \geq 3$. It is assumed that V and $r \in L^\infty(R^N)$ with $r \geq 0$. In this context the basic Hilbert space H is the usual Sobolev space $H^1(R^N)$ and the solutions of (1.2) are weak solutions of (1.3) in the usual sense. We discuss the case where $r(x) \to 0$ as $|x| \to \infty$ as well as the case where V and r are periodic with the same period.

As a second application of the general results we consider in section 10 the Hamiltonian system

$$Ju'(t) + Mu(t) - ||K(t)u(t)||^{p-2}K(t)^T K(t)u(t) = \lambda u(t) \tag{1.4}$$

$$u \in [H^1(R)]^{2N}$$

where $p > 2$. Here J, M and $K(t)$ are real $2N \times 2N$ matrices with $J^{-1} = J^T = -J$, $M = M^T$ and $||K(t)|| \in K^\infty(R)$. In this context the basic Hilbert space H used to cast (1.4) into the form (1.2) turns out to be the fractional Sobolev space $[H^{1/2}(R)]^{2N}$. Our results cover cases where K has compact support as well as cases where K is periodic.

The discussion of the problem (1.2) using a global Lyapunov-Schmidt reduction followed by an application of the Mountain Pass Theorem is not new. This approach was used in [17] to obtain an existence result and more extensively by Buffoni [14] for both existence and bifurcation, in the case where the potential φ has the property that $\varphi(u) > 0$ for all $u \neq 0$. In these notes we have taken the opportunity to carry out the more intricate analysis that is required to treat the case where we only have $\varphi \geq 0$ on H. Here is also some simplication and sharpening of the results on bifurcation. The condition $T(\delta)$ which plays a fundamental role in the analysis is a slight modification of the one introduced in earlier work with Kpper and Heinz [12,13].

The results obtained in section 9 concerning (1.3) are similar to those in [13] for the case where $\tau(x) \to 0$ as $|x| \to \infty$ and to those in [15,16] for the case where τ is periodic. The work by Buffoni and Jeanjean [15,16] also covers situations where τ satisfies neither of these conditions. An alternative method of establishing existence results was initiated by Alama and Li [19] and has been extended by Jeanjean [18]. Heinz [10,11] has made considerable progress in obtaining information about the multiplicity of solutions of (1.3).

In this thesis Buffoni [14] derived results about the existence of homoclinic solutions of Hamiltonian systems from general results about (1.2). The discussion in section 10 refines this approach and extends it to cover bifurcation.

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2 Posing the problem

Let $(H, < \cdot, > \cdot)$ denote a real Hilbert space with $||u|| =< u, u >^{1/2}$. The space of bounded linear operators from H to itself is denoted by $B(H, H)$. For $T \in B(H, H) \sigma(T)$ denotes its spectrum and T^* the adjoint operator.
$(H1) A \in B(H, H), A = A^*$ and $0 \notin \sigma(A)$.
$(H2) L \in B(H, H), L = L^*$ and $< Lu, u >> 0 \quad \forall u \in H\{0\}$.
$(H3) \varphi \in C^2(H, R), \lim_{||u|| \to 0} \frac{\varphi(u)}{||u||}^2 = 0$ and φ is convex.

Setting $N = \varphi$ we have that $N \in C'(H, H)$ and $\varphi'(u)v =< N(u), v >$ $\forall u, v \in H$. Clearly $(H3)$ implies that $N(0) = 0$, $N'(0) = 0$ and $N'(u) = [N'(u)]^* \forall u \in H$. The convexity of φ ensures that

$$< N'(u)v, v >\geq 0 \quad \forall u, v \in H. \tag{2.1}$$

Our results concern solutions $(\lambda; u) \in R \times H$ of the equation

$$Au - \lambda Lu - N(u) = 0. \tag{2.2}$$

Since $(\lambda, 0)$ is a solution for every $\lambda \in R$, we set

$$E = \{(\lambda, u) \in R \times H : u \neq 0 \text{ and } Au - \lambda Lu = N(u)\}.$$

Defining $J : R \times H \to R$ by

$$J(\lambda, u) = \frac{1}{2}\lambda < (A - \lambda L)u, u > -\varphi(u),$$

it is clear that $(H1)$ to $(H3)$ imply that $J \in C^2(R \times H, R)$ and that (2.2) is equivalent to ${}_uJ(\lambda, u) = 0$.

The operator $A - \lambda L$ gives the dominant contribution to ${}_uJ(\lambda, u)$ near $u = 0$. Let

$$\rho(A, L) = \{\lambda \in R : A - \lambda L : H \to H \text{ is an isomorphism}\}$$

and $\sigma(A, L) = R \backslash \rho(A, L)$. Note that $\rho(A, L)$ is an open subset of R and that $0 \in \rho(A, L)$ by $(H1)$. In fact, by the spectral theorem for self-adjoint operators the statement that $0 \notin \sigma(A)$ is equivalent to the following assertion.

There exist a closed subspace V of H and two constants α and $\beta \in (0, \infty)$ such that
$(i) A(V) \subset V$
$(ii) < Au, u >\geq \beta ||u||^2 \quad \forall u \in V$ (2.3)
$(iii) < Au, u >\leq -\alpha ||u||^2 \quad \forall v \in V^\perp$. (2.4)

Setting $W = V^\perp$, it follows that $A(V) = V$ and that $A(W) = W$. The orthogonal projection of H onto V will be denoted by $P : H \to H$ with $I - P = Q$. Then A commutes with P and Q.

Using this decomposition, $H = V \oplus W$, we introduce the following quantities which play a fundamental role in the discussion of (2.2)

$$a = \begin{cases} \sup & \{\frac{<Au,u>}{<Lu,u>} : u \in W \text{ and } u \neq 0\} \\ -\infty \text{ if } W = \{0\} \end{cases}$$

$$b = \begin{cases} \inf & \{\frac{<Au,u>}{<Lu,u>} : u \in V \text{ and } u \neq 0\} \\ +\infty \text{ if } W = \{0\} \end{cases}$$

$$n(\lambda) = \begin{cases} \alpha(1 - \lambda/a) & \text{for } \lambda \leq 0 \\ \alpha & \text{for } \lambda > 0, \end{cases}$$

$$m(\lambda) = \begin{cases} \beta & \text{for } \lambda \leq 0 \\ \beta(1 - \lambda/b) & \text{for } \lambda > 0 \end{cases}$$

and finally

$$M(\lambda) = \min\{m(\lambda), n(\lambda)\}.$$

The following result shows how these quantities relate to the properties of the operator $A - \lambda L$.

Lemma 2.1. *The hypotheses* $(H1)$ *and* $(H2)$ *imply the following statements.*
(1) $a < 0 < b$.
(2) $< (A - \lambda L)u, u >\geq m(\lambda)||u||^2 \quad \forall u \in V$ *and* $\lambda \leq b$,
$< (A - \lambda L)u, u >\leq -n(\lambda)||u||^2 \quad \forall u \in W$ *and* $\lambda \geq a$,
$< (A - \lambda L)u, Pu - Qu >\geq M(\lambda)||u||^2 \quad \forall u \in H$ *and* $a \leq \lambda \leq b$.
(3) $||(A - \lambda L)u|| \geq M||u|| \quad \forall u \in H$ *and* $a \leq \lambda \leq b$,
$(a, b) \subset \rho(A, L)$.
(4) *If* $PL = LP$ *then* $\{a, b\} \cap \rho(A, L) = \phi$.

Proof (1) To show that $b > 0$ we may assume that $V \neq \{0\}$. For $u \in V$ with $u \neq 0$, $0 << Lu, u >\leq ||L||||U||$ and so, by (2.3),

$$\frac{< Au, u >}{< Lu, u >} \geq \frac{\beta}{||L||}.$$

Hence $b \geq \frac{\beta}{||L||} > 0$.
Similarly, $a \leq \frac{-\alpha}{||L||} < 0$.

(2) For $u \in V$ and $\lambda \leq 0$, $< (A - \lambda L)u, u >\geq< Au, u >\geq \beta||u||^2$.
For $u \in V$ and $\lambda > 0$,

$$< (A - \lambda L)u, u >\geq< Au, u > -\lambda\frac{<Au,u>}{b} =< Au, u > (1 - \lambda/b)$$

$$\geq \beta||u||^2(1 - \lambda/b) \text{ for } 0 < \lambda \leq b.$$

Hence $< (A - \lambda L)u, u >\geq m(\lambda)||u||^2$ for all $u \in V$ and $\lambda = b$.
The result for W is proved similarly.

For $u \in H$ and $a \leq \lambda b$,

$$< (A - \lambda L)u, Pu - Qu > = < (A - \lambda L)Pu, Pu > - < (A - \lambda L)Qu, Qu >$$

since $(A - \lambda L)^* = (A - \lambda L)$. Hence

$$\begin{aligned} < (A - \lambda L)u, Pu - Qu \quad &\geq m(\lambda)\|Pu\|^2 + n(\lambda)\|Qu\|^2 \\ &\geq M(\lambda)\|u\|^2 \text{ for } u \in H \text{ and } a \leq \lambda \leq b. \end{aligned}$$

(3) For $u \in H$ and $a \leq \lambda \leq b$,

$$M(\lambda)\|u\|^2 \leq \|(A - \lambda L)u\| \quad \|Pu - Qu\| = \|(A - \lambda L)u\| \quad \|u\|.$$

Thus $\|(A - \lambda L)u\| \geq M(\lambda)\|u\| \quad \forall u \in H$ and $a \leq \lambda \leq b$.
Since $M(\lambda) > 0$ for $a < \lambda < b$, it follows that $A - \lambda L : H \to H$ is an isomorphism if $\lambda \in (a, b)$.

(4) Let us suppose that $V \neq \{0\}$ and show that $b \in \sigma(A, L)$. By the definition of b there is a sequence $\{u_n\} \subset V$ such that $\|u_n\| = 1 \quad \forall n \in N$ and $\frac{<Au_n, u_n>}{<Lu_n, u_n>} \to b$. Hence $< (A - bL)u_n, u_n > \to 0$ since $| < Lu_n, u_n > | \leq \|L\| \quad \forall n \in N$. Setting $S = (A - bL)|_V$ and using the assumption that $PL = LP$ we see that $S \in B(V, V)$, $S = S^*$ and $< SU, U > \geq 0 \quad \forall u \in V$.
But $< Su_n, u_n > \to 0$ and so $0 = \inf\{< Su, u >: u \in V \text{ and } \|u\| = 1\}$. This implies that $0 \in \sigma(S)$ which is turn implies that there is a sequence $\{v_n\}$ in V such that $\|v_n\| = 1$ for all $n \in N$ and $\|Sv_n\| \to 0$. Thus $A - bL : H \to H$ cannot have a bounded inverse and consequently $b \in \sigma(A, L)$. The point a is treated similarly.

3 Global Lyapunov-Schmidt reduction

Using the decomposition $H = V \oplus W$, which is introduced in section 2, we replace (2.2) by an equivalent problem posed on the subspace V. The following reformulation of (2.2) is standard.
$(\lambda, u) \in R \times H$ and $_uJ(\lambda, u) = 0$

$$\iff \begin{cases} \lambda \in R, u = v + w \text{ where } \quad v \in V \text{ and } w \in W \text{ and} \\ P_u J(\lambda, v + w) = 0 \quad (3.1) \\ Q_u J(\lambda, v + w) = 0 \quad (3.2) \end{cases}$$

The equations (3.1) and (3.2) can be written as

$$(A - \lambda PL)v - \lambda PLw - PN(v + w) = 0 \tag{3.3}$$

and

$$(A - \lambda QL)w - \lambda QLv - QN(v + w) = 0. \tag{3.4}$$

Given $(\lambda, v) \in (a, \infty) \times V$, the equation (3.4) has a unique solution $g(\lambda, v) \in W$, as is shown in the following result.

Lemma 3.1. *Under the hypotheses* $(H1)$ *to* $(H3)$ *there is a unique function* $g \in C^1((a, \infty) \times V, W)$ *such that* $(\lambda, v, w) \in (a, \infty) \times V \times W$ *satisfies (3.2) if an only if* $(\lambda, v) \in (a, \infty) \times V$ *and* $w = g(\lambda, v)$.

Remark For future reference we note some properties of g.
$(i) g(\lambda, 0) = 0 \quad \forall \lambda > a,$
$(ii)_u J(\lambda, v + g(\lambda, v)) = P_u J(\lambda, v + g(\lambda, v)) \quad \forall (\lambda, v) \in (a, \infty) \times V$
$(iii)(\lambda, u) \in (a, \infty) \times H$ satisfies (2.2) if and only if $\lambda > a, Pu = v$ and $Qu = g(\lambda, v)$ where v satisfies

$$P_u J(\lambda, v + g(\lambda, v)) = 0 \tag{3.5}$$

(iv) The equation (3.5) can be written as

$$(A - \lambda PL)v - \lambda PLg(\lambda, v) - PN(v + g(\lambda, v)) = 0. \tag{3.6}$$

Proof Setting $G(\lambda, v, w) = Q_u J(\lambda, v + w)$ we have that

$$G \in C^1(R \times V \times W, W) \text{ and for } \lambda \in R, v \in V, w \text{ and } z \in W,$$

$D_w F(\lambda, v, w)z = (A - \lambda QL)z - QN'(v + w)z.$
Hence

$$< D_w F(\lambda, v, w)z, z >=< (A - \lambda L)z, z > - < N'(v + w)z, z >$$

$$\leq < (A - \lambda)Lz, z >$$

for $(\lambda, v, w) \in R \times V \times W$ and $z \in W$ by the convexity of φ.

From Lemma 2.1 (2) it follows that

$$< D_w G(\lambda, v, w)z, z > \leq -n(\lambda)||z||^2 \text{ for } \lambda \geq a$$

and hence $||D_w G(\lambda, v, w)z|| \geq n(\lambda)||z||$ where $n(\lambda) > 0$ for $\lambda > a$. Thus for $(\lambda, v, w) \in (a, \infty) \times V \times W$, $D_w G(\lambda, v, w) : W \to W$ is an isomorphism and $||[D_w G(\lambda, v, w)]^{-1}|| \leq \frac{1}{n(\lambda)}$ It follows from the global inverse function theorem [2, Theorem 1.22] that $G(\lambda, v, \cdot) : W \to W$ is a homeomorphism for all $(\lambda, v) \in (a, \infty) \times V$. In particular, $\forall (\lambda, v) \in (a, \infty) \times V$ there is a unique element $g(\lambda, v)$ of W such that $G(\lambda, v, g(\lambda, v)) = 0$. The regularity of the function g follows from the implicit function theorem.

According to Lemma 3.1 the study of (2.2) for $\lambda > a$ has been reduced to the study of the equation (3.5) for $v \in V$. The variational nature of (3.5) is exposed as follows.

Define $F : (a, \infty) \times V \to R$ by

$$F(\lambda, v) = J(\lambda, v + g(\lambda, v)) \text{ for } (\lambda, v) \in (a, \infty) \times V \tag{3.7}$$

where g is given by Lemma 3.1.

Corollary 3.2. *Under the hypotheses (H1) to (H3),*
$F \in C^1((a, \infty) \times V, R)$ and ${}_vF(\lambda, v) = {}_u J(\lambda, v + g(\lambda, v))$
for all $(\lambda, v) \in (a, \infty) \times V$.

Proof For $\lambda > a$ and $v, z \in V$, $<_v F(\lambda, v), z >= D_v F(\lambda, v)z$
$= D_u J(\lambda, v + g(\lambda, v)[z + D_v g(\lambda, v)z)]$
$=<_u J(\lambda, v + g(\lambda, v)), z + D_v g(\lambda, v)z >$
$=< P_u J(\lambda, v + g(\lambda, v)), z >$ since $Q_u J(\lambda, v + g(\lambda, v)) = 0$.
Hence ${}_vF(\lambda, v) = P_u J(\lambda, v + g(\lambda, v)) =_u J(\lambda, v + g(\lambda, v))$.

In establishing the further properties of F the following expression are useful.

For $(\lambda, v, w) \in R \times V \times W$,

$$J(\lambda, v + w) = \frac{1}{2}\{< (A - \lambda L)v, v > + < (A - \lambda L)w, w >\}$$
$$-\lambda < Lv, w > -\varphi(v + w) \tag{3.8}$$

whence, for $(\lambda, v) \in (a, \infty) \times V$,

$$F(\lambda, v) = \frac{1}{2}\{< (A - \lambda L)v, v\} + < (A - \lambda L)g, g >\}$$
$$-\lambda < Lv, g > -\varphi(v + g) \tag{3.9}$$

where g denotes $g(\lambda, v)$.
But for all $(\lambda, v) \in (a, \infty) \times V, Q_u J(\lambda, v + g) = 0$ and so

$$\begin{aligned} 0 &=< Q_u J(\lambda, v + g), g > \\ &=< (A - \lambda L)g, g > -\lambda < Lv, g > - < N(v + g), g > \quad (3.10) \end{aligned}$$

Using (3.9) and (3.10) we obtain the following expressions $F(\lambda, v) = \frac{1}{2}\{< (A - \lambda L)v, v > - < (A - \lambda L)g, g >\}$

$$+ < N(v + g), g > - \varphi(v + g) \tag{3.11}$$

$$= \frac{1}{2}\{< (A - \lambda L)v, v > - \lambda < Lv, g >$$

$$+ < N(v + g), g >\} - \varphi(v + g) \tag{3.12}$$

for all $(\lambda, v) \in (a, \infty) \times V$ where g denotes $g(\lambda, v)$.

