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# Complex WKB method for 3-level scattering systems

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**Abstract.** In this note the $S$-matrix naturally associated with a singularly perturbed three-dimensional system of linear differential equations without turning point on the real axis is considered. It is shown that for a fairly large class of examples, the Complex WKB method gives results far better than what is proven under generic circumstances. In particular, we show how to compute asymptotically all exponentially small off-diagonal elements of the corresponding $S$-matrix.

Keywords: Singular perturbations, semiclassical analysis, adiabatic approximations, exponential asymptotics, $n$-level $S$-matrix, turning point theory.

## 1. Introduction

We consider the computation of the leading term of exponentially small elements of the $S$-matrix naturally associated with singularly perturbed 3-dimensional systems of linear ordinary differential equations without turning points on the real axis by means of the complex WKB method. Several progresses have been made during the last few years on general aspects of this method in several directions, such as the improvement of the asymptotics it yields [11] or its application to systems of ODE of higher dimension than two [7,3]. However, it is well known [2], that in general the $S$-matrix cannot be completely determined asymptotically for systems of dimension higher than 2.

In this note, we present a model whose study illustrates the fact that the complex WKB method can actually give results for specific cases going beyond those proven in [7] or [3] for "generic" three-dimensional systems. Indeed, for this model the *whole* $S$-matrix is computed asymptotically. Moreover, and this is the main point of this study, this is true for a whole class of systems we describe at the end of the paper.

Before introducing our model, let us mention that the complex WKB theory has a very long history which can be retraced in the classics [4,15,14,1] for example. More recent developpements as well as studies of non-generic situations can be found in [10,11,13,7,3] and references therein. The reader is directed to this non-exhaustive list for an historic point of view and precise references on the general aspects of the theory.

We now define our model and then explain in more details the strategy we will follow to determine the corresponding $S$-matrix. Consider the following system in the singular limit $\varepsilon \to 0$

$$i\varepsilon\psi'(t) = H(t,\delta)\psi(t), \quad t \in \mathbf{R},\ \varepsilon \to 0, \tag{1.1}$$

where the prime denotes the derivative with respect to $t$, $\psi(t) \in \mathbf{C}^3$, and

$$H(t,\delta) = H(t,0) + \delta V(t) = \begin{pmatrix} 1 & \tanh(t) & 0 \\ \tanh(t) & -1 & 0 \\ 0 & 0 & \alpha\tanh(t) \end{pmatrix} + \delta \begin{pmatrix} 0 & 0 & t\,\mathrm{e}^{-t^2} \\ 0 & 0 & 0 \\ t\,\mathrm{e}^{-t^2} & 0 & 0 \end{pmatrix} \tag{1.2}$$

for $\alpha > 3\sqrt{2}$. Here, $\delta$ denotes a small strictly positive parameter which will be fixed below. It is readily seen that for $\delta > 0$ the eigenvalues $e_j(t,\delta)$, $j = 1,2,3$, are non-degenerate for any $t \in \mathbf{R}$, including $t = \pm\infty$, see below. This is the statement of the absence of turning point on the real axis, which is an important hypothesis for the method. We note that the generator $H(t,\delta)$ is hermitian

$$H(t,\delta) = H^*(t,\delta) \tag{1.3}$$

and that it can be continued analytically in the strip $\Sigma = \{z \mid |\mathrm{Im}\, z| < \pi/2\}$. The $S$-matrix associated with this equation is defined as follows. Let $\varphi_j(t,\delta)$ be a complete set of normalized eigenvectors of $H(t,\delta)$ for $t \in \mathbf{R}$, associated with the eigenvalues $e_j(t,\delta)$, $j = 1,2,3$, which are uniquely determined (up to a constant factor) by the phase fixing condition

$$\langle \varphi_j(t,\delta) | \varphi_j'(t,\delta) \rangle \equiv 0, \quad \forall t \in \mathbf{R},\ j = 1,2,3. \tag{1.4}$$

Here $\langle \cdot \mid \cdot \rangle$ denotes the usual scalar product in $\mathbf{C}^3$.

It can be shown that these eigenvectors are analytic in $t$ in a neighbourhood of the real axis if $H(t,\delta)$ is analytic and self-adjoint on the real axis [12]. Hence, any solution $\psi(t)$ of (1.1) can be expanded as

$$\psi(t) = \sum_{j=1}^{3} c_j(t)\,\mathrm{e}^{-\mathrm{i}\int_0^t e_j(s,\delta)\,\mathrm{d}s/\varepsilon} \varphi_j(t,\delta) \tag{1.5}$$

by means of unknown coefficients $c_j(t)$, $j = 1,2,3$, to be determined (omitting the $\varepsilon$ and $\delta$ dependence in the notation). The phases $\mathrm{e}^{-\mathrm{i}\int_0^t e_j(s,\delta)\,\mathrm{d}s/\varepsilon}$ are introduced for convenience. By inserting (1.5) in (1.1) we get the following differential equation for the $c_j(t)$'s

$$c_j'(t) = \sum_{k=1}^{3} a_{jk}(t,\delta)\,\mathrm{e}^{\mathrm{i}\Delta_{jk}(t,\delta)/\varepsilon} c_k(t), \tag{1.6}$$

where

$$\Delta_{jk}(t,\delta) = \int_0^t \big(e_j(s,\delta) - e_k(s,\delta)\big)\,\mathrm{d}s \tag{1.7}$$

and

$$a_{jk}(t,\delta) = -\langle \varphi_j(t,\delta) | \varphi_k'(t,\delta) \rangle. \tag{1.8}$$

Since our generator (1.2) tends to limiting matrices $H(\pm)$ (independent of $\delta$) fast enough as $t \to \pm\infty$ (exponentially fast), we have the existence of the limits[1]

$$\lim_{t\pm\infty} c_j(t) = c_j(\pm\infty), \quad j = 1, 2, 3. \tag{1.9}$$

Then we define the $S$-matrix, $S \in M_3(\mathbf{C})$, by the identity

$$S \begin{pmatrix} c_1(-\infty) \\ c_2(-\infty) \\ c_3(-\infty) \end{pmatrix} = \begin{pmatrix} c_1(+\infty) \\ c_2(+\infty) \\ c_3(+\infty) \end{pmatrix}. \tag{1.10}$$

We can think of Eq. (1.1) as describing the adiabatic limit of the Schrödinger equation governed by the time-dependent hamiltonian (1.2) and the $S$-matrix describes the transition probabilities between the different energy levels.

Under our hypotheses, the elements $s_{jk}$, $j, k \in \{1, 2, 3\}$, of the $S$-matrix satisfy the estimates [1,2,4, 8,10,11,13–15]

$$s_{jj}(\varepsilon) = 1 + \mathcal{O}(\varepsilon), \quad j \in \{1, 2, 3\}, \tag{1.11}$$

$$s_{jk} = \mathcal{O}\left(\mathrm{e}^{-\Gamma/\varepsilon}\right), \quad \Gamma > 0,\ j \neq k \tag{1.12}$$

in the limit $\varepsilon \to 0$. We will compute the leading term of *all* exponentially small off-diagonal elements of the $S$-matrix, as $\varepsilon \to 0$.

This leading behaviour can be computed by shifting the path of integration of Eq. (1.1) from the real axis to the upper or lower half plane, a harmless procedure for the solution $\psi$ which is analytic in $\Sigma$. The leading term we are looking for is thus determined somehow by the turning points, or degeneracy points, defined as the set of $z_0 \in \mathbf{C}$ such that $e_j(z_0, \delta) = e_k(z_0, \delta)$ for $e_j$ and $e_k$, some analytic continuation in $\Sigma$ of the corresponding eigenvalues defined on the real axis. The idea is to make use of the generic multi-valuedness of the eigenvalues and eigenvectors of $H$ in the expansion (1.5) and thus get an exponentially small contribution as $\varepsilon \to 0$ from the analytic continuations of the phases $\mathrm{e}^{-\mathrm{i}\int_0^t e_j(s,\delta)\,\mathrm{d}s/\varepsilon}$. This is the so-called complex WKB method. Roughly speaking, this is some kind of steepest descent analysis which, in general, requires more than one path in the complex plane. Moreover, because of their global nature, the technical assumptions required to validate from a mathematical point of view this formal procedure yielding the asymptotics of the $S$-matrix are neither easy to check, nor always satisfied. This important step is sometimes neglected in applications, leading to incorrect results.

