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CORRECTIONS TO: EXTENDING THE MARTINGALE MEASURE STOCHASTIC INTEGRAL WITH APPLICATIONS TO SPATIALLY HOMOGENEOUS S.P.D.E.'S[1]

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The proof of Lemma 19 of the paper is not correct as given. In fact, in order to prove the lemma, a stronger hypothesis than the one given in the paper seems to be necessary.

Before stating the new hypothesis, we mention a stronger version of Theorem 3 of the paper (though the theorem is correct as stated). Indeed, its hypothesis (33) is *not* satisfied in the case where S is the fundamental solution of the heat equation. In order that this be the case, we replace (33) by the following.

$$\lim_{h\downarrow 0}\int_0^T dt \int_{\mathbb{R}^d}\mu(d\xi) \sup_{t<r<t+h} |\mathcal{F}S(r)(\xi)-\mathcal{F}S(t)(\xi)|^2 = 0. \tag{1}$$

Notice that the supremum is over r near but greater than t, whereas in (33), the supremum is two-sided.

The proof of Theorem 3 under this weaker hypothesis only requires the following changes. Instead of the definition in the paper, define $\varphi_n(t,x)$ as follows:

$$\varphi_n(t,x) = \sum_{k=0}^{2^n-1} \varphi(t_n^{k+1},x) 1_{[t_n^k,t_n^{k+1}[}(t),$$

The remainder of the proof is essentially unchanged, except that for $t \in [t_n^k, t_n^{k+1}[$, $\mathcal{F}\varphi_n(t,\cdot)(\xi) = \mathcal{F}\varphi(t_n^{k+1},\cdot)(\xi)$ for all ξ, so $\|\varphi-\varphi_n\|_0^2$ converges to 0 by (1).

[1]Electronic Journal of Probability Vol. 4, no.6 (1999), 1–29.

We shall replace Hypothesis B of the paper by the stronger Hypothesis C below.

Hypothesis C. Assume, in addition to Hypothesis B (with (33) replaced by the weaker condition (1) above), that:

(i) $t \mapsto \mathcal{F}\Gamma(t)(\xi)$ is continuous, for all $\xi \in \mathbb{R}^d$;

(ii) there is $\varepsilon > 0$ and a function $t \mapsto k(t)$ with values in the space of non-negative distributions with rapid decrease such that for all $t \geq 0$ and $h \in [0, \varepsilon]$,

$$|\mathcal{F}\Gamma(t+h)(\xi) - \mathcal{F}\Gamma(t)(\xi)| \leq |\mathcal{F}k(t)(\xi)|,$$

and

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}k(t)(\xi)|^2 < \infty.$$

We note that as in Remark 4, (ii) implies (1) above.

Proof that Lemma 19 holds under Hypothesis C.

Fix $n \geq 0$, assume by induction that u_n is L^2-continuous, and let $Z_n = \alpha(u_n(t,x))$. We begin with the time increments. For $t \in [0,T]$, $x \in \mathbb{R}^d$ and $h > 0$, observe from (52) and the definition of $\|\cdot\|_{0,Z}$ that

$$E((u_{n+1}(t,x) - u_{n+1}(t+h,x))^2) \leq 2(E_1 + E_2),$$

where

$$\begin{aligned}
E_1 &= \|\Gamma(t-\cdot, x-\cdot) - \Gamma(t+h-\cdot, x-\cdot)\|^2_{0,Z_n}, \qquad (2)\\
E_2 &= E\Big(\Big(\int_0^t ds \int_{\mathbb{R}^d} \beta(u_n(t-s, x-y))\ \Gamma(s, dy) \\
&\qquad - \int_0^{t+h} ds \int_{\mathbb{R}^d} \beta(u_n(t+h-s, x-y))\ \Gamma(s,dy)\Big)^2\Big).
\end{aligned}$$

Notice by Lemma 18 that the law of the time-increments does not depend on x, so the L^2-norm will not either. By definition of $\|\cdot\|_{0,Z_n}$,

$$\begin{aligned}
E_1 &= \int_0^t ds \int_{\mathbb{R}^d} \mu_s^{Z_n}(d\xi)\ |\mathcal{F}(\Gamma(t-s, x-\cdot) - \Gamma(t+h-s, x-\cdot))(\xi)|^2 \\
&\qquad + \int_t^{t+h} ds \int_{\mathbb{R}^d} \mu_s^{Z_n}(d\xi)\ |\mathcal{F}\Gamma(t+h-s, x-\cdot)(\xi)|^2 \\
&= \int_0^t ds \int_{\mathbb{R}^d} \mu_s^{Z_n}(d\xi)\ |\mathcal{F}(\Gamma(t-s,\cdot) - \Gamma(t+h-s,\cdot))(\xi)|^2 \\
&\qquad + \int_t^{t+h} ds \int_{\mathbb{R}^d} \mu_s^{Z_n}(d\xi)\ |\mathcal{F}\Gamma(t+h-s,\cdot)(\xi)|^2,
\end{aligned}$$

and so E_1 does not depend on x. The second integral has limit 0 by (28), (51) and (53). As for the first integral, observe that the integrand goes to 0 pointwise by (i) in Hypothesis C, and by