4 Applying the Mountain Pass Theorem

In this section we establish conditions under which the existence of solutions of (2.2) can be obtained by applying the mountain pass theorem to $F(\lambda, \cdot) : V \to R$. This first step in this direction is to observe that for $\lambda \in (a, b)$, $F(\lambda, \cdot)$ has a strict local minimum at $v = 0$.

Lemma 4.1. *Let the conditions* $(H1)$ *to* $(H3)$ *be satisfied and consider* $\lambda \in (a, b)$. *There exists a radius* $\rho(\lambda) > 0$ *such that* $F(\lambda, v) \geq \frac{1}{4} m(\lambda) ||v||^2$ *for all* $v \in V$ *such that* $||v|| \leq \rho(\lambda)$.

Proof From (3.11) and Lemma 2.1, we have that

$$F(\lambda, v) \geq \frac{1}{2} < (A - \lambda L)v, v > + < N(v + g), g > -\varphi(v + g)$$

for all $(\lambda, v) \in (a, \infty) \times V$ where g denotes $g(\lambda, v)$.
By the convexity of φ,

$$\varphi(u) \geq \varphi(z) + < \varphi(z), u - z > \text{ for all } u, z \in H \tag{4.1}$$

Setting $u = v$ and $z = v + g$ we obtain

$$\varphi(v) \geq \varphi(v + g) - < N(v + g), g > \quad \forall v \in V$$

and hence

$$\begin{aligned} F(\lambda, v) & \geq \tfrac{1}{2} < (A - \lambda L)v, v > -\varphi(v) \\ & \geq \tfrac{1}{2} m(\lambda) ||v||^2 - \varphi(v) \end{aligned}$$

where $m(\lambda) > 0$ since $\lambda < b$. Using $(H3)$ the result now follows.

Let us now introduce the quantities lying at the heart of the mountain pass method.
Given $(\lambda, v) \in (a, b) \times V$, set

$$c(\lambda, v) = \inf_{\lambda \in \Gamma} \max_{\in [0,1]} F(\lambda, \gamma(t)) \tag{4.2}$$

where $\Gamma = \{\gamma \in C([0, 1], V) : \gamma(0) = 0 \text{ and } \gamma(1) = v\}$.
If $F(\lambda, v) < 0$ the mountain pass results establish the existence of a sequence $\{v_n\}$ in V such that $F(\lambda, v_n) \to c(\lambda, v)$ and $||_v F(\lambda, v_n)|| \to 0$. To deduce that such sequences are bounded we introduce the following hypothesis.
$(H4)$ There exist constants $C, D > 0$ such that

$$||N(u)|| \leq C + D\varphi(u) \quad \forall u \in H.$$

This leads to the fundamental result about (2.2) based on the mountain pass approach.

Lemma 4.2. *Let the conditions $(H1)$ to $(H4)$ be satisfied and suppose also that*

$$\exists p > 2 \text{ and } R > 0 \text{ such that } \varphi'(u)u \geq p\varphi(u)$$
$$\text{for all } u \in H \text{ with } \|u\| \geq R. \tag{4.3}$$

If $(\lambda, v) \in (a, b) \times V$ *is such that* $F(\lambda, v) < 0$, *then*

(i) $c(\lambda, v) > 0$ and
(ii) $\exists$ a bounded sequence $\{u_n\}$ in H such that

$$J(\lambda, u_n) \to c(\lambda, v) \text{ and } \|_u J(\lambda, u_n)\| \to 0.$$

Proof We consider a point $(\lambda, v) \in (a, b) \times V$ such that $F(\lambda, v) < 0$. By Lemma 4.1, $\|v\| > r(\lambda)$ and $c(\lambda, v) \geq \frac{1}{4}m(\lambda)r(\lambda)^2 > 0$. Applying Theorem 1 of [3] to $F(\lambda,) : V \to R$ we obtain a sequence $\{v_n\} \subset V$ such that $F(\lambda, v_n) \to c(\lambda, v)$ and $\|_v F(\lambda, v_n)\| \to 0$. Set $u_n = v_n + g_n$ where g_n denotes $g(\lambda, v_n)$.
Then $J(\lambda, u_n) = F(\lambda, v_n) \to C(\lambda, v)$ and by Corollary 3.2
$\|_u J(\lambda, u_n)\| = \|_v F(\lambda, v_n)\| \to 0$.
Let us show that $\{\|u_n\|\}$ is a bounded sequence. Since $\lambda \in (a, b), M(\lambda) > 0$ and for all $n \in N$,

$$\begin{aligned} M(\lambda)\|u_n\| &\leq \|(A - \lambda L)u_n\| \quad \text{by Lemma 2.1} \\ &= \|_u J(\lambda, u_n) + N(u_n)\| \\ &\leq \epsilon_n + C + D\varphi(u_n) \quad \text{by } (H4), \end{aligned}$$

where $\epsilon_n = \|_u J(\lambda, u_n)\|$.
If $\|u_n\| \geq R$, it follows from (4.3) that

$$\begin{aligned} (p - 2)\varphi(u_n) &\leq < N(u_n), u_n > - \quad 2\varphi(u_n) \\ &= 2J(\lambda, u_n) - <_u J(\lambda, u_n), u_n > \\ &\leq 2J(\lambda, u_n) + \epsilon_n\|u_n\| \end{aligned}$$

and hence,

$$M(\lambda)\|u_n\| \leq \epsilon_n + C + \frac{D}{(p - 2)}\{2J(\lambda, u_n) + \epsilon_n\|u_n\|\}$$

Since $\epsilon_n \to 0$ and $J(\lambda, u_n) \to C(\lambda, v)$ it follows that $\{\|u_n\|\}$ is bounded.

The next issue to be confronted concerns the kind of compactness that is required to obtain a solution of (2.2) from the sequence given by Lemma 4.2. The most straight forward cases are those where the Palais-Smale condition is satisfied. However in other situations this fails because the functional is invariant with respect to a group whose orbits are not compact. To deal with some cases of this kind we introduce the following terminology.

Let $O(H)$ denote the groupe (with respect to composition) of all isometric isomorphisms of H. Given a subgroup G of $O(H)$, $\theta(u) = \{Tu : T \in G\}$ is the orbit containing $u \in H$ generated by G.

Consider a functional $K \in C^1(H,R)$. It is called *G-invariant* if and only if $K(Tu) = K(u) \quad \forall u \in H, T \in G$. In this case it follows that $K'(Tu)Tv = K'(u)v \quad \forall u,v \in H$ and so $T^*K(Tu) = K(u) \quad \forall u \in H, T \in G$. Thus K is G-equivariant and we note that $\forall u \in H$ and $v \in \theta(u)$, $K(u) = K(v)$ and $||K(u)|| = ||K(v)||$.

Definition. *Given $K \in C^1(H,R)$ and a subgroup G of $O(H)$, we say that K is* weakly G-compact *provided that*
(1)K is G-invariant,
(2) from every bounded sequence $\{u_n\}$ in H such that $K(u_n) \to c \neq K(0)$ and $||K(u_n)|| \to 0$ we can extract a subsequence $\{u_{n_i}\}$ and select elements $v_n \in \theta(u_n)$ such that $v_{n_i} \rightharpoonup v$ weakly in H where $v \neq 0$ and $K(V) = 0$

Remarks 1. In dealing with G-invariant functionals that do not satisfy the $P-S$ condition, one commonly adopted approach is to consider the subspace $H_g = \{u \in H : \theta(u) = \{u\}\}$ and to require that the restriction of the functional to H_g alors satisfy the $P-S$ condition. This method is often successful in treating problems with roational symmetry. However in other cases we can have $H_g = \{0\}$ and the method fails. Indeed in the case we treat in section 8 involving invariance under translation, it turns out that for all $u \in H\backslash\{0\}$, $\theta(u)$ is non-compact. In such cases we can deal with functionals that are weakly G-compact.

2. Let us emphasize that the essential point in part (3) of the above definition is the fact that the weak limit v is non-zero.

3. The relationship between cases (1) and (2) in the next result is clarified by the observation that in case (1), $J(\lambda,\cdot) : H \to R$ satisfies the Palais-Smale condition. Case (2) is a weakening of this condition.

Before stating the main result of this section it is convenient to make some observations about the function φ. By $(H3)$, $\varphi(0) = 0$ and so setting $z = 0$ in (4.1) we see that

$$\varphi(u) \geq 0 \quad \forall u \in H. \tag{4.4.}$$

Noting that

$$2J(\lambda,u) - <_u J(\lambda,u), u> = < N(u), u > -2\varphi(u) \tag{4.5}$$

for all $(\lambda,u) \in R \times H$, we define $\psi : H \to R$ by

$$\psi(u) = < N(u), u > -2\varphi(u) \quad \forall u \in H \tag{4.6}$$

Theorem 4.3. *Let the hypotheses of Lemma 4.2 be satisfied.*
(1) If $N : H \to H$ is compact, then for every $\lambda \in (a,b)$ there exists $u_\lambda \in H\backslash\{0\}$ such that ${}_uJ(\lambda,u_\lambda) = 0$ and $J(\lambda,u_\lambda) = c(\lambda,v)$.
(2) If there is a subgroup G of $O(H)$ such that $J(\lambda,\cdot) : H \to R$ is weakly G-compact for a $\lambda \in (a,b)$, then there exists $u_\lambda \in H\backslash\{0\}$ such that ${}_uJ(\lambda,u_A) = 0$.

If, in addition, $\psi : H \to R$ is weakly sequentially lower semi-continuous, then $J(\lambda, u_\lambda) \leq c(\lambda, v)$.

Proof We consider the sequence $\{u_n\}$ that is given by Lemma 4.2.
(1) By passing to a subsequence we may suppose that $u_n \rightharpoonup u$ weakly in H and that $\|N(u_n) - w\| \to 0$ where $u, w \in H$. Since $\lambda \in (a, b)$, it follows from Lemma 2.1 that $A - \lambda L : H \to H$ is an isomorphism and so $\exists z \in H$ such that $(A - \lambda L)z = w$. Also $M(\lambda) > 0$ and

$$M(\lambda)\|u_n - z\| \leq \|(A - \lambda L)(u_n - z)\| = \|(A - \lambda L)u_n - w\|$$

$$= \|{}_u J(\lambda, u_n) + N(u_n) - w\| \leq \|{}_u J(\lambda, u_n)\| + \|N(u_n) - w\|.$$

Hence $\|u_n - z\| \to 0$ and $z = u$. It follows that $\|{}_u J(\lambda, u)\| = \lim \|{}_u J(\lambda, u_n)\| = 0$ and that $J(\lambda, u) = \lim J(\lambda, u_n) = c(\lambda, v)$.

(2) Note that $J(\lambda, u_n) \to c(\lambda, v) > 0$ and that $J(\lambda, 0) = 0$.
By passing to a subsequence we may suppose that there exist $u_\lambda \in H \ \{0\}$ and $v_n \in \theta(u_n)$ such that $v_n \rightharpoonup u_\lambda$ weakly in H and ${}_u J(\lambda, u_\lambda) = 0$.
By (4.5), $2J(\lambda, u_\lambda) = \psi(u_\lambda)$.
If ψ is weakly sequentially lower semi-continuous,

$$\begin{aligned} \psi(u_\lambda) &\leq \liminf \psi(v_n) = \liminf\{2J(\lambda, v_n) - <_u J(\lambda, v_n), v_n >\} \\ &= 2c(\lambda, v) \end{aligned}$$

Since $J(\lambda, v_n) = J(\lambda, u_n) \to 0$, $\|J_u(\lambda, v_n)\| = \|J_u(\lambda, u_n)\| \to 0$ and $\|v_n\| = \|u_n\|$. Hence $J(\lambda, u_n) \leq c(\lambda, v)$.

5 Existence for asymptotically definite non-linearities

To complete the discussion in Section 4 we must give explicit conditions ensuring the existence of an element (λ, v) such that $F(\lambda, v) < 0$. This can be done by requiring φ to be asymptotically definite in the following sense.
$(AD) V \neq \{0\}$ and $\exists R > 0$ such that $\varphi(u) > 0 \quad \forall u \in H$ with $||u|| \geq R$.
We recall that $(H3)$ already implies that $\varphi(u) \geq 0 \quad \forall u \in H$.

Lemma 5.1. *Let the conditions $(H1)$ to $(H3)$, (AD) and (4.3) be satisfied. Choose $\xi \in (a, b)$ and a finite dimensional subspace Z of V. Then there exists $M > 0$ (depending only on ξ and Z) such that $F(\lambda, v) < 0$ for all $\lambda \in [\xi, b)$ and all $v \in Z$ with $||v|| \geq M$.*

Proof Since $V \neq \{0\}$, $b < \infty$ and we set $\delta = \max\{|\xi|, b\}$.
If the conclusion is false, $\forall n \in N \quad \exists v_n \in Z$ and $\lambda_n \in [\xi, b)$ such that $||v_n|| \geq n$ and $F(\lambda_n, v_n) \geq 0$.
Setting $g_n = g(\lambda_n, v_n)$, it follows from (3.9) that

$$-\frac{1}{2} < (A - \lambda_n L)g_n, g_n > +\varphi(v_n + g_n) \leq \frac{1}{2} < (A - \lambda_n L)v_n, v_n >$$
$$-\lambda_n < Lv_n, b_n > .$$

Using Lemma 2.1 this yields

$$\frac{1}{2} n(\lambda_n)||g_n||^2 + \varphi(v_n + g_n) \leq (||A|| + \delta||L||)||v_n||^2 + \delta||L|| \, ||v_n|| \, ||g_n||.$$

Since $n(\lambda_n) \geq n(\xi) > 0$, we deduce easily that $\exists C > 0$ such that

$$||g_n||^2 \leq ||g_n||^2 + \varphi(v_n + g_n) \leq C||v_n||^2 \quad \forall n \in N.$$

Let $R > 0$ be choosen so that both (AD) and (4.3) are satisfied for all $u \in H$ with $||u|| \geq R$. By passing to a subsequence we may suppose that

$$\frac{Rv_n}{||v_n||} \rightharpoonup v \text{ weakly in } H \text{ and } \frac{Rg_n}{||v_n||} \rightharpoonup w \text{ weakly in } H.$$

Then $v \in Z$, $w \in W$ and since $\dim Z < \infty$, $||\frac{Rv_n}{||v_n||} \rightharpoonup v|| \to 0$.
Hence $||v|| = R$ and $||v + w||^2 = ||v||^2 + ||w||^2 \geq R^2$. From (AD) we conclude that $\varphi(v + w) > 0$. However the convexity of φ implies that it is weakly sequentially lower semi-continuous on H and so $\varphi(v + w) \leq \liminf \varphi(\frac{R}{||v_n||}[v_n + g_n])$.
Hence $\exists n_0 \in N$ such that

$$\varphi(\frac{R[v_n + g_n]}{||v_n||} \geq \frac{1}{2}\varphi(v + w) > 0 \quad \forall n \geq n_0.$$

But (4.3) implies that for any $u \in H$ with $||u|| \geq R$, $\frac{\varphi(tu)}{t^p}$ is non-decreasing function of t on $[1, \infty)$.

Since $\quad ||\frac{R}{||v_n||}[v_n + g_n|||^2 = \frac{R^2}{||v_n||^2}[||v_n||^2 + ||g_n||^2] \geq R^2$
this means that for all $t \geq 1$,

$$\varphi(\frac{tR}{||v_n||}[v_n + g_n]) \geq t^p \varphi(\frac{R}{||v_n||}[v_n + g_n])$$

and since $||v_n|| \geq n$, for all large n we can set $t = \frac{||v_n||}{R}$ and thus obtain

$$\begin{array}{ll} \varphi(v_n + g_n) & \geq (\frac{||v_n||}{R})^p \quad \varphi(\frac{R}{||v_n||}[v_n + g_n]) \\ & \frac{1}{2}(\frac{||v_n||}{R})^p \quad \varphi(v + w) \text{ for all } n \geq n_0 \end{array}$$

provided that n_0 is choosen large enough.
However we showed earlier that $\varphi(v_n + g_n) \leq C||v_n||^2 \quad \forall n \in N$,
so we now have that

$$\varphi(v + w) \leq 2R^p ||v_n||^{2-p} C \quad \forall n \geq n_0.$$

This implies that $\varphi(v + w) \leq 0$ because $p > 2$ and $||v_n|| \geq n$, contradicting the fact that $\varphi(v + w) > 0$. This completes the proof.