In the case $n = 2$, the situation is nevertheless well understood now, at least in generic cases, see [8, 10,11]. However, the corresponding conditions required when $n \geqslant 3$ may be incompatible for a given generator, see [1,2] and [5]. This fact is expressed by M.V. Fedoryuk in the review [2] in the following way: “In short, at present there is no global asymptotic theory for [linear] equations of order $n \geqslant 3$ and, in the author’s opinion, it is impossible to construct one in general.” Nevertheless, some light has been cast recently on the case $n \geqslant 3$ in a perturbative context we describe in more details below. Roughly speaking, $H$ is assumed to depend on a supplementary parameter $\delta \geqslant 0$ such that for $\delta = 0$, the spectrum of $H(t, \delta = 0)$, $t \in \mathbf{R}$, displays real degeneracy points which, for $\delta > 0$, are turned into avoided crossings,

[1]Actually, a dependence in $\delta$ of the limiting matrices is allowed and a decay characterized by $\lim_{t\to\pm\infty} |t|^{1+a} \sup_{s|t+\mathrm{i}s\in\Sigma} \|H(t + \mathrm{i}s, \delta) - H(\pm, \delta)\| < \infty$, uniformly in $\delta$, for some $a > 0$ is enough [11,7].

i.e., degeneracy points close to the real axis. Under some genericity conditions, it is possible to get the asymptotic behaviour of *some* off-diagonal elements of the $S$-matrix, namely those governed by these avoided crossings.

The construction of our model (1.2) can now be explained. When $\delta = 0$, $H(t,0)$ consists in the direct sum of a 2-dimensional system and a 1-dimensional system. The 2-dimensional system is such that the off-diagonal elements of the associated $S$-matrix are governed by one degeneracy point, say $z_0$. The third eigenvalue admits one real crossing point with each eigenvalue of the 2-dimensional system. For $\delta > 0$ small, the two sub-systems get coupled and the real crossings mentionned above become avoided crossings. Thus, the results in [7] yield the elements of the $S$-matrix which are governed by the induced degeneracy points close to the real axis. We show that due to the properties of the unperturbed 2-dimensional sub-system and to the perturbative nature of the construction, it is also possible to compute the missing elements of the $S$-matrix associated with the perturbed degeneracy point which corresponds to $z_0$ and which is thus located far from the real axis in the complex plane. As will be clear from our analysis, a whole class of models sharing these same general properties can be dealt with in a similar fashion. This class of models is specified in Section 6.

It is also possible to improve the accuracy of the asymptotic formulae we derive by making use of the so-called superasymptotic renormalization procedure, see [9,11,7], but we will not deal with this systematic aspect of the theory.

In the next two sections, we recall the basics and the main results in the complex WKB method and make precise the hypotheses under which it works. The following sections contain the detailed study of our model whereas the definition of the class of systems for which equivalent results hold follows.

## 2. Analyticity properties

As a first step, we recall the analyticity properties of the quantities of interest. The proofs of the statements made here can be found in [12] and [6] for example.

The generator $H(z,\delta)$ being analytic in $\Sigma$, the solution of the linear equation (1.1) $\psi(z)$ is analytic in $\Sigma$ as well. The eigenvalues and the eigenvectors defined by (1.4) of $H$ can be analytically continued in $\Sigma$ but they may have isolated singularities, actually branching points at the set of degeneracies $\Omega(\delta)$, given by

$$\Omega(\delta) = \{z_0 \mid e_j(z_0,\delta) = e_k(z_0,\delta),\ \text{for some } k,j \text{ and some analytic continuation}\}. \tag{2.1}$$

This set is symmetric with respect to the real axis due to Schwarz's principle. We determine $\Omega(\delta)$ in a perturbative manner in the parameter $\delta$: when $\delta = 0$, the unperturbed eigenvalues are given for all $z \in \Sigma$ by

$$e_1(z,0) = \alpha \tanh(z), \tag{2.2}$$

$$e_2(z,0) = -\sqrt{1+\tanh^2(z)}, \tag{2.3}$$

$$e_3(z,0) = +\sqrt{1+\tanh^2(z)}. \tag{2.4}$$

They display two real eigenvalue crossings at the points

$$\{t_1 = -\operatorname{arctanh}(1/\sqrt{\alpha^2-1}),\ t_2 = \operatorname{arctanh}(1/\sqrt{\alpha^2-1})\}$$

such that

$$e_1(t_1,0) = e_2(t_1,0), \tag{2.5}$$

$$e_1(t_2,0) = e_3(t_2,0), \tag{2.6}$$

$$\frac{\partial}{\partial t}\big(e_1(t,0) - e_2(t,0)\big)\big|_{t=t_1} \neq 0, \tag{2.7}$$

$$\frac{\partial}{\partial t}\big(e_1(t,0) - e_3(t,0)\big)\big|_{t=t_2} \neq 0. \tag{2.8}$$

Note that all eigenvalues are analytic at the real degeneracy points $t_1$, $t_2$ due to the self-adjointness of $H(t,0)$ for real $t$. The only other degeneracy points in $\Sigma$ are $\{z_0 = \mathrm{i}\pi/4, \overline{z}_0\}$ such that

$$e_2(\pm z_0, 0) = e_3(\pm z_0, 0). \tag{2.9}$$

They are generic in the sense that $\pm z_0$ are square root branching points for $e_3(z,0) - e_2(z,0)$. Hence

$$\Omega(0) = \big\{\mathrm{arctanh}\big(1/\sqrt{\alpha^2-1}\big), -\mathrm{arctanh}\big(1/\sqrt{\alpha^2-1}\big), \mathrm{i}\pi/4, -\mathrm{i}\pi/4\big\}. \tag{2.10}$$

When $\delta > 0$ is small enough, the eigenvalues $e_j(t,\delta)$, $j = 1,2,3$, are nondegenerate for any $t \in \mathbf{R}$ and are analytic in a neighbourhood of the real axis. We have the following asymptotic relations with the unperturbed eigenvalues $e_j(t,0)$: we fix the indices of the non-crossing eigenvalues $e_j(z,\delta)$, $z$ real, by continuity and the condition

$$e_j(-\infty,\delta) = e_j(-\infty,0), \quad j = 1,2,3. \tag{2.11}$$

Then, we have

$$\begin{aligned} e_1(+\infty,\delta) &= e_2(+\infty,0), \\ e_2(+\infty,\delta) &= e_3(+\infty,0), \\ e_3(+\infty,\delta) &= e_1(+\infty,0), \end{aligned} \tag{2.12}$$

see Fig. 1.

More precisely, the real eigenvalue crossing $t_1$ becomes an avoided crossing for the eigenvalues $e_1(t,\delta)$ and $e_2(t,\delta)$. This means that for $\delta > 0$ small enough, there exists a unique point $z_1(\delta)$ with $\mathrm{Im}\, z_1(\delta) > 0$ and $z_1(\delta) - t_1 = \mathcal{O}(\delta)$ such that

$$e_1(z,\delta) - e_2(z,\delta) \simeq \sqrt{z - z_1(\delta)} \tag{2.13}$$

for any analytic continuations of $e_1$ and $e_2$ in a neighbourhood of $t_1$, as easily verified by perturbation theory. Hence $z_1(\delta)$ and $\overline{z_1(\delta)}$ belong to $\Omega(\delta)$. In the same neighbourhood, $e_1(z,\delta) + e_2(z,\delta)$ and $e_3(z,\delta)$ are analytic and $e_3(z,\delta)$ is distinct from the other two eigenvalues. Similarly, there exists a unique $z_2(\delta)$ with $\mathrm{Im}\, z_2(\delta) > 0$ and $z_2(\delta) - t_2 = \mathcal{O}(\delta)$ such that

$$e_2(z,\delta) - e_3(z,\delta) \simeq \sqrt{z - z_2(\delta)} \tag{2.14}$$

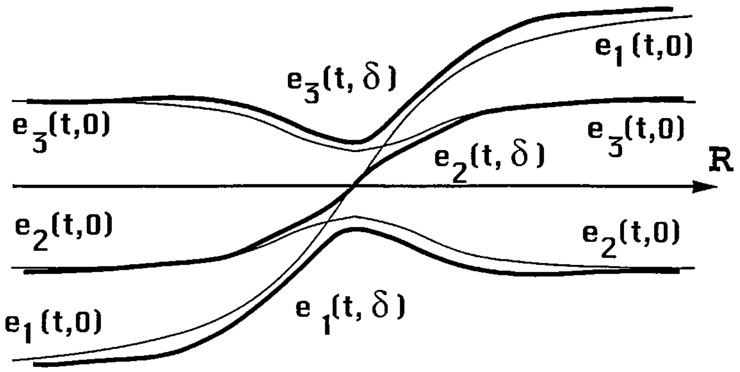


Fig. 1. The pattern of avoided crossings of the model.