(ii) in this hypothesis, the integrand is bounded by $|\mathcal{F}k(t-s)(\xi)|^2$, and by Theorem 2,

$$\begin{aligned}
&\int_0^t ds \int_{\mathbb{R}^d} \mu_s^{Z_n}(d\xi)\ |\mathcal{F}k(t-s)(\xi)|^2 \\
&\qquad \leq \sup_{0\leq s\leq T} \sup_{x\in\mathbb{R}^d} E(Z_n(s,x)^2) \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}k(t-s)(\xi)|^2 \\
&\qquad < +\infty
\end{aligned}$$

by Hypothesis C. Therefore, the Dominated Convergence Theorem applies and so E_1 has limit 0 as $h \downarrow 0$. Note that the calculation is similar if one considers increments of the form $E((u_{n+1}(t,x) - u_{n+1}(t-h,x))^2)$.

The proof that $E_2 \to 0$ as $h \downarrow 0$ is the same as in the original proof of Lemma 19.

We now consider spatial increments. Observe from (52) and the definition of $\|\cdot\|_{0,Z_n}$ that

$$E((u_{n+1}(t,x) - u_{n+1}(t,y))^2) \leq 2(F_1 + F_2),$$

where

$$\begin{aligned}
F_1 &= \|\Gamma(t-\cdot, x-\cdot) - \Gamma(t-\cdot, y-\cdot)\|^2_{0,Z_n}, \\
F_2 &= E\left(\left(\int_0^t ds \int_{\mathbb{R}^d} (\beta(u_n(t-s,x-z)) - \beta(u_n(t-s,y-z)))\,\Gamma(s,dz)\right)^2\right).
\end{aligned}$$

Note that

$$\begin{aligned}
F_1 &= \int_0^t ds \int_{\mathbb{R}^d} \mu_s^{Z_n}(d\xi)\ |\mathcal{F}(\Gamma(t-s,x-\cdot) - \Gamma(t-s,y-\cdot))(\xi)|^2 \\
&= \int_0^t ds \int_{\mathbb{R}^d} \mu_s^{Z_n}(d\xi)\ |1 - e^{i\,\xi\cdot(x-y)}|^2\ |\mathcal{F}\Gamma(t-s,\cdot)(\xi)|^2.
\end{aligned}$$

The integrand converges pointwise to 0 as $|x-y| \to 0$. Because $|1 - e^{i\,\xi\cdot(x-y)}|^2 \leq 4$, we use (28), (53) and (26) together with the Dominated Convergence Theorem to conclude that $F_1 \to 0$ as $|x-y| \to 0$. The proof that $F_2 \to 0$ as $|x-y| \to 0$ is the same as in the original proof of Lemma 19.

We have shown that $x \mapsto u_{n+1}(t,x)$ is L^2-continuous for t fixed, and $t \mapsto u_{n+1}(t,x)$ is L^2-equicontinuous for $x \in \mathbb{R}^d$, $0 \leq t \leq T$, so $(t,x) \mapsto u_{n+1}(t,x)$ is L^2-continuous. This proves Lemma 19. □

VERIFICATION OF HYPOTHESIS C FOR THE WAVE EQUATION.

Assume $d \leq 3$ and let $\Gamma = \Gamma_1$ be the fundamental solution of the wave equation (as in Example 6). Assuming that (40) holds, we only check (i) and (ii) in Hypothesis C, because the remaining conditions have already been checked in the paper.

Set $k(t)(x) = 2G_1(x)$, where G_1 is the modified Bessel function defined in [28, Chap.V§3, Prop.2, p.132]. This function is non-negative, with rapid decrease in x, and $\mathcal{F}k(t)(\xi) = 2(1+|\xi|^2)^{-1/2}$.