As an immediate consequence of Theorem 4.3 and Lemma 5.1 we obtain the following existence theorem for the equation (2.2).

Theorem 5.2. *Let the conditions $(H1)$ to $(H4)$, (AD) and (4.3) be satisfied. Suppose that either*
(1)$N : H \to H$ is compact, or
(2) there is a subgroup G of $O(H)$ such taht $J(\lambda, \cdot)$ is weakly G-compact for every $\lambda \in (a, b)$. Then for every $\lambda \in (a, b)$ there exists $u_\lambda \in H \backslash \{0\}$ such that ${}_u J(\lambda, u_\lambda) = 0$.

6 Existence for semi-definite non-linearities

In order to avoid the condition (AD) we introduce an alternative criterion based on test-functions. For $\delta > 0$ we say that the condition $T(\delta)$ is satisfied if

(1) $PL = LP$, and

(2) there is a sequence $\{u_n\}$ in H such that $||u_n|| = 1$, $\varphi(u_n) > 0$ for all $n \in N$ and

$$\lim_{n\to\infty} \frac{<(A-bL)u_n, u_n>}{\varphi(u_n)^\delta} = \lim_{n\to\infty} \frac{||(A-bL)u_n||^2}{\varphi(u_n)^\delta} = 0.$$

Remarks 1. Part (1) asserts that V is invariant for A and L. If there exists an eigenvector $u \in ker(A-bL)$ with $\varphi(u) > 0$ then part (2) is trivially satisfied for all $\delta > 0$. However the condition can also be satisfied for some values of δ even when $\ker(A-bL) = \{0\}$ as will be seen in section 7.

2. The test-functions $\{u_n\}$ in $T(\delta)$ will be used to establish the existence of $(\lambda, v) \in (a,b) \times V$ with $F(\lambda, v) < 0$.

To exploit the dondition $T(\delta)$ we replace the hypothesis (4.3) by a global pinching condition.

(P) There exist $q \geq p > 2$ such that

$$q\varphi(u) \geq \varphi'(u)u \geq p\varphi(u) \geq 0 \quad \forall u \in H.$$

From (P) it follows easily that for all $u \in H$,

$\frac{\varphi(tu)}{t^p}$ is non-decreasing on $(0,\infty)$ and $\frac{\varphi(tu)}{t^q}$ is non-increasing on $(0,\infty)$.

Lemma 6.1. *Let the conditions* $(H3)$ *and* (P) *be satisfied and set* $M = \sup\{\varphi(u) : u \in H$ *and* $||u|| = 1\}$.

(1) *We have* $M < \infty$.

For $u \in H$ *we set* $\rho(u) = \{\frac{\varphi(u)}{2^q M}\}^{1/p}$.

(2) *For all* $u \in H$ *with* $||u|| = 1$, *we have that* $0 \leq \rho(u) \leq \frac{1}{2}$ *and* $\varphi(v) \geq 2^{-q}\varphi(u) \quad \forall v \in H$ *with* $||u - v|| \leq \rho(u)$.

Proof (1) By $(H3), \exists\delta \in (0,1)$ such that $0 \leq \varphi(u) \leq ||ul^2$ for all $u \in H$ with $||u|| \leq \delta$. Using (P) it follows that $\varphi(\delta u) \geq \delta^q\varphi(u)$ and so for $u \in H$ with $||u|| = 1$, $0 \leq \varphi(u) \leq \delta^{-q}\varphi(\delta u) \leq \delta^{2-q}$. Hence $M \leq \delta^{2-q}$.

(2) Using (P) and the convexity of φ, for all $u, v \in H$, $0 \leq \varphi(u+v) \leq \frac{1}{2}\{\varphi(2u) + \varphi(2v)\} \leq 2^{q-1}\{\varphi(u) + \varphi(v)\}$

and so

$$\varphi(u) \geq 2^{1-q}\varphi(u+v) - \varphi(v) \quad \forall u, v \in H \tag{6.1}$$

For $u \in H$ with $||u|| = 1$, $0 \leq \rho(u)\rho(\frac{1}{2})^{q/p} \leq \frac{1}{2}$ and if $||z|| \leq \rho(u)$, it follows from (P) that

$$0 \leq \varphi(z) \leq ||z||^p M \leq \rho(u)^p M = 2^{-q}\varphi(u).$$

Thus for $u, z \in H$ with $||u|| = 1$ and $||z|| \leq \varphi(u)$,

$$\varphi(u-z) \geq 2^{1-q}\varphi(u) - \varphi(z) \geq 2^{-q}\varphi(u).$$

This proves the lemma.
Remarks 1. If (P) and $T(\delta)$ are satisfied the sequence $\{u_n\}$ in part (2) of $T(\delta)$ has the property that $0 < \varphi(u_n) \leq M$. Hence $T(\mu)$ is satisfied for all $\mu \in (0, \delta]$.

2. The condition $T(\delta)$ enables us to construct a useful family of test-functions in the subspace V.

Lemma 6.2. *Let the conditions $(H1)$ to $(H3)$ and (P) be satisfied. Suppose that the condition $T(\delta)$ holds for a $\delta \geq 2/p$ whre p is the constant in (P). Then there is a sequence $\{v_n\}$ in V such that $\|v_n\| = 1$, $\varphi(v_n) < 0 \quad \forall n \in N$ and* $\lim_{n\to\infty} \frac{<(A-bL)v_n, v_n>}{\varphi(v_n)^\delta} = 0$.

Proof Let $\{u_n\}$ be sequence in H given by the condition $T(\delta)$. Set $z_n = Pu_n$. For $n \in N$, $< (A - bL)u_n, Qu_n > = < (A - bL)Qu_n, Qu_n >$ since $QL = LQ$. From Lemma 2.1 it follows that

$$< (A - bL)u_n, Qu_n > \leq -n(b)\|Qu_n\|^2$$

and hence

$$\|(A - bL)u_n\| \geq n(b)\|Qu_n\| \text{ for all } n \in N. \tag{6.2}$$

Since the condition $T(\delta)$ is satisfied with $\delta = 2/p$ there is a sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and $\|(A - bL)u_n\| \leq \epsilon_n \varphi(u_n)^{1/p}$. Hence $\|Qu_n\| \leq \frac{\epsilon_n}{n(b)}\varphi(u_n)^{1/p}$ for all $n \in N$ and there exists $n_0 \in N$ such that $\|Qu_n\| \leq \rho(u_n)$ for all $n \geq n_0$, the $\rho(u)$ is the radius defined in Lemma 6.1. It follows from Lemma 6.1 (2) that

$$\varphi(Pu_n) \geq 2^{-q}\varphi(u_n) \text{ for all } n \geq n_0.$$

Furthermore since $0 \leq \varphi(u_n) \leq M$ it follows from $T(\delta)$ with $\delta = 2/p$ that $\|(A - bL)u_n\| \to 0$ as $n \to \infty$. Thus (6.2) implies that $\|Qu_n\| \to 0$ and we may suppose that $\|Pu_n\| \geq \frac{1}{2}$ for all $n \geq n_0$.
Hence forth we consider $n \geq n_0$ and we set $v_n = \frac{Pu_n}{\|Pu_n\|}$.
Then $\varphi(v_n) \leq \|Pv_n\|^{-p}\varphi(Pv_n)$ by (P)

$$\geq \varphi(Pv_n) \geq 2^{-q}\varphi(u_n) > 0$$

and $\frac{|<(A-bL)v_n, v_n>|}{\varphi(v_n)^\delta} \leq \frac{\|Pu_n\|^{-2}|<(A-bL)Pu_n, Pu_n>|}{2^{-q\delta}\varphi(u_n)^\delta}$

$$\leq\leq 2^{2+q\delta}\frac{< (A - bL)Pu_n, Pu_n > |}{\varphi(u_n)^\delta}.$$

But $| < (A - bL)Pu_n, Pu_n > |$

$$\leq | < (A - bL)u_n, u_n > | + | < (A - bL)Qu_n nQu_n > |$$

and $| < (A - bL)Qu_n, Qu_n > | \leq \|(A - bL)u_n\| \quad \|Qu_n\|$
$\leq \frac{\|(A-bL)u_n\|^2}{n(b)}$ by (6.2).
Hence it follows from $T(2/p)$ that $\frac{<(A-bL)Pu_n, Pu_n>}{\varphi(u_n)^\delta} \to 0$
and the proof is complete.

Lemma 6.3. *Suppose that the conditions $(H1)$ to $(H3)$, (P) and $T(2/p)$ are all satisfied where p is the constant in (P). Then there exist $\lambda^* \in (a,b), T > 0$ and $v \in V$ with $||v|| = 1$ such that $F(\lambda, tv) < 0$ for all $\lambda \in (\lambda^*, b)$ and $t > T$.*

Proof Let M be the constant defined in Lemma 6.1 and recall that $n(b) = \alpha > 0$. By Lemma 6.2 there exists a $v_n \in V$ such that $||v_n|| = 1$, $\varphi(v_n) > 0$ and

$$\frac{< (A - bL)v_n, v_n >}{\varphi(v_n)^{2/p}} \leq \frac{\alpha}{2} \frac{1}{2^\alpha M}^{2/p}.$$

We set $v = v_n$ and note that

$$0 \leq < (A - bL)v, v > \leq \frac{\alpha}{2} \{\varphi(v_n)^{2^q M}\}^{2/p} = \frac{\alpha}{2} \rho(v)^2$$

where $\rho(v)$ is the radius defined in Lemma 6.1.
Let $\lambda^+ = b - \frac{\gamma p(v)^2}{2||L||}$ where $\gamma = \min\{\alpha, \beta\}$.
Then $0 \leq \lambda^* < b$ since $b \geq \frac{\beta}{||L||}$ by Lemma 2.1, $0 \leq t(v) \leq \frac{1}{2}$ by Lemma 6.1 and $\varphi(v) > 0$.
If the conclusion of the lemma is false, there exist sequences $\{t_n\}$ and $\{\lambda_n\}$ such that

$$\lambda^* < \lambda_n < b, t_n \geq n \text{ and } F(\lambda_n, t_n v) \geq 0 \quad \forall n \in N.$$

Setting $g_n = g(\lambda_n, t_n v)$ and using (3.9) as in the proof of Lemma 5.1 we find that $\forall n \in N$

$$\frac{1}{2} n(\lambda_n) ||g_n||^2 + \varphi(t_n v + g_n) \leq \frac{1}{2} < (A - \lambda_n L)v, v > t_n^2 \tag{6.3}$$

since $< Lv_n, g_n >= 0$ by $T(2/p)$, part (1).
Thus

$$\begin{aligned} \frac{\alpha ||g_n||^2}{t_n^2} &\leq < (A - bL)v, v > + (b - \lambda_n) < Lv, v > \\ &\leq \frac{\alpha}{2} \rho(v)^2 + (b - \lambda_n) ||L|| \leq \alpha \rho(v)^2 \end{aligned}$$

since $(b - \lambda^*) ||L|| \leq \frac{\alpha}{2} (v)^2$ by the definition of λ^*. By passing to a subsequence we may assume that there exists $w \in W$ such that $\frac{g_n}{t_n} \rightharpoonup w$ weakly in H where $||w|| \leq \rho(v)$. It follows from Lemma 6.1 that $\varphi(v + w) \geq 2^{-q} \varphi(v) > 0$.
The convexity of φ implies tht it is weakly sequentially lower semi-continuous and so $\varphi(v + w) \leq \liminf \varphi(v + \frac{g_n}{t_n})$.
Hence there exists $n_0 \in N$ such that

$$\varphi(v + \frac{g_n}{t_n}) \geq \frac{1}{2} \varphi(v + w) \qquad \forall n \geq n_0.$$

Using (P) we also have $\varphi(t_n v + g_n) \geq t_n^p \varphi(v + \frac{g_n}{t_n})$ and by combining this inequalities we have that

$$\varphi(t_n v + g_n) \geq \frac{1}{2} t_n^p \varphi(v + w) \geq 2^{-(q+1)} \varphi(v) t_n^p \quad \forall n \geq n_0.$$

On the other hand, $\varphi(t_n v + g_n) \leq \frac{1}{2}||A||t_n^2$ by (6.3) since $\lambda_n \geq 0$ and $||v|| = 1$. Thus

$$2^{-(q+1)}\varphi(v) \geq \frac{1}{2}||A||t_n^{2-p} \quad \forall n \geq n_0$$

where $p > 2$ and $t_n \geq n$. This implies that $\varphi(v) \leq 0$ contradicting the fact that $\varphi(v) > 0$.
This proves the lemma.

The following result is an immediate consequence of Theorem 4.3 and Lemma 6.3.

Theorem 6.4. *Suppose that the conditions* $(H1)$ *to* $(H4)$, (P) *and* $T(2/p)$ *are satisfied where* p *is the constant in* (P) *suppose that either*
(1) $N : H \to H$ *is compact, or*
(2) *there is a subgroup* G *of* $O(H)$ *such that* $J(\lambda, \cdot) : H \to R$ *is weakly* G-*compact for all* $\lambda \in (a, b)$. *Then there is a* $\lambda^* \in (a, b)$ *such that for every* $\lambda \in (\lambda^*, b)$ *there is a* $u_\lambda \in H$ $\{0\}$ *such that* ${}_u J(\lambda, u_\lambda) = 0$.

7 Bifurcation

The conclusions of Lemma 6.3 can be strenghthened in a significant way if the condition $T(\delta)$ is satisfied for a number $\delta \geq 1$.

Lemma 7.1. *Let the conditions* $(H1)$ *to* $(H3)$ *and* (P) *be satisfied. Suppose that the condition* $T(\delta)$ *is satisfied for a number* $\delta \geq 1$. *Set*
$q_n = \frac{\min}{\|L\|}\{b\|L\|, \frac{\alpha}{2}(v_n)^2, 2^{-(q+1)}\varphi(v_n), \varphi(v_n)^\delta\}$
where $\{v_n\}$ *is the sequence given by Lemma 6.2,* $\rho(v_n)$ *is the radius defined in Lemma 6.1 and* α *is the constant in (2.4).*

(1) We have that $0 < q_n \leq b$ and there exists $n_0 \in N$ such that $F(\lambda, v_n) < 0$ for all $n \geq n_0$ and $\lambda \in (b - q_n, b)$.
(2) For $n \geq n_0$ and $\lambda \in (b - q_n, b)$,

$$0 < c(\lambda, v_n) \leq K(q) < (A - \lambda L)v_n, v_n >^{\frac{q}{q-2}} \varphi(v_r)^{\frac{-2}{q-2}}$$

where $C(\lambda, v_n)$ is defined by (4.2) and q is the constant in (P). The constant $K(q)$ depends only on q.
(3) There exists a sequence $\{\lambda_n\}$ such that $\lambda_n \in (b - q_n, b)$ for all $n \in N$, $\lambda_n \to b$ and

$$\lim_{n\to\infty} \frac{c(\lambda_n, v_n)}{(b - \lambda_n)^{1+\theta}} = 0 \text{ where } \theta = \frac{2}{(q-2)}[1 - \frac{1}{\delta}].$$

Proof (1) Since $\varphi(v_n) > 0$ it follows that $q_n > 0$ for all $n \in N$. By Lemmas 6.1 and 6.2,

$$\lim_{n\to\infty} \frac{< (A - bL)v_n, v_n >}{\varphi(v_n)^\mu} = 0 \text{ for all } \mu \in [0, \delta].$$

Hence there exists $n_0 \in N$ such that for all $n \geq n_0$,

$$\frac{< (A - bL)v_n, v_n >}{\varphi(v_n)} \leq 2^{-(q+1)}, \tag{7.1}$$

$$\frac{< (A - bL)v_n, v_n >}{\varphi(v_n)^{2/p}} \leq \frac{\alpha}{2}\{\frac{1}{2^q M}\}^{2/p} \tag{7.2}$$

where M is given by Lemma 6.1,

$$< (A - \lambda L)v_n, v_n > \leq b\|L\| \text{ and } \frac{< (A - bL)v_n, v_n >}{\varphi(v_n)^\delta} \leq 1. \tag{7.3}$$

Let $\lambda \in (b - q_n, b)$ and set $g_n = g(\lambda, v_n)$.
To prove the first part of the lemma we suppose that $F(\lambda, v_n) \geq 0$ and then obtain a contradiction.
As in the proof of Lemma 5.1, $F(\lambda, v_n) \geq 0$ implies that

$$\frac{1}{2}n(\lambda)\|g_n\|^2 + \varphi(v_n + g_n) \leq \frac{1}{2} < (A - \lambda L)v_n, v_n > .$$