for any analytic continuations of $e_2$ and $e_3$ in a neighbourhood of $t_2$. Here, $e_2(z,\delta)+e_3(z,\delta)$ and $e_1(z,\delta)$ are analytic in this neighbourhood. Finally, in a neighbourhood of $z_0 = \mathrm{i}\pi/4$ there exists a unique $z_0(\delta) = z_0 + \mathcal{O}(\delta)$ and a pair of indices $j \neq k \in \{1,2,3\}$, depending on the analytic continuation of the eigenvalues $\widetilde{e}_i(z,\delta)$, $i = 1,2,3$, from the real axis to this neighbourhood, such that

$$\widetilde{e}_j(z,\delta) - \widetilde{e}_k(z,\delta) \simeq \sqrt{z - z_0(\delta)}. \tag{2.15}$$

The other eigenvalue $\widetilde{e}_l(z,\delta)$ with $l \neq j$, $l \neq k$ and $\widetilde{e}_j(z,\delta) + \widetilde{e}_k(z,\delta)$ are analytic close to $z_0(\delta)$. This means that

$$\Omega(\delta) = \{z_1(\delta), \overline{z_1(\delta)}, z_2(\delta), \overline{z_2(\delta)}, z_0(\delta), \overline{z_0(\delta)}\}, \tag{2.16}$$

where all $z_j(\delta)$ are square root degeneracy points. By perturbation theory again, for any $z \in \Sigma \setminus \Omega(0)$ and $\delta > 0$ small enough, there exists a neighbourhood of $z$ in which any analytic continuation $\widetilde{e}_j(z,\delta)$ of the perturbed eigenvalues is analytic and tends in the limit $\delta \to 0$ to some corresponding analytic continuation $\widetilde{e}_k(z,0)$ of the unperturbed eigenvalues. Similarly, the eigenvectors $\varphi_j(t,\delta)$ defined on the real axis are analytic on the real axis and possess multivalued analytic continuations in $\Sigma$, with singularities at $\Omega(\delta)$. Of course, these multivalued eigenvalues and eigenvectors become single valued when defined on a suitable multi-sheeted Riemann surface. Since we shall need the values of these quantities along certain paths in $\Sigma \setminus \Omega(\delta)$ only, we do not need to introduce the Riemann surface explicitely.

For future reference, we need to see more precisely what happens to these multivalued functions when we turn around a subset of degeneracy points $\omega \subseteq \Omega(\delta)$. This amounts to make a comparison between the values of the eigenvalues on certain sheets of the Riemann surface. Let $\eta$ be a loop based at the origin which encircles $\omega$ and let $\{e_j(z,\delta)\}_{j=1}^3$ be a set of eigenvalues defined in a neighbourhood of the origin in $\Sigma \setminus \Omega(\delta)$. We perform the analytic continuation of this set along a path $\rho$ which is homotopic to $\eta$ in $\Sigma \setminus \Omega(\delta)$, see Fig. 2. We denote by $\{\widetilde{e}_j(z,\delta)\}_{j=1}^3$ the resulting set of analytic continuations which satisfies for $\delta > 0$ small enough

$$\widetilde{e}_j(z,\delta) = e_{\sigma(j)}(z,\delta), \quad j = 1,2,3, \tag{2.17}$$

where

$$\sigma : \{1,2,3\} \to \{1,2,3\} \tag{2.18}$$

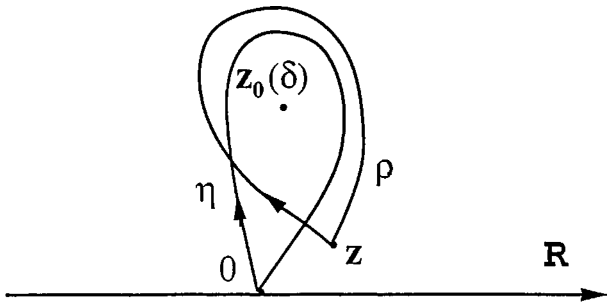


Fig. 2. The loop $\eta$ and the path $\rho$.

is a permutation which depends on $\eta$. For example, if the loop $\eta$ encircles $z_1(\delta)$ only, it follows from (2.13) that the corresponding permutation $\sigma$ is such that

$$\sigma\big(\{1,2,3\}\big) = \{2,1,3\}. \tag{2.19}$$

If $\eta$ encircles $z_0(\delta)$ only, we get

$$\sigma\big(\{1,2,3\}\big) = \{3,2,1\}, \tag{2.20}$$

since $e_1(0,\delta) \to e_2(0,0)$ and $e_3(0,\delta) \to e_3(0,0)$ as $\delta \to 0$, whereas, if $\eta$ encircles $z_1(\delta)$, $z_2(\delta)$ and $z_0(\delta)$, one sees using, e.g., (2.12) that the corresponding permutation $\sigma$ is

$$\sigma\big(\{1,2,3\}\big) = \{3,2,1\}. \tag{2.21}$$

Similarly, and with the same notations, we get for the analytic continuation of the eigenvector $\varphi_j(z,\delta)$ along $\rho$, the eigenvector $\widetilde{\varphi}_j(z,\delta)$ which must be proportional to $\varphi_{\sigma(j)}(z,\delta)$. We introduce the complex quantity $\theta_j(\eta;\delta) \in \mathbf{C}$ by the definition

$$\widetilde{\varphi}_j(z,\delta) = \mathrm{e}^{-\mathrm{i}\theta_j(\eta;\delta)}\varphi_{\sigma(j)}(z,\delta), \quad j = 1,2,3. \tag{2.22}$$

One can also show, see [11], that the couplings $a_{jk}(t)$ can be analytically continued in $\Sigma \setminus \Omega(\delta)$ so that the differential equation (1.6) can be analytically continued in $\Sigma \setminus \Omega(\delta)$ together with the coefficients $c_j(t)$ solutions to (1.6). They admit multivalued continuations in $\Sigma \setminus \Omega(\delta)$. In order that $\psi(z)$ be analytic throughout $\Sigma$, there must be a relation between the analytic continuations $\widetilde{c}_j(z)$ around $\rho$ of the $c_j(z)$, when $z$ belongs to a neighbourhood of the real axis, and the analytic continuations of the eigenvalues and eigenvectors. This is the key point of the method.

**Lemma 2.1.** *For any $j = 1,2,3$, we have*

$$\widetilde{c}_j(z)\,\mathrm{e}^{-\mathrm{i}\int_\eta e_j(u,\delta)\,\mathrm{d}u/\varepsilon}\,\mathrm{e}^{-\mathrm{i}\theta_j(\eta;\delta)} = c_{\sigma(j)}(z), \tag{2.23}$$

*where $\eta$, $\theta_j(\eta;\delta)$ and $\sigma(j)$ are defined as above.*

**Proof.** $\psi(z)$ is analytic in $\Sigma$ so that

$$\sum_{j=1}^{3} c_j(z)\,\mathrm{e}^{-\mathrm{i}\int_0^z e_j(u,\delta)\,\mathrm{d}u/\varepsilon}\varphi_j(z,\delta) = \sum_{j=1}^{3}\widetilde{c}_j(z)\,\mathrm{e}^{-\mathrm{i}\widetilde{\int_0^z e_j(u,\delta)\,\mathrm{d}u/\varepsilon}}\widetilde{\varphi}_j(z,\delta)$$

$$= \sum_{j=1}^{3}\widetilde{c}_j(z)\,\mathrm{e}^{-\mathrm{i}\int_\eta e_j(u,\delta)\,\mathrm{d}u/\varepsilon}\,\mathrm{e}^{-\mathrm{i}\int_0^z e_{\sigma(j)}(u,\delta)\,\mathrm{d}u/\varepsilon}\,\mathrm{e}^{-\mathrm{i}\theta_j(\eta;\delta)}\varphi_{\sigma(j)}(z,\delta). \tag{2.24}$$

We conclude by the fact that $\{\varphi_j(z,\delta)\}_{j=1}^3$ is a basis.

## 3. WKB estimates

We now come to the second essential point of the method. We see from Lemma 2.1 with $z=+\infty$ that if we take $c_k(-\infty)=\delta_{jk}$ as initial conditions at $-\infty$, we have access to the element $s_{\sigma(j)j}$ of the $S$-matrix provided we can control $\widetilde{c}_j(z)$ in the complex plane as $\varepsilon\to 0$. This section describes basic estimates on the coefficients $\widetilde{c}_k(z)$ in certain domains extending to infinity in both the positive and negative directions inside the strip $\Sigma$.