Further,

$$
\begin{aligned}
|\mathcal{F}\Gamma(t-s+h)(\xi) - \mathcal{F}\Gamma(t-s)(\xi)| &= \left| \frac{\sin(2\pi(t-s+h)|\xi|)}{2\pi|\xi|} - \frac{\sin(2\pi(t-s)|\xi|)}{2\pi|\xi|} \right| \\
&= \frac{|\sin(\pi h|\xi|)\cos(2\pi(t-s+h/2)|\xi|)|}{2\pi|\xi|} \\
&\leq \frac{|\sin(\pi h|\xi|)|}{2\pi|\xi|}.
\end{aligned}
$$

Observe that for $x \geq 0$ and $0 \leq h \leq 1$,

$$
\frac{|\sin(hx)|}{x} \leq \frac{2}{(1+x^2)^{1/2}}.
$$

Finally, notice that

$$
\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}k(t)(\xi)|^2 = T \int_{\mathbb{R}^d} \mu(d\xi)\ \frac{4}{1+|\xi|^2} < \infty
$$

by (40). □

VERIFICATION OF HYPOTHESIS C FOR THE HEAT EQUATION.

Let $\Gamma = \Gamma_3$ be the fundamental solution of the heat equation (as in Example 8). As above, we only check (i) and (ii) of Hypothesis C. Take $k(t)(x) = \Gamma(t)(x)$, where $\Gamma(t)(x)$ is the heat kernel. This is a non-negative function with rapid decrease in the space variables, and

$$
\begin{aligned}
|\mathcal{F}\Gamma(t-s+h)(\xi) - \mathcal{F}\Gamma(t-s)(\xi)| &= \exp(-4\pi^2(t-s)|\xi|^2) - \exp(-4\pi^2(t-s+h)|\xi|^2) \\
&= (1-\exp(-4\pi^2 h|\xi|^2))\exp(-4\pi^2(t-s)|\xi|^2) \\
&\leq |\mathcal{F}\Gamma(t-s)(\xi)|.
\end{aligned}
$$

Furthermore,

$$
\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}k(t-s)(\xi)|^2 < \infty
$$

by (40). □

OTHER MINOR CORRECTIONS

On page 22, $M_n(t)$ should be defined without the supremum over s:

$$
M_n(t) = \sup_{x\in\mathbb{R}^d} E(|u_{n+1}(t,x) - u_n(t,x)|^p).
$$

The remainder of the proof is correct, except that in order to have the desired uniform convergence of $(u_n(t,x))$, it is necessary to check that for $0 \leq t \leq T$,

$$
M_n(t) \leq a_n, \qquad \text{with } \sum_{n=1}^{\infty} a_n^{1/p} < \infty.
$$

For this, the statement of Lemma 15 should be strengthened as follows.

Lemma 15 *(Extension of Gronwall's Lemma.) Let $g : [0,T] \to \mathbb{R}_+$ be a non-negative function such that*

$$\int_0^T g(s)\, ds < +\infty.$$

Then there is a sequence $(a_n, n \in \mathbb{N})$ of non-negative real numbers such that for all $p \geq 1$, $\Sigma_{n=1}^{\infty} a_n^{1/p} < \infty$, and with the following property. Let $(f_n, n \in \mathbb{N})$ be a sequence of non-negative functions on $[0,T]$ and k_1, k_2 be non-negative numbers such that for $0 \leq t \leq T$,

$$f_n(t) \leq k_1 + \int_0^t (k_2 + f_{n-1}(s))g(t-s)\, ds. \tag{3}$$

If $\sup_{0 \leq s \leq T} f_0(s) = M$, then for $n \geq 1$,

$$f_n(t) \leq k_1 + (k_1 + k_2) \sum_{i=1}^{n-1} a_i + (k_2 + M) a_n. \tag{4}$$

In particular, $\sup_{n \geq 0} \sup_{0 \leq t \leq T} f_n(t) < \infty$, and if $k_1 = k_2 = 0$, then $\Sigma_{n \geq 0} f_n(t)^{1/p}$ converges uniformly on $[0,T]$.

The proof of the lemma is unchanged, except for the last line, which becomes "Finally, $\sum_{n=1}^{\infty} a_n^{1/p} < \infty$ by Lemma 17 below." The statement of Lemma 17 should be strengthened as follows.

Lemma 17 *Let F be the common distribution function of an i.i.d. sequence $(X_n,\ n \in \mathbb{N})$ of non-negative random variables and fix $p \geq 1$. Suppose that $F(0) = 0$ and set $S_n = X_1 + \cdots + X_n$. Then for any $a \geq 1$ (and trivially, for $0 \leq a < 1$) and $t > 0$,*

$$\sum_{n=1}^{\infty} a^{n/p} P\{S_n \leq t\}^{1/p} < +\infty. \tag{5}$$

The proof of Lemma 17 remains unchanged.

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