Since $n(\lambda)=\alpha$ and, by (7.2), $<(A-bL)v_n,v_n>\leq \frac{\alpha}{2}\rho(v_n)^2$
we obtain

$$\begin{aligned}\alpha\|g_n\|^2 &\leq <(A-bL)v_n,v_n>+(b-\lambda)<Lv_n,v_n> \\ &\leq \tfrac{\alpha}{2}(v_n)^2+q_n\|L\|\leq \alpha\rho(v_n)^2\end{aligned}$$

By Lemma 6.1 this implies that $\varphi(v_n+g_n)\geq 2^{-q}\varphi(v_n)>0$.
But then

$$\begin{aligned}0 &\leq F(\lambda,v_n)\\ &=\tfrac{1}{2}\{<(A-\lambda L)v_n,v_n>+<(A-\lambda F)g_n,g_n>\}-\varphi(v_n+g_n)\\ &\leq \tfrac{1}{2}<(A-L)v_n,v_n>-\varphi(v_n+g_n) \text{ by Lemma 2.1}\\ &\leq \tfrac{1}{2}<(A-b\lambda)v_n,v_n>+\tfrac{1}{2}(b-\lambda)\|L\|-2^{-q}\varphi(v_n)\\ &\leq 2^{-(q+2)}\varphi(v_n)+\tfrac{1}{2}q_n\|L\|-2^{-q}\varphi(v_n) \text{ by (7.1)}\\ &\leq -2^{-(q+1)}\varphi(v_n)<0 \text{ by the definition of } q_n.\end{aligned}$$

(2) Clearly $c(\lambda,v_n)\leq \max_{t\in[0,1]}F(\lambda,tv_n)$. By Lemma 4.1 and part (1) we have that $\max_{t\in[0,1]}F(\lambda,tv_n)>0$ and there exists $\hat{t}\in(0,1)$ such that $F(\lambda,\hat{t}v_\gamma)=\max_{t\in[0,1]}F(\lambda,tv_n)$.
As in part (1) from $F(\lambda,\hat{t}v_n)>0$ we deduce that

$$\alpha\|g_n\|^2\leq <(A-\lambda L)v_n,v_n>\hat{t}^2 \text{ where } g_n=g(\lambda,\hat{t}v_n)$$

and hence

$$\alpha(\frac{\|g_n\|}{\hat{t}})^2\leq <(A-bL)v_n,v_n>+(b-\lambda)\|L\|\leq \alpha\rho(v_n)^2$$

Thus $\varphi(v_n+\frac{g_n}{\hat{t}})\geq 2^{-q}\varphi(v_n)>0$ by Lemma 6.1.
Using (P) this yields $\varphi(\hat{t}v_n+g_n)\geq \hat{t}^q\varphi(v_n+g_n/\hat{t})\geq(\frac{\hat{t}}{2})^q\varphi(v_n)$ and so

$$\begin{aligned}F(\lambda,\hat{t}v_n) &\leq \tfrac{1}{2}<(A-\lambda L)v_n,v_n>\hat{t}^2-\varphi(\hat{t}v_n+g_n)\\ &\leq k\hat{t}^2-\ell\hat{t}^q\end{aligned}$$

where $k=\frac{1}{2}<(A-\lambda L)v_n,v_n>$ and $\ell=2^{-q}\varphi(v_n)$.

Hence $c(\lambda,v_n)\leq \max_{t\in[0,1]}\{kt^2-\ell t^q\}=Kk^{q/q-2}\ell^{-2/q-2}$

where $K=(\frac{2}{q}^{\frac{2}{q-2}})\{1-(\frac{2}{q})^{(\frac{q}{2})}\}$.
(3) For $n\geq n_0$ we define λ_n as follows,

$$\lambda_n=\begin{cases} b-\frac{1}{4\|L\|}<(A-bL)v_n,v_n> & \text{if } <(A-bL)v_n,v_n>>0\\ b-\frac{1}{n}q_n & \text{if } <(A-bL)v_n,v_n>=0\end{cases}$$

Clearly $\lambda_n<b$ and $\lambda_n\to b$. Recalling (7.1) to (7.3) we see that $\lambda_n>b-q_n$.
By part (2) we have that $0<c(\lambda_n,v_n)\leq Kk_n^{\frac{q}{q-2}}\ell_n^{\frac{2}{q-2}}$ where $k_n=<(A-\lambda_nL)v_n,v_n>$, $\ell_n=\varphi(v_n)^{-1}$ and K is independent of n.

Setting $\gamma = \frac{2}{(q-2)\delta}$ and $\theta = \frac{2}{(q-2)}[1-\frac{1}{\delta}]$, this yields $0 < c(\lambda_n, v_n) \leq K k_n^{H_\theta}\{k_n \ell_n\}^\delta$.
But

$$
\begin{aligned}
0 \leq k_n \quad & \leq < (A-bL)v_n, v_n > +(b-\lambda_n)||L|| \\
& \leq 5(b-\lambda_n)||L|| \text{ if } < (A-bL)v_n, v_n >> 0
\end{aligned}
$$

and $0 \leq k_n \leq (b-\lambda_n)||L|| \leq \frac{1}{n} q_n ||L||$ if $< (A-bL)v_n, v_n >= 0$.
Hence

$$
c(\lambda_n, v_n) \leq \begin{cases} K\{5(b-\lambda_n)||L||\}^{1+\theta}\{k_n \ell_n^\delta\}^\gamma \text{ if } < (A-bL)v_n, v_n >> 0 \\ K\{(b-\lambda_n)||L||\}^{1+\theta}\{\frac{1}{n}||L||\ell_n^\delta\}^\gamma \text{ if } < (A-bL)v_n, v_n > 0. \end{cases}
$$

Now $k_n \ell_n^\gamma \to 0$ by $T(\delta)$, whereas $q_n ||L|| \ell_n^\delta \leq 1$ by the definition of q_n. Since $\theta \geq 0$ and $\gamma > 0$ it follows that $\lim_{n\to\infty} \frac{C(\lambda_n, v_n)}{(b-\lambda_n)^{1+\theta}} = 0$.

Using these estimates we can now state conditioins ensuring that b is a bifurcation point for (2.2). We introduce one further hypothesis.
$(H5)\exists$ constants $\epsilon > 0$ and $K > 0$ such that

$$
||N(u)|| \leq K\varphi(u)^{\frac{1}{2}} \text{ for all } u \in H \text{ with } \varphi(u) < \epsilon.
$$

Theorem 7.2. *Let the conditions $(H1)$ to $(H5)$ and (P) be satisfied. Suppose also that the condition $T(\delta)$ is satisfied for a number $\delta \geq 1$ and that either*
$(i) N : H \to H$ is compact, or
(ii) for all $\lambda \in (a, b), J(\lambda, \cdot) : H \to R$ is weakly G-compact for a subgroup G of $O(H)$, and $\psi : H \to R$ is weakly sequentially lower semi-continuous. Set $\theta = \frac{2}{(q-2)}[1-\frac{1}{\delta}]$ where q is the constant in (P).
Then there is a sequence $\{(\lambda_n, u_n)\}$ in E such that $\lambda_n < b \quad \forall n \in N$, $\lambda_n \to b$ and $\lim_{n\to\infty}(b-\lambda_n)^{\theta/2}||u_n|| = 0$.

Proof Let $\{v_n\}$ be the sequence in V given by Lemma 6.2 and let $\{\lambda_n\}$ be the sequence given by part (3) of Lemma 7.1.
By Theorem 4.3, $\exists u_n \in H$ such that $(\lambda_n, u_n) \in E$ and $J(\lambda_n, u_n) \geq c(\lambda_n, v_n)$. In fact, if (i) holds $J(\lambda_n, u_n) = c(\lambda_n, v_n) > 0$. If (ii) holds, we note that $2J(\lambda_n, u_n) = \psi(u_n)$ by (4.5) and that

$$
\psi(u) \geq (p-2)\varphi(u) \geq 0 \quad \forall u \in H \text{ by } (P).
$$

Hence in both cases, $0 \leq J(\lambda_n, u_n) \leq c(\lambda_n, v_n)$. (7.4)
As in the proof of Lemma 2.1,

$$
< (A-\lambda_n L)u_n, Pu_n - Qu_n >\geq m(\lambda_n)||Pu_n||^2 + n(\lambda_n) l Qu_n||^2
$$

and $< (A-\lambda_n L)u_n, Pu_n - Q_n >=< N(u_n), Pu_n - Qu_n >$
$=< N(u_n), u_n > -2 < N(u_n), Qu_n >$
$\leq q\varphi(u_n) + 2||N(u_n)||||Qu_n||$ by (P)
$\leq q\varphi(u_n) + \frac{\alpha}{2}||Qu_n||^2 + \frac{2}{\alpha}||N(u_n)||^2$

where α is the constant in (2.4).
Since $\lambda_n \geq 0$, $n(\lambda_n) = \alpha$ and we obtain

$$m(\lambda_n)||Pu_n||^2 + \frac{\alpha}{2}||Qu_n||^2 \leq q\varphi(u_n) + \frac{2}{\alpha}||N(u_n)||^2 \tag{7.5}$$

By (P),

$$\begin{aligned}(p-2)\varphi(u_n) &\leq \psi(u_n) \\ &= 2J(\lambda_n, u_n) \text{ by (4.5)} \\ &\leq 2c(\lambda_n, u_n) \text{ by (7.4)}\end{aligned}$$

By Lemma 7.1, $c(\lambda_n, v_n) \to 0$ and by $(H5)$ $\exists n_0 \in N$ such that $||N(u_n)|| \leq K\varphi(u_n)^{\frac{1}{2}} \quad \forall n \geq n_0$.
Hence for $n \geq n_0$, we have that

$$\begin{aligned}\tfrac{1}{2}M(\lambda_n)||u_n||^2 &\leq \tfrac{1}{2}m(\lambda_n)||Pu_n||^2 + \tfrac{\alpha}{2}||Qu_n||^2 \\ &\leq q\varphi(u_n) + \tfrac{2}{\alpha}||N(u_n)||^2 \text{ by (7.5)} \\ &\leq \{q + \tfrac{2}{\alpha}K^2\}\varphi(u_n) \\ &\leq \tfrac{2}{(p-2)}\{q + \tfrac{2}{\alpha}K^2\}c(\lambda_n, v_n).\end{aligned}$$

Since $\lim_{n\to\infty} \frac{(b-\lambda_n)}{M(\lambda_n)} = \frac{b}{\beta}$, it now follows from part (3) of Lemma 7.1 that $\lim_{n\to\infty}(b-\lambda_n)^\theta ||u_n||^2 = 0$. This completes the proof.

Remark. It is convenient to summarise the conclusion of Theorem 7.2 by saying that b is a bifurcation point of order $\theta/2$ for (2.2).

8 Nonlinear perturbation of a self-adjoint operator

Equations involving nonlinear perturbations of an unbounded self-adjoint operator S can be cast in the form (1.2) by a suitable choice of the space H and the operators A and L. Let us give the precise conditions under which this can be done and show that $\sigma(S)=\sigma(A,L)$ where $\sigma(S)$, denotes the spectrum of S in the usual sense.

Let $(H,(\cdot,\cdot))$ be a real Hilbert space with norm $||u||=(u,u)^{1/2}$. We suppose that $S: A(S)\subset H\to H$ is a self-adjoint operator such that $0\notin\sigma(S)$.

The aim is to discuss equations of the form (1.1).

Early the assumption that $0\notin\sigma(S)$ involves no loss of generality provided that $\sigma(S)\neq R$.

Let $\{E(\lambda):\lambda\in R\}$ denote the resolution of the identity corresponding to S. Recall [20] that for any $f\in C(\sigma(S),R)$,

$$\mathcal{D}(f(S))=\{u\in H:\int f(\lambda)^g d(E(\lambda)u,u)<\infty\} \text{ and}$$

$$|f(S)u|^2=\int f(\lambda)^2 d(E(\lambda)u,u) \text{ for } u\in\mathcal{D}(f(S)).$$

Furthermore, if $u\in\mathcal{D}(f(S)), E(\lambda)u\in\mathcal{D}(f(S))$
with $f(S)E(\lambda)u=E(\lambda)f(S)u \quad \forall\lambda\in R$.
In particular,
(1) $\mathcal{D}(|S|)=\mathcal{D}(S)$ and $|Su|=||S|u|\forall u\in\mathcal{D}(S)$,
and
(2) $\mathcal{D}(S)\subset\mathcal{D}(|S|^{1/2})$ and $||S|^{1/2}u|^2=\int|\lambda|d(E(\lambda)u,u)$

$$\leq\{\int\lambda^2 d(E(\lambda)u,u)\}^{1/2}\{\int d(E(\lambda)u,u)\}^{1/2}=|Su||u|$$

for all $u\in\mathcal{D}(S)$.

We consider the Hilbert space $(H,<\cdot,\cdot>)$ defined by $H=\mathcal{D}(|S|^{1/2})$ and

$$<u,v>=(u,v)+(|S|^{\frac{1}{2}}u,|S|^{\frac{1}{2}}v) \text{ for } u,v\in H.$$

Then $||u||=<u,u>^{1/2}$ is the graph norm of $|S|^{\frac{1}{2}}$ and we recall that $\mathcal{D}(S)$ is a dense subset of H for this norm, [20]. In the terminology of page 183 of [1], H is called the form domain of S and so following this lead we call $(H,<\cdot,\cdot>)$ the form space of S.

If S is bounded below (i.e. $\exists x\in R$ such that $(Su,u)\geq\alpha(u,u) \quad \forall u\in\mathcal{D}(S))$ then $H=\mathcal{D}((S+\alpha I)^{\frac{1}{2}}$ and $||\cdot||$ is equivalent to the graph norm of $(S+\alpha I)^{\frac{1}{2}}$ for any α such that $S+\alpha I\geq 0$, [1, page 184].

Let (ℓ, k) be the maximal interval of $\rho(S) = R\backslash\sigma(S)$ containing 0 and set

$$\gamma = \begin{cases} \min\{||\ell||, k\} & \text{if } \ell > -\infty \\ k & \text{if } \ell = -\infty \end{cases}$$

For $u \in H$, $||S|^{\frac{1}{2}}u|^2 = \int |\lambda| d(E(\lambda)u, u)$
$= \int_{-\infty}^{\ell+} |\lambda| d(E(\lambda)u, u) + \int_{k-}^{\infty} |\lambda| d(E(\lambda)u, u)$
$\leq \gamma |u|^2$.
Hence $||S|^{\frac{1}{2}}u|$ is n norm which is equivalent to $||\cdot||$ on H since

$$||Su|^{\frac{1}{2}}| \leq ||u|| \leq \sqrt{\frac{\gamma+1}{\gamma}}||S|^{\frac{1}{2}}u| \quad \forall u \in H.$$

Furthermore for $u \in \mathcal{D}(S)$ and $v \in H$,
$(Su, v) = \int \lambda d(E(\lambda)u, v)$
$= -\int_{-\infty}^{\ell} |\lambda| d(E(\lambda))u, v) + \int_{k-}^{\infty} |\lambda| d(E(\lambda)u, v)$
$= -\int_{-\infty}^{0} |\lambda| d(E(\lambda)u, E(0)v) + \int_{0}^{\infty} |\lambda| d(E(\lambda)u, [I - E(0)]v)$
$= -(|S|u, E(0)v) + (|S|u, [I - E(0)]v)$
$= -(|S|^{\frac{1}{2}}u, E(0)|S|^{\frac{1}{2}}v) + (|S|^{\frac{1}{2}}u, [I - E(0)]|S|^{\frac{1}{2}}v)$
$= (|S|^{\frac{1}{2}}u, \{[I - E(0)] - E(0)\}|S|^{\frac{1}{2}}v)$ (8.1)
and hence

$$|(Su, v)| \leq ||S|^{\frac{1}{2}}u||S|^{\frac{1}{2}}v| \leq ||u||\,||v||.$$

Since $\mathcal{D}(S)$ is dense in H, there is a unique continuous symmetric bilinear form $B : H \times H \to R$ such that $B(u, v) = (Su, v) \forall u \in \mathcal{D}(S)$ and $v \in H$.
Also there is a unique operator $A \in B(H, H)$ such that

$$< Au, v >= B(u, v) \quad \forall u, v \in H.$$

Clearly $A = A^*$ and we shall now show that $0 \notin \sigma(A)$. Setting $Q = E(0)|_H$ we note that $Qu \in H$ for all $u \in H$ and so $Q^2 = Q$. Furthermore for all $u, v \in H$,

$$\begin{aligned} < Qu, v > &= (Qu, v) + (|S|^{\frac{1}{2}}Qu, |S|^{\frac{1}{2}}v) \\ &= (E(0)u, v) + (E(0)|S|^{\frac{1}{2}}u, |S|^{\frac{1}{2}}v) \\ &=< u, Qv > . \end{aligned}$$

Hence $Q : H \to H$ is the orthogonal projection onto $W = QH$ and $W = (E(0)H) \cap H$.