It is obvious from the differential equation

$$\widetilde{c}_k'(z) = \sum_{l=1}^{3}\widetilde{a}_{kl}(z)\,\mathrm{e}^{\mathrm{i}\widetilde{\Delta}_{kl}(z,\delta)/\varepsilon}\widetilde{c}_l(z) \tag{3.1}$$

the coefficients $\widetilde{c}_k(z)$ satisfy in $\Sigma\setminus\Omega(\delta)$ that sufficient control on $\widetilde{c}_k(z)$ in the complex plane as $\varepsilon\to 0$ can usually be achieved along special paths only, called dissipative paths, or inside special domains, called dissipative domains. Let $j\in\{1,2,3\}$ be fixed and consider the initial condition in (3.1)[2]

$$\lim_{\mathrm{Re}\,z\to-\infty}\widetilde{c}_k(z) = \lim_{t\to-\infty} c_k(t) = \delta_{jk},\quad k=1,2,3 \tag{3.2}$$

(where the analytic continuation of (3.1) for $\mathrm{Re}\,z<\mathrm{Re}\,z_1(\delta)$ is performed from the real axis vertically to $z$). We say that a path $\gamma_k\in\Sigma\setminus\Omega(\delta)$ parametrized by $u\in]-\infty,t]$ such that

$$\lim_{u\to-\infty}\mathrm{Re}\,\gamma_k(u) = -\infty,\qquad \gamma_k(t)=z \tag{3.3}$$

is a *dissipative path for* $\{jk\}$ if it satisfies the monotonicity condition[3]

$$\mathrm{Im}\,\widetilde{\Delta}_{jk}\big(\gamma_k(u),\delta\big)\text{ is a non-decreasing function of }u\in]-\infty,t]. \tag{3.4}$$

$\gamma_k$ is *strictly dissipative for* $\{jk\}$ if $\mathrm{Im}\,\widetilde{\Delta}_{jk}(\gamma_k(u),\delta)$ is increasing as a function of $u\in]-\infty,t]$. Here $\widetilde{\Delta}_{jk}$ denotes the analytic continuation of $\int_0^t(e_j(s,\delta)-e_k(s,\delta))\,\mathrm{d}s$ from 0 along the real axis to $-T\in\mathbf{R}$

[2]We note that due to the decay of $H(z,\delta)$ to its limits $H(\pm)$ as $\mathrm{Re}\,z\to\pm\infty$ in $\Sigma$ as well, the limits $\lim_{t\to\pm\infty}\widetilde{c}_j(t+\mathrm{i}s)$ are independent of $s\in]-\pi/2,\pi/2[$.

[3]Note that the quantity $\mathrm{Im}\,\widetilde{\Delta}_{jk}(z,\delta)$ is finite $\forall z\in\Sigma\setminus\Omega(\delta)$.

with $-T < \operatorname{Re} z_1(\delta)$, then up or down to $\gamma_k(-T)$ and finally along $\gamma_k$. A sufficient condition for $\gamma_k$ to be strictly dissipative for $\{jk\}$ is

$$\operatorname{Im} \dot{\gamma}_k(u)\bigl(e_j\bigl(\gamma_k(u),\delta\bigr) - e_k\bigl(\gamma_k(u),\delta\bigr)\bigr) > 0 \quad \forall u \in \left]-\infty, t\right], \tag{3.5}$$

where $\dot{\gamma}_k(u) = \mathrm{d}/\mathrm{d}u\, \gamma_k(u)$.

It is a standard manipulation to see by an integration by parts on the exponentials in the Volterra equation corresponding to (3.1) that if there exists a path $\gamma \in \Sigma\setminus\Omega(\delta)$ which links $-\infty$ to $+\infty$ and which is dissipative for *all* couples $\{jk\}$, $k \in \{1,2,3\}$, then, with the initial conditions (3.2), the following estimate is true

$$\widetilde{c}_j(\infty) = 1 + \mathcal{O}(\varepsilon). \tag{3.6}$$

We shall prove below the existence of such a path for our model.[4] However, in general, this notion is too restrictive, see, e.g., [2], and we have to resort to the notion of dissipative domain.

We call $D_j \subset \Sigma \setminus \Omega(\delta)$ a *dissipative domain for the index* $j$ if it stretches from $-\infty$ to $+\infty$ and if for any $z \in D_j$ and any $k \in \{1,2,3\}$, there exists a dissipative path $\gamma_k \subset D_j$ for $\{jk\}$ which links $-\infty$ to $z$. It is shown by similar methods that when such a dissipative domain exists for the index $j$, the solution of (3.1) subjected to the initial conditions (3.2) still satisfies (3.6) [8,7].

**Proposition 3.1.** *Assume there exists a dissipative domain $D_j$ for the index $j$. Let $\eta_j$ be a loop based at the origin which encircles all degeneracy points between the real axis and $D_j$ and let $\sigma_j$ be the permutation of labels associated with $\eta_j$. The loop $\eta_j$ is negatively, respectively positively, oriented if $D_j$ is above, respectively below, the real axis. Then the solution of* (1.6) *subjected to the initial conditions $c_k(-\infty) = \delta_{jk}$ satisfies*

$$c_{\sigma_j(j)}(+\infty) = \mathrm{e}^{-\mathrm{i}\theta_j(\eta_j)}\mathrm{e}^{-\mathrm{i}\int_{\eta_j} e_j(z)\,\mathrm{d}z/\varepsilon}\bigl(1 + \mathcal{O}(\varepsilon)\bigr). \tag{3.7}$$

**Proof.** Use Lemma 2.1 and estimate (3.6).

**Remark.** We can prove exponential estimates for the other coefficients $c_{\sigma_j(k)}(+\infty)$ by the same method, see, e.g., [7].

The difficult part of the problem, as stressed in the introduction, is to prove the existence of such domains $D_j$, which do not necessarily exist, and to have enough of them to compute the asymptotic of the whole $S$-matrix.

## 4. Avoided crossings

We apply the results of [7] to the avoided crossings of our model.

The main point of [7] is the proof of the existence of dissipative domains in an avoided crossing context, assuming some genericity properties of the unperturbed generator $H(t,0)$, to be checked for *real* $t$'s.[5] Essentially:

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[4] In particular, the real axis is a dissipative domain for all indices and we have $\widetilde{c}_j(\gamma(u)) \equiv c_j(u)$. Hence we get by applying this result for all indices successively that $S = \mathbf{I} + \mathcal{O}(\varepsilon)$.

[5] Mild regularity conditions on the behaviour of $H(z,\delta)$ in $(z,\delta) \in \Sigma\times]0,\delta^*]$, for some small positive $\delta^*$ (which are satisfied by our generator (1.2)) are also assumed.

(i) For each couple $\{j,k\}$, there exists at most one real crossing point $t_c$ of the eigenvalues $e_j(t,0)$ and $e_k(t,0)$ which must be generic in the sense that $(\mathrm{d}/\mathrm{d}t)(e_j(t,0)-e_k(t,0))|_{t_c}\neq 0$.
(ii) $\forall j\in\{1,2,3\}$, the eigenvalue $e_j(t,0)$ crosses eigenvalues whose indices are all superior to $j$ or all inferior to $j$.

Eqs (2.5) to (2.8) show that our $H(t,0)$ satisfies those requirements. Under hypotheses (i) and (ii), it is shown in Lemma 6.1 in [7] that: for each $j=1,2,3$, there exists a dissipative domain $D_j$ above or below the real axis which is close to the real axis and such that all avoided crossings are between $D_j$ and the real axis. The permutations $\sigma_j$ associated with these dissipative domains (see Proposition 3.1) are independent of $j$, $\sigma_j=\sigma$, and $\sigma$ can be read on the pattern of avoided crossings of the eigenvalues on the *real axis* (see Fig. 1) in the following way: for any $l=1,2,3$, at $t=\infty$, the eigenvalue $e_l(\infty,0)$ coincides with $e_k(\infty,\delta)$, for some $k\in\{1,2,3\}$. Then we have $\sigma(l)=k$, $l=1,2,3$ (see also below). The dissipative domain $D_j$ is above (resp. below) the real axis whenever $j<\sigma(j)$ (resp. $j>\sigma(j)$). Note that the actual shape of these domains is irrelevant for our purpose. It remains to apply the results of the previous section to get the desired exponentially small asymptotic formula of the off-diagonal elements $s_{\sigma(j)j}$, $j=1,2,3$, of the $S$-matrix. Note that we only get one off-diagonal element per line and per column from that result.