Let V be the orthogonal complement of W in H and set $P = I - Q = [I - E(0)]|_H$.
For $u \in \mathcal{D}(S)$, $< Au, Pu - Qu >= (Su, Pu - Qu)$
$= (|S|^{\frac{1}{2}}u, \{[I - E(0)] - E(0)\}|S|^{\frac{1}{2}}\{[I - E(0)]U - E(0)U\})$ by (8.1)
$= (|S|^{\frac{1}{2}}u, |S|^{\frac{1}{2}}u) = ||S|^{1/2}u|^2$ (8.2)

Hence $||S|^{\frac{1}{2}}u|^2 \leq ||Au||||Pu - Qu|| = ||Au||||u|| \quad \forall u \in \mathcal{D}(S)$.

It follows that $||Au|| \geq \frac{\gamma}{1+\gamma}||u|| \quad \forall u \in H$ since $\mathcal{D}(S)$ is dense in H and $A \in B(H,H)$. Recalling that $A = A^*$, we conclude that $A : H \to H$ is an isomorphism.
Let $L \in B(H,H)$ be the operator that is uniquely defined by the relation $< Lu, v >= (u,v)$ for all $u, v \in H$.
Clearly $L = L^*$ and $< Lu, u >= |u|^2 > 0 \quad \forall u \in H\backslash\{0\}$.
Thus we have shown that the operators A and L satisfy the hypotheses $(H1)$ and $(H2)$ of section 2.
Furthermore for all $a, v \in H, < LQu, v >= (E(0)u, v)$
$= (u, Qv) =< Lu, Qv >=< QLu, v >$
and so $PL = LP$ as required in part (1) of the condition $T(\delta)$.

To relate V and W to the decomposition of H introduced in section 2 we now show that $AQ = QA$ and

$$< Au, u > \leq -\alpha ||u||^2 \quad \forall u \in W \text{ where } \alpha = \frac{\gamma}{1+\gamma}.$$

In fact for any $u \in W$ there is a sequence $\{u_n\}$ in $\mathcal{D}(S)$ such that $||u_n - u|| \to$. Thus $Qu_n \in \mathcal{D}(S) \cap W$ and $||Qu_n - u|| = ||Q(u_n - u)|| \to 0$, showing that $\mathcal{D}(S) \cap W$ is dense in W. Now for $u \in \mathcal{D}S \cap W$, it follows from (8.2) that $< Au, u >= ||S|^{\frac{1}{2}}u| \geq \frac{\gamma}{1+\gamma}||u||^2$
and by the density of $\mathcal{D}(S) \cap W$ in W we obtain $< Au, u > \leq -\alpha||u||^2$ for all $u \in W$ as required.
On the other hand, for all $u \in \mathcal{D}(S)$ and $v \in H$,

$$< QAu, v >=< Au, Qv >= (Su, E(0)v) = (SE(0)u, v) =< AQu, v > .$$

Hence $QAu = AQu \quad \forall u \in \mathcal{D}(S)$ and by the density of $\mathcal{D}(S)$ in H we have $QA = AQ$.

Henceforth we can identify the above decomposition $H = V \oplus W$ with that introduced in section 2 via A.

Let a and b be the parameters defined in section 2 in terms of A and L. Since $PL = LP$ we know from Lemma 2.1 that (a, b) is a maximal interval in $\rho(A, L)$. In fact, as we now show, $(a, b) = (\ell, k)$ and so it is also a maximal interval in $R\ \sigma(S)$.

First we note that for $u \in \mathcal{D}(S)$ and $\lambda \in R$,

$$\begin{aligned} ||(A - \lambda L)u|| &= \sup\{| < (A - \lambda L)u, v > | : V \in H \text{ and } ||v|| = 1\} \\ &= \sup\{|((S - \lambda I)u, v)|/||v|| : v \in H \text{ and } v \neq O\} \end{aligned}$$

Since $|v| \leq ||v||$ and H is dense in H it follows that

$$||(A - \lambda L)u|| \leq |(S - \lambda I)u| \text{ for all } \quad u \in \mathcal{D}(S).$$

If $\lambda \in \rho(A, L)$ there exists $K > 0$ such that $||(A - \lambda I)u|| \geq K||u||$ for all $u \in H$ and we deduce that

$$K|u| \leq K||u|| \leq |(S - \lambda I)u| \text{ for all } u \in \mathcal{D}(S).$$

Hence $\rho(A,L) \subset R\backslash\sigma(S)$ and $(a,b) \subset (\ell,k)$.

But if $\ell > -\infty$, $\ell \in \sigma(S)$ and $\exists\{u_n\} \subset \mathcal{D}(S)$ such that $|u_n| = 1 \quad \forall n \in N$ and $|(S-\ell I)u_n| \to 0$.

Then

$$((S-\ell I)u_n, Pu_n) = ((S-\ell I)Pu_n, Pu_n) \geq (k-\ell)|Pu_n|^2$$

and so $(k-\ell)|Pu_n| \leq |(S-\ell I)u_n| \quad \forall n \in N$.

This imlplies that $|Pu_n| \to 0$ and hence that $|Qu_n| \to 1$.

But then

$$\frac{< Aw_n, w_n >}{< Lw_n, w_n >} - \ell = \frac{(Sw_n, w_n) - \ell|w_n|^2}{|w_n|^2}$$

$$= \frac{(S-QI)w_n, w_n}{|w_n|^2} \to 0 \text{ where } w_n = Qu_n,$$

since $|w_n| = |Qu_n| \to 1$ and

$$\begin{aligned} |((S-\ell I)w_n, w_n)| &= |((S-\ell I)u_n, Qu_n)| \\ &\leq |(S-\ell I)u_n||Qu_n| \leq |(S-\ell I)u_n|. \end{aligned}$$

This shows that if $\ell > -\infty$, then $a \geq \ell$.

It follows that $a = \ell$ in all cases and similarly we obtain $b = k$.

Now consider a (nonlinear) operator $R = \mathcal{D}(S) \to \mathcal{H}$ and the linear operator $\tilde{L} \in B(\mathcal{H}, H)$ such that

$$(u,v) =< \tilde{L}u, v > \text{ for all } v \in H.$$

Clearly $\tilde{L}_n = Lu \quad \forall u \in H$ and every solution of (1.1) satisfies $(A-\lambda L)u - \tilde{L}R(u) = 0$.

Thus we suppose that the functional φ in $(H3)$ of section 2 can be chosen in such a way that

$$N(u) = \tilde{C}R(u) \text{ for all } u \in \mathcal{D}(S).$$

In this case every solution of (1.1) will satisfy (1.2) and a solution of (1.2) will satisfy (1.1) provided that $u \in \mathcal{D}(S)$. In this sense solutions of (1.2) can be regarded as generalised solutions of (1.1).

9 Bound states for nonlinear Schrdinger operators

In this section we discuss the existence and bifurcation of solutions of (1.3). For this we introduce the following basic assumptions.
$(A1)$ $V \in L^\infty(R^N)$
$(A2)$ $\rho \in L^\infty(R^N)$ and $2 < p < 2^*$. Also $\tau \geq 0$. Here $2^* = +\infty$ for $N = 1, 2$ and $2^* = \frac{2N}{N-2}$ for $N \geq 3$.

Under the assumption $(A1)$ it is well-known that a self-adjoint operator $S : \mathcal{D}(\mathcal{S}) \subset L^2(R^N) \to L^2(R^N)$ is defined by $\mathcal{D}(S) = H^2(R^N)$ and

$$Su = -\Delta u + Vu \quad \forall u \in \mathcal{D}(S).$$

Clearly S is bounded below and in dealing with (1.3) we may assume that $0 \notin \sigma(S)$. Let $(H, < \cdot, \cdot >)$ be the form space associated with S in the sense of section 8. Using the results in [1,20].
It is easy to see that $H = H^1(R^N)$ and that the norm $\| \cdot \| =< \cdot, \cdot >^{\frac{1}{2}}$ is equivalent to the usual norm on $H^1(R^N)$. As is shown in section 8 there exist A and $L \in B(H, H)$ which satisfy the conditions $(H1)$ and $(H2)$ and for which

$$((S - \lambda I)u, v) =< (A - \lambda I)u, v > \quad \forall n \in \mathcal{D}(S) \text{ and } v \in H.$$

where $(\cdot, \cdot)$ is the usual scalar product in $H = L^2(R^N)$.
Furthermore, for a and b as defined in section 2, we have that (a, b) is a maximal interval in $R \backslash \sigma(S)$. Note also that $b < \infty$ since $\sup \sigma(S) = \infty$.

Turning to the nonlinear term in (1.3) we set

$$\varphi(u) = \int r|u|^p dx \text{ for } u \in H.$$

When no domain of integration is indicated the integral is taken over R^N. Clearly

$$0 \leq \varphi(u) \leq |\tau|_\infty |u|_p^p \leq |\tau|_\infty C \|u\|^p \quad \forall u \in H \tag{9.1}$$

where $| \cdot |_p$ denotes the usual norm in L^p. Furthermore φ is convex, and as is well known $\varphi \in C^2(H, R)$ with

$$\varphi'(u)v = \int \tau |u|^{p-2} uv dx \quad \forall u, v \in H.$$

Thus $\varphi'(u)u = p\varphi(u) \quad \forall u \in H$.
These observations show that, under the assumption $(A2)$, φ satisfies the conditions $(H3)$ and (P).
If, in addition, $\tau > 0$ a.e. on R^N then φ satisfies (AD) since $\varphi(u) > 0 \quad \forall u \in H \backslash \{0\}$ and $b < \infty$.

The general results concern solutions $(\lambda, u) \in R \times H$ of the equation (2.2) where $N = \varphi$. However in the present context they are in fact solutions of (1.3) as we see from the following remarks. Note the (2.2) corresponds to the usual notion of weak solution of (1.3).
(1) For $u \in H$ we set $q = \lambda - V + \tau|u|^{p-2}$ and observe that as is shown in [13] the results of Agmon [21] imply that if (λ, u) satisfies

$$\int u \cdot v - quvdx = 0 \quad \forall v \in C_o^\infty(R^N)$$

then $u \in L^\infty(R^N)$ and $\lim_{R\to\infty} inn \sup_{|x|\geq k} |u(x)| = 0$.
From this it follows that if (λ, u) satisfies (1.3) then $u \in H^2(R^N) \cap C^1(R^N) \cap L^\infty(R^N)$ and $\lim_{|x|\to\infty} u(x) = 0$.
(2) Furthermore if (λ, u) satisfies (1.3) and $\lambda \notin \sigma(S)$ then there is a constant $\mu > 0$ such that $\lim_{|x|\to\infty} e^{\mu|x|}u(x) = 0$. See [13].

With the aim of applying the general results to (1.3) in the weak form we establish the following properties of N.

Lemma 9.1. *Let (A1) and (A2) be satisfied.*
(1) There exists $K > 0$ such that

$$||N(u)|| \leq K\varphi(u)^{1/p'} \quad \forall u \in H \text{ where } \frac{1}{p} + \frac{1}{p'} = 1.$$

(2) $N : H \to H$ is weakly sequentially continuous
(3) If $\lim_{|x|\to\infty} \tau(x) = 0$ then $N : H \to H$ is compact.

Proof (1) For all $u \in H$, $||N(u)|| = \sup\{< N(u), v >: v \in H \text{ and } ||v|| = 1\}$. For $u, v \in H$,

$$\begin{aligned} |< N(u)v >| &= 1\textstyle\int \tau|u|^{p-2}uvdx| \leq |\int \tau|u|^{p-1}vdx| \\ &\leq |\tau|_\infty^{1/p} \textstyle\int \tau^{1/p'}|v|dx \\ &\leq |\tau|_{1/p}\{\textstyle\int \tau|u|^p dx\}^{1/p'}\{\int |x|^p\}^{1/p} \\ &\leq |\tau|_\infty^{1/p}\{p\varphi(u)\}^{1/p'}C||v||. \end{aligned}$$

(2) Let $\{u_n\} \subset H$ converge weakly to u in H. To show that $\{N(u_n)\}$ converges weakly to $N(u)$ in H it is sufficient to show that

(a) $\{||N(u_n)||\}$ is bounded and

(b) $< N(u_n) - N(u), v >\to 0 \quad \forall v \in C_0^\infty(R^N)$.

Since $\{||u_n||\}$ is bounded it follows from (9.1) and part (1) that $\{||N(u_n)||\}$ is bounded.
For (b) we fix $v \in C_o^\infty(R^N)$ and then $d > 0$ such that $v(x) = 0 \quad \forall |x| \geq d$. Then

$$\begin{aligned} |< N(u_n) - N(u), v >| &\leq |\tau|_\infty\{\textstyle\int_{|x|\leq d}[f(u_n) - f(u)]^{p'}dx\}^{\frac{1}{p'}}\{\int |v|^p\}^{\frac{1}{p}} \\ &= |\tau|_\infty|f(u_n) - f(u)|_{L^{p'}}(|x| \leq d)C||v|| \end{aligned}$$ where

$f(s) = |s|^{p-2}s$. Since $|f(s)| = |s|^{p-1} = |s|^{p/p'}$ it follows that the function $u \to$

$f(u)$ is continuous from $L^p(|x| \leq d)$ into $L^{p'}(|x| \leq d)$. But $|u_n - u|_{L^p(|x|\leq d)} \to 0$ by the compactness of the Sobolev embedding on bounded domains. Hence (b) is established.
(3) Again we consider a sequence $\{u_n\} \subset H$ converging weakly to u in H. For $v \in H$ and any $d > 0$,

$$\begin{aligned} |< N(u_n) - N(u), v >| &\leq |\tau|_\infty |f(u_n) - f(u)|_{L^{p'}}(|x| \leq d)|v|_{L^p(|x|\leq d)} \\ &+ \int_{|x|>d} \tau[|f(u_n)| + |f(u)|]|v|dx \end{aligned}$$

Put $\epsilon(d) = ss\sup\{\tau(x) : |x| > d\}$. Then

$$\begin{aligned} \int_{|x|>d} \tau|f(u_n)||v|dx &\leq \epsilon(d)|f(u_n)|_{p'}|v|_p = c(d)|u_n|_p^{\frac{p}{p'}}|v|_p \\ &\leq C\epsilon(d)||u_n||^{\frac{p}{p'}}||v|| \end{aligned}$$

and so $|< N(u_n) - N(u), v >|$

$$\leq \{|r|_\infty |f(u_n) - f(u)|_{L^{p'}(|x|\leq d)} + 2C\epsilon(d)K^{\frac{p}{p'}}\}||v||$$

where $K = \sup\{||u_n||, ||u|| : n \in N\} < \infty$.
Hence $||N(u_n) - N(u)|| \leq |\tau|_\infty |f(u_n) - f(u)||_{L^{p'}(|x|\leq d)}$

$$+2CK^{\frac{p}{p'}}\epsilon(d)$$

for all $n \in N$ and all $d > 0$. Thus for any $d > 0$, $\limsup_{n\to\infty} ||N(u_n) - N'u)|| \leq 2CK^{\frac{p}{p'}}\epsilon(d)$ since

$$|f(u_n) - f(u)|_{L^p(|x|\leq d)} \to 0 \text{ as } n \to \infty.$$

But, by hypothesis, $\lim_{d\to\infty} \epsilon(d) = 0$ and it follows that $||N(u_n) - N(u)|| \to 0$.
Remark It follows from part (1) that φ satisfies the conditions $(H4)$ and $(H5)$.

To deal with the case of periodic coefficients we introduce the following notation.
For $k \in Z^N$ we define $T_k \in O(H)$ by

$$(T_k u)(x) = u(x - k)$$

and set $G = \{T_k : k \in Z^N\}$. (9.2)
$(A3)$ $V(x + k) = V(x)$ $\forall x \in R^N$ and $\forall k \in Z^N$.
$(A4)\tau(x + k) = \tau(x)$ $\forall x \in R^N$ and $\forall k \in Z^N$.
This periodicity can be used in two ways. The condition $(A3)$ can be used to establish the condition $T(\delta)$ for some $\delta > 0$. The conditions $(A3)$ and $(A4)$ can be used to show that the functional, $J(\lambda, u) = \frac{1}{2} < (A - \lambda L)u, u > -\varphi(u)$ is weakly G-compact for all $\lambda \in R$.
We begin by establishing the latter property.

Lemma 9.2. *Suppose that the conditions (A1) to (A4) are satisfied. Let G be the group defined by (9.2). For all $\lambda \in R$ the functional $J(\lambda, \cdot) : H \to R$ is weakly G-compact.*

Proof Clearly $J(\lambda, \cdot)$ is G-invariant.

Since $J(\lambda, 0) = 0$ we consider a sequence $\{u_n\}$ such that $\{\|u_n\|\}$ is bounded, $J(\lambda, u_n) \to c \neq 0$ and $\|_u J(\lambda, u_n)\| \to 0$.

By (4.5), $(p-2)\varphi(u_n) = 2J(\lambda, u_n) - <_u J(\lambda, u_n), u_n >$ and so $\varphi(u_n) \to \frac{2c}{(p-2)}$. Next we recall a special kind of Sobolev inequality due to P.L. Lions, [4].

Since $p \in (2, 2^*)$, there exist $K > 0$ and $\theta \in (0,1)$ such that

$$|u|_p \leq K\|u\|^{1-\theta}\{\sup_{y\in R^N}\int_{B(y,1)} u(x)^2dx\}^\theta \quad \forall u \in H.$$

But $0 \leq \varphi(u) \leq |r|_\infty |u|_p^p \quad \forall u \in H$ and so there exists a constant D such that

$$0 \leq \varphi(u_n) \leq D\{\sup_{y\in R^N}\int_{B(y,1)} u_n(x)^2dx\}^{p\theta} \quad \forall n \in N$$

Recalling that $\varphi(u_n) \to \frac{2c}{(p-2)} \neq 0$ we conclude that $\exists\delta > 0$ and $n_0 \in N$ such that

$$\sup_{y\in R^N}\int_{B(y,1)} u_n(x)^2dx \geq \delta \text{ for all } n \geq n_0$$

and hence that $\exists y_n \in R^N$ such that

$$\int_{B(y_n,1)} u_n(x)^2dx \geq \delta/2 \text{ for all } n \geq n_0$$

Set $A = [0,1]^N$ and $D = [-1,2]^N$.
There exists $k_n \in Z^N$ such that $y_n + k_n \in A$.
Then $B(x_n, 1) \subset D$ where $x_n = y_n + k_n$.
Setting $v_n = T_{k_n}u_n$ we have that $v_n \in O(u_n)$ and

$$\int_D v_n(x)^2dx \geq \int_{B(x_n,1)} u_n(x-k_n)^2dx = \int_{B(y_n,1)} u_n(x)^2dx \geq \delta/2$$

for all $n \geq n_0$.
But $\|v_n\| = \|u_n\|$ and so there is a subsequence $\{v_{n_i}\}$ such that $v_{n_i} \rightharpoonup v$ weakly in H.
This implies that $|v_{n_i} - v|_{L^2(D)} \to 0$ and so $\int_D v(x)^2dx \geq \frac{\delta}{2}$. Hence $v \not\equiv 0$.
Furthermore, $\|_u J(\lambda, v_n)\| = \|_u J(\lambda, u_n)\| \to 0$
and by part (2) of Lemma 9.1, $\{_u J(\lambda, v_{n_i})\}$ converges weakly to $_u J(\lambda, v)$. Hence $_u J(\lambda, v) = 0$. This completes the proof.

We can now state the first result concerning the existence of solutions of (1.3).

Theorem 9.3. *Suppose that $(A1)$ and $(A2)$ are satisfied and that $\tau > 0$ a.e. on R^N. Suppose also that either (1) $\lim_{|x|\to\infty} \tau(x) = 0$ or (2) $(A3)$ and $(A4)$ hold.*
Then for every $\lambda \notin \sigma(S)$ there exists $u_\lambda \in H'\{0\}$ such that (λ, u_λ) is a solution of (1.3).

Proof This follows from Theorem 5.2.

When V satisfies the conditions $(A1)$ and $(A3)$ the spectrum of S consists of those $\lambda \in R$ for which the equation $-\Delta u + Vu = \lambda u$ admits a solution in the form of a Bloch wave, [22]. From this we deduce that for all $\lambda \in \sigma(S)$ there exists a non-trivial solution $\Psi \in H^2_{loc}(R^N) \cap C^1(R^N)$ of $\Delta u + Vu = \lambda u$ such that $\Psi : R^N \to R$ is uniformly almost-periodic in the sense of Bisicovich [23]. The functions Ψ and Ψ are bounded on R^N. For a uniformly almost-periodic function $f : R^N \to R$ its mean-value $M(f)$ is defined by

$$M(f) = \lim_{T\to\infty} \frac{1}{T^N} \int_0^T \dots \int_0^T f(x)dx.$$

The function Ψ does not belong to $L^2(R^N)$ so we use a truncation. Let $\eta \in C_o^\infty(R^N)$ be such that

$$\eta(x) = \begin{cases} 1 \text{ for } |x| \le 1 \\ 0 \text{ for } |x] \ge 2 \end{cases} \quad \text{and } \eta(x) \ge 0 \quad \forall x \in R^N$$

For $k > 0$ we set

$$z_k(x) = k^{N/2}\eta(\frac{x}{k})\Psi(x) \text{ for } x \in R^N \tag{9.3}$$

Lemma 9.4. *Let $(A1)$ and $(A3)$ be satisfied and consider $\lambda \in \sigma(S)$. The functions defined by (9.3) have the following properties.*
(1) $z_k \in H^2(R^N) \cap C^1(R^N)$
(2) $\lim_{k\to\infty} |z_k|_2^2 = M(\Psi^2)|\eta|_2^2$,
$\lim_{k\to\infty} k^2 < (S - \lambda I)z_k, z_k >= M(\Psi^2)|\eta|_2^2$,
$\lim_{k\to\infty} k^2|(S - \lambda I)z_k|_2^2 = 4\Sigma_{i,j=1}^N M(\partial_j\psi\partial_i\psi) \int \partial_j\eta(x)\partial_i\eta(x)dx$,
$\lim_{k\to\infty} |z_k|_2^2 = M((\lambda - V)\Psi^2)|\eta|_2^2$.

Proof These results follows from the Riemann-Lebesgue Lemma for uniformly almost-periodic functions, [13].
It follows that $\{||z_k||\}$ is bounded and there exists $k_0 > 0$ such that $||z_k|| \ge \gamma > 0$ for all $k \ge k_0$. For $k \ge k_0$ wet set

$$u_k = \frac{z_k}{||z_k||}.$$

We note that

$$< (A - \lambda L)u_k, u_k >= ((S - \lambda I)u_k, u_k)$$

and that there is a constant $C > 0$ such that

$$||(A - \lambda L)u_k|| \le C|(S - \lambda I)u_k|_2.$$

We can now give two situations in which the condition $T(\delta)$ is satisfied at every point of $\sigma(S)$. For $\tau \geq 0$ we say that τ satisfies the condition $D(I)$ provided there exist $L > 0$ and $x_0 \in R^N$ such that $\tau(x) \geq L|x|^{-I}$ for all $x \in C$ where $C = \{tx : t \geq 1 \text{ and } \|x - x_o\| \leq 1\}$.

Lemma 9.5. *Let (A1) to (A3) be satisfied.*
(a) It τ satisfies the condition $D(I)$ for some $I \geq 0$ then the condition $T(\delta)$ is satisfied for all $\delta < \frac{4}{N(p-2)+2I}$.
(b) If τ satisfies (A4) and $\tau \not\equiv$ a.e. on R^N then the condition $T(\delta)$ is satisfied for all $\delta < \frac{4}{N(p-2)}$.

Proof As is shown in section 8, part (1) of the condition $T(\delta)$ is always satisfied in the present context. The case (a) follows from Proposition 3.6 of [13] after appropriate normalisation.
(b) For $k \geq k_0, p\varphi(z_k) = \int \tau(x)k^{-\frac{Np}{2}}\eta(\frac{x}{k})^p|\Psi(x)|^p dx$

$$k^{N(1-p/2)} \int \tau(ky)|\Psi(ky)|^p \eta(y)^p \, dy$$

so $\lim_{k\to\infty} k^{\frac{N}{2}(p-2)}\varphi(z_k) = \frac{1}{p}M(\rho|\Psi|^p)|\eta|_p^p$ where $M(\rho|\Psi|^p) > 0$. It is now an easy matter to verify that for $k \geq k_0$, the sequence $\{u_k\}$ has the properties required in part (2) of the condition $T(\delta)$, provided that $\delta < \frac{4}{N(p-2)}$.

Theorem 9.6. *Let the conditions (A1) to (A3) be satisfied.*
(a) Suppose that $\lim_{|x|\to\infty} \tau(x) = 0$ and that the condition $D(I)$ is satisfied for a number $I \geq 0$ such that $2I + (p-2)(N-2) < 4$. Then $\exists \lambda^* \in (a, b)$ such that for all $\lambda \in (\lambda^*, b)$ there exists $u_\lambda \in H\backslash\{0\}$ such that (λ, u_λ) is a weak solution of (1.3).
Furthermore if $2I + (p-2)N < 4$ then b is a bifurcation point of order θ for (1.3) where
$0 \leq \theta < \frac{1}{(p-2)}[1 - \frac{(p-2)N+2I}{4}]$.
(b) Suppose that τ satisfies (A4) and $\tau \not\equiv 0$ a.e. on R^N. If $(p-2)(N-2) < 4$ then $\exists \lambda^* \in (a, b)$ such that for all $\lambda \in (\lambda^*, b)$ there exists $u_\lambda \in H\backslash\{0\}$ such that (λ, u_λ) is a weak solution of (1.3). If $(p-2)N < 4$ then b is a bifurcation point of order θ for (1.3) where $0 \leq \theta < \frac{1}{(p-2)} - \frac{N}{4}$.

10 Homoclinic solutions of Hamiltonian systems

In this section we discuss the existence and bifurcation of solutions of the equation (1.4). For this we introduce the following basic hypothesis. The transpose of a matrix M is denoted by M^T.

$(G1)$ Let J and M be real $2N \times 2N$ matrices such that

$$J^{-1} = J^T = -J \qquad \text{and } M = M^T.$$

Let $\mathcal{H} = [L^2(R)]^{2N}$ with the usual scalar product $(\cdot,\cdot)$. It is easy to check that the operator $S : \mathcal{D}(S) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$\mathcal{D}(S) = [H^1(R)]^{2N} \text{ and } Su = Ju' + Mu \quad \forall u \in \mathcal{D}(S)$$

is self-adjoint. In order to characterise the spectrum of S it is convenient to use to complexification.

We set $h = \mathcal{H} \oplus i\mathcal{H}$, $\mathcal{D}(\mathcal{S}) = \mathcal{D}(S) \oplus i\mathcal{D}(S)$ and $\mathcal{S}(u+iv) = Su + iSv$ for $u+iv \in \mathcal{D}(\mathcal{S})$.

The Fourier transform defines a unitary operator on h which we denote by $\wedge$. Let $\hat{\mathcal{S}} : \mathcal{D}(\hat{\mathcal{S}}) \subset h \to h$ be defined by $\mathcal{D}(\hat{\mathcal{S}}) = \widehat{\mathcal{D}(\mathcal{S})}$ and $\hat{\mathcal{S}}u = (\mathcal{S}\overset{\vee}{u})^\wedge$ for $u \in \mathcal{D}(\hat{\mathcal{S}})$ where V denotes the inverse Fourier transform.

We observe that

$$\mathcal{D}(\hat{\mathcal{S}}) = \{u \in h : (1+\xi^2)^{\frac{1}{2}} u(\xi) \in h\}$$

and that

$$\hat{\mathcal{S}}u(\xi) = (i\xi J + M)u(\xi) \quad \text{ for } u \in \mathcal{D}(\hat{\mathcal{S}})$$

Thus $\hat{\mathcal{S}}$ is a matrix multiplication operator and $\sigma(\hat{\mathcal{S}}) = \sigma(\mathcal{S}) = \sigma(\mathcal{S})$. Clearly $\hat{\mathcal{S}}$, and hence S, has no eigenvalues.

For $\xi \in R$ we denote by $w(\xi)$ the spectrum of the complex matrix $i\xi J + M$. Since $i\xi J + M$ is Hermitian, $w(\xi) \subset R$ and since

$$\det(i\xi J + M - \lambda I) = \det(i\xi J^T + M - \lambda I) = \det(-i\xi J + M - \lambda I)$$

we also have $w(\xi) = w(-\xi)$.

Lemma 10.1. *The assumption $(C1)$ implies that* $\sigma(S) = \cup_{\xi \in R} w(\xi)$.

Proof If $\lambda \in w(\xi_0)$ there exists $v \in C^{2N}$ such that $||v|| = 1$ and $[i\xi_o J + M - \lambda I]v = 0$. Setting $f(\xi) = \frac{v}{1+[\xi-\xi_o|}$ we have that $f \in C(R) \cap h$ and $f(\xi_o) = v$. If $f \in R(\hat{S} - \lambda I)$ there exists $u \in \mathcal{D}(\hat{\mathcal{S}})$ such that $[i\xi J + M - \lambda I]u(\xi) = f(\xi)$ a.e. on R. Then

$< i(\xi - \xi_o)Ju(\xi), f(\xi) >= ||f(\xi)||^2$ a.e. on R where $< \cdot,\cdot >$ is the usual scalar product on ℓ^{2N} since $< [i\xi_o J + M - \lambda I]u(\xi), f(\xi) >$

$$=< u(\xi), [i\xi_o J + M - \lambda I]f(\xi) >= 0 \quad \forall \xi \in R.$$

Hence for almost all $\xi \in R, |u(\xi)| \quad |f(\xi)| \geq$

$$| < iJu(\xi), f(\xi) > | = \frac{||f(\xi)||^2}{|\xi - \xi_o|}.$$

and, since u and $f \in h$, we conclude that $\frac{||f(\xi)||^2}{|\xi-\xi_o|} \in L^1(R)$. This is impossible because $f \in C(R)$ and $|f(\xi_o)| = ||v|| \neq 0$.
Therefore the assumption that $\lambda \in w(\xi_o)$ implies that $R(\hat{\mathcal{S}} - \lambda I) \neq h$. Hence $\cup_{\xi \in R} w(\xi) \subset \sigma(\hat{\mathcal{S}}) = \sigma(\mathcal{S})$.

For the converse we suppose that $\lambda \notin \cup_{\xi \in R} w(\xi)$. Then we claim that $\exists \delta > 0$ such that

$$|\det(i\xi J + M - \lambda I)| \geq \delta > 0 \text{ for all } \quad \xi \in R.$$

Otherwise there is a sequence $\{\xi_n\}$ such that $\det(i\xi_n J + M - \lambda I) \to 0$. But for $\xi \neq 0$

$$|\det(i\xi J + M - \lambda I)| = \xi^{2N} |\det(I - \frac{J(M - \lambda I)}{i\xi})|$$

and we use that $\{\xi_n\}$ must be bounded.
Passing to a subsequence we can suppose that $\xi_n \to \xi \in R$ and this implies that $\det(i\xi J + M - \lambda I) = 0$, contradicting the assumption that $\lambda \notin \cup_{\lambda \in R} w(\xi)$.
Having justified the claim it now follows from Cramer's formula that $\exists C > 0$ such that

$$||(i\xi J + M - \lambda I)^{-1})|| \leq \frac{C(1 + |\xi|^{2N-1})}{1 + |\xi|^{2N}} \text{ for all } \xi \in R.$$

Given $f \in h$ and setting

$$u(\xi) = (i\xi J + M - \lambda I)^{-1} f(\xi) \text{ for } \xi \in R,$$

we see that

$$(1 + \xi^2)^{\frac{1}{2}} ||u(\xi)|| \leq \frac{C(1 + \xi^2)^{\frac{1}{2}} (1 + |\xi|)^{2N-1}}{1 + |\xi|^{2N}} ||f(\xi)|| \leq 2\sqrt{2} C ||f(\xi)||.$$

Hence $u \in \mathcal{D}(\hat{\mathcal{S}})$, $(\hat{\mathcal{S}} - \lambda I)u = f$ and $\int ||u||^2 d\xi \leq 8C^2 \int ||f||^2 d\xi$.
Thus the assumption that $\lambda \notin \cup_{\xi \in R} w(\xi)$ implies that $\lambda \in \rho(\hat{\mathcal{S}} - \lambda I)$ and so

$$\sigma(S) = \sigma(\hat{\mathcal{S}}) \subset \cup_{\xi \in R} w(\xi).$$

Corollary 10.2. *Let the condition $(C1)$ be satisfied*
(1) $\inf \sigma(S) = -\infty$ *and* $\sup \sigma(S) = +\infty$
(2) $0 \in \sigma(S) \iff \sigma(JM) \cap iR \neq \phi$

(3) If $\lambda \in \sigma(S)$ then there exists a non-trivial periodic solution $\Psi \in C^\infty(R, R^{2N})$ of the equation $Ju' + Mu = \lambda u$.