In our case, see Fig. 1, $e_1(\infty,0)=e_3(\infty,\delta)$ so that $\sigma(1)=3$ and $e_2(\infty,0)=e_1(\infty,\delta)$ so that $\sigma(1)=3$. Thus we have

$$\sigma(\{1,2,3\})=\{3,1,2\} \tag{4.1}$$

so that $D_1$ is above the real axis whereas $D_2$ and $D_3$ are below the real axis in our case (actually, $D_2=D_3=\overline{D_1}$ [7]). See Fig. 3. Hence, with $\eta$ a negatively oriented loop based at the origin which encircles $z_1(\delta)$ and $z_2(\delta)$ and denoting by $\overline{\eta}$ its complex conjugate, we get for $\delta>0$ small enough[6]

$$s_{31}=\mathrm{e}^{-\mathrm{i}\theta_1(\eta;\delta)}\mathrm{e}^{-\mathrm{i}\int_\eta e_1(z,\delta)\,\mathrm{d}z/\varepsilon}\big(1+\mathcal{O}(\varepsilon)\big), \tag{4.2}$$

$$s_{12}=\mathrm{e}^{-\mathrm{i}\theta_2(\overline{\eta};\delta)}\mathrm{e}^{-\mathrm{i}\int_{\overline{\eta}} e_2(z,\delta)\,\mathrm{d}z/\varepsilon}\big(1+\mathcal{O}(\varepsilon)\big), \tag{4.3}$$

$$s_{23}=\mathrm{e}^{-\mathrm{i}\theta_3(\overline{\eta};\delta)}\mathrm{e}^{-\mathrm{i}\int_{\overline{\eta}} e_3(z,\delta)\,\mathrm{d}z/\varepsilon}\big(1+\mathcal{O}(\varepsilon)\big). \tag{4.4}$$

We can rewrite these formulae in terms of quantities related to each avoided crossing and computed in the upper half plane only in the following way. Let us introduce two negatively oriented loops based at the origin, $\eta_1$ and $\eta_2$, which encircle respectively $z_1(\delta)$ and $z_2(\delta)$ only. The analyticity properties reviewed above show that,

$$\int_{\overline{\eta}} e_2(z,\delta)\,\mathrm{d}z=\int_{\overline{\eta_1}} e_2(z,\delta)\,\mathrm{d}z=\overline{\int_{\eta_1} e_2(z,\delta)\,\mathrm{d}z}=-\overline{\int_{\eta_1} e_1(z,\delta)\,\mathrm{d}z}, \tag{4.5}$$

and similar identities for other loops and eigenvalues. Corresponding identities are true for the factors $\mathrm{e}^{-\mathrm{i}\theta_j(\eta,\delta)}$. Indeed, the eigenvectors $\varphi_j(z,\delta)$ can be expressed as $\varphi_j(z,\delta)=W(z)\varphi_j(0,\delta)$, where $W(z)\in$

[6] The remainders $\mathcal{O}(\varepsilon)$ are $\delta$-dependent at that point. However, it should be possible to prove that they are actually uniform in $\delta$, using the techniques of [6].

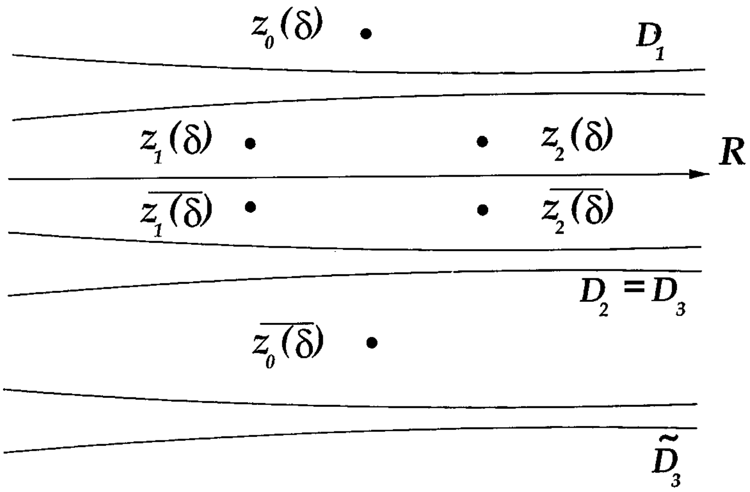


Fig. 3. The positions of the dissipative domains.

$M_3(\mathbf{C})$ is multivalued in $\Sigma \setminus \Omega(\delta)$. Moreover, when $H(t,\delta)$, $t \in \mathbf{R}$, is self-adjoint, $W(t)$ is unitary so that $W^*(\overline{z}) = W^{-1}(z)$, see [12,11,7]. As a consequence, we easily get the identities

$$\mathrm{e}^{-\mathrm{i}\theta_1(\eta_1;\delta)}\mathrm{e}^{-\mathrm{i}\theta_2(\eta_1;\delta)} = -1, \tag{4.6}$$

$$\mathrm{e}^{-\mathrm{i}\theta_1(\overline{\eta}_1;\delta)} = \mathrm{e}^{-\mathrm{i}\overline{\theta_1(\eta_1;\delta)}} \tag{4.7}$$

and their equivalents for other indices. Hence,

$$s_{31} = \mathrm{e}^{-\mathrm{i}\theta_1(\eta_1;\delta)}\mathrm{e}^{-\mathrm{i}\theta_2(\eta_2;\delta)}\mathrm{e}^{-\mathrm{i}\int_{\eta_1} e_1(z,\delta)\,\mathrm{d}z/\varepsilon}\mathrm{e}^{-\mathrm{i}\int_{\eta_2} e_2(z,\delta)\,\mathrm{d}z/\varepsilon}\big(1+\mathcal{O}(\varepsilon)\big), \tag{4.8}$$

$$s_{12} = -\mathrm{e}^{\mathrm{i}\overline{\theta_1(\eta_1;\delta)}}\mathrm{e}^{\mathrm{i}\overline{\int_{\eta_1} e_1(z,\delta)\,\mathrm{d}z/\varepsilon}}\big(1+\mathcal{O}(\varepsilon)\big), \tag{4.9}$$

$$s_{23} = -\mathrm{e}^{\mathrm{i}\overline{\theta_2(\eta_2;\delta)}}\mathrm{e}^{\mathrm{i}\overline{\int_{\eta_2} e_2(z,\delta)\,\mathrm{d}z/\varepsilon}}\big(1+\mathcal{O}(\varepsilon)\big). \tag{4.10}$$

Note here that the positive exponential decay rates

$$\Gamma_1(\delta) \equiv -\mathrm{Im}\int_{\eta_1} e_1(z,\delta)\,\mathrm{d}z, \tag{4.11}$$

$$\Gamma_2(\delta) \equiv -\mathrm{Im}\int_{\eta_2} e_2(z,\delta)\,\mathrm{d}z \tag{4.12}$$

are such that

$$\lim_{\delta\to 0}\Gamma_j(\delta) = 0, \quad j = 1,2, \tag{4.13}$$

by continuity. The actual computation of the different prefactors $\mathrm{e}^{-\mathrm{i}\theta_j}$ is addressed in an appendix of [9] when the generator is self-adjoint and it can also be shown that they have vanishing modulus as $\delta \to 0$, see [6].

We can get some more information by making use the unitarity of the $S$-matrix, which is a consequence of (1.3). It is shown in [7] (making use of exponential estimates mentionned after Proposition 3.1) that there exists a $K > 0$ independent of $\delta$ such that for $\delta$ small enough

$$S = \begin{pmatrix} s_{11} & s_{12} & \mathcal{O}\big(\mathrm{e}^{-(\Gamma_2+\Gamma_2+K)/\varepsilon}\big) \\ -\overline{s}_{12}\,\dfrac{s_{11}}{\overline{s}_{22}}\big(1+\mathcal{O}\big(\mathrm{e}^{-2\Gamma_2/\varepsilon}\big)\big) & s_{22} & s_{23} \\ s_{31} & -\overline{s}_{23}\,\dfrac{s_{33}}{\overline{s}_{22}}\big(1+\mathcal{O}\big(\mathrm{e}^{-2\Gamma_1/\varepsilon}\big)\big) & s_{33} \end{pmatrix}, \tag{4.14}$$

where all $s_{jk}$ above can be computed asymptotically, see (1.11).