Proof (1) There exists $v \in C^{2N}$ such that $\|v\| = 1$ and $iJv = v$. For $\xi \neq 0$,

$$< (iJ + \frac{M}{\xi}v, v >\geq \|v\|^2 - \frac{\|M\|}{|\xi|}\|v\|^2 = 1 - \frac{\|M\|}{|\xi|}.$$

Hence $\sup \sigma(iJ + \frac{M}{\xi}) \geq \frac{1}{2}$ for all ξ such that $|\xi| \geq 2\|M\|$. Thus there exists $\mu(\xi) \geq \frac{1}{2}$ such that $\mu(\xi) \in \sigma(iJ + \frac{M}{\xi})$ if $|\xi| \geq 2\|M\|$, and so

$$\eta = \det(iJ + \frac{M}{\xi} - \mu(\xi)I) = \xi^{-2N} \det(i\xi J + M - \xi\mu(\xi)I)$$

This means that for all $|\xi| > 2\|M\|$,

$$\mu(\xi) \in w(\xi) \text{ where } \mu(\xi) \geq \frac{1}{2}.$$

Thus $\forall \xi > 2\|M\|, \xi\mu(\xi) \in \sigma(S)$ with $\xi\mu(\xi) \geq \frac{\xi}{2}$
and $-\xi\mu(-\xi) \in \sigma(S)$ with $-\xi\mu(-\xi) \leq -\frac{\xi}{2}$.
Hence $\sup \sigma(S) = +\infty$ and $\inf \sigma(S) = -\infty$.
(2) $0 \in \sigma(S) \iff \exists \in R$ such that $0 \in w(\xi)$
$\iff \exists \xi \in R$ such that $\det(i\xi J + M) = 0$
$\iff \exists \xi \in R$ such that $\det(i\xi I - JM) = 0$
$\iff \sigma(JM) \cap iR \neq \phi$.
(3) If $\lambda \in \sigma(S) \exists \xi \in R$ and $w \in C^{2N}$ such that $\|w\| = 1$ and $(i\xi J + M - \lambda I)w = 0$.
Let $z(t) = e^{i\xi t}w$.
Clearly $z'(t) = i\xi z(t)$ and $Jz'(t) = i\xi e^{i\xi t}Jw$
$= e^{i\xi t}[\lambda w - Mw] = \lambda z(t) - Mz(t)$.
Setting $u(t) = Rez(t) = \cos \xi t \Re w - \sin \xi t I_m W$
we see that $u \in C^\infty(R, R^{2N})$, u is periodic and $u \not\equiv 0$. Since J, λ and M are real, u satisfies $Ju' = \lambda u - Mu$.

In view of part (2) we introduce the following condition
$(C2)$ $\quad \sigma(JM) \cap iR = \phi$.

In the study of the existence of homoclinic solutions of systems like (1.4) using a variational approach the condition $(C2)$ was first introduced by Coti-Zelati, Ekeland and Sr in [24].

The next step in our discussion of (1.4) involves the introduction of the form space $(H, L, \cdot >)$ associated with S in the sense of section 8. Recalling that $\mathcal{D}(S)$ is the Sobolev space $[H^1(R)]^{2N}$, we shall show that, up to equivalence of norms, $(H, < \cdot, \cdot >)$ is characterised as the fractional Sobolev space $[H^{\frac{1}{2}}(R)]^{2N}$. To establish the fundamental properties of nonlinear operators defined on H we consider its embedding in $[L^p(R)]^{2N}$. For this it is convenient to introduce the following scale of Hilbert space $(h_t, <\cdot,\cdot>_t)$ for $t \geq 0$ where $h_t = \{u \in h : (1+\xi^2)^{\frac{t}{2}}\hat{u}(\xi) \in h\}$ and

$$(u,v)_t = \int (1+\xi^2)^t < \hat{u}(\xi), \hat{v}(\xi) > d\xi.$$

Note that $h_0 = h$ and $h_1 = \mathcal{D}(\hat{\mathcal{S}})$. For any $f \in C(\sigma(S), R)$ we know that $\mathcal{D}(f(\mathcal{S})) = [\mathcal{D}(f(\hat{\mathcal{S}}))]^\vee$ and

$$f(\mathcal{S})u = [f(\hat{\mathcal{S}})\hat{u}]^\vee \text{ for } u \in \mathcal{D}(\mathcal{S}).$$

But $\hat{\mathcal{S}}$ is a matrix multiplication operator so it follows easily that

$$\mathcal{D}(|\hat{\mathcal{S}}|^{\frac{1}{2}}) = h_{\frac{1}{2}} \text{ and that}$$

$$|\hat{\mathcal{S}}|^{\frac{1}{2}}u(\xi) = |i\xi J + M|^{\frac{1}{2}}u(\xi) \text{ for } u \in h_{\frac{1}{2}}$$

where $|i\xi J + M|^{\frac{1}{2}}$ denotes the positive self-adjoint square-root of the Hermitian matrix $|i\xi J + M|$. Thus we see that $H = \Re h_{\frac{1}{2}}$ and that

$$|S|^{\frac{1}{2}}u = [|i\xi J + M|^{\frac{1}{2}}\hat{u}]^\vee \text{ for all } \quad u \in H.$$

In particular, for $u \in H$, we have that

$$||S|^{\frac{1}{2}}u|^2 = |[|S|^{\frac{1}{2}}u]^\vee|^2$$

$$-||\hat{\mathcal{S}}|^{\frac{1}{2}}\hat{u}|^2 = \int |||i\xi J + M|^{\frac{1}{2}}\hat{u}(\xi)||^2 d\xi. \tag{10.1}$$

These observations lead to the following result.

Lemma 10.3. *Let the conditions (C1) and (C2) be satisfied.*
Then, up to equivalence of norms, the form space $(H, <\cdot,\cdot,>)$ of S is equal to $(\mathcal{R}e \quad h_{\frac{1}{2}}, (\cdot,\cdot)_{\frac{1}{2}})$.

Proof Since $0 \notin \sigma(S)$, by the arguments used to prove Lemma 10.1 we see that $\exists C > 0$

$$|||i\xi J + M|^{-1}|| \leq \frac{C(1+|\xi|^{2N-1})}{1+|\xi|^{2N}} \text{ for all } \xi \in R.$$

and hence $|||i\xi J + M|^{-\frac{1}{2}}|| \leq [\frac{C(1+|\xi|^{2N-1})}{1+|\xi|^{2N}}]^{\frac{1}{2}}$.
It follows from (10.1) that for all $u \in H$,

$$||S|^{\frac{1}{2}}u|^2 \geq \int \frac{(1+\xi^{2N})}{C(1+|\xi|^{2N-1})}||\hat{u}(\xi)||^2 d\xi$$

$$\geq \frac{1}{C2\sqrt{2}} \int (1+\xi^2)^{\frac{1}{2}}||\hat{u}(\xi)||^2 d\xi.$$

Thus $||u||^2 \geq \frac{1}{C2\sqrt{2}} \int (1+\xi^2)^{\frac{1}{2}} ||\hat{u}(\xi)||^{\frac{1}{2}} d\xi \quad \forall n \in H$,
where $||u||^2 = |u|^2 + ||S^{\frac{1}{2}} u|^2|^2$

$$= \int ||\hat{u}(\xi)||^2 ds + \int |||i\xi J + M|^{\frac{1}{2}} \hat{u}(\xi)||^2 d\xi.$$

But $|i\xi J + M|^{\frac{1}{2}} V||^2 = | < |i\xi J + M|V, V > |$

$$\leq |||i\xi J + M|V|| \quad ||V|| \leq ||(i\xi J + M)V|| \quad ||V||$$
$$\leq (|\xi| + ||M||)||V||^2 \text{ for all} v \in \ell^N \text{ and } \xi \in R.$$

Hence $\exists D > 0$ such that $||u||^2 \leq D \int (1+\xi^2)^{\frac{1}{2}} l\hat{u}(\xi)||^2 d\xi$.
The proves the lemma.
In the next result we use $\mathcal{L}^p(R)$ to denote the complexification of the real space $L^p(R)$.

In the following estimates $|\cdot|_p$ denotes the usual norm on $[\mathcal{L}^p(R)]^{2N}$.

Lemma 10.4. *For $t > 0$ the Hilbert space $(h_t, (\cdot,\cdot)_t)$ is continuously embedded in $[L^p(R)]^{2N}$ for $p \in [2, T)$ where $T = +\infty$ if $t \geq \frac{1}{2}$ and $T = \frac{2}{1-2t}$ for $0 < t < \frac{1}{2}$.*

Proof Since $(h_t, (\cdot,\cdot)_t)$ is continuously embedded in $(h_S, (\cdot,\cdot)_S)$ for $s < t$, it is enough to treat the case where $0 < t < \frac{1}{2}$ and $p \in (2, \frac{2}{1-2t})$.

We set $\alpha = \frac{p}{p-2}$. Let $u, v \in [C_0^1(R)]^{2N}$. Then
$| \int < u, v > dx| = | \int < \hat{u}, \hat{v} > d\xi|$
$\leq \{\int (1+\xi^2)^t ||\hat{u}(\xi)||^2 d\xi\}^{\frac{1}{2}} \{\int (1+\xi^2)^{-t} ||\hat{v}(\xi)||^2 d\xi\}^{\frac{1}{2}}$
$\leq (u,u)_t^{\frac{1}{2}} \{\int (1+\xi^2)^{-\alpha t} d\xi\}^{\frac{1}{\alpha}} |\hat{v}|_{2\alpha'}$
$\leq C(\alpha)(u,u)_t^{\frac{1}{2}} |\hat{v}|_{2\alpha'}$ where $C(\alpha) < \infty$ since $2\alpha t > 1$.

But $2\alpha' > 2$ and so, by the Hausdorff-Young inequality, $\exists L > 0$ such that $|\hat{v}|_{2\alpha'} \leq L|v|_\gamma$ where $\frac{1}{\gamma} + \frac{1}{2\alpha'} = 1$. Using the definition of α we obtain $\gamma = p'$ and

$$| \int < u, v > d\alpha| \leq C(\alpha) L (u,u)_t^{\frac{1}{2}} |v|_\gamma \text{ for all } u, v \in [C_0^1(R)]^2$$

Given $u \in [C_0^1(R)]^{2N}$ and setting $v = |u|^{p-2}u$ we have $v \in [C_0^1(R)]^{2N}$ since $p > 2$, and

$$|v|_\gamma = \{\int ||u||^{(p-1)p'} dx\}^{1/p'} = |u|_p^{p/p'}.$$

Hence $\int ||u||^p dx \leq C(\alpha) L (u,u)_t^{\frac{1}{2}} |u|_p^{\frac{p}{p'}}$ and consequently $|u|_p \leq C(\alpha) L (u,u)_t^{\frac{1}{2}} \quad \forall u \in [C_0^1(R)]^{2N}$.
By the density of $[C_0^1(R)]^{2N}$ we obtain the final conclusion.

Corollary 10.5. *Let the conditions (C1) and (C2) be satisfied.*
(1) For all $p \in [2,\infty)$, H *is continuously embedded in* $[L^p(R)]^{2N}$.
(2) For all $p \in [2,\infty)$ *and every* $z > 0$, H *is compactly embedded in* $[L^p(-z,z)]^{2N}$.

Proof We need only prove part (2) and for this we may suppose that $2 < p < \infty$.
Choose $t \in (0, \frac{1}{2})$ such that $p < \frac{2}{1-2t}$.
For $u \in H$ and $R > 0$ we have that $u = v^R + w^R$ where
$v^R(x) = \frac{1}{\sqrt{2\pi}} \int_{\xi \leq R}^{e^{-i\xi n}} \hat{u}(\xi)d\xi$ and
$w^R(x) = \frac{1}{\sqrt{2\pi}} \int_{|\xi| \geq R} e^{-i\xi x} \hat{u}(\xi)d\xi$.
Clearly $v^R \in C^\infty(R)$ and $\|v^R(x)\| + \|(v^R)'(x)\| \leq \frac{1}{\sqrt{2\pi}} \int_{|\xi| \leq R} [1 + |\xi|] \|\hat{u}(\xi)\| d\xi \leq$

$$\frac{1}{2\pi}\{2\int_0^R (1+\xi)^2 d\xi\}^{\frac{1}{2}} \{\int \|\hat{u}(\xi)\|^2 d\xi\}^{\frac{1}{2}} \leq (R+1)\frac{\sqrt{2R}}{\pi}\|u\| \qquad (10,2)$$

Also $\int (1+\xi^2)^t \|\widehat{w^R(\xi)}\|^2 d\xi$

$$= \int (1+\xi^2)^t \chi_R(\xi) \|\hat{u}(\xi)\|^2 d\xi$$

where χ_R is the characteristic function of the set $|\xi| \geq R$. Here we use the fact that w^R can be considered as the inverse Fourier transform of $\hat{u}\chi_R$.
Now $\int (1+\xi^2)^t \chi_R(\xi) \|\hat{u}(\xi)\|^2 d\xi$
$\leq \int (1+R^2)^{t-\frac{1}{2}} \chi_R(\xi)(1+\xi^2)^{\frac{1}{2}} \|\hat{u}(\xi)\|^2 d\xi$ since $t < \frac{1}{2}$
$\leq (1+R^2)^{t-\frac{1}{2}}(u,u)_t$
and so $w^R \in h_t$ with $(w^R, w^R)_t \leq (1+R^2)^{t-\frac{1}{2}}(u,u)_t$
Then for any $z > 0$, $|u|_{L^p(-z,z)}$
$\leq |v^R|_{Z^p(-z,z)} + |w^R|_{Z^p(-z,z)}$
$\leq \max_{|x| \leq z} \|v^R(x)\|(2z)^{1/p} + C(p,t)(w^R, w^R)_t^{\frac{1}{2}}$ by Lemma 10.4
$\leq \max_{|x| \leq z} \|v^R(x)\|(2z)^{1/p} + C(p,t)(1+R^2)^{t-\frac{1}{2}}\|u\|$ since $t < \frac{1}{2}$.
Let $\{u_n\}$ be a sequence in H such that $\|u_n\| \leq M \quad \forall n \in N$. Setting $u = u_m - u_n$ in the above estimte we obtain, $|u_m - u_n|_{L^p(-z,z)}$

$$\leq \max_{|x| \leq z} \|u_m^R(x) - u_n^R(x)\|(2z)^{1/p} + C(p,t)(1+R^2)^{t-\frac{1}{2}} 2M$$

But $\max_{|x| \leq z}\{\|u_n^R(x)\| + \|(u_n^R)^1(x)\|\} \leq (R+1)\frac{R}{\pi}M$ by (10.2) and so using the Ascoli-Arzela Theorem,$\{u_n\}$ has a subsequence $\{u_{n_i}\}$ so converging uniformly on $[-z,z]$ to u.
It follows that

$$|u - u_{n_i}|_{L^{p^c}}(-z,z) \leq \max_{|x| \leq z} \|u(x) - u_{n_i}^R(x)\|(2z)^{\frac{1}{p}} + KR^{2t-1}$$

where $K = C(p,t)2M$, and so

$$\limsup |u - u_{n_i}|_{L^p(-z,z)} \leq KR^{2t-1} \text{ for any } R > 0.$$

Thus $|u - u_{n_i}|_{L^p(-z,z)} \to 0$ and the proof is complete.

Let M_{2N} denote the spare of all real $2N \times 2N$ matrices
$(C3) K \in L^{\infty}(R, M_{2N})$ and $p > 2$.

Under the condition $(C3)$ we define

$$\varphi(u) = \frac{1}{p} \int ||Ku||^p dx \text{ for } u \in H.$$

Using Corollary 10.5 it is easy to check that satisfies the hypothesis $(H3)$ and that

$$\varphi'(u)v = \int ||Ku||^{p-2} < Ku, Kv > dx \quad \forall u, v \in H.$$

Hence $N(u) = \tilde{L}||Ku||^{p-2} K^T Ku$ for $u \in H$ where, following the notation introduced in section 8, $\tilde{L} \in B(H, H)$ is defined by the relation that $(u, v) =< \tilde{L}u, v >$ for all $u \in H$ and $v \in H$. In the present context we clearly have that

$$\tilde{L}u = [(1 + \xi^2)^{-\frac{1}{2}} \hat{u}]^V.$$

Under the assumptions $(C1)$ to $(C3)$ the procedure of section 8 now associates with (1.4) the equation

$$(A - \lambda L)u - N(u) = 0 \text{ for } (\lambda, u) \in R \tag{10.3}$$

where $H = [H^{\frac{1}{2}}(R)]^{2N}$,

$$(A - \lambda L)u = \{(1 + \xi^2)^{\frac{1}{2}} [i\xi J + M]\hat{u}\}^v.$$

Lemma 10.6. *Let the conditions $(C1)$ to $(C3)$ be satisfied and suppose that (λ, u) is a solution of (10.3). Then $u \in [H^1(R)]^{2N}$ and (λ, u) satisfies (1.4).*

Remarks 1. Recalling that $u(x) \to 0$ as $|x| \to \infty$ for all $u \in H^1(R)$, we see that solutions of (10.3) do correspond to homoclinic solutions of (1.4).