The smallest asymptotically computable element $s_{31}$ describes the transition from $e_1(-\infty, \delta)$ to $e_3(+\infty, \delta)$. The result we get for this element is in agreement with the rule of the thumb claiming that the transitions take place locally at the avoided crossings and can be considered as independent. Accordingly, we can only *estimate* the smallest element of all, $s_{13}$, which describes the transition from $e_3(-\infty, \delta)$ to $e_1(+\infty, \delta)$, for which the avoided crossings are not encountered in "right order", as discussed in [5]. This is all we can say about the $S$-matrix under generic circumstances. We prove that it is possible however to get an asymptotic expression for this element on our model, which must be governed by a degeneracy point which is located *far* in the complex plane. This turning point corresponds to the degeneracy point $-\mathrm{i}\pi/4$ of the unperturbed levels $e_2(z, 0)$ and $e_3(z, 0)$, as explained in the introduction. We show in the next section that $s_{13}$ can be computed asymptotically for $\delta > 0$ small enough, using the techniques presented above, by proving the existence of a dissipative domain $\widetilde{D}_3$ for the index 3 in the lower half plane which passes below all degeneracy points $\overline{z_0(\delta)}$, $\overline{z_1(\delta)}$, $\overline{z_2(\delta)}$ in $\Sigma$. It is the fact that the relevant turning point for the computation of $s_{13}$ lies far away from the real axis which makes the existence of a dissipative domain $\widetilde{D}_3$ non generic, in some sense.

## 5. Existence of $\widetilde{D}_3$

As a first step, we show that there exists a dissipative domain $\widetilde{D}_3$ with respect to the unperturbed eigenvalues, located below the line $\operatorname{Im} z = -\pi/4$. The second step consists in proving that when $\delta > 0$, this domain remains dissipative, with respect to the perturbed eigenvalues now, provided $\delta$ is small enough. The associated permutation of indices is, see (2.21),

$$\sigma\big(\{1,2,3\}\big) = \{3,2,1\}. \tag{5.1}$$

Actually, we show that any horizontal path parametrized by

$$\tau \mapsto \gamma(\tau) = \tau + \mathrm{i}s; \quad \tau \in\, ]-\infty, +\infty[;\ s \in\, ]-\pi/2, -\pi/4[ \tag{5.2}$$

is strictly dissipative for {32} and {31} and thus defines a dissipative domain $\widetilde{D}_3$. Indeed, the strict dissipativity conditions to fulfill are in such a case, see (3.5),

$$\begin{cases} \operatorname{Im}\big(e_3\big(\gamma(\tau), 0\big) - e_2\big(\gamma(\tau), 0\big)\big) > 0, \\ \operatorname{Im}\big(e_3\big(\gamma(\tau), 0\big) - e_1\big(\gamma(\tau), 0\big)\big) > 0, \end{cases} \quad \forall \tau \in\, ]-\infty, +\infty[;\ s \in\, ]-\pi/2, -\pi/4[ \tag{5.3}$$

which are equivalent for our model to

$$\begin{cases} \operatorname{Im} 2\sqrt{1+\tanh^2(\tau+\mathrm{i}s)} > 0, \\ \operatorname{Im} \sqrt{1+\tanh^2(\tau+\mathrm{i}s)} - \alpha\tanh(\tau+\mathrm{i}s) > 0, \end{cases} \qquad \forall\tau \in\, ]-\infty,+\infty[;\ s\in\, ]-\pi/2,-\pi/4[. \tag{5.4}$$

Using the identities

$$\tanh(\tau+\mathrm{i}s) = \frac{\sinh(2\tau)+\mathrm{i}\sin(2s)}{\cosh(2\tau)+\cos(2s)},$$

$$1+\tanh^2(\tau+\mathrm{i}s) = 2\frac{(\cosh^2(2\tau)+\cosh(2\tau)\cos(2s)+\cos^2(2s)-1)+\mathrm{i}\sinh(2\tau)\sin(2s)}{(\cosh(2\tau)+\cos(2s))^2} \tag{5.5}$$

we see that the image of $\gamma(\tau) = \tau + \mathrm{i}s$ by $1+\tanh^2(z)$ is a loop which looks like Fig. 4 with $s \in\, ]-\pi/2,-\pi/4[$, so that the image of $\gamma$ by $\sqrt{1+\tanh^2(z)}$ lies in the upper half plane, thus insuring that $\operatorname{Im}\sqrt{1+\tanh^2(\gamma(\tau))} > 0$, $\forall\tau\in\, ]-\infty,+\infty[$. As $\operatorname{Im}\tanh(\tau+\mathrm{i}s) = \sin(2s)/(\cosh(2\tau)+\cos(2s)) < 0$, we see that both conditions (5.4) are satisfied along the horizontal path $\gamma(\tau)$.

It remains to prove that $\gamma(\tau)$ is still dissipative when $\delta > 0$ and small enough. We have

$$e_j(z,\delta) = e_j(z,0) + \mathcal{O}(\delta\|V(z)\|), \quad j=1,2,3, \tag{5.6}$$

for all $z\in\gamma$ and, in particular,

$$\begin{aligned} &\operatorname{Im}\big(e_3(\gamma(\tau),\delta) - e_j(\gamma(\tau),\delta)\big) \\ &\quad = \operatorname{Im}\big(e_3(\gamma(\tau),0) - e_j(\gamma(\tau),0)\big) + \mathcal{O}\big(\delta\|V(\gamma(\tau))\|\big), \quad j=1,2, \end{aligned} \tag{5.7}$$

with

$$\|V(\tau+\mathrm{i}s)\| = \mathcal{O}\big(\sqrt{\tau^2+s^2}\,\mathrm{e}^{-\tau^2+s^2}\big) = \mathcal{O}\big(\tau\,\mathrm{e}^{-\tau^2}\big). \tag{5.8}$$

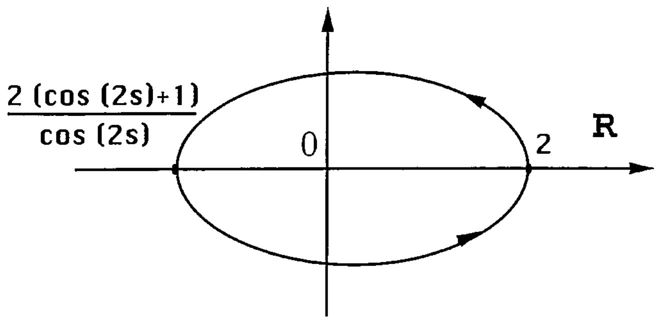


Fig. 4. The image of $\gamma$ by the function $1+\tanh^2(z)$.

For $|\tau|$ large,

$$\begin{cases} \operatorname{Im}\sqrt{1+\tanh^2(\tau+\mathrm{i}s)} = -\dfrac{\sin(2s)}{\sqrt{2}\cosh(2\tau)}\big(1+\mathcal{O}\big(\mathrm{e}^{-2|\tau|}\big)\big), \\ -\operatorname{Im}\tanh(\tau+\mathrm{i}s) = \dfrac{\sin(2s)}{\cosh(2\tau)}\big(1+\mathcal{O}\big(\mathrm{e}^{-2|\tau|}\big)\big) \end{cases} \tag{5.9}$$

so that we can write

$$\operatorname{Im}\big(e_3\big(\gamma(\tau),\delta\big) - e_j\big(\gamma(\tau),\delta\big)\big) = \operatorname{Im}\big(e_3\big(\gamma(\tau),0\big) - e_j\big(\gamma(\tau),0\big)\big)\big(1+\mathcal{O}\big(\delta\tau\,\mathrm{e}^{-(\tau^2-2|\tau|)}\big)\big). \tag{5.10}$$

Hence, for $\delta > 0$ small enough, $\gamma(\tau)$ is dissipative and we can apply Proposition 3.1 to compute $s_{13}$. Introducing two negatively oriented loops based at the origin $\eta_0$, resp. $\eta_3$ which encircle $z_0(\delta)$ only, resp. $z_0(\delta)$, $z_1(\delta)$ and $z_2(\delta)$, we get by similar considerations as above

$$\begin{aligned} s_{13} &= \mathrm{e}^{-\mathrm{i}\theta_3(\overline{\eta}_3;\delta)}\mathrm{e}^{-\mathrm{i}\int_{\overline{\eta}_3} e_3(z,\delta)\,\mathrm{d}z/\varepsilon}\big(1+\mathcal{O}(\varepsilon)\big) = \mathrm{e}^{-\mathrm{i}\theta_3(\overline{\eta}_0;\delta)}\mathrm{e}^{-\mathrm{i}\int_{\overline{\eta}_0} e_3(z,\delta)\,\mathrm{d}z/\varepsilon}\big(1+\mathcal{O}(\varepsilon)\big) \\ &= -\mathrm{e}^{\mathrm{i}\overline{\theta_1(\eta_0;\delta)}}\mathrm{e}^{\mathrm{i}\overline{\int_{\eta_0} e_1(z,\delta)\,\mathrm{d}z}/\varepsilon}\big(1+\mathcal{O}(\varepsilon)\big). \end{aligned} \tag{5.11}$$

It is clear from these formulae that the corresponding positive exponential decay rate

$$\Gamma_0(\delta) \equiv -\operatorname{Im}\int_{\eta_0} e_1(z,\delta)\,\mathrm{d}z \tag{5.12}$$

tends to a fixed positive value as $\delta \to 0$ which can be computed by means of the unperturbed eigenvalues:

$$\lim_{\delta\to 0}\Gamma_0(\delta) = -\operatorname{Im}\int_{\eta_0} e_2(z,0)\,\mathrm{d}z > 0, \tag{5.13}$$

since $e_1(z,\delta) \to e_2(z,0)$ as $\delta \to 0$ when the analytic continuation is performed along $\eta_0$.

## 6. From the model to a class of self-adjoint generators

The only property of the perturbation $V(z)$ we use in the previous proof is the fact that $\|V(\gamma(\tau))\|$ decays faster to zero as $\tau \to \pm\infty$ than $\operatorname{Im}(e_3(\gamma(\tau),0) - e_j(\gamma(\tau),0))$, $j = 1,2$. Thus the present construction of a three-dimensional model whose $S$-matrix is completely computable asymptotically can clearly be adapted to generate a whole class of models. Let us mention briefly what the main steps to take are:

1. Let $h(z)$ be a two-dimensional matrix depending analytically on $z$ in some strip $\Sigma$ including the real axis such that:
   (a) $h(t)$ is non-degenerate and self-adjoint for any $t \in \mathbf{R}$, including $t = \pm\infty$;
   (b) $\lim_{t\to\pm\infty} |t|^{1+a} \sup_{s|t+\mathrm{i}s\in\Sigma} \|h(t+\mathrm{i}s) - h(\pm)\| < \infty$, for some $a > 0$;

(c) The eigenvalues of $h(z)$, denoted by $e_2(z,0)$ and $e_3(z,0)$, possess two generic degeneracy non-real points $z_0$ and $\overline{z}_0$ such that there exists a strict dissipative path $\gamma^7$ for $\{32\}$ in the lower half plane, parameterized by $\tau \in ]-\infty,\infty[$, which goes from $-\infty$ to $+\infty$ in $\Sigma$ and passes below $\overline{z}_0$.

2. Let $e_1(z,0)$, $z$ in $\Sigma$, be an analytic function such that:

(a) $e_1(z,0)$ is real on the real axis and $\lim_{t\to\pm\infty} |t|^{1+a} \sup_{s|t+\mathrm{i}s\in\Sigma} |e_1(t+\mathrm{i}s,0) - e_1(\pm,0)| < \infty$, for some $a > 0$;
(b) $e_1(-\infty,0) < e_2(-\infty,0) < e_3(-\infty,0)$ and there exists one generic real crossing point $t_1 < 0$ for $e_1(t,0)$ and $e_2(t,0)$ and one generic real crossing point $t_2 > 0$ for $e_1(t,0)$ and $e_3(t,0)$. Moreover $e_1(+\infty,0) > e_3(+\infty,0)$;
(c) $e_1(z,0)$ is such that $\gamma$ is a dissipative path for $\{31\}$ as well.

3. Define the unperturbed self-adjoint generator $H(z,0)$ as

$$\begin{pmatrix} \begin{matrix} h(z) \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & e_1(z,0) \end{pmatrix} \in M_3(\mathbf{C}).$$

4. Let $V(z) \in M_3(\mathbf{C})$, $z$ in $\Sigma$, be an analytic matrix such that:

(a) $V(t)$ is self-adjoint for real $t$'s and $\lim_{t\to\pm\infty} |t|^{1+a} \sup_{s|t+\mathrm{i}s\in\Sigma} \|V(t+\mathrm{i}s)\| < \infty$, for some $a > 0$;
(b) The perturbation $\delta V(t)$ turns the real crossing points $t_1$ and $t_2$ of $H(t,0)$ into avoided crossings for $\delta > 0$ small enough and $t \in \mathbf{R}$;
(c) $\|V(\gamma(\tau))\|$ tends to zero as $\tau \to \pm\infty$ sufficiently fast so that, recall (3.5),

$$\lim_{\tau\to 0} \frac{|\dot\gamma(\tau)|\, \|V(\gamma(\tau))\|}{\operatorname{Im} \dot\gamma(\tau)(e_3(\gamma(\tau),0) - e_j(\gamma(\tau),0))} = 0 \quad \text{for } j = 1,2. \tag{6.1}$$

Then, by mimicking the proofs above, we have the following

**Proposition 6.1.** *Assume the above hypotheses. Then, the $S$-matrix corresponding to the self-adjoint generator $H(t,\delta) = H(t,0) + \delta V(t)$ so constructed is completely computable asymptotically as $\varepsilon \to 0$, provided $\delta$ is small enough, and it is given, by*

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ -\overline{s}_{12}\, \dfrac{s_{11}}{\overline{s}_{22}} \big(1 + \mathcal{O}\big(\mathrm{e}^{-2\Gamma_2/\varepsilon}\big)\big) & s_{22} & s_{23} \\ s_{31} & -\overline{s}_{23}\, \dfrac{s_{33}}{\overline{s}_{22}} \big(1 + \mathcal{O}\big(\mathrm{e}^{-2\Gamma_1/\varepsilon}\big)\big) & s_{33} \end{pmatrix}, \tag{6.2}$$

*where $s_{jj} = 1 + \mathcal{O}(\varepsilon)$, $s_{31}, s_{12}, s_{23}$ are given by* (4.8) *to* (4.10) *and $s_{13}$ by* (5.11).

[7] Not necessarily horizontal.

**Remarks.** We can of course deal with the case where the supplementary eigenvalue is above those of $h(-\infty)$ at $t = -\infty$ by obvious changes in the computations above.

The locus of the unperturbed degeneracy point $z_0$ with respect to $t_1$ and $t_2$ is irrelevant. However, formula (5.13) is true when $t_1 < 0 < t_2$ only.

## 7. Example of difficulty in the construction of a dissipative domain

Let us finally show on the model (1.2) that the symmetry of the $S$-matrix, here the unitarity, is necessary to compute asymptotically all its entries. Using the result of [7], we can compute in absence of any symmetry, the elements $s_{31}$, $s_{12}$, $s_{23}$ only. We have also access to the element $s_{13}$ in our model by the construction explained above. The unitarity of $S$ allows then to get the missing off-diagonal elements. If $S$ were not unitary, in order to compute say $s_{21}$ by our method, we should show the existence of a dissipative domain $\widetilde{D}_1$ with associated permutation $\sigma$ satisfying $\sigma(1) = 2$. The simplest domain with the required permutation is such that $z_1$ is the only turning point between the real axis and $\widetilde{D}_1$.[8] However, we show below that such a domain does not exist, thus showing the importance of symmetries of $S$ for the computation of the remaining elements. This illustrates the difficulty in the construction of dissipative domains.

In order to do this, we introduce the set of level lines $\mathrm{Im}\,\Delta_{12}(z,\delta) = cst$, where the analytic continuations are performed from the real axis, vertically. In a neighbourhood of $z_1(\delta)$, this set of lines is independent of the analytic continuation (see (2.13)). Let us consider the set of lines defined by

$$\mathrm{Im}\,\Delta_{12}(z,\delta) = \mathrm{Im}\,\Delta_{12}\big(z_1(\delta),\delta\big) \neq 0 \tag{7.1}$$

for $\mathrm{Im}\,z \geqslant 0$, which we call Stokes lines.[9] A local analysis shows that there are three branches $l_1, l_2, l_3$ emanating from $z_1(\delta)$. These Stokes lines are of interest since a dissipative path for $\{12\}$ cannot cross more than one Stokes line emanating from $z_1(\delta)$ in a simply connected set of $\Sigma \setminus \Omega(\delta)$. As the dissipative domain $\widetilde{D}_1$ contains such dissipative paths by definition, the Stokes lines become the borders of certain sectors in $\Sigma \setminus \Omega(\delta)$ where the dissipative domain $\widetilde{D}_1$ is constrained to lie (see below). A similar argument with the Stokes lines of (the suitable analytic continuation of) $\mathrm{Im}\,\Delta_{13}(z,\delta)$ emanating form $z_2(\delta)$ is true. We show below that the global behaviour of these Stokes lines prevents the domain $\widetilde{D}_1$ to extend from $-\infty$ to $+\infty$. Some of the claims we make below on the global behaviour of the Stokes lines or their pertubative behaviour as $\delta \to 0$ are non-trivial. The reader is directed to the proposed references for complete proofs.