2. In fact since $(C2)$ ensures that 0 is a hyperbolic point of (1.4) we can even assert that u delays to (1.4) we can even assert that u delays to 0 exponentially as $|x| \to \infty$.

Proof For all $v \in H$ we have that
$< (A - \lambda L)u, v >=< N(u), v >$ where
$< (A - \lambda L)u, v >= \int < [i\xi J + M]\hat{u}(\xi), hatv(\xi) > d\xi$ and

$$< N(u), v >= \varphi'(u)v = \int ||Ku||^{p-2} < K^T Ku, v > dt.$$

But $u \in H$ and so by Corollary 10.5 (1), $\exists z \in [L^2(R)]^{2N}$ such that $||Ku||^{p-2} K^T Ku = z$. Hence

$$\int < (i\xi J + M]\hat{u}(\xi), \hat{(}\xi) > d\xi = \int < \hat{z}(\xi), \hat{v}(\xi) > d\xi \quad \forall v \in H$$

From this it is easy to see that

$$i\xi\hat{u}(\xi) = J^{-1}[\hat{z}(\xi) - M\hat{u}(\xi)] \in [L^2(R)]^{2N}$$

and hence $u \in [H^1(R^N)]^{2N} = \mathcal{D}(S)$. Thus

$$< (A - \lambda L)u, v >= \int < J\dot{u} + Mu, v > dt \quad \forall v \in H$$

and so (1.4) is satisfied a.e. on R.

Let us now consider some of the other hypothesis of the general theory.

Lemma 10.7. *Let the conditions* $(C1)$ *to* $(C3)$ *be satisfied. (1)* $\exists C > 0$*such that*
$||N(u)|| \leq C\varphi(u)^{1/p'} \quad \forall u \in H,$ *where* $\frac{1}{p} + \frac{1}{p'} = 1$.
(2) $N : H \to H$ *is weakly sequentially continuous.*
(3) If $\lim_{|x|\to\infty} ||K(x)|| = 0$, *then* $N : H \to H$ *is compact.*

Proof Using Corollary 10.5 this result can be established by minimizing step by step the proof of Lemma 9.1. It shows that φ satisfies the conditions $(H4)$ and $(H5)$.

As we have already remarked, the conditions $(C1)$ and $(C2)$ ensure that the operator S satisfies the conditions of section δ and hence there exist A and $L \in B(H, H)$ satisfying the hypotheses $(H1)$ and $(H2)$ such that

$$((S - \lambda I)u, v) =< (A - \lambda I)u, v > \quad \forall u \in \mathcal{D}(S) \text{ and } v \in H.$$

Furthermore, for a and b as defined in section 2, we have that (a, b) is a maximal interval in $R\backslash\sigma(S)$.
Note that by Lemma 2.1, $-\infty < a < 0 < b < \infty$. Following the general solution we set

$$J(\lambda, u) = \frac{1}{2} < (A - \lambda L)u, u > -\varphi(u) \text{ for } (\lambda, u) \in R \times H.$$

In dealing with cases where $\lim_{|x|\to\infty} ||K(x)|| \neq 0$ we use the following hypothesis of periodicity.
$(C4) \quad K(x + 1) = K(x)$ a.e. on R.

For $k \in Z$ we define $T_k \in O(H)$ by $T_k u(x) = u(x - k)$ and we set

$$G = \{T_k : k \in Z\} \tag{10.4}$$

By the some arguments used to prove Lemma 9.2 we obtain the following result.

Lemma 10.8. *Suppose that the conditions* $(C1)$ *to* $(C4)$ *are satisfied. Let* G *be the group defined by (10.4). For all* $\lambda \in R$ *the function* $J(\lambda, \cdot) : H \to R$ *in weakly G-compact.*

We can now state the first result concerning the existence of solutions of (10.3).

Theorem 10.9. *Suppose that the conditions (C1) to (C3) are satisfied and that* $\det K(x) \neq 0$ *a.e. on* R. *Suppose also that either (1)* $\lim_{|x|\to\infty} \|K(x)\| = 0$ *or (2) (C4) holds.*
Then for every $\lambda \notin \sigma(S)$ *there exists* $u_\lambda \in H\backslash\{0\}$ *such that* (λ, u_λ) *is a solution of (10.3).*

Proof This is an immediate consequence of Theorem 5.2.

To obtain a criterion for bifurcatioon we now construct a family of test-functions as required by the condition $T(\delta)$, at points in the spectrum of S.

Given $\lambda \in \sigma(S)$, let $\Psi \in C^\infty(R, R^{2N})$ be a non-trivial periodic solution of $Ju' + Mu = \lambda u$, as established in Corollary 10.2 (3). For $k > 0$ and $x \in R$, define z_k by

$$z_k(x) = k^{-\frac{1}{2}}\eta(\frac{x}{k})\Psi(x) \qquad (10.5)$$

where $\eta \in C_0^\infty(R)$ with $\eta \geq 0$ on R, $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$. Clearly $\int \|z_k(x)\|^2 dx = \int \eta(y)^2 \|\Psi(y)^2\|\Psi(ky)\|^2 dy$ and

$$\int \|z_k'(x)\|^2 dx = k^{-2}\int \eta'(y)^2\|\Psi(ky)\|^2 dy + \int \eta(y)^2\|\Psi'(ky)\|^2 dy.$$

By the Riemann-Lebesgue Lemma,
$\int \|z_k(x)\|^2 dx \to M(\|\Psi\|^2)\int \eta(y)^2 dy$ and
$\int \|z_k'(x)\|^2 dx \to M(\|\Psi'\|^2)\int \eta(y)^2 dy$ as $k \to \infty$,
where $M(f)$ denotes the mean-value of the periodic function f.

On the form space $(H, <\cdot,\cdot>)$ associated with $S, <\cdot,\cdot>^{\frac{1}{2}}$ and $(\cdot,\cdot)^{\frac{1}{2}}_{\frac{1}{2}}$ are equivalent norms as is shown in Lemma 10.3. Also

$$\int \|z_k(x)\|^2 dx \leq (z_k, z_k)_{\frac{1}{2}} \leq (z_k, z_k)_1 = \int \|z_k(x)\|^2 + \|z_k'(x)\|^2 dx$$

and so there exist positive constants k_o, A and B such that

$$0 < A \leq \|z_k\| \leq B \text{ for all } k \geq k_o$$

where $\|z_k\|$ denotes the Hilbert space norm of $(H, <\cdot,\cdot>)$.
Furthermore, $z_k \in \mathcal{D}(S)$ and $(S - \lambda I)z_k(x) = k^{\frac{\lambda}{2}}\eta'(\frac{x}{k})\bar{v}\bar{\Psi}(x)$
from which it follows that
$<(A - \lambda L)z_k, z_k> = ((S - \lambda I)z_k, z_k) = 0$ and

$$\|(A - \lambda L)z_k\|^2 \leq |(S - \lambda I)z_k|^2 = k^2\int \eta'(y)^2\|\Psi(ky\|^2 dy.$$

Finally we note that

$$\begin{aligned} p\varphi(z_k) &= k^{-\frac{p}{2}}\int \eta(\tfrac{x}{k})^p\|K(x)\Psi(x)\|^p dx \\ &= k^{1-\frac{p}{2}}\int \eta(y)^p\|K(ky)\Psi(ky)\|^p dy \end{aligned}$$

From these formulae it is now easy to obtain the following result.

Lemma 10.10. *Suppose that the conditions $(C1)$ to $(C3)$ are satisfied and that* $\det K(x) \neq 0$ *for all x in a set of non zero measure. Then the condition $T(\delta)$ is satisfied for any $\delta < \frac{4}{p}$. If in addition $(C4)$ is satisfied, then $T(\delta)$ is satisfied for any $\delta < \frac{4}{p-2}$.*

Proof As noted in section 8 the first part of condition $T(\delta)$ is automatically satisfied in this context. Let $b \in \sigma(S)$ and consider the functions defined by (10.5). By hypothesis there is a bounded interval T such that $\int_T \|K(x)\Psi(x)\|^p dx > 0$ and for k large enough $\eta(\frac{x}{k}) \equiv 1$ for all $x \in T$. Hence there exist $k_o > 0$ and $\gamma > 0$ such that $\varphi(z_k) \geq \gamma k^{-\frac{p}{2}}$ for all $k \geq k_o$.
Setting $u_n = \frac{z_n}{\|z_n\|}$, it is now easy to check that for n large enough the sequence $\{u_n\}$ satisfies part (u) of condition $T(\delta)$ for any $\delta < \frac{4}{p}$.
If in addition $(C4)$ holds, there exist $k_0 > 0$ and $\gamma > 0$ such that $\varphi(z_k) \geq \gamma k^{1-\frac{p}{2}}$ for all $k \geq k_0$.
It then follows that part $(\ddot{u})$ of $T(\delta)$ is satisfied for any $\delta < \frac{4}{p-2}$.

From Theorems 6.4 and 7.2 we now obtain the following result about the system (1.4).

Theorem 10.11. *Suppose that the conditions $(C1)$ to $(C3)$ are satisfied and that* $\det K(x) \neq 0$ *for all x in a set of non-zero measure.*
(i) If $\lim_{|x|\to\infty} \|K(x)\| = 0$ then there exist $\lambda^* \in [0, b)$ such that, for all $\lambda \in (\lambda^*, b)$ there exists a solution $u_\lambda \in H\backslash\{0\}$ of (10.3). Furthermore if $p < 4$ then b is a bifurcation point of order θ for any $\theta < \frac{1}{p-2}(1 - \frac{p}{4})$.
(ii) If $(C4)$ is satisfied then again there exists $\lambda^* \in [0, b)$ such that, for all $\lambda \in (\lambda^*, b)$, there exists a solution $u_\lambda \in H\backslash\{0\}$ of (10.3). Furthermore if $p < b$ then b is a bifurcation point of order θ for any $\theta < \frac{1}{p-2} - \frac{1}{4}$.

Remark By Lemma 10.6 the solutions u_λ of (10.3) satisfy (1.4) a.e. on R.

In order to deal with situations where $\det K(x)$ may be zero for all $x \in R$ we suppose that the linear part of (1.4) has the following block structure.

$(B1)$ $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$ where I is the $N \times N$ matrix such that $\det B \neq 0$.

This assumption leads to the following properties of S.

Lemma 10.12. *The condition $(B1)$ implies that $(C1)$ and $(C2)$ are satisfied and that $\sigma(S) = (-\infty, -b] \cup [b, \infty)$ where $b = \min\{|\lambda| : \lambda \in \sigma(B)\}$. Furthermore $b \in \sigma(M)$.*

Proof Checking $(C1)$ is trivial. For $\xi \in R$, $\lambda \in w(\xi) \iff \exists v, w \in C^N$ such that $\|v\|^2 + \|w\|^2 = 1$ and

$$(i\xi J + M - \lambda I)\begin{pmatrix} v \\ w \end{pmatrix} = 0$$

$\iff v, w \in C^N$ such that $\|v\|^2 + \|w\|^2 = 1$ and

$$\begin{cases} (B - i\xi I)w &= \lambda v \\ (B + i\xi I)v &= \lambda w \end{cases}$$

But this system implies that $(B^2+\xi^2 I)v = \lambda^2 v$ and that $(B^2+\xi^2 I)w = \lambda^2 w$.
Hence $\lambda \in w(\xi)$ implies that $\lambda^2 - \xi^2 \in \sigma(B^2)$.
Now if $\lambda \in \sigma(S)$, Lemma 10.1 shows that $\exists \xi \in R$ such that $\lambda \in w(\xi)$ and hence $\lambda^2 \geq \inf \sigma(B^2) = b^2$. Thus $\sigma(S) \subset (-\infty, -b] \cup [b, \infty)$.
For the converse we observe that
either (i) $\exists w \in R^N$ such that $||w|| = 1$ and $Bw = bw$
or (ii) $(\dot{u})\exists w \in R$ such that $||v|| = 1$ and $Bv = -bv$ (perhaps both).

Given $\lambda \in R$ such that $\lambda^2 \geq b^2$ we set $\xi = \sqrt{\lambda^2 - b^2}$. In the first case we set $v = \frac{(b-i\xi)w}{\lambda}$ and that $(B - i\xi I)w = (b - i\xi)w = \lambda v$ whereas

$$(B - i\xi I)v = \frac{(b-ic)(B+i\xi I)w}{\lambda} = \frac{(b^2+\xi^2)w}{\lambda} = \lambda w$$

Hence we see that $\lambda \in w(\xi) \subset \sigma(S)$.
In the second case we set $w = \frac{(-b+i\xi)}{v}\lambda v$ and again we find that $\lambda \in w(\lambda) \subset \sigma(S)$.
Thus $\sigma(S) = (-\infty, -b] \cup [b, \infty)$ and $(C2)$ follows from Lemma 10.2 (2).
Furthermore we have shown that in case (i),

$$b \in w(0) \text{ and } (M - bI)\begin{pmatrix} w \\ w \end{pmatrix} = 0$$

whereas in case (ii), $b \in w(0)$ and $(M - bI)\begin{pmatrix} v \\ -v \end{pmatrix} = 0$.

We now turn to the test-functions z_k defined by (10.5) at $\lambda = b$. By Lemma 10.12, Ψ can be any vector in $\ker(M - bI)$ with $||\Psi|| = 1$. To ensure that we can choose such a vector Ψ in such a way that the potential φ does not vanish on these test-functions we introduce the following hypothesis.
$(B2)$ The condition $(C3)$ is satisfied and there exists $\Psi \in \ker(M - bI)$ such that $||K(x)\Psi|| > 0$ for all x in a set of non-zero measure.

Under the hypotheses $(B1)$ and $(B2)$ the conclusions of Lemma 10.10 remain valid. Hence we also obtain the following result.

Theorem 10.13. *Let the conditions $(B1)$ and $(B2)$ be satisfied. Then the statements in parts (i) and (ii) of Theorem 10.11 remain valid.*

We close this discussion of (44) with an example which shows that the conclusions of Theorem 10.13 are sharp.
Exemple Consider the problem (1.4) where $N = 1$,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, K(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \forall t \in R.$$

Clearly conditions $(B1)$ and $(C4)$ are satisfied. Also $b = 1$ and $\ker(M - bI) = span\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$. Hence $(B2)$ is satisfied and by Theorem 10.13 *(ii)* we conclude that $\exists\lambda^* \in (-1,1)$ such that for all $\lambda \in (\lambda^*, 1)$ the system

$$\begin{cases} x'(t) + \lambda(t) = \lambda y(t) \\ -y'(t) + y(t) - |x(t)|^{p-2}x(t) = \lambda x(t) \end{cases} \tag{10.6}$$

has a solution $u_\lambda = (x_\lambda, y_1) \in [H^1(R)]^2$.
Furthermore if $p < 6$, then I is a bifurcation point of order θ for any $\theta < \frac{1}{p-2} - \frac{1}{4}$.
However the system (10.6) can be solved explicitly and we find that the above conclusions are sharp.
For the latter purpose it is enough to make the following observations. The system (10.6) is equivalent to the equation

$$x''(t) = (1-\lambda^2)x(t) - \lambda|x(t)|^{p-2}x(t)$$

and so $x'(t)^2 - (1-\lambda^2)x(t)^2 + \frac{2\lambda}{p}|x(t)|^p$ is constant for all solutions. Hence there is a solution $x_\lambda \not\equiv 0$ with $|x(t)| \to 0$ as $|t| \to \infty$ if and only if $0 < \lambda < 1$.
Furthermore for $0 < \lambda < 1$, x_λ is unique up to translation and $|x_\lambda|_\infty = \{\frac{p(1-\lambda^2)}{2\lambda}\}^{\frac{1}{p-2}}$. Setting $y_\lambda = \frac{1}{\lambda}[x' + x]$, we see that $u_\lambda = (x_\lambda, y_\lambda)$ is a solution of (10.6) and u_λ tends to zero exponentially as $|t| \to \infty$, since x_λ and x'_λ do so. By rescaling the variables t and x we find that $\exists C > 0$ such that

$$\int x_\lambda(t)^2 dt = C\frac{(1-\lambda^2)^{\frac{2}{p-2}-\frac{1}{2}}}{\lambda^{2/p-2}} \text{ for } 0 < \lambda < 1$$

Since $||u_\lambda|| \geq \{\int x_\lambda(t)^2 dt\}^{\frac{1}{2}}$ we deduce that $||u_\lambda|| \to \infty$ as $\lambda \to 0$, for all $p > 2$. Also if $\frac{1}{p-2} - \frac{1}{4} \leq 0$ there cannot be bifurcation at $\lambda = 1$, and for $\frac{1}{p-2} - \frac{1}{4} > 0$ the order of bifurcation, θ, must satisfy $\theta < \ frac1p - 2 - \frac{1}{4}$.

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