By virtue of the perturbative nature of the whole construction, the Stokes lines

$$\mathrm{Im}\,\Delta_{12}(z,\delta) = \mathrm{Im}\,\Delta_{12}\big(z_1(\delta),\delta\big) \neq 0 \tag{7.2}$$

are close to the corresponding Stokes lines for $\delta = 0$ [6]. When $\delta = 0$, the Stokes lines emanating from $z_1(0) = t_1$ consist in the real axis together with a line $l$ crossing $\mathbf{R}$ perpendicularly at $t_1$, see Fig. 5. By construction, the dissipative domain $D_1$ in the upper half plane we considered in section 4 is close to the real axis for $\delta > 0$ small enough so that it crosses $l$, as explained in [7]. Hence, when $\delta > 0$, one of the

[8] Actually any other more complicated domain satisfying the condition on the permutation of indices would do. However it is very unlikely that such a more complicated domain could be dissipative.

[9] We use the terminology of [8]; these lines are also called anti-Stokes lines.

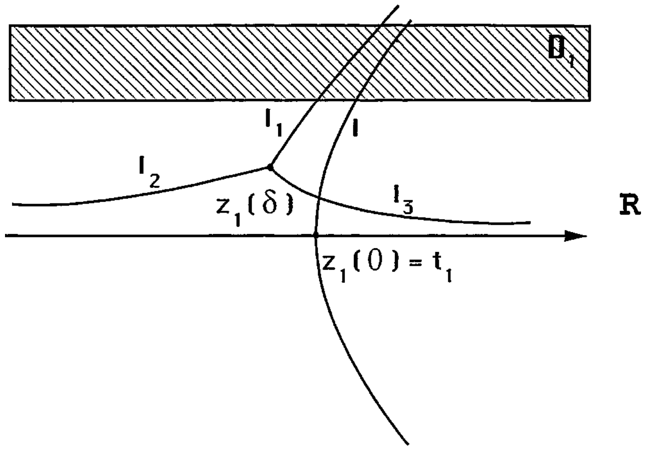


Fig. 5. Stokes lines associated with $z_1(\delta)$.

three branches, say $l_1$, must cross $D_1$ as well, by perturbation [6]. Moreover, another branch $l_2$ goes from $z_1(\delta)$ to $-\infty$ below $D_1$, for $\delta$ small, by perturbation again. It follows from the detailed analysis of Stokes lines performed in [8] that the third branch $l_3$ cannot go from $z_1(\delta)$ to $-\infty$ unless there is a singularity of $e_1(z,\delta) - e_2(z,\delta)$ between the branches $l_1$ and $l_3$, which not the case since both $e_1$ and $e_2$ are analytic for $\mathrm{Re}\, z < \mathrm{Re}\, z_1(\delta)$. Hence we have the situation described in Fig. 5. Note however, that $z_2(\delta)$ is a branching point for $e_2(z,\delta)$.

If $\widetilde{D}_1$ is a dissipative domain for {12} and {13}, there exists by definition a dissipative path for {12} which goes from $-\infty$ to $+\infty$ and passes between $z_1(\delta)$ and $z_2(\delta)$. Hence it crosses the branch $l_1$. To be dissipative, such a path can cross neither $l_3$, nor $l_2$, so that $\widetilde{D}_1$ must lie above $l_3$ which becomes a boundary for $\widetilde{D}_1$, see Fig. 6. Consequently, $l_3$ must pass below $z_2(\delta)$ if we want $\widetilde{D}_1$ to pass between $z_1(\delta)$ and $z_2(\delta)$.

Then we note that due to the property

$$H(z,\delta) = -H(-z,\delta), \quad \forall \delta \geqslant 0,\ z \in \Sigma, \tag{7.3}$$

we have $e_2(z,\delta) = -e_2(-z,\delta)$ and $e_1(z,\delta) = -e_3(-z,\delta)$, where the analytic continuations are performed from the real axis, vertically. Hence, the pattern of level lines $\mathrm{Im}\,\Delta_{12}(z,\delta) = cst$ for $\mathrm{Re}\, z \leqslant 0$ is symmetric with respect to the imaginary axis to the pattern of level lines

$$\mathrm{Im}\,\Delta_{23}(z,\delta) = cst, \qquad \mathrm{Re}\, z \geqslant 0, \tag{7.4}$$

with analytic continuation performed as above. Hence, by symmetry, we can draw the branches $l_1'$, $l_2'$, $l_3'$ of the Stokes lines defined as

$$\mathrm{Im}\,\Delta_{23}(z,\delta) = \mathrm{Im}\,\Delta_{23}\big(z_2(\delta),\delta\big) \neq 0, \tag{7.5}$$

with analytic continuations chosen as above, see Fig. 6.

Since $\widetilde{D}_1$ is to pass above $z_1(\delta)$ the set of level lines (7.4) is equivalent to the set of lines $\mathrm{Im}\,\widetilde{\Delta}_{13}(z,\delta) = cst$ where the analytic continuations are performed along $\widetilde{D}_1$. By definition, there must also exist a dissipative path for {13}, passing between $z_1(\delta)$ and $z_2(\delta)$ which goes from $-\infty$ to $+\infty$. Noting that $e_1$

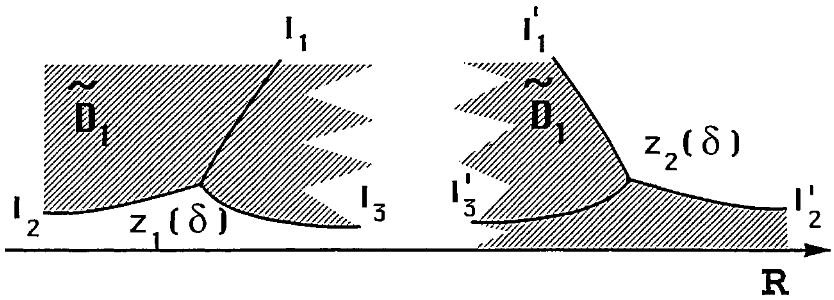


Fig. 6. The Stokes lines and the possible locus of the domain $\widetilde{D}_1$ (dashed).

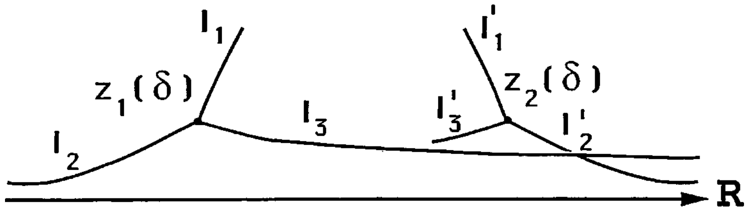


Fig. 7. Intersections of Stokes lines.

is changed to $e_2$ when passing above $z_1(\delta)$, and for the same reasons as above, this path crosses $l_3'$ and can cross neither $l_2'$, nor $l_1'$. Thus $\widetilde{D}_1$ must lie below $l_2'$ which becomes a boundary for $\widetilde{D}_1$. Hence we must have the situation described in Fig. 6.

However, we check below that $l_3$ and $l_2'$ have an intersection, which prevents $\widetilde{D}_1$ to link $-\infty$ to $+\infty$, see Fig. 7. Indeed, at $-\infty$ we have

$$\operatorname{Im}\Delta_{21}\big(z_1(\delta),\delta\big) = h_2(-\sqrt{2}+\alpha), \tag{7.6}$$

where $h_2$ is the height of $l_2$ above the real axis at $-\infty$, and, similarly at $+\infty$,

$$\operatorname{Im}\Delta_{21}\big(z_1(\delta),\delta\big) = h_3 2\sqrt{2} \tag{7.7}$$

with $h_3$ the height of $l_3$ above the real axis at $+\infty$. As

$$\frac{h_3}{h_2} = \frac{\alpha-\sqrt{2}}{2\sqrt{2}} > 1 \quad \text{since } \alpha > 3\sqrt{2}, \tag{7.8}$$

the branches $l_3$ and $l_2'$ must have an intersection by continuity due to the symmetry between $l_2'$ and $l_2$.

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